SOME COMBINATORIAL ARRAYS RELATED TO THE LOTKA-VOLTERRA SYSTEM

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Abstract. The purpose of this paper is to investigate the connection between the Lotka-Volterra system and combinatorics. We study several context-free grammars associated with the Lotka-Volterra system. Some combinatorial arrays, involving the Stirling numbers of the second kind and Eulerian numbers, are generated by these context-free grammars. In particular, we present grammatical characterization of some statistics on cyclically ordered partitions.

Keywords: Lotka-Volterra system; Context-free grammars; Eulerian numbers; Cyclically ordered partitions

1. Introduction

One of the most commonly used models of two species predator-prey interaction is the classical *Lotka-Volterra model*:

$$\frac{dx}{dt} = x(a - by), \frac{dy}{dt} = y(-c + dx), \tag{1}$$

where $y(t)$ and $x(t)$ represent, respectively, the predator population and the prey population as functions of time, and a, b, c, d are positive constants. In general, an n-th order *Lotka-Volterra system* takes the form

$$\frac{dx_i}{dt} = \lambda_i x_i + x_i \sum_{j=1}^{n} M_{i,j} x_j, i = 1, 2, \ldots, n, \tag{2}$$

where $\lambda_i, M_{i,j}$ are real real constants. The differential system (2) is ubiquitous and arises often in mathematical ecology, dynamical system theory and other branches of mathematics (see [2, 3, 8, 16, 18, 19]). In this paper, we study several context-free grammars associated with (2).

In his study [4] of exponential structures in combinatorics, Chen introduced the grammatical method systematically. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. Following Chen, a *context-free grammar* G over A is defined as a set of substitution rules that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G.

Let $[n] = \{1, 2, \ldots, n\}$. *The Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ is the number of ways to partition $[n]$ into k blocks. It is well known that $S(n, k) = S(n-1, k-1) + kS(n-1, k)$, $S(0, 0) = 1$ and $S(n, 0) = 0$ for $n \geq 1$ (see [20, A008277]). Let $\mathfrak{S}_n$ be the symmetric group of all permutations of $[n]$. A *descent* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i) > \pi(i+1)$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. The *Eulerian number* $\left\langle {n \atop k} \right\rangle$ is the number of permutations in $\mathfrak{S}_n$ with $k-1$ descents, where $1 \leq k \leq n$ (see [20, A008292]). Let us now recall two classical results on grammars.

Proposition 1 ([4, Eq. 4.8]). *If $G = \{x \to xy, y \to y\}$, then*

$$D^n(x) = x \sum_{k=1}^{n} {n \brace k} y^k \quad \textit{for } n \geq 1.$$

Proposition 2 ([7, Section 2.1]). *If $G = \{x \to xy, y \to xy\}$, then*

$$D^n(x) = \sum_{k=1}^{n} \left\langle {n \atop k} \right\rangle x^k y^{n-k+1} \quad \textit{for } n \geq 1.$$

This paper is a continuation of [4, 7]. Throughout this paper, arrays are indexed by n, i and j. Call $(a_{n,i,j})$ a combinatorial array if the numbers $a_{n,i,j}$ are nonnegative integers. For any function $H(x, p, q)$, we denote by H_y the partial derivative of H with respect to y, where $y \in \{x, p, q\}$. In the next section, we present grammatical characterization of some statistics on cyclically ordered partitions.

2. Relationship to cyclically ordered partitions

Recall that a partition π of $[n]$, written $\pi \vdash [n]$, is a collection of disjoint and nonempty subsets $B_1, B_2, \ldots, B_k$ of $[n]$ such that $\bigcup_{i=1}^{k} B_i = [n]$, where each B_i $(1 \leq i \leq k)$ is called a block of π. A *cyclically ordered partition* of $[n]$ is a partition of $[n]$ whose blocks are endowed with a cyclic order. We always use a canonical representation for cyclically ordered partitions: a list of blocks in which the first block contains the element 1 and each block is an increasing list. For example, (123), (12)(3), (13)(2), (1)(23), (1)(2)(3), (1)(3)(2) are all cyclically ordered partitions of [3]. The *opener* of a block is its least element. For example, the list of openers of (13)(2) and (1)(3)(2) are respectively given by 12 and 132. In the following, we shall study some statistics on the list of openers.

