ON COMPOSITIONS WITH $x^2/(1-x)$

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Abstract. In the past, empirical evidence has been presented that Hilbert series of symplectic quotients of unitary representations obey a certain universal system of infinitely many constraints. Formal series with this property have been called *symplectic*. Here we show that a formal power series is symplectic if and only if it is a formal composite with the formal power series $x^2/(1-x)$. Hence the set of symplectic power series forms a subalgebra of the algebra of formal power series. The subalgebra property is translated into an identity for the coefficients of the even Euler polynomials, which can be interpreted as a cubic identity for the Bernoulli numbers. Furthermore we show that a rational power series is symplectic if and only if it is invariant under the idempotent Möbius transformation $x \mapsto x/(x-1)$. It follows that the Hilbert series of a graded Cohen-Macaulay algebra A is symplectic if and only if A is Gorenstein with its a-invariant and its Krull dimension adding up to zero. It is shown that this is the case for algebras of regular functions on symplectic quotients of unitary representations of tori.

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1. Introduction

Let $G \to \mathrm{U}(V)$ be a unitary representation of a compact Lie group G on a hermitian vector space $(V, \langle\ ,\ \rangle)$. Here, V is viewed as a symplectic manifold or real variety. The $\mathbb{R}$-algebra of smooth functions on V is denoted $\mathcal{C}^\infty(V)$, and its subalgebra of real regular functions is denoted $\mathbb{R}[V]$. Note that $\mathbb{R}[V]$ is actually a Poisson subalgebra of $\mathcal{C}^\infty(V)$. The symplectic form on V is given by the imaginary

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part of the scalar product $\langle\, ,\, \rangle$, and the G-action on V is Hamiltonian with moment map

$$J : V \to \mathfrak{g}^*, \quad J_\xi(v) := (J(v), \xi) := \frac{\sqrt{-1}}{2}\langle v, \xi.v\rangle.$$

Here, $\xi.v := d/dt_{t=0}\,(\exp(-t\xi).v)$ denotes the infinitesimal action of $\xi \in \mathfrak{g}$ on $v \in V$ and $(\, ,\,)$ stands for the dual pairing between the dual space $\mathfrak{g}^*$ and the Lie algebra $\mathfrak{g}$ of G.

Let us denote by $Z := J^{-1}(0)$ the zero fibre of the moment map. If G is finite, then $J = 0$ by convention and $Z = V$. Since J is G-equivariant, we can consider the space $M_0 := Z/G$ of G-orbits in Z, the so-called *symplectic quotient*. It is a stratified symplectic space and can be viewed in a natural way as a semialgebraic set (for more information the reader may consult [10]). In order to define the smooth structure on M_0, one introduces the vanishing ideal I_Z of Z inside $\mathcal{C}^\infty(V)$. Then the algebra of smooth functions on M_0 is given by $\mathcal{C}^\infty(M_0) := \mathcal{C}^\infty(V)^G/(I_Z \cap \mathcal{C}^\infty(V)^G)$. Note that $\mathcal{C}^\infty(M_0)$ carries a canonical Poisson bracket. The $\mathbb{N}$-graded $\mathbb{R}$-algebra of regular functions $\mathbb{R}[M_0] := \mathbb{R}[V]^G/(I_Z \cap \mathbb{R}[V]^G)$ is a Poisson subalgebra of $\mathcal{C}^\infty(M_0)$.

In this paper, we are concerned with the Hilbert series of the $\mathbb{N}$-graded algebra $\mathbb{R}[M_0]$. This is the generating function counting the dimensions $\dim_{\mathbb{R}}(\mathbb{R}[M_0]_i)$ of the spaces of regular functions of degree $i \in \mathbb{N}$:

$$\mathrm{Hilb}_{\mathbb{R}[M_0]}(t) := \sum_{i\geq 0} \dim_{\mathbb{R}}(\mathbb{R}[M_0]_i)\, t^i \in \mathbb{N}[\![t]\!] \subset \mathbb{C}[\![t]\!].$$

The Poisson brackets will play no role in the considerations to follow.

The main motivation for our investigation is Conjecture 1.2 below, that has been formulated in [8]. We recall the following definition from [8].

Definition 1.1. For a formal power series $\varphi(x) = \sum_{i\geq 0} \gamma_i\, x^i \in \mathbb{C}[\![x]\!]$ and $m \geq 1$ we introduce the linear constraint

$$(\mathrm{S}_m) \qquad \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} \gamma_{m+k} = 0.$$

We say that $\varphi(x)$ is *symplectic* if condition (S_m) holds for each $m \geq 1$. A meromorphic function $\psi(t)$ in the variable t is said to be *symplectic at* $a \in \mathbb{C}$ *of pole order* $d \in \mathbb{Z}$ if the formal power series $x^d\psi(a - x) \in \mathbb{C}[\![x]\!]$ is symplectic. Here we assume that the order of the pole of $\psi(t)$ at $t = a$ is $\leq d$.

The reader is invited to check that in a symplectic power series $\varphi(x) = \sum_{i\geq 0} \gamma_i x^i \in \mathbb{C}[\![x]\!]$ the odd coefficients $\gamma_1, \gamma_3, \gamma_5, \ldots$ are uniquely determined by the even ones $\gamma_0, \gamma_2, \gamma_4, \ldots$. Moreover, for each choice of the even coefficients $\gamma_0, \gamma_2, \gamma_4, \ldots$ there is a uniquely determined symplectic power series $\varphi(x) = \sum_{i\geq 0} \gamma_i\, x^i$.

The curious sign convention (we expand in powers of $(a - x)$ instead of $(x - a)$) appears to be more natural, because in this way our typical examples render non-negative coefficients. When we say a meromorphic function is symplectic at $x = a$ of order $d = 0$, we mean that it is analytic at $x = a$ and symplectic as a series expanded in $(a - x)$. Note that we only use this sign convention for a formal power series in the context of Lemma 3.1.

