A linear birth-death process on a star graph and its diffusion approximation

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Abstract

Consider a continuous-time random walk on a lattice formed by the integers of d semiaxes joined at the origin, i.e. a star graph. The motion on each ray behaves as a one-dimensional linear birth-death process with immigration. When the walk reaches the origin, then it may jumps toward any semiaxis. We investigate transient and asymptotic behaviours of the resulting stochastic process, as well as its diffusion approximation.

As a byproduct, we obtain a closed form of the number of permutations with a fixed number of components, and a new series form of the polylogarithm function involving the Gauss hypergeometric function.

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1 Introduction

A relevant stochastic model in the study of population processes of biological and ecological systems is the linear birth-death process with immigration (see, for instance, Section 5 of Ricciardi [26]). This process is also employed in queueing theory for capacity expansion problems (cf. Nucho [23]). An usual approach for the transient analysis of such kind of processes is the method of characteristics (see Zheng *et al.* [30]). Moreover, a graphical argument and a binary tree representation of births and deaths has been used by Branson [3] and [4] to study inhomogeneous birth-death-immigration processes. An extension of the

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birth-death-immigration process has been discussed recently by Jakeman and Hopcraft [21], to describe a family of Markovian population models that include the possibility of multiple immigrations.

Aiming to consider a more general model, in this paper we propose a birth-death process with immigration over a lattice formed by the integers of d semiaxes joined at the origin, i.e. a star graph. Random walks on graphs are often described by Markov processes and deserve interest in many applied fields, such as biology (see, e.g., the review by Volchenkov [29] and references therein). We recall for instance the application of birth-death processes on graphs to evolutionary models of spatially structured populations. See Allen and Tarnita [2] for a comprehensive investigation on population models with fixed population size and structure, which involve state-dependent birth-death processes. In particular, star graphs are worthy of interest in mathematical biology and other applied areas. Recall, for instance, Broom and Rychtář [5], who studied evolutionary dynamics of populations on graphs, with special attention to fixation probabilities.

The object of our investigation, i.e. the birth-death-immigration process on a star graph, is suitable to describe the dynamics of a population formed by d species into a given habitat that is initially empty. As soon as the habitat is occupied by an individual of a certain species (by effect of immigration) then the dynamics evolves according to a linear birth-death-immigration process until extinction. Next the habitat can be occupied again due to immigration of an individual of a possibly different species, and so on. For the considered model we develop a generating function-based approach in order to carry out the transient analysis of its probability law, whereas its asymptotic behaviour is studied making use of the Laplace transform. We also perform a diffusion approximation, which leads to a diffusion process defined on the star graph, having linear drift and infinitesimal variance.

Diffusion processes on graphs have been studied by several authors. We recall for instance Freidlin and Wentzell [14], that is one of the first contributions on this topic. Recently, some results for Brownian motion on a general oriented metric graph have been given in Hajri and Raimond [17]. An investigation involving a diffusion process on star graph has been performed in Papanicolaou *et al.* [25], where the authors obtain exit probabilities and certain other quantities involving exit and occupation times for a Brownian Motion on star graph. Other examples of diffusion processes on star graphs have been studied in Mugnolo *et al.* [22].

Let us now provide the plan of the paper. In Section 2 we describe the stochastic model and introduce the state probabilities. Some formal relations for the related generating functions are also provided. This allows to obtain a formal expression for the transient probability that the walk is located in the origin, whose proof is provided in Appendix A. In Section 3 we perform the transient analysis of the process in two cases: (i) when the birth, death and immigration parameters are equal, and (ii) when the birth and death parameters are different, whereas the immigration and birth parameters are equal. In both cases we obtain

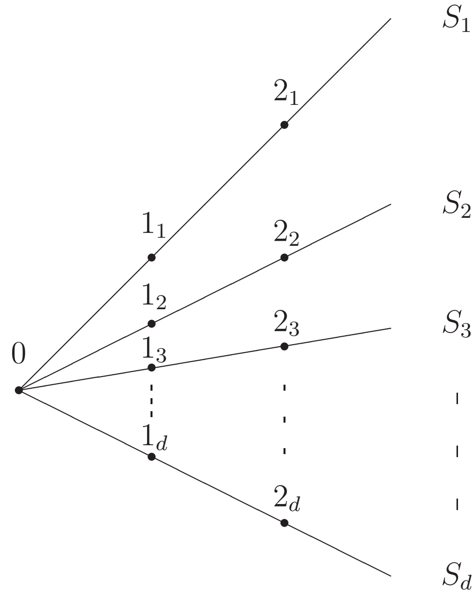

Figure 1: Schematic representation of the state space S.

some series-form expressions for the relevant transient probabilities. Section 4 is devoted to obtain the asymptotic expression of the state probabilities, that involves a zero-modified negative binomial distribution. Section 5 deals with the diffusion approximation of the process. We adopt a suitable scaling leading to a time-homogeneous diffusion process on the star graph, characterized by linear infinitesimal moments. A gamma-type stationary density is also obtained under suitable assumptions.

It is worth pointing out that, as a byproduct of our investigations, we are able to provide some new results which are of interest in other fields of mathematics. Indeed, in Section 3
(i) we obtain a closed form of the number of permutations of $\{1,\dots,n\}$ with k components, also known as number of permutations with $k-1$ global descents;
(ii) we prove a new series form of the polylogarithm function expressed in terms of the Gauss hypergeometric function.

2 The stochastic model

Consider a particle moving randomly on a set S consisting of the integers of d semiaxes $S_1, S_2, \dots, S_d$ ($d \in \mathbb{N}$) with a common origin 0. Let $\{N(t), t \geq 0\}$ be a continuous-time stochastic process, with state space S, describing the position of the particle at time t. Throughout this paper we denote by k_j the k-th state, $k \in \mathbb{N}^+$, located on the semiaxis S_j (see Figure 1) and by

$$p(k_j,t) = \mathbb{P}\{N(t) = k_j\}, \qquad t > 0, \tag{1}$$

the probability that the particle at time t is located in k_j, for $j = 1, 2, \ldots, d$. Moreover, we set $0_j = 0$ and assume that

$$p(0,0) = 1, \tag{2}$$

i.e. initially the particle is located a.s. in the origin 0. Let $N(t)$, $t \geq 0$, be a Markov process. Denoting its transition rates by

$$q(u,v) = \lim_{h \to 0^+} \frac{1}{h} \mathbb{P}[N(t+h) = v \,|\, N(t) = u], \qquad u, v \in S,$$

for all $j = 1, 2, \ldots, d$ and $k \in \mathbb{N}^+$ we assume that

$$\begin{aligned} q(0, 1_j) &= \alpha, \\ q(k_j, (k+1)_j) &= \alpha + \lambda k, \\ q(k_j, (k-1)_j) &= \mu k, \end{aligned} \tag{3}$$

where α, λ and μ are positive constants. We remark that if $d = 1$ then $N(t)$ identifies with the linear birth-death process with immigration, where α is the immigration rate, λ is the birth rate, and μ is the death rate per individual. Due to (1),

$$P(k,t) := \sum_{j=1}^{d} p(k_j, t), \qquad k \in \mathbb{N}^+, \quad t \geq 0 \tag{4}$$

is the probability that at time t the particle is located in the k-th state of any semiaxis.

In order to investigate the probability law of $N(t)$, hereafter we adopt a probability generating function-based approach and set

$$F(z,t) = p(0,t) + \sum_{k=1}^{+\infty} z^k P(k,t), \qquad 0 \leq z \leq 1, \quad t \geq 0. \tag{5}$$

Such generating function, by virtue of (2), satisfies the initial condition

$$F(z,0) = 1, \qquad 0 \leq z \leq 1. \tag{6}$$

Moreover, the following boundary conditions hold:

$$F(0,t) = p(0,t), \qquad t \geq 0, \tag{7}$$

$$F(1,t) = 1, \qquad t \geq 0. \tag{8}$$

Proposition 2.1 *The generating function (5) satisfies the following differential equation for* $0 \leq z \leq 1$ *and* $t \geq 0$*:*

$$\frac{\partial}{\partial t} F(z,t) = -\alpha(d-1)(1-z)p(0,t) - \alpha(1-z)F(z,t) - (\lambda z - \mu)(1-z)\frac{\partial}{\partial z} F(z,t). \tag{9}$$