2.1. **Descent statistic.**

Consider the grammar

$$G = \{x \to x + xy, y \to y + xy\}. \tag{3}$$

From (3), we have

$$\begin{aligned} D(x) &= x + xy, \\ D^2(x) &= x + 3xy + xy^2 + x^2y, \\ D^3(x) &= x + 7xy + 6xy^2 + xy^3 + 6x^2y + 4x^2y^2 + x^3y. \end{aligned}$$

For $n \geq 0$, we define

$$D^n(x) = \sum_{i \geq 1, j \geq 0} a_{n,i,j} x^i y^j.$$

Since

$$\begin{aligned} D^{n+1}(x) &= D\left(\sum_{i,j} a_{n,i,j} x^i y^j\right) \\ &= \sum_{i,j} (i+j) a_{n,i,j} x^i y^j + \sum_{i,j} i a_{n,i,j} x^i y^{j+1} + \sum_{i,j} j a_{n,i,j} x^{i+1} y^j, \end{aligned}$$

we get

$$a_{n+1,i,j} = (i+j)a_{n,i,j} + ia_{n,i,j-1} + ja_{n,i-1,j} \tag{4}$$

for $i, j \geq 1$, with the initial conditions $a_{0,i,j}$ to be 1 if $(i,j) = (1,0)$, and to be 0 otherwise. Clearly, $a_{n,1,0} = 1$ and $a_{n,i,0} = 0$ for $i \geq 2$.

Define

$$A = A(x,p,q) = \sum_{n,i,j\geq 0} a_{n,i,j}\frac{x^n}{n!}p^i q^j.$$

We now present the first main result of this paper.

Theorem 3. *The generating function A is given by*

$$A = \frac{p(p-q)e^x}{p - qe^{(p-q)(e^x-1)}}.$$

Moreover, for all $n, i, j \geq 1$,

$$a_{n,i,j} = \left\{ {n+1 \atop i+j} \right\} \left\langle {i+j-1 \atop i} \right\rangle. \tag{5}$$

Proof. By rewriting (4) in terms of generating function A, we have

$$A_x = p(1+q)A_p + q(1+p)A_q. \tag{6}$$

It is routine to check that the generating function

$$\widetilde{A}(x,p,q) = \frac{p(p-q)e^x}{p - qe^{(p-q)(e^x-1)}}$$

satisfies (6). Also, this generating function gives $\widetilde{A}(0,p,q) = p$, $\widetilde{A}(x,p,0) = pe^x$ and $\widetilde{A}(x,0,q) = 0$ with $q \neq 0$. Hence, $A = \widetilde{A}$. Now let us prove that $a_{n,i,j} = \left\{ {n+1 \atop i+j} \right\} \left\langle {i+j-1 \atop i} \right\rangle$. Note that

$$\begin{aligned} \frac{d}{dx}\sum_{n,i,k\geq 0} a_{n,i,k+1-i}\frac{x^{n+1}}{(n+1)!}v^{i+1}w^k &= v\frac{d}{dx}\sum_{k\geq 0}\left(\sum_{n\geq k+1}\left\{ {n+1 \atop k+1} \right\}\frac{x^{n+1}}{(n+1)!}\sum_{i=0}^{k}\left\langle {k \atop i} \right\rangle v^i\right)w^k \\ &= v\frac{d}{dx}\sum_{k\geq 0}\left(\sum_{i=0}^{k}\left\langle {k \atop i} \right\rangle v^i\right)\frac{(e^x-1)^{k+1}}{(k+1)!}w^k. \end{aligned}$$

By using the fact that

$$\sum_{k\geq 0}\left(\sum_{i=0}^{k}\left\langle {k \atop i} \right\rangle p^i\right)u^k = \int_0^u \frac{p-1}{p-e^{u'(p-1)}}du' = \frac{1}{p}(u(p-1) - \ln(e^{u(p-1)} - p) + \ln(1-p)),$$

we obtain that

$$v\frac{d}{dx}\sum_{n,i,k\geq 0} a_{n,i,k+1-i}\frac{x^{n+1}}{(n+1)!}v^i w^k = \frac{wv(v-1)e^x}{v - e^{(e^x-1)w(v-1)}},$$

which implies

$$A(x,vw,w) = \frac{wv(v-1)e^x}{v - e^{(e^x-1)w(v-1)}},$$

as required. □

Define

$$a_n = \sum_{i\geq 1, j\geq 0} a_{n,i,j}.$$

Clearly, $a_n = \sum_{k=0}^{n} k!\left\{ {n+1 \atop k+1} \right\}$.

Proposition 4. *$\left\{ {n \atop k} \right\}\left\langle {k-1 \atop i} \right\rangle$ is the number of cyclically ordered partitions of $[n]$ with k blocks whose list of openers contains $i-1$ descents.*

Proof. To form such a cyclically ordered partition, start with a partition of $[n]$ into k blocks in canonical form, each block increasing and blocks arranged in order of increasing first entries (there are $\left\{ {n \atop k} \right\}$ choices). The first opener is thus 1. Then leave the first block in place and rearrange the $k-1$ remaining blocks so that their openers, viewed as a list, contain $i-1$ descents (there are $\left\langle {k-1 \atop i} \right\rangle$ choices).

We can now conclude the following corollary from the discussion above.