Conjecture 1.2 ([8]). *Let $G \to U(V)$ be a unitary representation of a compact Lie group G and let $\mathbb{R}[M_0]$ be the graded $\mathbb{R}$-algebra of regular functions on the* *corresponding symplectic quotient M_0. Then* $\mathrm{Hilb}_{\mathbb{R}[M_0]}(t)$ *is symplectic at $t = 1$ of order $d = \dim_{\mathbb{R}}(M_0)$.*

There is an analogue of this conjecture for *cotangent lifted representations* of reductive complex Lie groups. Certainly, over the complex numbers, there exist also symplectic quotients that arise from non-cotangent lifted representations whose Hilbert series are symplectic. For instance, the invariant ring of any unimodular representation of a finite group has a symplectic Hilbert series; for more details, see Section 6. To name a specific example, for $n \geq 2$ the action of the binary dihedral group $\mathbb{D}_n \subset \mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ cannot be cotangent lifted as there are no quadratic invariants.

Our aim is to give a simple proof of the following statement.

Theorem 1.3. *Conjecture 1.2 holds if G is a torus.*

The crucial insight that helps us to achieve our goal is the following reformulation of what it means for a generating function to be symplectic.

Proposition 1.4. *A formal power series $\varphi(x)$ is symplectic if and only if it is a formal composite with $x^2/(1-x)$, i.e., if there exists a formal power series $\rho(y) \in \mathbb{C}[\![y]\!]$ such that $\varphi(x) = \rho(x^2/(1-x))$.*

As a corollary, we obtain the following.

Corollary 1.5. *The space of symplectic power series forms a subalgebra of $\mathbb{C}[\![x]\!]$. A meromorphic function $\psi(t)$ is symplectic of order d at $a \in \mathbb{C}$ if and only if there exists a formal power series $\rho(y) \in \mathbb{C}[\![y]\!]$ such that the Laurent expansion of $\psi(t)$ at $t = a$ is*

$$\frac{1}{(a-t)^d}\,\rho\left(\frac{(a-t)^2}{1-a+t}\right).$$

If $\psi_1(t)$ is symplectic at $a \in \mathbb{C}$ of order d_1 and $\psi_2(t)$ is symplectic at $a \in \mathbb{C}$ of order d_2, then the product $\psi_1(t)\psi_2(t)$ is symplectic at $a \in \mathbb{C}$ of order $d_1 + d_2$.

It is tempting to think of $x^2/(1-x)$ as some sort of fundamental (rational or formal) invariant of a group action. In fact, the requisite transformation is provided by the order two Möbius transformation $x \mapsto x/(x-1)$.

Theorem 1.6. *A formal power series $\varphi(x)$ is symplectic if and only if it is invariant under the substitution $x \mapsto x/(x-1)$. If $\varphi(x)$ is rational, then the following statements are equivalent:*

(1) $\varphi(x)$ is symplectic,
(2) there exists a rational function $\rho(y)$ such that $\varphi(x) = \rho(x^2/(1-x))$,
(3) $\varphi(x) = \varphi(x/(x-1))$.

Corollary 1.7. *A rational function $\psi(t)$ is symplectic of order d at $t = a$ if and only if*

$$\psi\left(\frac{a^2-2a+(1-a)t}{a-1-t}\right) = (a-1-t)^d\,\psi(t). \tag{1.1}$$

This type of functional equation one encounters in the theory of *Gorenstein algebras* (cf. [2, Section 4.4]). Namely, by a theorem of Richard P. Stanley [12], an $\mathbb{N}$-graded Cohen-Macaulay algebra $R = \oplus_{i\geq 0} R_i$ is Gorenstein if and only if its Hilbert series $\mathrm{Hilb}_R(t) = \sum_{i\geq 0} \dim(R_i)\, t^i$ fulfills

$$\mathrm{Hilb}_R(t^{-1}) = (-1)^d t^{-a(R)}\,\mathrm{Hilb}_R(t), \tag{1.2}$$

where $d = \dim(R)$ and $a(R)$ is the so-called a-invariant. By comparison with (1.1) for $a = 1$ we finally obtain the following result.

Corollary 1.8. *The Hilbert series* $\mathrm{Hilb}_R(t)$ *of a graded Cohen-Macaulay algebra* R *is symplectic of order* $d = \dim R$ *if and only if* R *is Gorenstein with a-invariant* $a(R) = -d$.

Remark 1.9. In particular, this implies that if the graded ring $R = \oplus_{i\geq 0} R_i$ is Gorenstein of Krull dimension d and in the Laurent expansion

$$\mathrm{Hilb}_R(t) = \sum_{i\geq 0} \frac{\gamma_i}{(1-t)^{d-i}}, \tag{1.3}$$

the coefficient $\gamma_1 = 0$, then $\mathrm{Hilb}_R(t)$ is symplectic of order d. Here we make use of the fact [15, Equation (3.32)] that $-2\gamma_1/\gamma_0 = a(R) + d$.

Let us give an outline of the paper. In Section 2 we prove Proposition 1.4 and, as a side remark, discuss relations to the sequence of Genocchi numbers. In Section 3 we use Proposition 1.4 to show Theorem 1.6. The latter is used in Section 4 to give a proof of our main result, Theorem 1.3, that is based on Molien's formula and the fact that a moment map of a faithful torus representation forms a regular sequence in the ring of invariants [7, 5]. In Section 5 we deduce from Corollary 1.5 an identity for the coefficients of the even Euler polynomials. In Section 6 we illustrate our results by discussing specific examples.

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2. Proof of Proposition 1.4

As, for each $i \geq 1$, the alternating sum over the ith row of the Pascal triangle is zero, the power series $x^2/(1-x)$ is symplectic. Based on this observation we are able to find more examples.