Proof. Due to (3), for $k_j \in \mathbb{N}^+$ and $j = 1, 2, \dots, d$, the following system of differential-difference equations holds for $t > 0$:

$$\frac{\partial}{\partial t}\, p(0,t) = \mu \sum_{j=1}^{d} p(1_j, t) - d\alpha\, p(0,t),$$
$$\frac{\partial}{\partial t}\, p(k_j,t) = [\alpha + \lambda(k-1)]\, p((k-1)_j, t) + \mu(k+1)\, p((k+1)_j, t) - [\alpha + (\lambda+\mu)k]\, p(k_j, t). \tag{10}$$

Hence, the probability generating function

$$G_j(z,t) := \sum_{k=1}^{+\infty} z^k p(k_j, t), \qquad 0 \le z \le 1, \quad t \ge 0, \tag{11}$$

for $j = 1, 2, \dots, d$ satisfies the following differential equation:

$$\frac{\partial}{\partial t}\, G_j(z,t) = \alpha z[p(0,t) + G_j(z,t)] + \lambda z^2 \frac{\partial}{\partial z} G_j(z,t)$$
$$+\mu \frac{\partial}{\partial z}[G_j(z,t) - z p(1_j,t)] - \alpha G_j(z,t) - (\lambda+\mu) z \frac{\partial}{\partial z} G_j(z,t). \tag{12}$$

Due to (5) and (11) we have

$$F(z,t) = p(0,t) + \sum_{j=1}^{d} G_j(z,t), \qquad 0 \le z \le 1, \quad t \ge 0.$$

Hence, the proof of (9) follows from Eq. (12). □

In the following proposition we solve the partial differential equation (9). Here, and in the following, f' denotes the derivative of any function f.

Proposition 2.2 *Eq. (9), with conditions (6) and (8), admits the following solution for* $0 \le z \le 1$ *and* $t \ge 0$*:*

$$F(z,t) = H(t) + (d-1)\int_0^t H'(t-y) p(0,y)\mathrm{d}y, \tag{13}$$

where

$$H(t) = h(t, z; \lambda, \mu) := \begin{cases} \dfrac{(\lambda-\mu)^{\frac{\alpha}{\lambda}} \mathrm{e}^{-\frac{\alpha}{\lambda}(\lambda-\mu)t}}{\left[\lambda(1-z) - (\mu - \lambda z)\, \mathrm{e}^{-(\lambda-\mu)t}\right]^{\frac{\alpha}{\lambda}}}, & \lambda \ne \mu, \\ \dfrac{1}{[1+\lambda t(1-z)]^{\frac{\alpha}{\lambda}}}, & \lambda = \mu. \end{cases} \tag{14}$$

Proof. Let us adopt the method of characteristics. If $\lambda \ne \mu$, Eq. (9) can be rewritten as

$$(\lambda z - \mu)(1-z)\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} + \alpha(1-z)F + \alpha(d-1)(1-z)p(0,t) = 0, \tag{15}$$

which gives the following characteristic equations for the original system

$$\begin{cases} \dfrac{\partial z}{\partial s} = (\lambda z - \mu)(1 - z), \\ \dfrac{\partial t}{\partial s} = 1, \\ \dfrac{\partial F}{\partial s} = -\alpha(1 - z)F - \alpha(d - 1)(1 - z)p(0, t). \end{cases} \tag{16}$$

From Eq. (16), along the characteristic curves

$$z = 1 - \frac{(\lambda - \mu)(1 - \tau)}{\lambda(1 - \tau) - (\mu - \lambda\tau)\mathrm{e}^{(\lambda-\mu)s}}, \qquad t = s, \qquad \tau \in \mathbb{R},$$

the partial differential equation (15) and conditions (6) and (8) yield

$$\begin{cases} \dfrac{\mathrm{d}F}{\mathrm{d}s} + \alpha \left[\dfrac{(\lambda - \mu)(1 - \tau)}{\lambda(1 - \tau) - (\mu - \lambda\tau)\mathrm{e}^{(\lambda-\mu)s}}\right] F + \alpha(d - 1) \left[\dfrac{(\lambda - \mu)(1 - \tau)}{\lambda(1 - \tau) - (\mu - \lambda\tau)\mathrm{e}^{(\lambda-\mu)s}}\right] p(0, s) = 0, \\ F(0) = 1. \end{cases}$$

Hence, Eqs. (13) and (14) follow after some calculations. If $\lambda = \mu$, the proof is similar. □

Hereafter we show that probability $p(0, t)$ satisfies a linear Volterra integral equation of the second kind.

Corollary 2.1 *The following integral equation holds, for* $t > 0$*,*

$$p(0, t) = 1 - G(t) - (d - 1) \int_0^t G'(t - y)p(0, y)\mathrm{d}y, \tag{17}$$

where

$$G(t) = 1 - h(t, 0; \lambda, \mu), \tag{18}$$

with $h(t, z; \lambda, \mu)$ *defined in (14).*

Proof. Eqs. (17) and (18) follow from Eqs. (13) and (14), for $z = 0$ and recalling initial condition (7). □

We remark that the function $G(t)$, given in (18), is a proper distribution function when $\lambda \geq \mu$.

Hereafter we consider the distribution function

$$F_Y^{(j)}(t) := \mathbb{P}(Y_1 + Y_2 + \cdots + Y_j \leq t), \tag{19}$$

where $Y_1, Y_2, \ldots, Y_j$ is a sequence of arbitrary i.i.d. random variables. In the following theorem we give a formal representation of probability $p(0, t)$ in terms of (19) when Y_i's have certain specific distribution.

Theorem 2.1 *For $t \geq 0$ we have*

$$p(0,t) = \begin{cases} 1 - d\displaystyle\sum_{j=1}^{+\infty}(1-d)^{j-1}F_Y^{(j)}(t), & \lambda \geq \mu, \\ 1 - d\displaystyle\sum_{j=1}^{+\infty}(1-d)^{j-1}\left[1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right]^j F_Y^{(j)}(t), & \lambda < \mu, \end{cases} \tag{20}$$

where

$$F_Y^{(1)}(t) = \begin{cases} 1-\left(\dfrac{\lambda-\mu}{\lambda \mathrm{e}^{(\lambda-\mu)t}-\mu}\right)^{\frac{\alpha}{\lambda}}, & \lambda > \mu, \\ 1-\dfrac{1}{(1+\lambda t)^{\frac{\alpha}{\lambda}}}, & \lambda = \mu, \\ \dfrac{1}{1-\left(1-\frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}\left[1-\left(\dfrac{\mu-\lambda}{\mu-\lambda \mathrm{e}^{-(\mu-\lambda)t}}\right)^{\frac{\alpha}{\lambda}}\right], & \lambda < \mu. \end{cases} \tag{21}$$

The proof of Theorem 2.1 is given in Appendix A.

Remark 2.1 The right-hand-side of Eq. (21), in each of the three cases, identifies with the distribution function of suitable transformations of a random variable, say Z, having Pareto type II (Lomax) distribution with shape and scale parameters $\tilde{\alpha}$ and $\tilde{\lambda}$, respectively. Namely,
- if $\lambda > \mu$, then $F_Y^{(1)}(t)$ is the distribution function of $\log(Z+1)/(\lambda-\mu)$, with $\tilde{\alpha} = \alpha/\lambda$ and $\tilde{\lambda} = (\lambda-\mu)/\lambda$,
- if $\lambda = \mu$, then $F_Y^{(1)}(t)$ is the distribution function of Z for $\tilde{\alpha} = \alpha/\lambda$ and $\tilde{\lambda} = 1/\lambda$,
- if $\lambda < \mu$, then $F_Y^{(1)}(t)$ is the distribution function of $-\log(1-Z)/(\mu-\lambda)$, assuming that Z has support $(0,1)$ and parameters $\tilde{\alpha} = \alpha/\lambda$ and $\tilde{\lambda} = (\mu-\lambda)/\lambda$.

3 Transient analysis

In this section, for t ranging over specified intervals of $\mathbb{R}$, we obtain explicit expressions of $p(0,t)$, generating function $F(z,t)$ and cumulative probability $P(k,t)$. We consider 2 cases:
(i) $\mu = \alpha = \lambda$,
(ii) $\mu \neq \lambda$ and $\alpha = \lambda$.