Corollary 5. *For all $n, i, j \geq 1$, $a_{n,i,j}$ is the number of cyclically ordered partitions of $[n+1]$ with $i+j$ blocks whose list of openers contains $i-1$ descents.*

2.2. **Peak statistics.**

The idea of a peak (resp. valley) in a list of integers $(w_i)_{i=1}^n$ is an entry that is greater (resp. smaller) than its neighbors. The number of peaks in a permutation is an important combinatorial statistic. See, e.g., [1, 6, 9, 12] and the references therein. However, the question of whether the first and/or last entry may qualify as a peak (or valley) gives rise to several different definitions. In this paper, we consider only left peaks and right valleys. A *left peak index* is an index $i \in [n-1]$ such that $w_{i-1} < w_i > w_{i+1}$, where we take $w_0 = 0$, and the entry w_i is a *left peak*. Similarly, a *right valley* is an entry w_i with $i \in [2, n]$ such that $w_{i-1} > w_i < w_{i+1}$, where we take $w_{n+1} = \infty$. Thus the last entry may be a right valley but not a left peak. For example, the list 64713258 has 3 left peaks and 3 right valleys. Clearly, left peaks and right valleys in a list are equinumerous: they alternate with a peak first and a valley last. Peaks and valleys were considered in [9]. The left peak statistic first appeared in [1, Definition 3.1].

Let $P(n,k)$ be the number of permutations in $\mathfrak{S}_n$ with k left peaks. Let $P_n(x) = \sum_{i\geq 0} P(n,k)x^k$. It is well known [20, A008971] that

$$\begin{aligned} P(x,z) &= 1 + \sum_{n\geq 1} P_n(x)\frac{z^n}{n!} \\ &= \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(z\sqrt{1-x}) - \sinh(z\sqrt{1-x})} \end{aligned}$$

Let D be the differential operator $\frac{d}{d\theta}$. Set $x = \sec\theta$ and $y = \tan\theta$. Then

$$D(x) = xy, D(y) = x^2. \tag{7}$$

There is a large literature devoted to the repeated differentiation of the secant and tangent functions (see [10, 11, 12] for instance). As a variation of (3), it is natural to consider the grammar

$$G = \{x \to x + xy, y \to y + x^2\}. \tag{8}$$

From (8), we have

$$\begin{aligned} D(x) &= x + xy, \\ D^2(x) &= x + 3xy + xy^2 + x^3, \\ D^3(x) &= x + 7xy + 6xy^2 + xy^3 + 6x^3 + 5x^3y. \end{aligned}$$

Define

$$D^n(x) = \sum_{i\geq 1, j\geq 0} b_{n,i,j} x^i y^j.$$

Since

$$\begin{aligned} D^{n+1}(x) &= D\left(\sum_{i\geq 1, j\geq 0} b_{n,i,j} x^i y^j\right) \\ &= \sum_{i,j}(i+j)b_{n,i,j}x^iy^j + \sum_{i,j} ib_{n,i,j}x^iy^{j+1} + \sum_{i,j} jb_{n,i,j}x^{i+2}y^{j-1}, \end{aligned}$$

we get

$$b_{n+1,i,j} = (i+j)b_{n,i,j} + ib_{n,i,j-1} + (j+1)b_{n,i-2,j+1} \tag{9}$$

for $i \geq 1$ and $j \geq 0$, with the initial conditions $b_{0,i,j}$ to be 1 if $(i,j) = (1,0)$, and to be 0 otherwise. Clearly, $b_{n,1,0} = 1$ for $n \geq 1$.

Define

$$B = B(x,p,q) = \sum_{n,i,j\geq 0} b_{n,i,j} p^i q^j \frac{x^n}{n!}.$$

We now present the second main result of this paper.

Theorem 6. *The generating function B is given by*

$$B(x,p,q) = \frac{p\sqrt{q^2-p^2}e^x}{\sqrt{q^2-p^2}\cosh(\sqrt{q^2-p^2}(e^x-1)) - q\sinh(\sqrt{q^2-p^2}(e^x-1))}.$$

Moreover, for all $n,i,j \geq 1$,

$$b_{n,2i-1,j} = \left\{ {n+1 \atop 2i-1+j} \right\} P(2i-2+j, i-1). \tag{10}$$

Proof. The recurrence (9) can be written as

$$B_x = p(1+q)B_p + (p^2+q)B_q. \tag{11}$$

It is routine to check that the generating function

$$\widetilde{B} = \widetilde{B}(x,p,q) = \frac{p\sqrt{q^2-p^2}e^x}{\sqrt{q^2-p^2}\cosh(\sqrt{q^2-p^2}(e^x-1)) - q\sinh(\sqrt{q^2-p^2}(e^x-1))}$$

satisfies (11)). Also, this generating function gives $\widetilde{B}(0,p,q) = p$ and $\widetilde{B}(x,0,q) = 0$. Hence, $B = \widetilde{B}$.

