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Nested sets, set partitions and Kirkman-Cayley dissection numbers

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Abstract

In this paper we show a a proof by explicit bijections of the famous Kirkman-Cayley formula for the number of dissections of a convex polygon. Our starting point is the bijective correspondence between the set of nested sets made by k subsets of $\{1, 2, ..., n\}$ with cardinality ≥ 2 and the set of partitions of $\{1, 2, ..., n+k-1\}$ into k parts with cardinality ≥ 2. A bijection between these two sets can be obtained from Péter L Erdős and L.A Székely result in [10]; to make this paper self contained we describe another explicit bijection that is a variant of their bijection.

1 Introduction

Let $D_{n+1,k-1}$ be the number of dissections of a convex polygon with $n+1$ labelled edges by $k-1$ diagonals, such that no two of the diagonals intersect in their interior. The formula for $D_{n+1,k-1}$ dates back to Kirkman (see [19]) and Cayley (who gave the first complete proof in [3]):

$$D_{n+1,k-1} = \frac{1}{k}\binom{n-2}{k-1}\binom{n+k-1}{k-1}$$

In [23] and [25] one can find two elegant proofs of this formula via bijections. In this paper we show another proof that consists in the description of two explicit bijections and is based on the combinatorics of nested sets and set partitions. We also provide a proof of the classical formula (see Chapter 13 of [17]) for the number of the dissections such that the types of the internal polygons are prescribed.

We start a more detailed outline of the content of the present paper by recalling the notion of nested set. This notion appeared in geometry in connection with models of configuration spaces, first in Fulton and MacPherson paper [13], then, with more generality, in De Concini and Procesi's papers [5], [6] (and later in [7]) on wonderful models of subspace arrangements.

Some generalizations of De Concini and Procesi's definition successively appeared in various combinatorial contexts. We refer the reader to [12] where the case of meet semilattices is dealt with, or to [21] and [22] where some polytopes named nestohedra are constructed, and finally to [20] where one can find a comparison among various definitions in the literature.

Let us denote by $\mathcal{P}_2(\{1, 2, ..., n\})$ the subset of the power set $\mathcal{P}(\{1, 2, ..., n\})$ whose elements have cardinality greater than or equal to 2. The following definition of nested set of $\mathcal{P}_2(\{1, 2, ..., n\})$ is a special case of the more general combinatorial definition.

Definition 1.1. *Let $n \geq 2$. A subset S of $\mathcal{P}_2(\{1, 2, ..., n\})$ is a nested set if and only if it contains $\{1, 2, ..., n\}$ and for any $I, J \in S$ we have that either $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$. We will denote by $\mathcal{S}_2(n, k)$ the set of the nested sets S of $\mathcal{P}_2(\{1, 2, ..., n\})$ such that $|S| = k$.*

Now we observe that there is a bijective correspondence between $\mathcal{S}_2(n, k)$ and the set $\mathcal{T}_2(n+k-1, k)$ of partitions of the set $\{1, 2, ..., n+k-1\}$ into k parts of cardinality greater than or equal to 2. A bijection between these two sets can be obtained as a particular case of the bijection between rooted trees on n leaves and partitions proven by Péter L Erdős and L.A Székely (see Theorem 1 of [10]).

To make our paper self contained we show in Section 2 (Theorem 2.1) an explicit bijection between $\mathcal{S}_2(n,k)$ and $\mathcal{T}_2(n+k-1,k)$ that is a variant of the bijection obtained from [10].

We notice that, for fixed n, the cardinalities $|\mathcal{S}_2(n,k)| = |\mathcal{T}_2(n+k-1,k)|$ are the Ward numbers (see [27], and the sequence A134991 of OEIS) and can be read along the diagonals in the table of the 2-associated Stirling numbers of the second kind at page 222 of [4]. They can as well be interpreted as the face numbers in the tropical Grassmannian $G(2,n+1)$, i.e. the space of phylogenetic trees T_{n+1} (see [1],[11],[24]). For a description of generating formulas for these numbers one can also see Chapter 5 of [26] (in particular Section 5.2.5 and Exercise 5.40).