Lemma 2.1. *For each* $n \geq 0$ *the formal power series* $(x^2/(1-x))^n$ *is symplectic.*

Proof. First let us observe that for a formal power series $\varphi(x)$ we have

$$(\mathrm{S}_m) \quad \Longleftrightarrow \quad \left.\frac{d^{2m-1}}{d^{2m-1}x}\right|_{x=0} \Big((1-x)^{m-1}\varphi(x)\Big) = 0.$$

Let us introduce the shorthand notation $f_{n,m}(x) := (1-x)^{m-1}\left(x^2/(1-x)\right)^n$. The rational function $f_{n,m}(x)$ is regular at $x = 0$ and vanishes there to the order $2n$. So if $m \leq n$, then

$$f_{n,m}^{(2m-1)}(0) = 0.$$

On the other hand, if $m > n$, then $f_{n,m}(x)$ is a polynomial of degree $n + m - 1 < 2m - 1$ and hence the $(2m-1)$-fold derivative of $f_{n,m}$ vanishes identically. □

It will be convenient to introduce some terminology.

Definition 2.2. By a *symplectic basis* we mean a sequence $(\varphi_n(x))_{n\in\mathbb{N}}$ of symplectic power series $\varphi_n(x) \in \mathbb{C}[\![x]\!]$ such that each $\varphi_n(x) \in \mathfrak{m}^{2n}$ and its class in $\mathfrak{m}^{2n}/\mathfrak{m}^{2n+1}$ is nonzero. Here $\mathfrak{m}$ denotes the maximal ideal $x\,\mathbb{C}[\![x]\!]$ of the complete local ring $\mathbb{C}[\![x]\!]$.

Lemma 2.3. *Let $(\varphi_n(x))_{n\in\mathbb{N}}$ be a symplectic basis. Then for each symplectic power series $\varphi(x)$ there exists a unique sequence $(a_n)_{n\in\mathbb{N}}$ of numbers such that for each $k \geq 0$*

$$\varphi(x) - \sum_{i=0}^{k} a_i\,\varphi_i(x) \in \mathfrak{m}^{2k+2}. \tag{2.1}$$

It follows that $\varphi(x) = \sum_{i\geq 0} a_i\,\varphi_i(x)$, where the sum converges in the $\mathfrak{m}$-adic topology of $\mathbb{C}[\![x]\!]$.

Proof. We start with a preparatory observation. Suppose that $k \geq 0$ and $f(x) = \sum_{i\geq 0} \alpha_i\,x^i$ is symplectic and in $\mathfrak{m}^{2k+1}$, i.e., $\alpha_0 = \alpha_1 = \cdots = \alpha_{2k} = 0$. Then (S_{k+1}) implies that $\alpha_{2k+1} = 0$ as well, that is $f(x) \in \mathfrak{m}^{2k+2}$.

Assume now for induction that

$$\varphi(x) - \sum_{i=0}^{k-1} a_i\,\varphi_i(x) \in \mathfrak{m}^{2k}.$$

It follows that there is a unique number a_k such that $\varphi(x) - \sum_{i=0}^{k} a_i\,\varphi_i(x) \in \mathfrak{m}^{2k+1}$. Since the latter series is symplectic, the above argument tells us that it is in fact in $\mathfrak{m}^{2k+2}$. □

As a consequence, with the choice of the symplectic basis

$$\left(\left(\frac{x^2}{1-x}\right)^n\right)_{n\in\mathbb{N}} \tag{2.2}$$

we can write each symplectic series $\varphi(x)$ as a formal composite $\rho(x^2/(1-x))$, where $\rho(y) = \sum_{i\geq 0} a_i\,y^i \in \mathbb{C}[\![y]\!]$. This proves Proposition 1.4.

Remark 2.4. There are of course plenty of other symplectic bases. In fact, any symplectic power series $\varphi_1(x)$ that is in $\mathfrak{m}^2$ and whose class in $\mathfrak{m}^2/\mathfrak{m}^3$ does not vanish generates a symplectic basis $((\varphi_1(x))^n)_{n\in\mathbb{N}}$. A choice different from $x^2/(1-x)$ is provided by the sequence of *Genocchi numbers*. The sequence of Genocchi numbers $(G_n)_{n\in\mathbb{N}}$ (cf. entry A036968 in the online encyclopedia [11]) is defined by the exponential generating function

$$\begin{aligned} \frac{2z}{e^z+1} &= \sum_{n\geq 0} G_n \frac{z^n}{n!} \\ &= z - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{3z^6}{6!} + \frac{17z^8}{8!} - \frac{155z^{10}}{10!} + \frac{2073z^{12}}{12!} - \dots \quad \in \mathbb{C}[\![z]\!]. \end{aligned}$$

Setting $\mathrm{Gen}(x) := \sum_{i\geq 0} G_{n+1} x^n$, it follows from [6] (see also Section 5) that

$$\varphi_1(x) := x^2\,\mathrm{Gen}(-x) \tag{2.3}$$

is symplectic, and hence generates a symplectic basis as described above. As $G_n = O(n!/\pi^n)$, $\mathrm{Gen}(x)$ as well as $\varphi_1(x)$ cannot be rational. Also note that the only even monomial occurring in the expansion of $\varphi_1(x) := x^2\,\mathrm{Gen}(-x)$ is x^2.