3.1 Transient analysis for $\mu = \alpha = \lambda$

Let us denote by $t_{n,k}$ the number of permutations of $\{1,\dots,n\}$, $n \geq 1$, with $k \geq 1$ components (see, for instance, Comtet [8], p. 262 and [24]). Alternatively, $t_{n,k}$ is the number of permutations of $\{1,\dots,n\}$ with $k-1$ global descents. Permutations with one component, i.e. $t_{n,1}$, are known as indecomposable permutations (we recall that a permutation is called indecomposable if its one-line notation cannot be split into two parts such that every number in the first part is smaller than every number in the second part). An implicit recursion formula for $t_{n,k}$ is given by (see Propositions 2.4 and 2.7 of Hegarty and Martinsson [19], for cases $k = 1$ and $2 \le k \le n$, respectively)

$$t_{n,k} = \begin{cases} n! - \sum_{j=1}^{n-1} (n-j)! \cdot t_{j,1}, & k = 1, \\ \sum_{j=1}^{n-k+1} t_{j,1} \cdot t_{n-j,k-1}, & 2 \le k \le n, \\ 0, & n < k. \end{cases} \tag{22}$$

Proposition 3.1 *For $\mu = \alpha = \lambda$ and $0 < t < 1/\lambda$ the integral equation (17) admits the following solution:*

$$p(0,t) = 1 + \sum_{n=1}^{+\infty} \frac{(-\lambda t)^n}{n!} \sum_{j=1}^{n} t_{n,j}\, d^j. \tag{23}$$

Proof. The proof is based on the coupling of the homotopy perturbation method and the expansion of the involved functions as Taylor series (see Biazar and Eslami [9]). From (17), for $\alpha = \lambda = \mu$, we can construct the following homotopy

$$H(p,q) = p(0,t) - \frac{1}{(1+\lambda t)} + \lambda(d-1) q \int_0^t \frac{p(0,y)}{[1+\lambda(t-y)]^2} \mathrm{d}y = 0, \tag{24}$$

with the embedding parameter q. By assuming that $p(0,t) = \sum_{n=0}^{+\infty} q_n(t) q^n$ and substituting functions $\frac{1}{(1+\lambda t)}$ and $\frac{1}{[1+\lambda(t-y)]^2}$ by their Taylor series forms, in agreement with Eq. (24) we define

$$\widetilde{H}(p,q) = \sum_{n=0}^{+\infty} q_n(t) q^n - \sum_{n=0}^{+\infty} (-\lambda t)^n q^n + \lambda(d-1) \sum_{n=0}^{+\infty} q^{n+1} \int_0^t \alpha_n(y,t) \mathrm{d}y = 0, \tag{25}$$

for $0 < t < 1/\lambda$, with

$$\alpha_n(y,t) = \sum_{k=0}^{n} q_k(y)(n-k+1)[-\lambda(t-y)]^{n-k}. \tag{26}$$

Hence, equating the coefficients of the terms with identical powers of q, we find that function $q_n(x)$ is solution of the following recursive equation:

$$q_n(x) = (-\lambda x)^n - \lambda(d-1) \int_0^x \alpha_{n-1}(y,x) \mathrm{d}y, \qquad n \in \mathbb{N}. \tag{27}$$

with $q_0(x) = 1$. By direct calculations, from Eq. (27) one immediately gets

$$q_1(x) = -d\lambda x.$$

Hereafter we make use of the strong induction principle to show that

$$q_n(x) = \frac{(-\lambda x)^n}{n!} \sum_{j=1}^{n} t_{n,j}\, d^j, \qquad n \in \mathbb{N}. \tag{28}$$

Being $t_{k,k} = 1$ for all $k \geq 1$ (see [8], p. 262), Eq. (28) holds for $n = 1$. Assuming that (28) holds for all $k = 1, 2, \ldots, n$ we now prove that it holds true for $k = n + 1$. From Eq. (26), due to the induction hypothesis, we have

$$\int_0^x \alpha_n(y, x) \mathrm{d}y = x \frac{(-\lambda x)^n}{(n+1)!} \sum_{j=1}^{n} (n+1-j)! \sum_{r=1}^{j} t_{j,r}\, d^r + x(-\lambda x)^n, \qquad n \in \mathbb{N}.$$

Hence, recalling Eq. (27) and using $t_{k,k} = 1\ \forall k \geq 1$, we obtain

$$\begin{aligned} q_{n+1}(x) &= \frac{(-\lambda x)^{n+1}}{(n+1)!} \left\{ d\,(n+1)! + (d-1) \sum_{j=1}^{n} (n+1-j)! \sum_{r=1}^{j} t_{j,r}\, d^r \right\} \\ &= \frac{(-\lambda x)^{n+1}}{(n+1)!} \left\{ d \left[(n+1)! - \sum_{j=1}^{n} (n+1-j)!\, t_{j,1} \right] + d^{n+1} \right. \\ &\quad \left. + \sum_{s=2}^{n} d^s \left[(n-s+2)! + \sum_{j=s}^{n} (n+1-j)!(t_{j,s-1} - t_{j,s}) \right] \right\}. \end{aligned} \tag{29}$$

Recalling Eq. (22), we note that

$$\sum_{j=s}^{n} (n+1-j)! \cdot t_{j,s} = \sum_{r=s-1}^{n-1} t_{r,s-1} \sum_{j=r+1}^{n} (n+1-j)! \cdot t_{j-r,1}.$$

Hence, repeated applications of Eq. (22) yield

$$\begin{aligned} &(n-s+2)! t_{s-1,s-1} + \sum_{j=s}^{n} (n+1-j)!(t_{j,s-1} - t_{j,s}) \\ &\quad = (n-s+2)! + \sum_{r=s}^{n-1} t_{r,s-1} \left[(n+1-r)! - \sum_{j=r+1}^{n} (n+1-j)! t_{j-r,1} \right] \\ &\quad + t_{n,s-1} - \sum_{j=s}^{n} (n+1-j)! t_{j-s+1,1} \\ &\quad = (n-s+2)! + \sum_{r=s}^{n} t_{r,s-1} \cdot t_{n+1-r,1} - \sum_{j=s}^{n} (n+1-j)! t_{j-s+1,1} \\ &\quad = t_{n-s+2,1} + \sum_{r=s}^{n} t_{r,s-1} \cdot t_{n+1-r,1} = t_{n+1,s}. \end{aligned} \tag{30}$$

From Eqs. (29) and (30) we thus obtain Eq. (28). Finally, by taking $q = 1$ in assumption $p(0,t) = \sum_{n=0}^{+\infty} q_n(t) q^n$ we get Eq. (23). □

Proposition 3.2 *If $\mu = \alpha = \lambda$, then for $0 < t < 1/\lambda$ we have*

$$F(z,t) = \frac{1}{1+\lambda t(1-z)} - \frac{\lambda t(d-1)(1-z)}{1+\lambda t(1-z)} \left[1 + \sum_{n=1}^{+\infty} \frac{(-\lambda t)^n}{n!} \sum_{j=1}^{n} t_{n,j}\, d^j\right]$$
$$-\frac{(d-1)(1-z)}{[1+\lambda t(1-z)]^2} \sum_{n=1}^{+\infty} n \frac{(-\lambda t)^{n+1}}{(n+1)!}\, {}_2F_1\left(1, n+1; n+2; 1 - \frac{1}{1+\lambda t(1-z)}\right) \sum_{j=1}^{n} t_{n,j}\, d^j, \quad (31)$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (32)$$

is the Gauss hypergeometric function. (Here, and in the remainder of the paper, $(d)_n = d(d+1)(d+2)\cdots(d+n-1)$, $n \geq 1$, denotes the Pochhammer symbol, with $(d)_0 = 1$ for $d \neq 0$.)

Proof. From Eqs. (13), (14) and (23), we obtain

$$F(z,t) = \frac{1}{1+\lambda t(1-z)} - \frac{\lambda t(d-1)(1-z)}{1+\lambda t(1-z)} - (d-1) \sum_{n=1}^{+\infty} \frac{(-\lambda t)^n}{n!} \sum_{j=1}^{n} t_{n,j}\, d^j$$
$$+\frac{(d-1)}{1+\lambda t(1-z)} \sum_{n=1}^{+\infty} \frac{(-\lambda t)^n}{n!}\, {}_2F_1\left(1, n; n+1; 1 - \frac{1}{1+\lambda t(1-z)}\right) \sum_{j=1}^{n} t_{n,j}\, d^j.$$

Hence, making use of (see, for instance, Eqs. 15.2.25 and 15.1.8 of [1]),

$$-c\,{}_2F_1(a,b;c;z) + (c-a)z\,{}_2F_1(a,b+1;c+1;z) - c(z-1)\,{}_2F_1(a,b+1;c;z) = 0, \quad (33)$$

$${}_2F_1(a,b;b;z) = (1-z)^{-a},$$

after some calculations we obtain Eq. (31). □

In the following proposition we obtain probability (4) under the assumptions of Propositions 3.1 and 3.2.