It follows from (9) that $b_{n,2i,j} = 0$ for all $(i,j) \neq (0,0)$. Now let us prove that

$$b_{n,2i-1,j} = \left\{ {n+1 \atop 2i-1+j} \right\} P(2i-2+j, i-1).$$

Note that

$$\sum_{n,i,j\geq 0} b_{n,i,j+1-2i}p^iq^j\frac{x^n}{n!} = \sum_{n\geq 0,i,j\geq 1} b_{n,2i-1,j+1-2i}p^iq^j\frac{x^n}{n!} = p\sum_{n\geq 0,j\geq 1}\left\{ {n+1 \atop j} \right\} P_{j-1}(p)q^j\frac{x^n}{n!}$$

$$= pe^x\sum_{j\geq 1}\frac{(e^x-1)^{j-1}}{(j-1)!}P_{j-1}(p)q^j = pqe^xP(p,q(e^x-1)),$$

Hence,

$$\sum_{n,i,j\geq 0} b_{n,i,j}p^iq^j\frac{x^n}{n!} = pe^xP(p^2/q^2,q(e^x-1)) = B(x,p,q),$$

as required. □

Let $b_n = \sum_{i\geq 1,j\geq 0} b_{n,i,j}$. It follows from (20) that $b_n = a_n$. In the following discussion, we shall present a combinatorial interpretation for $b_{n,i,j}$.

Lemma 7. *Suppose that $(w_i)_{i=1}^k$ is a list of distinct integers containing ℓ right valleys and that $w_1 = 1$. Then, among the k ways to insert a new entry $m > \max(w_i)$ into the list in a noninitial position, $2\ell+1$ of them will not change the number of right valleys and $k-(2\ell+1)$ will increase it by 1.*

Proof. As observed above, peaks and valleys alternate, a peak occurring first, and a valley occurring last. Thus there are ℓ peaks. If m is inserted immediately before or after a peak or at the very end, the number of valleys is unchanged, otherwise it is increased by 1.

Proposition 8. *The number $u_{n,k,\ell}$ of cyclically ordered partitions on $[n]$ with k blocks and ℓ right valleys in the list of openers satisfies the recurrence*

$$u_{n,k,\ell} = ku_{n-1,k,\ell} + (2\ell+1)u_{n-1,k-1,\ell} + (k-2\ell)u_{n-1,k-1,\ell-1} \tag{12}$$

for $n\geq 2,\ \ell\geq 0,\ 2\ell+1\leq k\leq n$.

Proof. Each cyclically ordered partition of size n is obtained by inserting n into one of size $n-1$, either as the last entry in an existing block or as a new singleton block. Let $\mathcal{U}_{n,k,\ell}$ denote the set of cyclically ordered partitions counted by $u_{n,k,\ell}$. To obtain an element of $\mathcal{U}_{n,k,\ell}$ we can insert n into any existing block of an element of $\mathcal{U}_{n-1,k,\ell}$ (this gives $ku_{n-1,k,\ell}$ choices), or insert n as a singleton block into an element of $\mathcal{U}_{n-1,k-1,\ell}$ so that the number of right valleys is unchanged (this gives $(2\ell+1)u_{n-1,k-1,\ell}$ choices), or insert n as a singleton block into an element of $\mathcal{U}_{n-1,k-1,\ell-1}$ so that the number of right valleys is increased by 1 (this gives $(k-2\ell)u_{n-1,k-1,\ell-1}$ choices). The last two counts of choices follow from Lemma 7.

Corollary 9. *For all $n,i,j\geq 1$, $b_{n,i,j}$ is the number of cyclically ordered partitions on $[n+1]$ with $i+j$ blocks and $\frac{i-1}{2}$ right valleys (equivalently, $\frac{i-1}{2}$ left peaks) in the list of openers.*

Proof. Comparing recurrence relations (9) and (12), we see that $b_{n,i,j} = u_{n+1,i+j,(i-1)/2}$.

Remark 10. *A cyclically ordered partition of size n with k blocks and ℓ right valleys in the list of openers is obtained by selecting a partition of $[n]$ with k blocks in $\left\{ {n \atop k} \right\}$ ways, and then arranging the blocks suitably, in $P(k,\ell)$ ways. Hence $u_{n,k,\ell} = \left\{ {n \atop k} \right\}P(k,\ell)$ and we get a combinatorial proof that $c_{n,2i-1,j} = \left\{ {n+1 \atop 2i-1+j} \right\}P(2i-2+j,i-1)$.*

2.3. The longest alternating subsequences.

Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. An *alternating subsequence* of π is a subsequence $\pi(i_1)\cdots\pi(i_k)$ satisfying

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k).$$

The study of the distribution of the length of the longest alternating subsequences of permutations was recently initiated by Stanley [21, 22].