3. Proof of Theorem 1.6

First let us prove that a formal power series $\varphi(x) = \sum_{i\geq 0} \gamma_i \, x^i$ is symplectic if and only if

$$\varphi(x) = \varphi(x/(x-1)). \tag{3.1}$$

The implication $\Longrightarrow$ is a consequence of Proposition 1.4. Conversely, let us assume that $\varphi(x)$ fulfills Equation (3.1). Using the identity

$$\left(\frac{x}{x-1}\right)^n = \sum_{i\geq 0} (-1)^n \binom{n+i-1}{n-1} x^{n+i},$$

for $n \geq 1$, we see that Equation (3.1) is tantamount to

$$\gamma_m = \sum_{i=1}^{m} (-1)^n \binom{m-1}{n-1} \gamma_n \tag{3.2}$$

for all $m \geq 1$. Without loss of generality we can assume that $\gamma_{2n} = 0$ for all $n \geq 0$. This can be achieved by subtracting a suitable symplectic power series. With this assumption it follows recursively from (3.2) that $\gamma_n = 0$ for all $n \geq 0$. Since $\varphi(x) = 0$ is symplectic, this shows implication $\Longleftarrow$.

This establishes the first claim of Theorem 1.6. The rest of the statement will follow from the following.

Lemma 3.1. *Let $\varphi(x) = P(x)/Q(x)$ be a rational symplectic function. Then there exists a rational function $\rho(y)$ such that $\varphi(x) = \rho(x^2/(1-x))$.*

Proof. Assume that φ is nonzero, and then we may express

$$\varphi(x) = Cx^k (x-1)^\ell (x-2)^m \prod_{i=1}^{r} (x-\lambda_i)^{n_i} \tag{3.3}$$

where $C \in \mathbb{C}^\times$, each $\lambda_i \in \mathbb{C} \setminus \{0, 1, 2\}$, r is a nonnegative integer, and k, ℓ, m, and n_i for $i = 1, \dots, r$ are integers. Let $q = \deg(Q(x)) - \deg(P(x))$, and then we have

$$q = -k - \ell - m - \sum_{i=1}^{r} n_i. \tag{3.4}$$

By a simple computation,

$$\varphi\left(\frac{x}{x-1}\right) = C(-1)^m x^k (x-1)^q (x-2)^m \prod_{i=1}^{r} (1-\lambda_i)^{n_i} \left(x - \frac{\lambda_i}{\lambda_i - 1}\right)^{n_i}. \tag{3.5}$$

We have that $\varphi(x)$ is symplectic by hypothesis so that by the substitution theorem [1, Theorem 9.25], Equation (3.1) holds for $\varphi(x)$. Hence, a comparison of Equations (3.3) and (3.5) yields

$$(-1)^m \prod_{i=1}^{r} (1-\lambda_i)^{n_i} = 1, \tag{3.6}$$

$$q = \ell, \qquad \text{and} \tag{3.7}$$

$$\prod_{i=1}^{r} \left(x - \frac{\lambda_i}{\lambda_i - 1}\right)^{n_i} = \prod_{i=1}^{r} (x-\lambda_i)^{n_i}. \tag{3.8}$$

From Equation (3.8), we have that for each factor $x-\lambda_i$ in $\prod_{i=1}^{r}(x-\lambda_i)^{n_i}$, a factor $x-\lambda_i/(\lambda_i-1)$ must also appear. Hence we may rewrite

$$\prod_{i=1}^{r}(x-\lambda_i)^{n_i} = \prod_{j=1}^{r'}(x-\mu_j)^{n'_j}\left(x-\frac{\mu_j}{\mu_j-1}\right)^{n'_j} \tag{3.9}$$

for a nonnegative integer r', nonnegative integers n'_j and $\mu_j \in \mathbb{C}\smallsetminus\{0,1,2\}$ for $j=1,\ldots,r'$. Combining Equations (3.6) and (3.9) and observing that for each j, $(1-\mu_j)(1-\mu_j/(\mu_j-1))=1$, we obtain

$$1 = (-1)^m \prod_{j=1}^{r'}(1-\mu_j)^{n'_j}\left(1-\frac{\mu_j}{\mu_j-1}\right)^{n'_j} = (-1)^m$$

so that $m=2m'$ for some $m'\in\mathbb{Z}$. Similarly, Equations (3.4) and (3.7) can now be used to express

$$k = -2\ell - 2m' - \sum_{j=1}^{r'} 2n'_j$$

so that $k=2k'$ for some $k'\in\mathbb{Z}$, and then we have

$$\ell = -k' - m' - \sum_{j=1}^{r'} n'_j.$$

Substituting (3.9) into (3.3) and applying the above observations yields

$$\begin{aligned}\varphi(x) &= Cx^{2k'}(x-1)^{-k'-m'-\sum_{j=1}^{r'} n'_j}(x-2)^{2m'}\prod_{j=1}^{r'}(x-\mu_j)^{n'_j}\left(x-\frac{\mu_j}{\mu_j-1}\right)^{n'_j}\\ &= C\left(-\frac{x^2}{1-x}\right)^{k'}\left(-\frac{x^2}{1-x}-4\right)^{m'}\prod_{j=1}^{r'}\left(-\frac{x^2}{1-x}-\frac{\mu_j^2}{\mu_j-1}\right)^{n'_j},\end{aligned}$$

a rational function of $x^2/(1-x)$, completing the proof. □

4. Proof of Theorem 1.3

In this section, we let $G=\mathbb{T}^\ell=(\mathbb{S}^1)^\ell$, let V be a unitary representation of G with $\dim_{\mathbb{C}} V = n$, and let M_0 denote the corresponding symplectic quotient. We choose a (complex) basis for V with respect to which the G-action is diagonal, and then the action of G is described by a *weight matrix* $A\in\mathbb{Z}^{\ell\times n}$. Specifically, we let $\boldsymbol{z} := (z_1,\ldots,z_\ell)\in G$ with each $z_i\in\mathbb{S}^1$ and introduce the notation $\boldsymbol{z}^{\boldsymbol{a}_j} := z_1^{a_{1,j}} z_2^{a_{2,j}}\cdots z_\ell^{a_{\ell,j}}$ for each $j=1,\ldots,n$. Then the action of $\boldsymbol{z}$ on V as a unitary transformation is given with respect to this basis by

$$\boldsymbol{z}\mapsto \operatorname{diag}(\boldsymbol{z}^{\boldsymbol{a}_1},\ldots,\boldsymbol{z}^{\boldsymbol{a}_n}).$$