Proposition 3.3 *If $\mu = \alpha = \lambda$, for $0 < t < 1/\lambda$ we have*

$$P(k,t) = \frac{(\lambda t)^k}{(\lambda t+1)^{k+1}} \left\{ d + (d-1) \sum_{n=1}^{+\infty} \frac{(-\lambda t)^n}{n!} \sum_{j=1}^{n} t_{n,j}\, d^j \right.$$
$$\left. \times \sum_{r=0}^{+\infty} \frac{(n)_r}{(n+1)_r} \left(1 - \frac{1}{1+\lambda t}\right)^r {}_2F_1\left(-r, -k; 1; -\frac{1}{\lambda t}\right) \right\}. \quad (34)$$

Proof. From Eq. (31), and recalling (23) and (32), we have

$$
\begin{aligned}
F(z,t) = &\frac{1}{1+\lambda t(1-z)} - (d-1)p(0,t)\frac{\lambda t(1-z)}{1+\lambda t(1-z)} \\
&-\frac{(d-1)(1-z)}{[1+\lambda t(1-z)]^2}\sum_{k=0}^{+\infty}\left[1-\frac{1}{1+\lambda t(1-z)}\right]^k \sum_{n=1}^{+\infty}\frac{(-\lambda t)^{n+1}}{(n-1)!(n+k+1)}\sum_{j=1}^{n} t_{n,j}\, d^j \\
= &\, p(0,t) + \sum_{m=0}^{+\infty} z^m \frac{(\lambda t)^m}{(\lambda t+1)^{m+1}} - p(0,t)\left[\frac{1+d\lambda t}{1+\lambda t} - (d-1)\sum_{m=1}^{+\infty} z^m \frac{(\lambda t)^m}{(\lambda t+1)^{m+1}}\right] \\
&-(d-1)\sum_{m=0}^{+\infty} z^m \frac{(\lambda t)^m}{(\lambda t+1)^{m+2}}\sum_{n=1}^{+\infty}\frac{(-\lambda t)^{n+1}}{(n-1)!}\sum_{j=1}^{n} t_{n,j}\, d^j \\
&\times \sum_{k=0}^{+\infty}\frac{\left(\frac{\lambda t}{\lambda t+1}\right)^k}{n+k+1}\, {}_2F_1\left(-k-1,-m;1;-\frac{1}{\lambda t}\right).
\end{aligned}
$$

Hence, since $p(0,t)$ satisfies the integral equation (17), it results

$$
\begin{aligned}
F(z,t) = &\, p(0,t) + \sum_{m=1}^{+\infty} z^m \frac{(\lambda t)^m}{(\lambda t+1)^{m+1}} + p(0,t)(d-1)\sum_{m=1}^{+\infty} z^m \frac{(\lambda t)^m}{(\lambda t+1)^{m+1}} \\
&+(d-1)\sum_{m=1}^{+\infty} z^m \frac{(\lambda t)^{m+1}}{(\lambda t+1)^{m+2}}\sum_{n=1}^{+\infty}\frac{(-\lambda t)^{n}}{(n-1)!}\sum_{j=1}^{n} t_{n,j}\, d^j \\
&\times \sum_{k=0}^{+\infty}\frac{(n+1)_k}{(n+2)_k\,(n+1)}\left(\frac{\lambda t}{\lambda t+1}\right)^k {}_2F_1\left(-k-1,-m;1;-\frac{1}{\lambda t}\right).
\end{aligned}
$$

Finally, recalling Eq. (5) and equating the coefficients of the terms with identical powers of z we obtain Eq. (34). $\square$

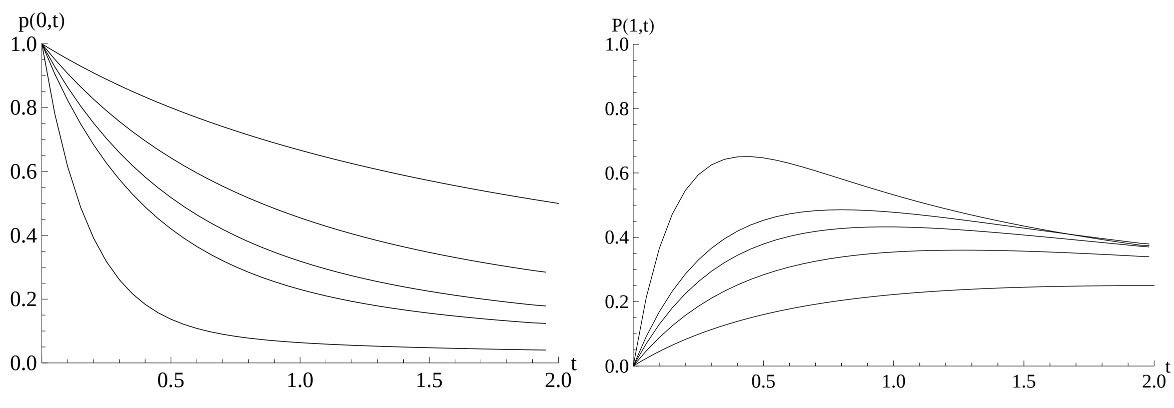

Figure 2: Plot of $p(0,t)$ and $P(1,t)$ for $\lambda = 0.5$, $\mu = 0.5$ and $\alpha = 0.5$, for $d = 1,2,3,4,10$, from top to bottom for $p(0,t)$, and from bottom to top for $P(1,t)$.

Figure 2 shows some plots obtained by means of the expressions given in Proposition 3.1 and Proposition 3.3.

3.2 Transient analysis for $\mu \neq \lambda$ and $\alpha = \lambda$

In order to investigate the case $\mu \neq \lambda$ and $\alpha = \lambda$, for brevity we set

$$Q_{j,m} \equiv Q_{j,m}\left(\frac{\mu}{\lambda}\right) := \sum_{\substack{s_1,\dots,s_j \geq 2 \\ s_1+\cdots+s_j=m}} A_{s_1}\left(\frac{\mu}{\lambda}\right) \times \cdots \times A_{s_j}\left(\frac{\mu}{\lambda}\right), \tag{35}$$

where $A_n(t)$ are the Eulerian Polynomials (see, for instance, Foata [13] or Hirzebruch [20]).

Proposition 3.4 *If $\alpha = \lambda$ and $\mu \neq \lambda$, then for $0 < t < \log(\mu/\lambda)/(\mu - \lambda)$ the integral equation (17) admits the following solution*

$$p(0,t) = 1 - d\left\{1 - \frac{\mu-\lambda}{\mu - \lambda \mathrm{e}^{-(\mu-\lambda)t}} - \frac{1}{d-1}\left[\mathrm{e}^{-\lambda t(d-1)} - 1 + \lambda t(d-1)\right]\right.$$
$$\left. - \sum_{n=3}^{\infty} \frac{(-\lambda t)^n}{n!} \sum_{k=1}^{n-2} (d-1)^k \sum_{j=1}^{n-k-1} \binom{k+1}{j} Q_{j,j+n-k-1}\right\}. \tag{36}$$

Proof. The proof proceeds similarly as that of Proposition 3.1. Recalling Eq. (18), for $0 < t < \log(\mu/\lambda)/(\mu - \lambda)$ we have

$$1 - G(t) = \frac{\frac{\mu}{\lambda} - 1}{\frac{\mu}{\lambda} - \mathrm{e}^{-\lambda t\left(\frac{\mu}{\lambda}-1\right)}} = \sum_{n=0}^{\infty} A_n\left(\frac{\mu}{\lambda}\right)\frac{(-\lambda t)^n}{n!}. \tag{37}$$

Note that the radius of convergence of the power series (37) has been determined finding the location (in the complex plane) of the singularity nearest to the origin. We can construct the following homotopy

$$H(p,q) = \sum_{n=0}^{+\infty} q_n(t) q^n - \sum_{n=0}^{+\infty} A_n\left(\frac{\mu}{\lambda}\right)\frac{(-\lambda t)^n}{n!} q^n$$
$$+\lambda(d-1)\sum_{n=0}^{+\infty} q^{n+1} \sum_{j=0}^{n} A_{j+1}\left(\frac{\mu}{\lambda}\right)\int_0^t \frac{(-\lambda y)^j}{j!} q_{n-j}(t-y)\mathrm{d}y = 0, \tag{38}$$

where q is the embedding parameter and we have set $p(0,t) = \sum_{n=0}^{+\infty} q_n(t) q^n$. We thus find that $q_n(t)$ satisfies the following recursive equation:

$$q_n(t) = A_n\left(\frac{\mu}{\lambda}\right)\frac{(-\lambda t)^n}{n!} - \lambda(d-1)\sum_{j=0}^{n-1} A_{j+1}\left(\frac{\mu}{\lambda}\right)\int_0^t \frac{(-\lambda y)^j}{j!} q_{n-1-j}(t-y)\mathrm{d}y. \tag{39}$$