As it is well known (see Section 4, Figure 3), the dissections of a convex polygon with $n+1$ edges by $k-1$ diagonals are in bijection with the parenthesizations with k couples of parentheses of a list of n distinct numbers $a_1, a_2, ..., a_n$ (the maximal couple of parentheses is included and every couple of parentheses contains at least two numbers). We will denote by $\mathcal{S}_2((a_1,a_2,...,a_n),k)$ the set of all these parenthesizations: as a consequence of the remark above, $|\mathcal{S}_2((a_1,a_2,...,a_n),k)| = D_{n+1,k-1}$.

Now, an 'ordered' variant of Theorem 2.1 (stated in Section 2 as Theorem 2.2) describes a bijection between $\bigcup_{\sigma\in S_n} \mathcal{S}_2((a_{\sigma(1)},a_{\sigma(2)},...,a_{\sigma(n)}),k)$ and the set of *admissible internally ordered* partitions of $\{1,2,...,n+k-1\}$ into k parts, i.e. partitions whose parts have cardinality ≥ 2 and are equipped with an internal total ordering.

This leads to a proof (in Section 4, Corollary 4.1) of Kirkman-Cayley formula, since the enumeration of the admissible internally ordered partitions is provided, again by an explicit bijection, by Theorem 3.1 in Section 3.

Even if our proof is purely combinatorial, in the end of Section 4 we sketch out a geometric interpretation: our argument corresponds to counting in two different ways the boundary components of a spherical model of $\overline{M}_{0,n+1}$, the moduli space of real stable $n+1$-pointed curves of genus 0.

The classical formula for the number of dissections of a convex polygon with $n+1$ edges such that the types of the internal polygons are prescribed also follows, in this combinatorial picture, as an another quick application of Theorem 2.2 (see Section 4, Corollary 4.2).

2 Nested sets and set partitions

For every $i \in \mathbb{Z}$ and $k \in \mathbb{N}$ let us denote by $[i,i+k]$ the interval of integers $\{i,i+1,...,i+k\}$.

As we pointed out in the Introduction, from Theorem 1 in [10] one obtains a bijection between $\mathcal{S}_2(n,k)$ and the set $\mathcal{T}_2(n+k-1,k)$ of partitions of $[1,n+k-1]$ into k parts of cardinality greater than or equal to 2. To make our paper self contained, the first part of this section is devoted to showing an explicit bijection between $\mathcal{S}_2(n,k)$ and $\mathcal{T}_2(n+k-1,k)$ that is a variant of the one that can be deduced from [10]. In the second part of the present section an 'ordered' version of this bijection, that involves partitions into ordered sets, is described (Theorem 2.2).

Definition 2.1. *We fix the following (strict) partial ordering in $\mathcal{P}_2(\{1,2,...,n\})$: given two sets I and J in $\mathcal{P}_2(\{1,2,...,n\})$ we put $I < J$ if the minimal element in I is less than the minimal element in J.*

Theorem 2.1. *Let us consider two integers n,k with $n \geq 2$, $n-1 \geq k \geq 1$. There is a bijection between $\mathcal{S}_2(n,k)$ and the set $\mathcal{T}_2(n+k-1,k)$ of partitions of $[1,n+k-1]$ into k parts of cardinality greater than or equal to 2.*

Proof. Let us consider a nested set $S \in \mathcal{S}_2(n,k)$. It can be represented by an oriented rooted tree on n leaves labelled by the numbers $1,2,...,n$ in the following way. We consider the set $\tilde{S} = S \cup \{1\} \cup \{2\} \cup \cdots \cup \{n\}$. Then the tree coincides with the Hasse diagram of $\tilde{S}$ viewed as a poset by the inclusion relation: the root is $\{1,2,...,n\}$ and the orientation goes from the root to the leaves, that are the vertices $\{1\},\{2\},\ldots,\{n\}$ labelled respectively by the numbers $1,2,...,n$.

We observe that we can partition the set of vertices of the tree into *levels*: level 0 is made by the leaves, and in general, level j is made by the vertices v such that the maximal length of an oriented path that connects v to a leaf is j. We notice that, since the nested set has k elements, there are k internal vertices of this tree, including the root. Now we can label the internal vertices of the tree in the following way. Let us suppose that there are q vertices in level 1. These vertices correspond, by the nested property, to pairwise disjoint elements of $\mathcal{P}_2(\{1, 2, ..., n\})$ and therefore we can totally order them using the ordering of Definition 2.1. Then we label them with the numbers from $n+1$ to $n+q$ (the label $n+1$ goes to the minimum, while $n+q$ goes to the maximum).