Concatenating our basis for V with its complex conjugate to produce a real basis for V, the action of $\boldsymbol{z}$ on V as real linear transformations is given by

$$\boldsymbol{z}\mapsto \operatorname{diag}(\boldsymbol{z}^{\boldsymbol{a}_1},\ldots,\boldsymbol{z}^{\boldsymbol{a}_n},\boldsymbol{z}^{-\boldsymbol{a}_1},\ldots,\boldsymbol{z}^{-\boldsymbol{a}_n}).$$

Let $J\colon V\to\mathfrak{g}^*$ denote the homogeneous quadratic moment map, let $Z := J^{-1}(0)$, and let $M_0 := Z/G$ denote the symplectic quotient; see Section 1. As G is abelian, the components of J are elements of $\mathbb{R}[V]^G$. We may assume without loss of generality that 0 is in the convex hull of the columns of A in $\mathbb{R}^\ell$ and the rank of A is ℓ; see [7, Section 2] or [5, Section 3].

Using Molien's formula, see [3, Section 4.6.1], the Hilbert series of the invariant ring $\mathbb{R}[V]^G$ is given by

$$\operatorname{Hilb}_{\mathbb{R}[V]^G}(t) = \frac{1}{(2\pi i)^\ell} \int\limits_{\boldsymbol{z}\in\mathbb{T}^\ell} \frac{dz_1 dz_2 \cdots dz_\ell}{\left(\prod_{j=1}^n z_j\right)\prod_{j=1}^n (1 - t\boldsymbol{z}^{\boldsymbol{a}_j})(1 - t\boldsymbol{z}^{-\boldsymbol{a}_j})}.$$

Then by [8, Proposition 2.1], the Hilbert series of the real regular functions on the symplectic quotient M_0 is given by

$$\operatorname{Hilb}_{\mathbb{R}[M_0]}(t) = \frac{1}{(2\pi i)^\ell} \int\limits_{\boldsymbol{z}\in\mathbb{T}^\ell} \frac{(1-t^2)^\ell dz_1 dz_2 \cdots dz_\ell}{\left(\prod_{j=1}^n z_j\right)\prod_{j=1}^n (1 - t\boldsymbol{z}^{\boldsymbol{a}_j})(1 - t\boldsymbol{z}^{-\boldsymbol{a}_j})}.$$

Define the function

$$h(\boldsymbol{z}, t) = \frac{(1-t^2)^\ell}{\left(\prod_{j=1}^n z_j\right)\prod_{j=1}^n (1 - t\boldsymbol{z}^{\boldsymbol{a}_j})(1 - t\boldsymbol{z}^{-\boldsymbol{a}_j})},$$

and then we have

$$\begin{aligned} h(\boldsymbol{z}, t^{-1}) &= \frac{(1-t^{-2})^\ell}{\left(\prod_{j=1}^n z_j\right)\prod_{j=1}^n (1 - t^{-1}\boldsymbol{z}^{\boldsymbol{a}_j})(1 - t^{-1}\boldsymbol{z}^{-\boldsymbol{a}_j})} \\ &= \frac{t^{2(n-\ell)}(t^2-1)^\ell}{\left(\prod_{j=1}^n z_j\right)\prod_{j=1}^n (1 - t\boldsymbol{z}^{\boldsymbol{a}_j})(1 - t\boldsymbol{z}^{-\boldsymbol{a}_j})} \\ &= (-1)^\ell t^{2(n-\ell)} h(\boldsymbol{z}, t). \end{aligned}$$

Fix $t \in \mathbb{C}$ with $|t| < 1$, and then

$$\begin{aligned} \operatorname{Hilb}_{\mathbb{R}[M_0]}(t^{-1}) &= \frac{1}{(2\pi i)^\ell} \int\limits_{\boldsymbol{z}\in\mathbb{T}^\ell} h(\boldsymbol{z}, t^{-1}) dz_1 dz_2 \cdots dz_\ell \\ &= \frac{t^{2(n-\ell)}}{(2\pi i)^\ell} \int\limits_{\boldsymbol{z}\in\mathbb{T}^\ell} (-1)^\ell h(\boldsymbol{z}, t) dz_1 dz_2 \cdots dz_\ell. \end{aligned}$$

Choose an i and fix arbitrary values $z_k \in \mathbb{S}^1$ for $k \neq i$. Dividing the numerator and denominator by $z_i^{a_{i,j}}$ for each $a_{i,j} < 0$ to express $h(\boldsymbol{z}, t)$ in terms of positive powers of z_i, and using the fact that each row of A contains at least one nonzero entry, it is easy to see that

$$\operatorname{Res}_{z_i=\infty} h(\boldsymbol{z}, t) = -\operatorname{Res}_{z_i=0} \frac{1}{z_i^2} h(z_1, \ldots, 1/z_i, \ldots, z_n, t) = 0.$$

A computation demonstrates that the transformation $t \mapsto t^{-1}$ induces a bijection between the poles in z_i inside the unit disk with those outside the unit disk. Then considering each $\mathbb{S}^1$-factor of $\mathbb{T}^\ell$ as a negatively-oriented curve about the point at infinity, introducing a factor of $(-1)^\ell$, we have

$$\operatorname{Hilb}_{\mathbb{R}[M_0]}(t^{-1}) = t^{2(n-\ell)} \operatorname{Hilb}_{\mathbb{R}[M_0]}(t).$$

Then Theorem 1.3 follows from Corollary 1.7 and the fact that $\mathbb{R}[M_0]$ has dimension $d = 2(n - \ell)$.