By straightforward calculations, from Eq. (39) one immediately gets

$$q_0(t) = 1, \qquad q_1(t) = -d\lambda t, \qquad q_2(t) = \lambda d(\mu + \lambda d)\frac{t^2}{2}. \tag{40}$$

Let us now make use of the strong induction principle to show that, for $n \geq 3$,

$$q_n(t) = d\,\frac{(-\lambda t)^n}{n!}\left\{\sum_{k=1}^{n-2}(d-1)^k \sum_{j=1}^{n-k-1}\binom{k+1}{j} Q_{j,j+n-k-1} + A_n\left(\frac{\mu}{\lambda}\right) + (d-1)^{n-1}\right\}. \tag{41}$$

By direct calculations, it follows from (39) and (40) that

$$q_3(t) = -\lambda d[\lambda^2 d^2 + 2\lambda\mu(d+1) + \mu^2]\frac{t^3}{3!},$$

which is equal to Eq. (41) for $n = 3$, being $A_3(z) = 1 + 4z + z^2$. Let us consider $n \geq 3$ and assume that Eq. (41) holds for all $r = 2, \dots, n-1$. We shall prove that identity (41) holds also for $r = n$. From Eq. (39), recalling (40) we have

$$\begin{aligned} q_n(t) = {} & dA_n\left(\frac{\mu}{\lambda}\right)\frac{(-\lambda t)^n}{n!} + d(d-1)A_{n-1}\left(\frac{\mu}{\lambda}\right)\frac{(-\lambda t)^n}{n!} \\ & -\lambda(d-1)\sum_{r=2}^{n-1} A_{n-r}\left(\frac{\mu}{\lambda}\right)\int_0^t \frac{(-\lambda y)^{n-1-r}}{(n-1-r)!} q_r(t-y)\mathrm{d}y. \end{aligned}$$

Hence, due to the induction hypothesis (41), we obtain

$$\begin{aligned} q_n(t) = {} & d\frac{(-\lambda t)^n}{n!}\left\{A_n\left(\frac{\mu}{\lambda}\right) + (d-1)A_{n-1}\left(\frac{\mu}{\lambda}\right) + (d-1)\sum_{r=2}^{n-1} A_{n-r}\left(\frac{\mu}{\lambda}\right)A_r\left(\frac{\mu}{\lambda}\right)\right. \\ & \left. + \sum_{r=2}^{n-1}(d-1)^r A_{n-r}\left(\frac{\mu}{\lambda}\right) + \sum_{r=3}^{n-1} A_{n-r}\left(\frac{\mu}{\lambda}\right)\sum_{k=1}^{r-2}(d-1)^{k+1}\sum_{j=1}^{r-k-1}\binom{k+1}{j}Q_{j,j+r-k-1}\right\} \\ = {} & d\frac{(-\lambda t)^n}{n!}\left\{A_n\left(\frac{\mu}{\lambda}\right) + \sum_{r=1}^{n-1}(d-1)^r A_{n-r}\left(\frac{\mu}{\lambda}\right)\right. \\ & \left. + \sum_{h=1}^{n-2}(d-1)^h\sum_{j=1}^{n-1-h}\binom{h}{j}\sum_{r=1}^{n-j-h} A_r\left(\frac{\mu}{\lambda}\right)Q_{j,n+j-h-r}\right\}. \end{aligned} \tag{42}$$

Noting that

$$\sum_{r=1}^{n-j-h} A_r\left(\frac{\mu}{\lambda}\right)Q_{j,n+j-h-r} = Q_{j,n+j-h-1} + Q_{j+1,n+j-h},$$

from Eq. (42) we obtain

$$\begin{aligned} q_n(t) = {} & d\frac{(-\lambda t)^n}{n!}\left\{A_n\left(\frac{\mu}{\lambda}\right) + \sum_{r=1}^{n-1}(d-1)^{n-r}A_r\left(\frac{\mu}{\lambda}\right)\right. \\ & \left. + \sum_{r=2}^{n-1}(d-1)^{n-r}\sum_{k=1}^{r-1}\binom{n-r}{k}Q_{k,r+k-1} + \sum_{r=2}^{n-1}(d-1)^{n-r}\sum_{k=2}^{r}\binom{n-r}{k-1}Q_{k,r+k-1}\right\} \\ = {} & d\frac{(-\lambda t)^n}{n!}\left\{A_n\left(\frac{\mu}{\lambda}\right) + \sum_{r=1}^{n-1}(d-1)^{n-r}A_r\left(\frac{\mu}{\lambda}\right) + \sum_{r=2}^{n-1}(d-1)^{n-r}(n-r)A_r\left(\frac{\mu}{\lambda}\right)\right. \\ & \left. + \sum_{r=2}^{n-1}(d-1)^{n-r}\sum_{k=2}^{r-1}\binom{n-r+1}{k}Q_{k,r+k-1}\right\}. \\ = {} & d\frac{(-\lambda t)^n}{n!}\left\{A_n\left(\frac{\mu}{\lambda}\right) + (d-1)^{n-1} + \sum_{r=2}^{n-1}(d-1)^{n-r}\sum_{k=1}^{r-1}\binom{n-r+1}{k}Q_{k,r+k-1}\right\}, \end{aligned}$$

which gives Eq. (41). By setting $q = 1$ in assumption $p(0,t) = \sum_{n=0}^{+\infty} q_n(t)q^n$, and recalling (41), we finally obtain Eq. (36). □

Remark 3.1 If $d = 1$ the expressions for $p(0,t)$ given in Theorem 2.1, Proposition 3.1 and Proposition 3.4 are in agreement with the well-known results for the linear birth-death process with immigration (see, for instance, Section 2.3 of Nucho [23]).

Hereafter we obtain an explicit expression for $t_{n,k}$ in terms of multinomial coefficients, for $n \geq 2$ and $1 \leq k \leq n$. It is worth pointing out that a closed form expression for such numbers does not appear to have been obtained before.

Corollary 3.1 *The following equalities hold for $n \geq 2$:*

$$t_{n,1} = n! + (-1)^{n-1} + \sum_{k=1}^{n-2}(-1)^k \sum_{j=1}^{n-k-1} \binom{k+1}{j} (n-k-1+j)! \sum_{\substack{s_1,\dots,s_j \geq 2 \\ s_1+\dots+s_j = n-k-1+j}} \frac{1}{\binom{n-k-1+j}{s_1,\dots,s_j}};$$

$$t_{n,k} = \binom{n-1}{k-1}(-1)^{n-k} + \sum_{r=k-1}^{n-2} \binom{r}{k-1}(-1)^{r-k+1} \sum_{j=1}^{n-r-1} \binom{r+1}{j}(n-r-1+j)!$$
$$\times \sum_{\substack{s_1,\dots,s_j \geq 2 \\ s_1+\dots+s_j = n-r-1+j}} \frac{1}{\binom{n-r-1+j}{s_1,\dots,s_j}}, \qquad 2 \leq k \leq n-1;$$

$$t_{n,n} = 1.$$

Proof. The proof follows from Propositions 3.1 and 3.4, by letting $\mu \to \lambda$ and noting that $A_j(1) = j!$. □

In the following proposition we obtain the probability generating function when $\mu \neq \lambda$ and $\alpha = \lambda$. In the sequel we shall denote by

$$\theta_n \equiv \theta_n(\lambda, \mu, d) := \sum_{i=1}^{n-2}(d-1)^i \sum_{j=1}^{n-i-1} \binom{i+1}{j} Q_{j,j+n-i-1} + A_n\left(\frac{\mu}{\lambda}\right) + (d-1)^{n-1}, \tag{43}$$

where $Q_{j,m}$ is defined in Eq. (35).