At the same way, if there are t vertices in level 2, we can label them with the numbers from $n+q+1$ to $n+q+t$, and so on. At the end of the process, the root is labelled with the number $n+k$: we have obtained an oriented labelled tree with $n+k$ vertices, where at least two edges stem from each internal vertex and the leaves are labelled by the numbers from 1 to n.

We can now associate to such a tree a partition in $\mathcal{T}_2(n+k-1, k)$ by assigning to every internal vertex v (including the root) the set of the labels of the vertices covered by v. An example of this process is provided by Figure 1.

We have therefore described a map $\phi : \mathcal{S}_2(n, k) \to \mathcal{T}_2(n + k - 1, k)$. Let us now describe its inverse.

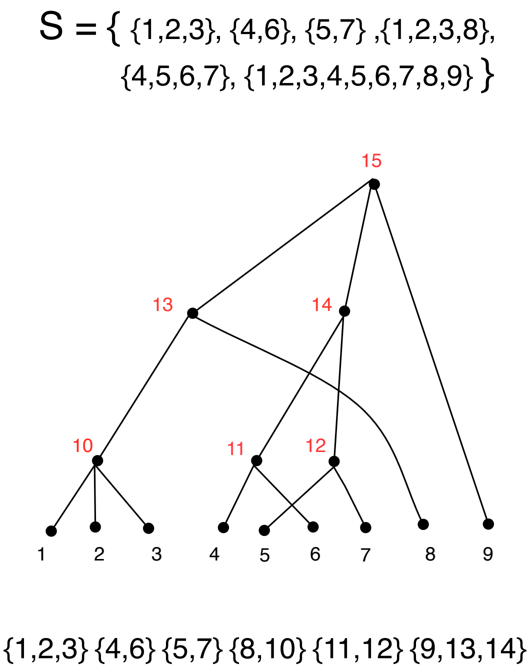

Figure 1: A nested set S with 6 elements in $\mathcal{P}_2(\{1, 2, ..., 9\})$ (top of the picture), its associated oriented rooted tree and its associated partition in $\mathcal{T}_2(14, 6)$ (bottom of the picture).

Let us consider a partition P in $\mathcal{T}_2(n+k-1, k)$ with $k > 1$ (the case $k = 1$ is trivial). We observe that, since there are k parts in the partition, at least one of these parts is a subset of $[1, n]$. Let $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_i$ (with $i \le k-1$) be the parts that are subsets of $[1, n]$; here we have indexed these sets according to the ordering of Definition 2.1.

Now, if $i = k-1$ the set $S = \{\{1, 2, ..., n\}, \tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_i\}$ is the only one nested set in $\mathcal{S}_2(n, k)$ such that $\phi(S) = P$. We observe that in the tree associated with this nested set the vertices in level 1 correspond to the sets $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_i$.

If $i < k-1$ we notice that at least one of the remaining $k-i$ parts in the initial partition P is a subset of the complement of $[n+i+1, n+k-1]$ in $[1, n+k-1]$, since the set $[n+i+1, n+k-1]$ has cardinality

$k-i-1$.

Let us then denote by $A_{i+1},...,A_{i+s}$ (with $1 \le s \le k-1-i$) these remaining parts included in the complement of $[n+i+1, n+k-1]$. For every $t = 1,...,s$ we associate to A_{i+t} the set

$$\tilde{A}_{i+t} = (A_{i+t} \cap [1,n]) \cup \bigcup_{h \in A_{i+t} \cap [n+1,n+i]} \tilde{A}_{h-n}$$

The indices have been chosen in such a way that, according to the ordering of Definition 2.1, we have $\tilde{A}_{i+1} < \tilde{A}_{i+2} < \cdots < \tilde{A}_{i+s}$.

Now, if $i+s = k-1$ the process stops, and the set $S = \{\{1,2,...,n\}, \tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_i, \tilde{A}_{i+1}, \tilde{A}_{i+2}, \cdots, \tilde{A}_{i+s}\}$ is the only one nested set in $\mathcal{S}_2(n,k)$ such that $\phi(S) = P$. We observe that in the tree associated with this nested set the vertices in level 1 correspond to the sets $\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_i$ and the vertices in level 2 correspond to the sets $\tilde{A}_{i+1}, \tilde{A}_{i+2}, \cdots, \tilde{A}_{i+s}$.