Denote by $\mathrm{as}\,(\pi)$ the length of the longest alternating subsequence of π. Let

$$a_k(n) = \#\{\pi \in \mathfrak{S}_n : \mathrm{as}\,(\pi) = k\},$$

and let $L_n(x) = \sum_{k=1}^{n} a_k(n)x^k$. Define

$$L(x,z) = \sum_{n\geq 0} L_n(x)\frac{z^n}{n!}.$$

Stanley [22, Theorem 2.3] obtained the following closed-form formula:

$$L(x,z) = (1-x)\frac{1+\rho+2xe^{\rho z}+(1-\rho)e^{2\rho z}}{1+\rho-x^2+(1-\rho-x^2)e^{2\rho z}},$$

where $\rho = \sqrt{1-x^2}$. Moreover, it follows from [15, Corollary 8] that

$$L(x,z) = -\sqrt{\frac{x-1}{x+1}}\left(\frac{\sqrt{x^2-1}+x\sin(z\sqrt{x^2-1})}{1-x\cos(z\sqrt{x^2-1})}\right). \tag{13}$$

As an extension of (8), it is natural to consider the grammar

$$G = \{w \to w+wx, x \to x+xy, y \to y+x^2\}. \tag{14}$$

From (14), we have

$$\begin{aligned}
D(w) &= w(1+x),\\
D^2(w) &= w(1+3x+xy+x^2);\\
D^3(w) &= w(1+7x+6xy+xy^2+6x^2+3x^2y+2x^3).
\end{aligned}$$

Define

$$D^n(w) = w\sum_{i,j\geq 0} t_{n,i,j}x^iy^j.$$

Since

$$\begin{aligned}
D^{n+1}(w) &= D\left(w\sum_{i,j\geq 0} t_{n,i,j}x^iy^j\right)\\
&= \sum_{i,j}(1+i+j)t_{n,i,j}x^iy^j + \sum_{i,j} t_{n,i,j}x^{i+1}y^j + \sum_{i,j} it_{n,i,j}x^iy^{j+1} + \sum_{i,j} jt_{n,i,j}x^{i+2}y^{j-1}.
\end{aligned}$$

we get

$$t_{n+1,i,j} = (1+i+j)t_{n,i,j} + t_{n,i-1,j} + ib_{n,i,j-1} + (j+1)b_{n,i-2,j+1} \tag{15}$$

for $i,j \geq 0$, with the initial conditions $t_{0,i,j}$ to be 1 if $(i,j) = (0,0)$ or $(i,j) = (1,0)$, and to be 0 otherwise. Clearly, $t_{n,0,0} = 1$ for $n \geq 0$.

Define

$$T = T(x,p,q) = \sum_{n,i,j\geq 0} t_{n,i,j} p^i q^j \frac{x^n}{n!}.$$

We now present the third main result of this paper.

Theorem 11. *The generating function T is given by*

$$T(x,p,q) = e^x \sqrt{\frac{p-q}{p+q}} \frac{\sqrt{p^2-q^2} + p\sin((e^x-1)\sqrt{p^2-q^2})}{p\cos((e^x-1)\sqrt{p^2-q^2})-q}.$$

Moreover, for all $n \geq 1, i \geq 1$ and $j \geq 0$,

$$t_{n,i,j} = \left\{ {n+1 \atop i+j+1} \right\} a_i(i+j). \tag{16}$$

Proof. The recurrence (15) can be written as

$$T_x = T + p(1+q)T_p + (p^2+q)T_q. \tag{17}$$

It is routine to check that the generating function

$$\widetilde{T} = \widetilde{T}(x,p,q) = e^x \sqrt{\frac{p-q}{p+q}} \frac{\sqrt{p^2-q^2} + p\sin((e^x-1)\sqrt{p^2-q^2})}{p\cos((e^x-1)\sqrt{p^2-q^2})-q}$$

satisfies (17)). Also, this generating function gives $\widetilde{T}(0,p,q) = 1$ and $\widetilde{T}(x,0,q) = e^x$. Hence, $T = \widetilde{T}$.

Now let us prove that $t_{n,2i-1,j} = \left\{ {n+1 \atop i+j+1} \right\} a_i(i+j)$. Note that

$$\begin{aligned} \sum_{n,i,j\geq 0} t_{n,i,j-i} p^i q^j \frac{x^n}{n!} &= \sum_{n,i,j\geq 0} t_{n,i,j-i} p^i q^j \frac{x^n}{n!} = \sum_{n,j\geq 0} \left\{ {n+1 \atop j+1} \right\} L_j(p) q^j \frac{x^n}{n!} \\ &= e^x \sum_{j\geq 0} \frac{(e^x-1)^j}{(j)!} L_j(p) q^j = e^x L(p, q(e^x-1)), \end{aligned}$$

Hence,

$$\sum_{n,i,j\geq 0} t_{n,i,j} p^i q^j \frac{x^n}{n!} = e^x L(p/q, q(e^x-1)) = T(x,p,q),$$

as required. □

Let $t_n = \sum_{i\geq 1, j\geq 0} t_{n,i,j}$. It follows from (16) that $t_n = a_n$. Along the same lines as the proof of Corollary 5, we get the following.