Proposition 3.5 *If $\mu \neq \lambda$ and $\alpha = \lambda$, for $t < \log(\mu/\lambda)/(\mu - \lambda)$, it is*

$$F(z,t) = 1 - d + \frac{d(\mu-\lambda)}{\mu - \lambda z - \lambda(1-z)\mathrm{e}^{-(\mu-\lambda)t}} - \frac{d(d-1)(\mu-\lambda)^2}{\mu - \lambda z}$$
$$\times \Big[-\lambda g_1(z) + \sum_{n=2}^{+\infty} \frac{(-\lambda)^n}{n!} g_n(z)\theta_n\Big] \tag{44}$$

where

$$g_k(z)=\begin{cases}\frac{1}{(\lambda-\mu)^{k+1}}\Big\{\lambda(1-z)(\lambda-\mu)^{k-1}t^k+k!\Big[\mathrm{Li}_k\left(\frac{\lambda(1-z)\mathrm{e}^{-(\mu-\lambda)t}}{\mu-\lambda z}\right)\\ \qquad -\mathrm{Li}_k\left(\frac{\lambda(1-z)}{\mu-\lambda z}\right)\Big]-\sum_{r=1}^{k-1}\frac{k!}{r!}[(\lambda-\mu)t]^r\mathrm{Li}_{k-r}\left(\frac{\lambda(1-z)}{\mu-\lambda z}\right)\Big\}, & \text{if } \mu>\lambda,\\ \frac{1}{(\mu-\lambda)^{k+1}}\Big\{(\mu-\lambda z)(\mu-\lambda)^{k-1}t^k+k!\Big[\mathrm{Li}_k\left(\frac{(\mu-\lambda z)\mathrm{e}^{-(\lambda-\mu)t}}{\lambda(1-z)}\right)\\ \qquad -\mathrm{Li}_k\left(\frac{\mu-\lambda z}{\lambda(1-z)}\right)\Big]-\sum_{r=1}^{k-1}\frac{k!}{r!}[(\mu-\lambda)t]^r\,\mathrm{Li}_{k-r}\left(\frac{\mu-\lambda z}{\lambda(1-z)}\right)\Big\}, & \text{if } \mu<\lambda,\end{cases}$$

and where

$$\mathrm{Li}_k(z)=\sum_{j=1}^{+\infty}\frac{z^j}{j^k} \tag{45}$$

is the polylogarithm function.

Proof. It immediately follows from Eqs. (13) and (14), recalling Eq. (36). □

We conclude this section by evaluating the probability (4).

Proposition 3.6 *Let $k\in\mathbb{N}$. If $\alpha=\lambda$, for $t<\log(\mu/\lambda)/(\mu-\lambda)$, we have*

- *for $\mu>\lambda$*

$$\begin{aligned}P(k,t)=&\frac{\lambda^k}{\mu^{k+1}}\Bigg\{d(\mu-\lambda)\mu^{k+1}\frac{[1-\mathrm{e}^{-(\mu-\lambda)t}]^k}{[\mu-\lambda\mathrm{e}^{-(\mu-\lambda)t}]^{k+1}}-d(d-1)(\lambda-\mu)\\ &\times\left[-\lambda t+\sum_{n=2}^{+\infty}\frac{(-\lambda t)^n}{n!}\theta_n\right]-\lambda d(d-1)\sum_{l=1}^{+\infty}\frac{(1-\mathrm{e}^{-l(\mu-\lambda)t})}{l}{}_2F_1\left(-l,k+1;1;1-\frac{\lambda}{\mu}\right)\\ &+d(d-1)\sum_{n=2}^{+\infty}\frac{(-\lambda)^n}{(\lambda-\mu)^{n-1}}\theta_n\sum_{r=1}^{n-1}\frac{[(\lambda-\mu)t]^r}{r!}\sum_{l=1}^{+\infty}\frac{1}{l^{n-r}}{}_2F_1\left(-l,k+1;1;1-\frac{\lambda}{\mu}\right)\\ &+d(d-1)\sum_{n=2}^{+\infty}\frac{(-\lambda)^n}{(\lambda-\mu)^{n-1}}\theta_n\sum_{l=1}^{+\infty}\frac{1-\mathrm{e}^{-l(\mu-\lambda)t}}{l^n}{}_2F_1\left(-l,k+1;1;1-\frac{\lambda}{\mu}\right)\Bigg\};\end{aligned} \tag{46}$$

- *for $\mu<\lambda$*

$$\begin{aligned}P(k,t)=&d(\lambda-\mu)\mathrm{e}^{-(\lambda-\mu)t}\frac{[\lambda\,(1-\mathrm{e}^{-(\lambda-\mu)t})]^k}{[\lambda-\mu\mathrm{e}^{-(\lambda-\mu)t}]^{k+1}}-d(d-1)\frac{\lambda}{\mu}\,\mathrm{Li}_{k+1}\left(\frac{\mu}{\lambda}\right)\\ &+d(d-1)\sum_{s=1}^{+\infty}\frac{\mathrm{e}^{-s(\lambda-\mu)t}}{s}{}_2F_1\left(1-s,k+1;1;1-\frac{\mu}{\lambda}\right)\\ &+\frac{d(d-1)}{\lambda}\sum_{n=2}^{+\infty}\frac{(-\lambda)^n}{(\mu-\lambda)^{n-1}}\theta_n\sum_{s=1}^{+\infty}\frac{(1-\mathrm{e}^{-s(\lambda-\mu)t})}{s^n}{}_2F_1\left(1-s,k+1;1;1-\frac{\mu}{\lambda}\right)\\ &+\frac{d(d-1)}{\lambda}\sum_{n=2}^{+\infty}\frac{(-\lambda)^n}{(\mu-\lambda)^{n-1}}\theta_n\sum_{r=1}^{n-1}\frac{[(\mu-\lambda)t]^r}{r!}\sum_{s=1}^{+\infty}\frac{1}{s^{n-r}}{}_2F_1\left(1-s,k+1;1;1-\frac{\mu}{\lambda}\right),\end{aligned} \tag{47}$$

where θ_n is defined in Eq. (43).

Proof. The proof follows from Eqs. (5) and (44), recalling that $p(0,t)$ satisfies the integral equation (17), noting that (see Eq. (65.1.3) of [18], for instance)

$$\sum_{k=0}^{+\infty} \frac{(d)_k}{k!} y^k {}_2F_1\left(-k, b; c; x\right) = (1-y)^{-d} {}_2F_1\left(d, b; c; \frac{xy}{y-1}\right),$$

and making use of (45). □

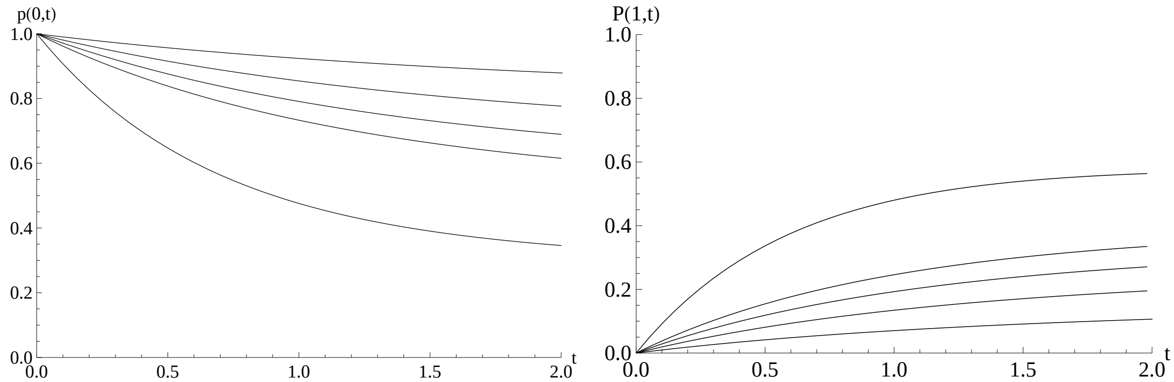

Figure 3: Plot of $p(0,t)$ and $P(1,t)$ for $\lambda = 0.1$, $\mu = 0.5$ and $\alpha = 0.1$, for $d = 1, 2, 3, 4, 10$, from top to bottom for $p(0,t)$, and from bottom to top for $P(1,t)$.

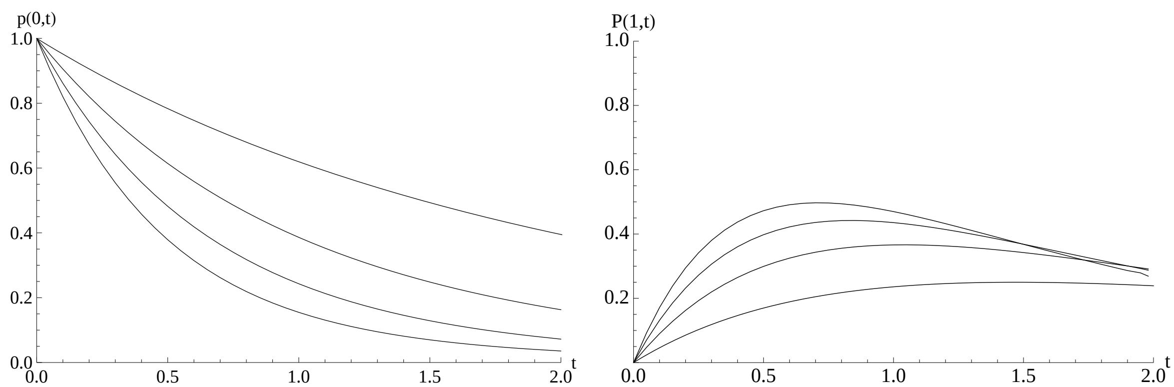

Figure 4: Plot of $p(0,t)$ and $P(1,t)$ for $\lambda = 0.5$, $\mu = 0.1$ and $\alpha = 0.5$, for $d = 1, 2, 3, 4$, from top to bottom for $p(0,t)$, and from bottom to top for $P(1,t)$.