If $i+s < k-1$ we can continue and we can construct, level after level, the only one nested set in $\mathcal{S}_2(n,k)$ such that $\phi(S) = P$.

□

Remark 2.1. *We notice that this bijection introduces on $\mathcal{S}_2(n,k)$ an action of the symmetric group S_{n+k-1}. As one can easily check, when $k > 3$ this action is not compatible with the usual S_n action. A geometric application of this remark will be discussed in the paper [2].*

We now state a variant of the theorem above, where nested sets of ordered lists and internally ordered partitions come into play.

Definition 2.2. *Let us consider a list of distinct numbers $a_1, a_2, ..., a_n$ (with $n \ge 2$), and a parenthesization of $a_1, a_2, ..., a_n$ that includes the maximal couple of parentheses $(a_1, a_2, ..., a_n)$ and such that every couple of parentheses contains at least two numbers. We will call this parenthesization* a nested set of the list *$a_1, a_2, ..., a_n$, and we will denote by $\mathcal{S}_2((a_1, a_2, ..., a_n), k)$ the set whose elements are the nested sets S of the list $a_1, a_2, ..., a_n$ with $|S| = k$. Furthermore, we will denote by $\mathcal{OS}_2((a_1, a_2, ..., a_n), k)$ the set:*

$$\mathcal{OS}_2((a_1, a_2, ..., a_n), k) = \bigcup_{\sigma \in S_n} \mathcal{S}_2((a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}), k)$$

We observe that if $\{a_1, a_2, ..., a_n\} = \{1, 2, .., n\}$ we can associate to a nested set of the list $a_1, a_2, ..., a_n$ a nested set in $\mathcal{S}_2(n,k)$ in the following obvious way: we associate to every couple of parentheses the set of the numbers contained in it.

Definition 2.3. *An* internally ordered k-partition *(with $k \ge 1$) of a finite set X is given by*

- *an unordered partition $\mathcal{P} = \{P_1, P_2, ..., P_k\}$ of X into k parts;*
- *a complete ordering of the elements of P_i for every $1 \le i \le k$.*

If the cardinality of all the subsets $P_1, P_2, ..., P_k$ is greater than or equal to 2 we say that the internally ordered partition is admissible.

Theorem 2.2. *Let us consider two integers n, k with $n \ge 2$, $n-1 \ge k \ge 1$. There is a bijection between $\mathcal{OS}_2((1, 2..., n), k)$ and the set $\mathcal{IT}_2(n+k-1, k)$ of the admissible internally ordered k-partitions of $[1, n+k-1]$.*

Proof. Let us put $a_1 = 1, a_2 = 2, ..., a_n = n$. Starting from a nested set S in $\mathcal{S}_2((a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}), k)$ we can construct, in a similar way as in the proof of Theorem 2.1, an oriented rooted labelled tree.

The only difference from the construction described in the proof of Theorem 2.1 is the following one: we draw the leaves labelled by $a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}$ from left to right, and, since we are dealing with a nested set of the list $a_{\sigma(1)}, a_{\sigma(2)}, ..., a_{\sigma(n)}$, in the picture of the tree no two of the edges intersect in their interiors.

Finally we construct an internally ordered admissible k-partition of $[1, n+k-1]$ according to the following rule. The parts $P_1, ..., P_k$ of the partition are produced by the internal vertices of the tree. The part P_i is obtained from the vertex labelled by $n+i$ and it is an ordered set constructed in this way: if, from left to right, the vertices covered by the vertex labelled by $n+i$ are labelled by $b_1, b_2, ..., b_r$, then P_i is the set $\{b_1, b_2, ..., b_r\}$ ordered by $b_1 \prec b_2 \prec \cdots \prec b_r$.

We have described a map $\Gamma : \mathcal{OS}_2((1,2,...,n),k) \to \mathcal{IT}_2(n+k-1,k)$. The inverse map is constructed by associating a tree to an internally ordered admissible k-partition $\{P_1, ..., P_k\}$ of $[1, n+k-1]$. The idea is essentially the same as in the proof of Theorem 2.1, with the only difference that we keep into account the internal orderings of the sets $P_1, ..., P_k$ and we draw the edges from left to right according to this ordering. At the end we produce a nested set of a permutation of the list $1, 2, ..., n$ as it is illustrated in Figure 2.