Corollary 12. *For all $n \geq 1, i \geq 1$ and $j \geq 0$, $t_{n,i,j}$ is the number of cyclically ordered partitions of $[n+1]$ with $i+j+1$ blocks and the length of the longest alternating subsequence of the list of openers equals i.*

3. A product of the Stirling numbers of the second kind and binomial coefficients

Consider the grammar

$$G = \{x \to x + x^2 + xy, y \to y + y^2 + xy\}. \tag{18}$$

From (18), we have

$$\begin{aligned} D(x) &= x + xy + x^2, \\ D^2(x) &= x + 3xy + 2xy^2 + 3x^2 + 4x^2y + 2x^3, \\ D^3(x) &= x + 7xy + 12xy^2 + 6xy^3 + 7x^2 + 24x^2y + 18x^2y^2 + 12x^3 + 18x^3y + 6x^4. \end{aligned}$$

For $n \geq 0$, we define

$$D^n(x) = \sum_{i \geq 1, j \geq 0} c_{n,i,j} x^i y^j.$$

Since

$$\begin{aligned} D^{n+1}(x) &= D\left(\sum_{i,j} c_{n,i,j} x^i y^j\right) \\ &= \sum_{i,j} (i+j) c_{n,i,j} x^i y^j + \sum_{i,j} (i+j) c_{n,i,j} x^i y^{j+1} + \sum_{i,j} (i+j) c_{n,i,j} x^{i+1} y^j, \end{aligned}$$

we get

$$c_{n+1,i,j} = (i+j)c_{n,i,j} + (i+j-1)c_{n,i,j-1} + (i+j-1)c_{n,i-1,j} \tag{19}$$

for $i \geq 1$ and $j \geq 0$, with the initial conditions $c_{0,i,j}$ to be 1 if $(i,j) = (1,0)$, and to be 0 otherwise. Clearly, $c_{n,1,0} = 1$ for $n \geq 1$.

Define

$$C = C(x,p,q) = \sum_{n,i,j \geq 0} c_{n,i,j} \frac{x^n}{n!} p^i q^j.$$

We now present the fourth main result of this paper.

Theorem 13. *The generating function C is given by*

$$C(x,p,q) = \frac{pe^x}{1-(p+q)(e^x-1)}.$$

Moreover, for all $n,i,j \geq 1$,

$$c_{n,i,j} = (i+j-1)! \left\{ {n+1 \atop i+j} \right\} \binom{i+j-1}{j}. \tag{20}$$

Proof. The recurrence (19) can be written as

$$C_x = p(1+p+q)C_p + q(1+p+q)C_q. \tag{21}$$

It is routine to check that the generating function

$$\widetilde{C}(x,p,q) = \frac{pe^x}{1-(p+q)(e^x-1)}$$

satisfies (21). Also, this generating function gives $\widetilde{C}(0,p,q)=p$ and $\widetilde{C}(x,0,q)=0$. Hence, $C=\widetilde{C}$. Now let us prove that $c_{n,i,j}=(i+j-1)!\left\{{n+1 \atop i+j}\right\}\binom{i+j-1}{j}$. Note that

$$\begin{aligned}\frac{d}{dx}\sum_{n,i,k\geq 0} c_{n,i,k+1-i}\frac{x^{n+1}}{(n+1)!}v^{i+1}w^k &= v\frac{d}{dx}\sum_{k\geq 0}\left(\sum_{n\geq k+1}\left\{{n+1 \atop k+1}\right\}\frac{x^{n+1}}{(n+1)!}\sum_{i=0}^{k}\binom{k}{i}v^i\right)k!w^k\\ &= v\frac{d}{dx}\sum_{k\geq 0}\frac{(e^x-1)^{k+1}}{k+1}(1+v)^k w^k\\ &= \frac{ve^x}{1-w(1+v)(e^x-1)},\end{aligned}$$

which implies

$$C(x,vw,w)=\frac{vwe^x}{1-w(1+v)(e^x-1)},$$

as required. □

Let $c_n=\sum_{i\geq 1,j\geq 0}c_{n,i,j}$. It follows from (20) that $c_n=\sum_{k=0}^{n}2^k k!\left\{{n+1 \atop k+1}\right\}$.

4. Descent statistic of hyperoctahedral group and perfect matchings

Let us first recall some definitions. The *Whitney numbers of the second kind* $W_m(n,k)$ can be explicitly defined by

$$W_m(n,k)=\sum_{i=k}^{n}\binom{n}{i}m^{i-k}\left\{{i \atop k}\right\}.$$

They satisfy the recurrence

$$W_m(n,k)=W_m(n-1,k-1)+(1+mk)W_m(n-1,k),$$

with initial conditions $W_m(0,0)=1$ and $W_m(n,0)=0$ for $n\geq 1$ (see [5]). In particular, $W_2(n,k)$ also known as the type B analogue of Stirling numbers of the second kind (see [20, A039755]).