Figures 3 and 4 show some plots of $p(0,t)$ and $P(1,t)$ obtained by evaluating the expressions given in Proposition 3.4 and Proposition 3.6.

Let us now show a simple relation between the polylogarithm function and a series of Gauss hypergeometric functions, which does not appear to have been given before. This result immediately follows from Proposition 3.6.

Corollary 3.2 *For all $k \in \mathbb{N}$ and $x \in (0,1)$ we have*

$$\mathrm{Li}_{k+1}(x) = x \sum_{s=1}^{+\infty} \frac{1}{s} \left[{}_2F_1 \left(1-s, k+1; 1; 1-x\right) \right]. \tag{48}$$

Proof. The proof of (48) follows from (47), by taking into account that $P(k,0) = 0$. □

A classical problem in biological models described by birth-death processes is the extinction, i.e. the first passage through the zero state (see, for instance, Van Doorn and Zeifman [28]). However, due to the specific structure of the state space S, in our model the first passage through the origin (when the initial state is not zero) reduces to a classical one-dimensional problem. Then we conclude the analysis of $N(t)$ by discussing some asymptotic results.

4 Asymptotic results

In the following proposition we obtain the asymptotic expressions of probabilities $p(0,t)$ and $P(k,t)$.

Proposition 4.1 *If $\lambda < \mu$, then*

$$\lim_{t \to +\infty} p(0,t) = \frac{1}{d} \frac{\left(1 - \frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}{1 - \left(1 - \frac{1}{d}\right)\left(1 - \frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}, \tag{49}$$

$$\lim_{t \to +\infty} P(k,t) = \frac{\left(1 - \frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}{1 - \left(1 - \frac{1}{d}\right)\left(1 - \frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}} \frac{(\frac{\alpha}{\lambda})_k}{k!} \left(\frac{\lambda}{\mu}\right)^k, \qquad k \in \mathbb{N}. \tag{50}$$

If $\lambda \geq \mu$, then $p(0,t) \to 0$ and $P(k,t) \to 0$ when $t \to +\infty$.

Proof. Eq. (49) follows from Theorem 2.1. Denoting by

$$\mathcal{L}_s[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t) \mathrm{d}t, \qquad s \geq 0, \tag{51}$$

the Laplace transform of an arbitrary function $f(t)$, from Proposition 2.2 we have

$$\mathcal{L}_s[F(z,t)] = \mathcal{L}_s[H(t)] + (d-1)(s\,\mathcal{L}_s[H(t)] - 1)\,\mathcal{L}_s[p(0,t)]. \tag{52}$$

Note that, due to Eq. (10) of Section 2.1.3, p. 59, of Erdélyi *et al.* [12],

$$\mathcal{L}_s[H(t)] = \begin{cases} \dfrac{\lambda(\lambda-\mu)^{\frac{\alpha}{\lambda}} {}_2F_1(\frac{\alpha}{\lambda}, \frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu}; \frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu}+1; \frac{\mu-\lambda z}{\lambda(1-z)})}{[\lambda(1-z)]^{\frac{\alpha}{\lambda}}\,[\lambda s+\alpha(\lambda-\mu)]}, & \lambda>\mu, \\[2ex] \dfrac{(\mu-\lambda)^{\frac{\alpha}{\lambda}-1}}{(\mu-\lambda z)^{\frac{\alpha}{\lambda}}} \dfrac{\mu-\lambda}{s} {}_2F_1\left(\dfrac{\alpha}{\lambda}, \dfrac{s}{\mu-\lambda}; \dfrac{s}{\mu-\lambda}+1; \dfrac{\lambda(1-z)}{\mu-\lambda z}\right), & \lambda<\mu, \\[2ex] \dfrac{\mathrm{e}^{\frac{s}{\lambda(1-z)}} E\left(\frac{\alpha}{\lambda}, \frac{s}{\lambda(1-z)}\right)}{\lambda(1-z)}, & \lambda=\mu, \end{cases}$$

where

$$E(\nu,z) := \int_1^{+\infty} \frac{\mathrm{e}^{-zt}}{t^{\nu}}\mathrm{d}t, \quad \nu\in\mathbb{R}, \quad z>0, \tag{53}$$

denotes the generalized exponential integral function and ${}_2F_1$ is defined in (32). Hence, recalling that $\lim_{t\to+\infty} F(z,t) = \lim_{s\to 0} s\,\mathcal{L}_s[F(z,t)]$ by the Tauberian theorem, and making use of Eqs. (49) and (52), we have

$$\lim_{t\to+\infty} F(z,t) = \begin{cases} 0, & \lambda\geq\mu, \\[2ex] \dfrac{(\mu-\lambda)^{\frac{\alpha}{\lambda}}}{(\mu-\lambda z)^{\frac{\alpha}{\lambda}}} \left[1+\dfrac{(d-1)\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}{d-(d-1)\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}\right] - \dfrac{(d-1)\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}{d-(d-1)\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}, & \lambda<\mu. \end{cases} \tag{54}$$

If $\lambda<\mu$, making use of

$$\frac{(\mu-\lambda)^{\frac{\alpha}{\lambda}}}{(\mu-\lambda z)^{\frac{\alpha}{\lambda}}} = \sum_{k=0}^{+\infty} \frac{(\frac{\alpha}{\lambda})_k}{k!} \left(1-\frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}} \left(\frac{\lambda}{\mu}\right)^k z^k$$

and recalling Eq. (5), after some calculations we obtain (50) from Eqs. (49) and (54). □

Remark 4.1 Denoting by N the random variable whose distribution is given in the right-hand-sides of Eqs. (49) and (50), for $\lambda<\mu$, the following mixture holds:

$$\mathbb{P}(N=k) = \vartheta_d\,\pi(k) + (1-\vartheta_d)\,\mathbf{1}_{\{k=0\}}, \qquad k\in\mathbb{N}_0, \tag{55}$$

where

$$\pi(k) = \left(1-\frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}} \frac{(\frac{\alpha}{\lambda})_k}{k!} \left(\frac{\lambda}{\mu}\right)^k, \qquad k\in\mathbb{N}_0$$

is a negative binomial distribution, and where the mixing parameter is given by

$$\vartheta_d = \frac{1}{1-\left(1-\frac{1}{d}\right)\left(1-\frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}.$$

Hence, after some calculations we obtain mean and variance of N, for $\lambda < \mu$:

$$E[N] = \vartheta_d \frac{\alpha}{\mu - \lambda}, \qquad Var[N] = \vartheta_d^2 \frac{\alpha\mu}{(\mu-\lambda)^2} \left[1 - \left(1 - \frac{1}{d}\right)\left(\frac{\alpha}{\mu} + 1\right)\left(1 - \frac{\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right].$$

We note that $E[N]$ is increasing in d, and $Var[N]$ is decreasing in d, both having a finite limit when $d \to +\infty$.

5 The diffusion approximation

In this section we construct a diffusion approximation starting from the process $N(t)$. We adopt a scaling procedure that is customary in queueing theory contexts (see, for instance, Di Crescenzo *et al.* [10]). First of all, we perform a different parameterization of the model studied in Section 2 by setting

$$\alpha = \tilde{\gamma} \frac{\tilde{\mu}}{\epsilon}, \qquad \lambda = \frac{\tilde{\mu}}{\epsilon} + \tilde{\beta}, \qquad \mu = \frac{\tilde{\mu}}{\epsilon}, \tag{56}$$

with $\tilde{\gamma} > 0$, $\tilde{\mu} > 0$, $\tilde{\beta} \in \mathbb{R}$ and $\epsilon > 0$. Note that ϵ is a positive constant that can be viewed as a measure of the size of $\tilde{\mu}$. It plays a crucial role in the approximating procedure indicated below, where we let $\epsilon \to 0^+$.