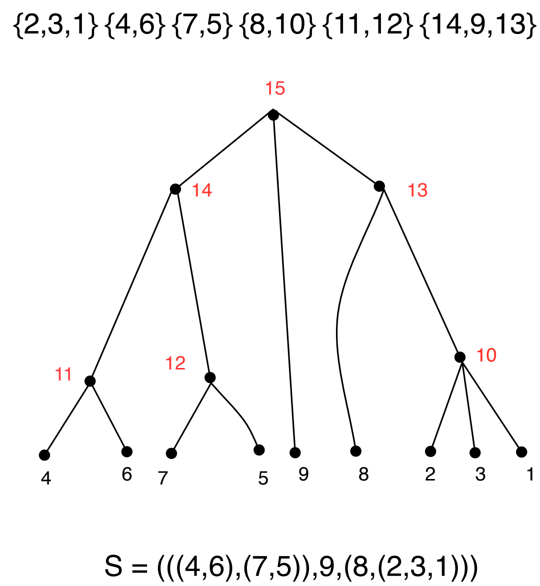

Figure 2: The internally ordered partition of $[1,14]$ into 6 parts that is on top of the picture (the internal orderings $\prec$ are obtained reading from left to right), produces, via its associated oriented rooted tree, the parenthesization of the list $4,6,7,5,9,8,2,3,1$ written on the bottom of the picture.

$\square$

Remark 2.2. *A slight modification of the proof of Theorem 2.1 shows that, as in [10], we can extend the bijection in the statement to a bijection between the set of rooted trees that have k vertices (including the root) and n labelled leaves and the set of all the partitions of $[1, n+k-1]$ into k parts. A similar remark applies to Theorem 2.2.*

3 Enumeration of internally ordered k-partitions

In the preceding section, Theorem 2.2 has pointed out an interesting combinatorial aspect of the admissible internally ordered k-partitions of $[1, n+k-1]$. In this section we are going to count them by showing an explicit bijection. It is useful to add to these partitions a further structure: we mark one of the parts, that becomes a *distinguished* part.

Definition 3.1. *A* distinguished internally ordered k-partition *(with $k \geq 1$) of a finite set X is an admissible internally ordered k-partition where one of the parts of the partition is distinguished. We will denote by X_1 this distinguished set, therefore the partition is made by the parts $X_1, P_2, ..., P_k$.*

Remark 3.1. *In the definition above, the partition of $X - X_1$ provided by the sets P_i is unordered. Let p_{i1} be, for every i, the smallest element of P_i with respect to its internal ordering. From now on we will consider the case when $X \subset \mathbb{Z}$ and we will assign, by convention, the indices to the sets P_i in such a way that if $i < j$ than $p_{i1} < p_{j1}$.*

Theorem 3.1. *Given two integers n, k with $n \geq 2$, $n-1 \geq k \geq 1$, there is a bijection between the set of distinguished internally ordered k-partitions of $[1, n+k-1]$ and the triples (I, σ, D) where*

- *$I = i_1, i_2, ..., i_n$ is a sublist of cardinality n extracted from the list $L = 1, 2, ..., n+k-1$,*
- *σ is a permutation in the symmetric group S_n,*
- *D is a sublist of cardinality $k-1$ extracted from the list $i_{\sigma(2)}, ..., i_{\sigma(n-1)}$ (in particular, for every $n \geq 2$, if $k = 1$ then D is the empty list).*

Therefore the number of distinguished internally ordered k-partitions of $[1, n+k-1]$ is

$$n!\binom{n-2}{k-1}\binom{n+k-1}{k-1}$$

Proof. Let us consider a triple (I, σ, D) as in the statement of the theorem.

Then $I = i_1, i_2, ..., i_n$ (with $i_1 < i_2 < \cdots < i_n$) and we denote by $J = j_1, j_2, ..., j_{k-1}$ the sublist of L made by the numbers that do not belong to I (notice that $j_1 < j_2 < \cdots < j_{k-1}$). Now we use σ to permute the list I, obtaining a list

$$\sigma I = i_{\sigma(1)}, i_{\sigma(2)}, ..., i_{\sigma(n)}$$

To fix the notation we put $D = i_{\sigma(d_1)}, ..., i_{\sigma(d_{k-1})}$ with $d_1 \geq 2, d_{k-1} \leq n-1$.