The *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i)=-\pi(i)$ for all i, where $\pm[n]=\{\pm 1,\pm 2,\ldots,\pm n\}$. For each $\pi\in B_n$, we define

$$\operatorname{des}_B(\pi):=\#\{i\in\{0,1,2,\ldots,n-1\}|\pi(i)>\pi(i+1)\},$$

where $\pi(0)=0$.

Let

$$B_n(x)=\sum_{\pi\in B_n}x^{\operatorname{des}_B(\pi)}=\sum_{k=0}^{n}B(n,k)x^k.$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type* B, while $B(n,k)$ is called an *Eulerian number of type* B (see [20, A060187]). The numbers $B(n,k)$ satisfy the recurrence relation

$$B(n+1,k)=(2k+1)B(n,k)+(2n-2k+3)B(n,k-1)$$

for $n,k\geq 0$, where $B(0,0)=1$ and $B(0,k)=0$ for $k\geq 1$. The first few of the polynomials $B_n(x)$ are

$$B_1(x)=1+x,$$
$$B_2(x)=1+6x+x^2,$$
$$B_3(x)=1+23x+23x^2+x^3.$$

Recall that a *perfect matching* of $[2n]$ is a partition of $[2n]$ into n blocks of size 2. Denote by $N(n,k)$ the number of perfect matchings of $[2n]$ with the restriction that only k matching pairs have odd smaller entries (see [20, A185411]). It is easy to verify that

$$N(n+1,k)=2kN(n,k)+(2n-2k+3)N(n,k-1). \tag{22}$$

Let $N_n(x)=\sum_{k=1}^{n}N(n,k)x^k$. It follows from (22) that

$$N_{n+1}(x)=(2n+1)xN_n(x)+2x(1-x)\frac{d}{dx}N_n(x),$$

with initial values $N_0(x)=1$, $N_1(x)=x$ and $N_2(x)=2x+x^2$. Let

$$N(x,t)=\sum_{n\geq 0}N_n(x)\frac{t^n}{n!}.$$

It is well known that

$$N(x,t)=e^{xt}\sqrt{\frac{1-x}{e^{2xt}-xe^{2t}}}. \tag{23}$$

There is a grammatical characterization of the numbers $N(n,k)$ and $B(n,k)$: if

$$G=\{x\rightarrow xy^2,y\rightarrow x^2y\}, \tag{24}$$

then

$$D^n(x)=\sum_{k=0}^{n}N(n,k)x^{2n-2k+1}y^{2k},\ D^n(xy)=\sum_{k=0}^{n}B(n,k)x^{2n-2k+1}y^{2k+1},$$

which was obtained in [14, Theorem 10].

As an extension of (24), it is natural to consider the grammar

$$G=\{x\rightarrow x+xy^2,y\rightarrow y+x^2y\}. \tag{25}$$

From (25), we have

$$D(x)=x+xy^2,$$
$$D^2(x)=x+4xy^2+xy^4+2x^3y^2;$$
$$D(xy)=2xy+xy^3+x^3y,$$
$$D^2(xy)=4xy+6xy^3+xy^5+6x^3y+6x^3y^3+x^5y.$$

Define

$$D^n(x)=\sum_{i,j\geq 0}e_{n,i,j}x^{2i+1}y^{2j},$$
$$D^n(xy)=\sum_{i,j\geq 0}f_{n,i,j}x^{2i+1}y^{2j+1},$$

Since

$$D^{n+1}(x) = D\left(\sum_{i,j\geq 0} e_{n,i,j}x^{2i+1}y^{2j}\right)$$
$$= \sum_{i,j}(2i+2j+1)e_{n,i,j}x^{2i+1}y^{2j} + \sum_{i,j}(2i+1)e_{n,i,j}x^{2i+1}y^{2j+2} + \sum_{i,j} 2je_{n,i,j}x^{2i+3}y^{2j},$$

we get

$$e_{n+1,i,j} = (2i+2j+1)e_{n,i,j} + (2i+1)e_{n,i,j-1} + 2je_{n,i-1,j} \tag{26}$$

for $i, j \geq 0$, with the initial conditions $e_{0,0,0} = 1$ and $e_{0,i,j} = 0$ if $(i, j) \neq (0, 0)$.

Similarly, since

$$D^{n+1}(xy) = D\left(\sum_{i,j\geq 0} f_{n,i,j}x^{2i+1}y^{2j+1}\right)$$
$$= \sum_{i,j}(2i+2j+2)f_{n,i,j}x^{2i+1}y^{2j+1} + \sum_{i,j}(2i+1)f_{n,i,j}x^{2i+1}y^{2j+3}+$$
$$\sum_{i,j}(2j+1)f_{n,i,j}x^{2i+3}y^{2j+1},$$

we get

$$f_{n+1,i,j} = (2i+2j+2)f_{n,i,j} + (2i+1)f_{n,i,j-1} + (2j+1)f_{n,i-1,j} \tag{27}$$

for $i, j \geq 0$, with the initial conditions $f_{0,0,0} = 1$ and $f_{0,i,j} = 0$ if $(i, j) \neq (0, 0)$.