5. Applications to even Euler polynomials and Bernoulli Numbers

Our aim is to derive from the fact that the space of symplectic power series forms a subalgebra (cf. Corollary 1.5) a certain combinatorial identity, Equation (5.3), that might be of independent interest. We recall that the Euler polynomials $E_n(x)$, $n = 0, 1, 2, \ldots$, are defined by the expansion

$$\frac{2e^{xt}}{e^t+1} = \sum_{n\geq 0} E_n(x)\frac{t^n}{n!}. \tag{5.1}$$

We introduce numbers $\begin{bmatrix} n \\ i \end{bmatrix}$ via

$$x(x^{2n} - E_{2n}(x)) =: \sum_i \begin{bmatrix} n \\ i \end{bmatrix} x^{2i},$$

observing that the even Euler polynomials, apart from their leading monomials, contain only monomials that are odd powers of x. Note that $\begin{bmatrix} n \\ i \end{bmatrix}$ are integers and $\begin{bmatrix} n \\ i \end{bmatrix} = 0$ for $i \leq 0$ or $i > n$. The coefficients of the even indexed Euler polynomials are listed in the online encyclopedia [11] as entry A060083. The first six lines in the triangle of $\begin{bmatrix} n \\ i \end{bmatrix}$ are:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & -1 & & 2 & & & & \\
 & & & 3 & & -5 & & 3 & & & \\
 & & -17 & & 28 & & -14 & & 4 & & \\
 & 155 & & -255 & & 126 & & -30 & & 5 & \\
-2073 & & 3410 & & -1683 & & 396 & & -55 & & 6
\end{array},$$

where the line and diagonal indexing starts with $n = 1$ (read from top to bottom) and $i = 1$ (read from left to right). We warn the reader that the recursions for the $\begin{bmatrix} n \\ i \end{bmatrix}$ have no resemblance to those for the binomial coefficients.

Theorem 5.1. *Let $\varphi(x) = \sum_{i\geq 0} \gamma_i x^i$ be a formal power series. Then $\varphi(x)$ is symplectic if and only if for each $n \geq 0$,*

$$\gamma_{2n+1} = \sum_i \begin{bmatrix} n \\ i \end{bmatrix} \gamma_{2i}. \tag{5.2}$$

In particular, for each choice of $\gamma_0, \gamma_2, \gamma_4, \gamma_6, \ldots$ there is a uniquely defined symplectic power series $\varphi(x)$ determined by the above rule.

Corollary 5.2. *For all integers n, k, ℓ we have*

$$\begin{bmatrix} n-k \\ \ell \end{bmatrix} + \begin{bmatrix} n-\ell \\ k \end{bmatrix} = \begin{bmatrix} n \\ k+\ell \end{bmatrix} + \sum_i \sum_r \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} r-1 \\ k \end{bmatrix} \begin{bmatrix} i-r \\ \ell \end{bmatrix}. \tag{5.3}$$

We would like to mention that the Euler coefficients are related to the Genocchi numbers G_n, respectively the Bernoulli numbers B_n, using the formula

$$\begin{bmatrix} n \\ i \end{bmatrix} = -\frac{G_{2(n-i+1)}}{2(n-i+1)}\binom{2n}{2i-1} = \frac{4^{n-i+1}-1}{n-i+1} B_{2(n-i+1)} \binom{2n}{2i-1}.$$

This means that Equation (5.3) can be interpreted as a cubic relation for the Bernoulli numbers.

Proof of Corollary 5.2. Let $\sum_{i\geq 0} \gamma_i\, x^i$ and $\sum_{j\geq 0} \delta_j\, x^j$ be symplectic power series. From Corollary 1.5 we know that their Cauchy product $\sum_{m\geq 0} \vartheta_m\, x^m$ is symplectic with, for each $m \in \mathbb{N}$, $\vartheta_m = \sum_{i+j=m} \gamma_i\, \delta_j$. The left hand side of Equation (5.3) arises from expressing

$$\vartheta_{2n+1} = \sum_{r+s=2n+1} \gamma_r\, \delta_s$$

in terms of even γ's and δ's using Equation (5.2). Similarly, the right hand side of Equation (5.3) arises from expressing

$$\vartheta_{2n+1} = \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \vartheta_{2i} = \sum_{i=1}^{n} \sum_{r+s=2i} \begin{bmatrix} n \\ i \end{bmatrix} \gamma_r\, \delta_s$$

in terms of even γ's and δ's. In the argument, we also use the fact that the even γ's and δ's can be chosen freely. □

Proof of Theorem 5.1. The argument is inspired by [6, Section 7]. There the situation is studied when two sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ are related by

$$d_n = \sum_{i=0}^{n} \binom{n}{i} c_i \tag{5.4}$$

for all n. Then [6, Theorem 7.4] states that for all nonnegative integers m and n,

$$\sum_{i=0}^{m} \binom{m}{i} c_{n+i} = \sum_{j=0}^{n} (-1)^{n+j} \binom{n}{j} d_{m+j}. \tag{5.5}$$

By inspection of the generating function (5.1), we derive the recursion

$$E_n(x) + \sum_{i=0}^{n} \binom{n}{i} E_i(x) = 2x^n.$$

The idea is to put $c_i := E_i(x)$ and $d_n := 2x^n - E_n(x)$ and observe that condition (5.4) holds. As the special case $n = m \geq 0$ of (5.5), we find

$$\sum_{i=0}^{n} \binom{n}{i} E_{n+i}(x) = \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} \left(2x^{n+i} - E_{n+i}(x)\right). \tag{5.6}$$