Let us show how to associate to (I, σ, D) a distinguished internally ordered k-partition of $[1, n+k-1]$. First we put X_1 to be equal to the set $\{i_{\sigma(n)}, i_{\sigma(1)}, ..., i_{\sigma(d_1-1)}\}$ equipped with the ordering $i_{\sigma(n)} \prec i_{\sigma(1)} \prec \cdots \prec i_{\sigma(d_1-1)}$. Then we build the sets

$$P_2 = \{j_1, i_{\sigma(d_1)}, i_{\sigma(d_1+1)}, ..., i_{\sigma(d_2-1)}\}$$

$$P_3 = \{j_2, i_{\sigma(d_2)}, i_{\sigma(d_2+1)}, ..., i_{\sigma(d_3-1)}\}$$

$$...$$

$$P_k = \{j_{k-1}, i_{\sigma(d_{k-1})}, i_{\sigma(d_{k-1}+1)}, ..., i_{\sigma(n-1)}\}.$$

and for every $2 \leq i \leq k$ we order the elements of P_i as displayed above (for instance, in P_2 we have $j_1 \prec i_{\sigma(d_1)} \prec i_{\sigma(d_1+1)} \prec \cdots \prec i_{\sigma(d_2-1)}$).

We notice that the sets $P_2, ..., P_k$ form an unordered partition of $[1, n+k-1] - X_1$, indexed according to Remark 3.1. In conclusion, starting from the triple (I, σ, D) we have produced a distinguished internally ordered k-partition of $[1, n+k-1]$ (see also the Example 3.1 at the end of this section). It is easy to check that this map is injective.

Let us now show that all the distinguished internally ordered k-partitions of $[1, n+k-1]$ can be obtained starting from a triple. If $k = 1$ this is trivial. Let then $k \geq 2$ and let $X_1, P_2, ..., P_k$ be such a partition. More in detail, let X_1 be the ordered set $\{x_1, x_2, ..., x_{|X_1|}\}$, and, for every i with $2 \leq i \leq k$, let P_i be the ordered set $P_i = \{p_{i1}, p_{i2}, ..., p_{i|P_i|}\}$. According to the convention described in Remark 3.1, the indices of the P_i's satisfy $p_{21} < \cdots < p_{k1}$. Let us denote by J the list $p_{21}, \ldots, p_{k1}$ and by $I = i_1, ..., i_n$ the sublist of L that is complementary to J. Then we choose the permutation $\sigma \in S_n$ defined as the permutation such that the list

$$\sigma I = i_{\sigma(1)}, ..., i_{\sigma(n-1)}, i_{\sigma(n)}$$

coincides with the list

$$x_2, .., x_{|X_1|}, p_{22}, p_{23}, ..., p_{2|P_2|}, p_{32}, p_{33}, ..., p_{3|P_3|}, p_{42}, ..., p_{4|P_4|}, ..., p_{k,2}, ..., p_{k|P_k|}, x_1$$

As a last step, we extract from the list σI, after cutting out its first element and its last element, the sublist of cardinality $k-1$

$$D = i_{\sigma(|X_1|)}, i_{\sigma(|X_1|+|P_2|-1)}, i_{\sigma(|X_1|+|P_2|+|P_3|-2)}, ..., i_{\sigma(|X_1|+|P_2|+\cdots+|P_{k-1}|-(k-2))}$$

(in particular, if $k=2$ this list is $D = i_{\sigma(|X_1|)}$).

We observe that, by construction, the triple (I, σ, D) is associated with the distinguished internally ordered partition $X_1, P_2, ..., P_k$.

$\square$

Example 3.1. *Let $n=7$, $k=4$, and let us consider the triple (I, σ, D) where:*

- $I = \overset{i_1}{1}, \overset{i_2}{2}, \overset{i_3}{3}, \overset{i_4}{4}, \overset{i_5}{6}, \overset{i_6}{9}, \overset{i_7}{10}$ *is a sublist of* $L = 1,2,3,4,5,6,7,8,9,10$.
- σ *is the permutation in* S_7 *such that*

$$\sigma I = \overset{i_{\sigma(1)}}{2}, \overset{i_{\sigma(2)}}{6}, \overset{i_{\sigma(3)}}{1}, \overset{i_{\sigma(4)}}{10}, \overset{i_{\sigma(5)}}{3}, \overset{i_{\sigma(6)}}{9}, \overset{i_{\sigma(7)}}{4}$$

- $D = \overset{i_{\sigma(3)}}{1}, \overset{i_{\sigma(5)}}{3}, \overset{i_{\sigma(6)}}{9}$ *is a sublist of the list* $6,1,10,3,9$ *(i.e. the list σI without its initial and final term).*

Let us associate to this triple a distinguished internally ordered partition X_1, P_2, P_3, P_4 of $[1,10]$ according to the bijection of Theorem 3.1.