Define

$$E = E(x, p, q) = \sum_{n,i,j\geq 0} e_{n,i,j}p^iq^j\frac{x^n}{n!},$$

$$F = F(x, p, q) = \sum_{n,i,j\geq 0} f_{n,i,j}p^iq^j\frac{x^n}{n!}.$$

We now present the fifth main result of this paper.

Theorem 14. *The generating functions E and F are respectively given by*

$$E(x, p, q) = e^{x+q(e^{2x}-1)/2}\sqrt{\frac{p-q}{pe^{q(e^{2x}-1)} - qe^{p(e^{2x}-1)}}}$$

and

$$F(x, p, q) = \frac{(p-q)e^{(p-q)(e^{2x}-1)/2+2x}}{p-qe^{(p-q)(e^{2x}-1)}}.$$

Moreover, for all $n, i, j \geq 1$,

$$e_{n,i,j} = W_2(n, i+j)N(i+j, j),$$

$$f_{n,i,j} = 2^{n-i-j}\left\{\begin{matrix} n+1 \\ i+j+1 \end{matrix}\right\}B(i+j, j).$$

Proof. By rewriting (26) and (27) in terms of generating functions, we obtain

$$E_x = 2p(1+q)E_p + 2q(1+p)E_q + (1+q)E, \tag{28}$$

and

$$F_x = 2p(1+q)F_p + 2q(1+p)F_q + (2+p+q)F. \tag{29}$$

It is routine to check that the generating functions

$$\widetilde{E} = \widetilde{E}(x,p,q) = e^{x+q(e^{2x}-1)/2}\sqrt{\frac{p-q}{pe^{q(e^{2x}-1)} - qe^{p(e^{2x}-1)}}}$$

and

$$\widetilde{F} = \widetilde{F}(x,p,q) = \frac{(p-q)e^{(p-q)(e^{2x}-1)/2+2x}}{p-qe^{(p-q)(e^{2x}-1)}}$$

satisfies (28) and (29), respectively. Also, this generating functions give $\widetilde{F}(0,p,q) = \widetilde{E}(0,p,q) = 1$, $\widetilde{E}(x,0,q) = \widetilde{F}(x,0,q) = e^{q(2^{2x}-1)/2+2x}$, $\widetilde{E}(x,p,0) = e^x$ and $\widetilde{F}(x,p,0) = e^{p(2^{2x}-1)/2+2x}$. Hence, $E = \widetilde{E}$ and $F = \widetilde{F}$.

Now let us prove that the generating function for the sequences $e_{n,i-j,j} = W_2(n,i)N(i,j)$ and $f_{n,i-j,j} = 2^{n-i}\left\{ {n+1 \atop i+1} \right\} B(i,j)$ are given by $E(x,p,pq)$ and $F(x,p,pq)$. By (23) and [20, A039755], we have

$$\begin{aligned}
\sum_{n,i,k\geq 0} e_{n,i-j,j}\frac{x^n}{n!}p^iq^j &= \sum_{i\geq 0}\left(\sum_{n\geq i} W_2(n,i)\frac{x^n}{n!}\right)N_i(q) \\
&= e^x\sum_{i\geq 0}\frac{p^i(e^x-1)^i}{2^i i!}N_i(q) \\
&= E(x,p,pq),
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dx}\sum_{n,i,k\geq 0} f_{n,i-j,j}\frac{x^{n+1}}{(n+1)!}p^iq^j &= \frac{d}{dx}\sum_{i\geq 0}\left(\sum_{n\geq i}\left\{ {n+1 \atop i+1} \right\}\frac{(2x)^{n+1}p^i}{2^{i+1}(n+1)!}\right)B_i(q) \\
&= e^{2x}\sum_{i\geq 0}\frac{p^i(e^x-1)^i}{2^i i!}B_i(q) \\
&= F(x,p,pq),
\end{aligned}$$

which completes the proof. □

5. Concluding remarks

In this paper, we explore some combinatorial structures associated with (2). In fact, there are many other extension of (1). For example, many authors investigated the following generalized Lotka-Volterra system (see [17]):

$$\frac{dx}{dt} = x(Cy+z), \frac{dy}{dt} = y(Az+x), \frac{dz}{dt} = z(Bx+y).$$

Consider the grammar

$$G = \{x \rightarrow x(y+z), y \rightarrow y(z+x), z \rightarrow z(x+y)\}.$$

Define

$$D^n(x) = \sum_{i \geq 1, j \geq 0} g_{n,i,j} x^i y^j z^{n+1-i-j}.$$

By induction, one can easily verify the the following: for all $n \geq 1, i \geq 1$ and $j \geq 0$, we have

$$g_{n,i,0} = \left\langle {n \atop i} \right\rangle, g_{n,i,n+1-i} = \left\langle {n \atop i} \right\rangle, g_{n,1,j} = \left\langle {n+1 \atop j+1} \right\rangle.$$

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