This can be rewritten as

$$\sum_{\substack{i=0 \\ n+i \text{ even}}}^{n} \binom{n}{i} \left(x^{n+i} - E_{n+i}(x)\right) = \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{n} \binom{n}{i} x^{n+i},$$

which is equivalent to

$$\sum_{\substack{i=0 \\ n+i \text{ even}}}^{n} (-1)^i \binom{n}{i} \sum_{j} \begin{bmatrix} \frac{n+i}{2} \\ j \end{bmatrix} x^{2j-1} + \sum_{\substack{i=0 \\ n+i \text{ odd}}}^{n} (-1)^i \binom{n}{i} x^{n+i} = 0. \tag{5.7}$$

Let now $(\gamma_n)_{n\in\mathbb{N}}$ be a number sequence such that (5.2) holds. It will be enough to show that $\sum_{i\geq 0} \gamma_i\, x^i$ is symplectic. We interpret x as an umbral variable (cf. for example [6]) and define the functional $\Gamma : \mathbb{C}[x] \to \mathbb{C}$ by $\Gamma(x^n) = \gamma_{n+1}$ for n

odd. For even n, we put $\Gamma(x^n) = 0$ (this choice will not affect the considerations to follow). Applying Γ to Equation (5.7), we end up with

$$\sum_{\substack{i=0\\ n+i \text{ even}}}^{n} (-1)^i \binom{n}{i} \underbrace{\sum_j \begin{bmatrix} \frac{n+i}{2} \\ j \end{bmatrix} \gamma_{2j}}_{=\gamma_{n+i+1}} + \sum_{\substack{i=0\\ n+i \text{ odd}}}^{n} (-1)^i \binom{n}{i} \gamma_{n+i+1} = 0,$$

showing that $\sum_{i=0}^{n}(-1)^i \binom{n}{i}\gamma_{n+i+1} = 0$, i.e., condition (S_{n+1}) of Definition 1.1. □

To complete this section, we use the above observations to indicate an alternate symplectic basis than those considered in Section 2.

Lemma 5.3. *For all integers $k \geq 0$, the formal power series*

$$\psi_k(x) := \frac{1}{(2k-1)!} \sum_{i=0}^{\infty} (-1)^{i-1} E_{i-1}^{(2k-1)}(0)\, x^i = -x^{2k} - \sum_{i=k}^{\infty} \begin{bmatrix} i \\ k \end{bmatrix} x^{2i+1} \tag{5.8}$$

is symplectic.

Proof. Using Equation (5.6), we write

$$\begin{aligned}
\sum_{i=0}^{n} \binom{n}{i} E_{n+i}(x) &= \frac{1}{2} \sum_{i=0}^{n} \binom{n}{i} E_{n+i}(x) + \frac{1}{2} \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} \left(2x^{n+i} - E_{n+i}(x)\right) \\
&= \frac{1}{2} \sum_{i=0}^{n} \left(1 - (-1)^{n+i}\right) \binom{n}{i} E_{n+i}(x) + \sum_{i=0}^{n} \binom{n}{i} (-x)^{n+i} \\
&= \sum_{\substack{i=0\\ n+i \text{ odd}}}^{n} \binom{n}{i} \underbrace{\left(E_{n+i}(x) - x^{n+i}\right)}_{=:\,(*)} + \sum_{\substack{i=0\\ n+i \text{ even}}}^{n} \binom{n}{i} x^{n+i},
\end{aligned}$$

where $(*)$ contains only even powers of x. Thus in $\sum_i \binom{n}{i} E_{n+i}(x)$, all coefficients of odd degree vanish, meaning that for all $k \geq 1$,

$$\frac{1}{(2k-1)!} \sum_{i=0}^{n} \binom{n}{i} E_{n+i}^{(2k-1)}(0) = 0.$$

It follows that $\sum\limits_{i=0}^{\infty} (-1)^{i-1} E_{n-1}^{(2k-1)}(0)\, x^i$ fulfills (S_{n+1}). □

As a consequence of Lemma 5.3, we have that $(\psi_n(x))_{n\in\mathbb{N}}$ forms a symplectic basis in the sense of Definition 2.2. Note that ψ_1 is essentially the generating function of the Genocchi sequence (2.3), namely we have $\psi_1(x) = -\varphi_1(x)$. The idea of the above proof can be used to argue that for each $\lambda \in \mathbb{C}$ the power series

$$\sum_{i\geq 0} (-1)^{i-1} \left(E_{i-1}(\lambda) - E_{i-1}(-\lambda)\right) x^i \in \mathbb{C}[\![x]\!]$$

is symplectic.

6. Sample Calculations

In this section, we survey a few special cases of Conjecture 1.2 and Theorem 1.3 that can be verified by direct computations. We first consider the case of a unitary representation of a finite group. In this case, as a consequence of Corollary 1.7, Conjecture 1.2 is a special case of Watanabe's Theorem [16, 17], see in particular [13, Theorem 7.1].

6.1. **Quotients by finite unitary group representations.** Let G be a finite group and $G \to \mathrm{U}(V)$ a unitary representation. For $g \in G$, we let $g_V\colon V \to V$ denote the corresponding linear transformation. Let $W := V \times \overline{V}$, and then G acts on W via $g_W\colon (u, \overline{v}) \mapsto (g_V u, (g_V^{-1})^t \overline{v})$. We identify $\mathbb{R}[V]$ with the subring $\mathbb{C}[W]^-$ of $\mathbb{C}[W]$ given by those elements fixed by complex conjugation, and then by Molien's formula [9], see also [3, 14], the Hilbert series of real regular invariants is given by

$$\mathrm{Hilb}_{\mathbb{R}[V]^G|\mathbb{R}}(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(\mathrm{id}_W - g_W^{-1} t)}. \tag{6.1}$$

Fix $g \in G$ and choose a basis for V with respect to which g_V is diagonal, say $g_V = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where $|\lambda_i| = 1$ for each i. Choosing the conjugate basis for $\overline{V}$ and concatenating to form a basis for W, we have $g_W = \mathrm{diag}(\lambda_1, \ldots, \lambda_n, \overline{\lambda_1}, \ldots, \overline{\lambda_n})$. It then follows that

$$\frac{1}{\det(\mathrm{id}_W - g_W^{-1} t)} = \prod_{i=1}^{n} \frac{1}{(1 - \lambda_i t)(1 - \lambda_i^{-1} t)}. \tag{6.2}$$

In this case, each term of the sum in Equation (6.1) is symplectic of order $2n$ at $t = 1$. Specifically, for $\lambda \in \mathbb{C}$, define

$$f_\lambda(t) := \frac{1}{(1 - \lambda t)(1 - \lambda^{-1} t)}.$$

Then by the above observations, we have that each term in Molien's formula is given by a product of $f_{\lambda_i}(t)$. We claim the following.