The set X_1 is $\{i_{\sigma(7)}, i_{\sigma(1)}, i_{\sigma(2)}\}$, i.e. $X_1 = \{4,2,6\}$, ordered from left to right: $4 \prec 2 \prec 6$.

Now we notice that the complement of $I = 1,2,3,4,6,9,10$ in $L = 1,2,3,4,5,6,7,8,9,10$, is the list $J = 5,7,8$. Then $P_2 = \{5, i_{\sigma(3)}, i_{\sigma(4)}\}$, i.e. $P_2 = \{5,1,10\}$ equipped with the ordering $5 \prec 1 \prec 10$. In an analogous way one finds $P_3 = \{7,3\}$ and $P_4 = \{8,9\}$.

4 Kirkman-Cayley dissection numbers and numbers of dissections of a prescribed type

As it is well known, $D_{n+1,k-1}$ coincides with $|\mathcal{S}_2((1,2,...,n),k)|$: an explicit bijection between $\mathcal{S}_2((1,2,...,n),k)$ and the set of the dissections of a convex polygon with $n+1$ labelled edges by $k-1$ non intersecting diagonals is illustrated by Figure 3.

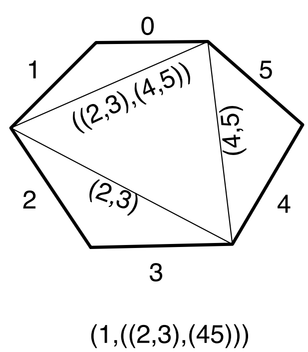

Figure 3: This dissection of the hexagon with 3 diagonals produces the parenthesized list $(1,((2,3),(4,5)))$ that has 4 couples of parentheses.

In view of this, a proof of Kirkman-Cayley formula via bijections immediately follows as a corollary of the Theorems 2.2 and 3.1.

Corollary 4.1 (A proof of Kirkman-Cayley formula)**.**

$$D_{n+1,k-1} = \frac{1}{k}\binom{n-2}{k-1}\binom{n+k-1}{k-1}$$

Proof. We can count the cardinality of $\mathcal{OS}_2((1,2,...,n),k)$ in two different ways. On one hand it is equal to

$$n!D_{n+1,k-1}$$

since there are $n!$ different lists based on the set of numbers $\{1,2,...,n\}$ and for every list we take into account all its nested sets with k couples of parentheses.

On the other hand, by the Theorem 2.2, the cardinality of $\mathcal{OS}_2((1,2,...,n),k)$ is equal to the number of admissible internally ordered k-partitions of $[1,n+k-1]$. This can be obtained dividing by k the number of distinguished internally ordered k-partitions of $[1,n+k-1]$.

Therefore by Theorem 3.1 we have:

$$n!D_{n+1,k-1} = \frac{1}{k}n!\binom{n-2}{k-1}\binom{n+k-1}{k-1}$$

that, after dividing by $n!$, gives Kirkman-Cayley formula. □

As another application of Theorem 2.2 we show a proof of the formula that counts the number of dissections of a prescribed type. This is a classical result (for another proof see for instance Chapter 13 of [17]; see also Section 2.3. of [9]).