Lemma 6.1. *For nonzero $\lambda \in \mathbb{C}$, the function $f_\lambda(t)$ is symplectic at $t = 1$ of order 2.*

Proof. If $\lambda = 1$, then $f_1(t) = (1 - t)^{-2}$ and the result is trivial, so assume not. Define

$$\rho_\lambda(y) = \frac{\lambda y}{1 + 2\lambda + \lambda^2 + \lambda y},$$

and then by a simple computation,

$$\frac{1}{(1 - t)^2}\, \rho_\lambda\!\left(\frac{(1 - t)^2}{t} \right) = f_\lambda(t).$$

The result then follows from Corollary 1.5. □

We remark that Lemma 6.1 can also be seen using the expansion

$$f_\lambda(t) = \sum_{k=-2}^{\infty} \frac{-\lambda(\lambda^{k+1} - (-1)^{k+1})}{(\lambda^2 - 1)(\lambda - 1)^{k+1}} (1 - t)^k$$

and verifying (S_m) directly, or by checking that $f_\lambda(t)$ satisfies (1.1) for $a = 1$ and $d = 2$.

By Lemma 6.1 and Corollary 1.5, it follows that the expression in Equation (6.2) is symplectic of order $2n$ at $t = 1$.

6.2. **Symplectic quotients by $\mathbb{S}^1$.** We observe that Corollary 1.5 (in the case $a=1$) is a sufficient tool to verify Conjecture 1.2 for symplectic $\mathbb{S}^1$-quotients case-by-case. When the action is *generic*, i.e. no two weights have the same absolute value, an algorithm for computing the Hilbert series is described in [8, Section 4] and has been implemented on *Mathematica* [18]. To check that a concrete Hilbert series is symplectic of order $d=\dim(M_0)$, we use the substitution

$$t \mapsto \frac{1}{2}\left(y+2\pm\sqrt{y(y+4)}\right)$$

as a heuristic to find a rational function $\rho(y)$ as in Corollary 1.5.

As an example, if M_0 is a reduced space associated to the weight vector $(\pm 1,\pm 2,\pm 3)$ where not all weights have the same sign, then

$$\mathrm{Hilb}_{\mathbb{R}[M_0]}(t)=\frac{t^{10}+t^8+3t^7+4t^6+4t^5+4t^4+3t^3+t^2+1}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)}.$$

One easily checks that

$$\rho(y)=\frac{y^5+10y^4+36y^3+59y^2+50y+22}{(y+2)(y+3)(y+4)(y^2+5y+5)}$$

satisfies the condition of Corollary 1.5.

Similarly, let $\mathbb{S}^1$ act on $\mathbb{C}^n$ with weight vector $(\pm 1,\dots,\pm 1)$ and let M_0 denote the symplectic quotient, which we note has dimension $2n-2$. We assume that $n\geq 2$ and not all weights have the same sign, for otherwise M_0 is a point and the result is trivial. By [8, Section 5.3], the Hilbert series $\mathbb{R}[M_0]$ is given by

$$\mathrm{Hilb}_{\mathbb{R}[M_0]}(t)=(1-t^2)\,{}_2F_1(n,n,1,t^2)=\frac{1}{(1-t^2)^{2n-2}}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 t^{2k}$$

where ${}_2F_1$ denotes the hypergeometric function; see [4]. Theorem 1.3 can be verified directly using Corollary 1.5 for this case as follows.

Define

$$\rho(y)=\frac{1}{(y+4)^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{2k}{k}y^{n-k-1}.$$

Let $(a)_b$ denote the Pochhammer symbol, i.e. $(a)_b := a(a+1)\cdots(a+b-1)$ for $b>0$ and $(a)_0=1$, and note that $(1/2)_k=(2k)!/(4^k k!)$. Applying the definition of ${}_2F_1$, we compute

$$\begin{aligned}\frac{1}{(1-t)^{2n-2}}\,\rho\!\left(\frac{(1-t)^2}{t}\right)&=\frac{1}{(1+t)^{2n-2}}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{2k}{k}\left(\frac{t}{(1-t)^2}\right)^k\\&=\frac{1}{(1+t)^{2n-2}}\sum_{k=0}^{n-1}\frac{(1-n)_k(1/2)_k}{(1)_k k!}\left(\frac{-4t}{(1-t)^2}\right)^k\\&=\frac{1}{(1+t)^{2n-2}}\,{}_2F_1(1-n,1/2,1,-4t/(1-t)^2).\end{aligned}$$

We then apply [4, page 113, Equation (36)] and continue

$$
\begin{aligned}
&= \frac{1}{(1-t^2)^{2n-2}}\, {}_2F_1(1-n, 1-n, 1, t^2) = \frac{1}{(1-t^2)^{2n-2}} \sum_{k=0}^{n-1} \frac{(1-n)_k (1-n)_k}{(1)_k k!} t^{2k} \\
&\qquad = \frac{1}{(1-t^2)^{2n-2}} \sum_{k=0}^{n-1} \binom{n-1}{k}^2 t^{2k} \\
&\qquad = \mathrm{Hilb}_{\mathbb{R}[M_0]}(t).
\end{aligned}
$$

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