Definition 4.1. *Given a dissection of a convex polygon with $n+1$ labelled edges by $k-1$ diagonals, such that no two of the diagonals intersect in their interior, we say that the dissection is of type $(i_1^{m_1}, i_2^{m_2}, ..., i_s^{m_s})$, with $3 \leq i_1 < i_2 < \cdots < i_s \leq n+1$, if the dissection is made by m_j polygons with i_j edges, for every $j = 1,2,...,s$.*

Remark 4.1. *As one can immediately check, the numbers that appear in the above definition satisfy the relations $\sum_{j=1}^{s} m_j = k$ and $\sum_{j=1}^{s} i_j m_j = n + 2k - 1$.*

Corollary 4.2 (A proof of the formula for the dissections of a prescribed type)**.** *Let n,k be two integers such that $n \geq 2$ and $n-1 \geq k \geq 1$. Given a convex polygon with $n+1$ labelled edges, the number of its dissections of type $(i_1^{m_1}, i_2^{m_2}, ..., i_s^{m_s})$ by $k-1$ diagonals is*

$$\frac{(n+k-1)!}{n!\ m_1!m_2!\cdots m_s!} \tag{1}$$

Proof. Let us consider a convex polygon with $n+1$ edges labelled counterclockwise from 0 to n, as in the example of Figure 3. As we know, a dissection of this polygon by $k-1$ diagonals corresponds to a nested set of the list $1,2,...,n$ and therefore, in view of Theorem 2.2, to an admissible internally ordered k-partition of $[1,n+k-1]$. By construction of the bijection of Theorem 2.2, each part of this partition describes one of the polygons of the dissection, and if it has cardinality a this polygon has $a+1$ edges. Therefore each dissection of type $(i_1^{m_1}, i_2^{m_2}, ..., i_s^{m_s})$ corresponds to an admissible internally ordered partition of $[1,n+k-1]$ that has m_j parts of cardinality $i_j - 1$ for every $j = 1,...,s$.

Let us then denote by A the set of all the admissible internally ordered partitions of $[1,n+k-1]$ that have m_j parts of cardinality $i_j - 1$, for every $j = 1,...,s$. A quick and elementary computation shows that

$$|A| = \frac{(n+k-1)!}{m_1!m_2!\cdots m_s!}$$

Now by Theorem 2.2 we know that each of the partitions in A corresponds to a nested set of a list that is a permutation of the list $1, 2, ..., n$. Then, by S_n-symmetry, the partitions in A that correspond to a nested set of the list $1, 2, ..., n$, i.e., to a dissection of the prescribed type, are exactly

$$\frac{1}{n!}|A| = \frac{1}{n!}\frac{(n+k-1)!}{m_1!m_2!\cdots m_s!}$$

$\square$

Even if our proof of Kirkman-Cayley formula is purely combinatorial, here we sketch out a geometric interpretation. Let us consider the real moduli space $\overline{M}_{0,n+1}$ of stable $n+1$-pointed genus 0 curves.

In [14], [15] some spherical models of subspace arrangements are described, in the spirit of De Concini and Procesi construction of wonderful models in [6]. These spherical models are manifolds with corners, and the minimal spherical model associated with the root arrangement of type A_{n-1} is made by the disjoint union of $n!$ copies of the $(n-2)$-dimensional Stasheff's associahedron (for a concrete realization see [16]). Furthermore, there is a surjective map Γ from this minimal spherical model to the minimal real compact De Concini-Procesi model of type A_{n-1}, that is isomorphic to $\overline{M}_{0,n+1}$.

This map Γ sends the $k-1$-codimensional faces of the associahedra into the $k-1$-codimensional strata of the boundary of $\overline{M}_{0,n+1}$. We notice that these strata are indexed by the elements of $\mathcal{S}_2(n,k)$ and the resulting tessellation coincides with the one previously described in [18], [8] and [9].

The picture above leads to our computation since one observes that the $k-1$-codimensional faces in the minimal spherical model are indexed by $\mathcal{OS}_2((1, 2, ..., n), k)$. On one hand they are $n!D_{n+1,k-1}$, given that $D_{n+1,k-1}$ counts the $k-1$-codimensional faces of a $(n-2)$-dimensional Stasheff's associahedron. On the other hand, one can count them by regrouping the ones whose images via Γ lie in the same isomorphism class of boundary strata of $\overline{M}_{0,n+1}$. Now, two strata belong to the same isomorphism class if and only if their associated nested sets give rise, under the bijection of Theorem 2.1, to two partitions whose parts have the same sizes. This remark points out the bijection, shown in Theorem 2.2, between $\mathcal{OS}_2((1, 2, ..., n), k)$ and $\mathcal{IT}_2(n+k-1, k)$. Then the Kirkman-Cayley formula needs only a last step, i.e. the computation of the cardinality of $\mathcal{IT}_2(n+k-1, k)$, that is provided by the bijection of Theorem 3.1.

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