For all $t > 0$, consider the scaling $N_\epsilon^*(t) = N(t)\,\epsilon$, so that $\{N_\epsilon^*(t);\ t \geq 0\}$ is a continuous-time stochastic process having state space

$$S_\epsilon^* = \{0\} \cup \left(\bigcup_{j=1}^d \{\epsilon_j, (2\epsilon)_j, (3\epsilon)_j, \ldots\} \right),$$

and transient probabilities

$$\begin{aligned} p_\epsilon^*(k_j, t) &:= \mathbb{P}\left\{N_\epsilon^*(t) = (k\epsilon)_j\right\} \\ &= \mathbb{P}\left\{(k\epsilon)_j \leq N_\epsilon^*(t) < ((k+1)\epsilon)_j\right\}, \qquad t \geq 0,\ \ k \in \mathbb{N},\ \ j = 1, 2, \ldots, d. \end{aligned}$$

In the limit as $\epsilon \to 0^+$, the scaled process $\{N_\epsilon^*(t);\ t \geq 0\}$ is shown to converge weakly to a diffusion process $\{X(t);\ t \geq 0\}$, whose state space is the star graph

$$S_X := \{0\} \cup \left(\bigcup_{j=1}^d \{x_j : 0 < x_j < +\infty\} \right).$$

For $x \in \mathbb{R}^+$, $t \geq 0$ and $j = 1, 2, \ldots, d$, let $\mathbb{P}\{x_j \leq X(t) < (x+\epsilon)_j\} = f(x_j, t)\epsilon + o(\epsilon)$, where x_j denotes the state x located on the ray S_j (as indicated in Section 2).

Proposition 5.1 *For $x \in \mathbb{R}^+$, $t \geq 0$ and $j = 1, 2, \ldots, d$, the following differential equation holds:*

$$\frac{\partial}{\partial t} f(x_j, t) = -\frac{\partial}{\partial x_j} \left\{ (\tilde{\beta}\, x_j + \tilde{\gamma}\, \tilde{\mu})\, f(x_j, t) \right\} + \frac{1}{2} \frac{\partial^2}{\partial x_j^2} \left\{ 2\, \tilde{\mu}\, x_j\, f(x_j, t) \right\}, \tag{57}$$

with boundary condition

$$\sum_{j=1}^{d} \lim_{x_j \to 0^+} \Big\{ (\tilde{\beta}\, x_j + \tilde{\gamma}\, \tilde{\mu})\, f(x_j,t) - \frac{1}{2}\frac{\partial}{\partial x_j}\big[2\,\tilde{\mu}\, x_j\, f(x_j,t)\big]\Big\} = 0. \tag{58}$$

Proof. Since $p^*_\epsilon(k_j,t) = p(k_j,t)$, due to (56) and in analogy with system (10), for $j = 1,2,\dots,d$ and $t \geq 0$ we have

$$p^*_\epsilon(0,t+\Delta t) = \sum_{j=1}^{d} p^*_\epsilon(1_j,t)\,\frac{\tilde{\mu}}{\epsilon}\Delta t + p^*_\epsilon(0,t)\,\left(1 - d\tilde{\gamma}\,\frac{\tilde{\mu}}{\epsilon}\Delta t\right) + o(\Delta t), \tag{59}$$

$$\begin{aligned} p^*_\epsilon(k_j,t+\Delta t) &= p^*_\epsilon((k-1)_j,t)\,\left[\tilde{\gamma}\,\frac{\tilde{\mu}}{\epsilon} + \left(\frac{\tilde{\mu}}{\epsilon} + \tilde{\beta}\right)(k-1)\right]\Delta t + p^*_\epsilon((k+1)_j,t)\,\frac{\tilde{\mu}}{\epsilon}\,(k+1)\Delta t \\ &+ p^*_\epsilon(k_j,t)\,\left\{1 - \left[\tilde{\gamma}\,\frac{\tilde{\mu}}{\epsilon} + \left(2\frac{\tilde{\mu}}{\epsilon} + \tilde{\beta}\right)k\right]\Delta t\right\} + o(\Delta t), \qquad k \in \mathbb{N}. \end{aligned} \tag{60}$$

Let $p^*_\epsilon(k_j,t) \simeq f((k\epsilon)_j,t)\,\epsilon$ for ϵ close to 0. Hence, for $x_j = (k\epsilon)_j$, from Eq. (60) we have

$$\begin{aligned} f(x_j,t+\Delta t) &= f(x_j-\epsilon,t)\,\left[\tilde{\gamma}\,\tilde{\mu} + \left(\frac{\tilde{\mu}}{\epsilon} + \tilde{\beta}\right)(x_j - \epsilon)\right]\frac{\Delta t}{\epsilon} + f(x_j+\epsilon,t)\,\tilde{\mu}\,(x_j+\epsilon)\frac{\Delta t}{\epsilon} \\ &+ f(x_j,t)\,\left\{1 - \left[\tilde{\gamma}\,\tilde{\mu} + \left(2\frac{\tilde{\mu}}{\epsilon} + \tilde{\beta}\right)x_j\right]\frac{\Delta t}{\epsilon}\right\} + o(\Delta t). \end{aligned}$$

Expanding f as Taylor series, by setting $\Delta t = A\,\epsilon^2$, with $A > 0$, and passing to the limit as $\epsilon \to 0^+$, we obtain Eq. (57). Similarly, Eq. (59) yields

$$f(0,t+\Delta t) = \sum_{j=1}^{d} f(\epsilon_j,t)\,\tilde{\mu}\,\frac{\Delta t}{\epsilon} + f(0,t)\,\left(1 - d\tilde{\gamma}\tilde{\mu}\,\frac{\Delta t}{\epsilon}\right) + o(\Delta t),$$

so that (58) holds. □

From the above procedure, the following approximation holds for small ϵ and for $x = k\epsilon$: $\mathbb{P}\{N(t) < k\} \simeq \mathbb{P}\{X(t) < x\}$, the approximation being expected to improve as ϵ goes to zero and as k grows larger.

Let us now introduce the density

$$h(x,t) := \sum_{j=1}^{d} f(x_j,t), \qquad x \in \mathbb{R}^+, \quad t \geq 0. \tag{61}$$

Proposition 5.2 *For $x \in \mathbb{R}^+$ and $t \geq 0$, the transition density (61) satisfies the following differential equation:*

$$\frac{\partial}{\partial t}\, h(x,t) = -\frac{\partial}{\partial x}\,\Big\{(\tilde{\beta}\, x + \tilde{\gamma}\,\tilde{\mu})\, h(x,t)\Big\} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big\{2\,\tilde{\mu}\, x\, h(x,t)\Big\}, \tag{62}$$

with boundary condition

$$\lim_{x\to 0^+}\Big\{(\tilde\beta\, x+\tilde\gamma\,\tilde\mu)\, h(x,t)-\frac{1}{2}\,\frac{\partial}{\partial x}\big[2\,\tilde\mu\, x\, h(x,t)\big]\Big\}=0 \tag{63}$$

and initial condition

$$\lim_{t\to 0^+} h(x,t)=\delta(x). \tag{64}$$

Proof. The proof of Eqs. (62) and (63) follows immediately from Proposition 5.1, and recalling position (61). The condition (64) can be obtained from (2). □

Note that Eq. (62) is the Fokker-Planck equation for a temporally homogeneous diffusion process on $\mathbb{R}^+$ with linear drift and linear infinitesimal variance, while Eq. (63) expresses a zero-flux condition in the state $x=0$. We remark that various results on such kind of diffusion process have been given in Buonocore *et al.* [6], Giorno *et al.* [15], and Sacerdote [27], for instance.

Hereafter we show that $h(x,t)$ is a gamma density with shape parameter $\tilde\gamma$ and rate $\psi(t)$.

Proposition 5.3 *The density (61) is given by*

$$h(x,t)=\frac{[\psi(t)]^{\tilde\gamma}}{\Gamma(\tilde\gamma)}\, x^{\tilde\gamma-1}\, \mathrm{e}^{-x\psi(t)}, \qquad x\in\mathbb{R}^+, \quad t\geq 0, \tag{65}$$

where

$$\psi(t)=\frac{\tilde\beta}{\tilde\mu}\cdot\frac{1}{\mathrm{e}^{\tilde\beta t}-1}, \qquad t\geq 0.$$

Proof. The transformation (see Capocelli and Ricciardi [7])

$$x'=x\,\mathrm{e}^{-\tilde\beta t}, \qquad t'=\frac{\tilde\mu}{\tilde\beta}\left(1-\mathrm{e}^{-\tilde\beta t}\right), \qquad h(x,t)=\mathrm{e}^{-\tilde\beta t}\, h'(x',t'),$$

changes equation (62) and condition (63) respectively into a Fokker-Planck equation for the time-homogeneous diffusion process on $\mathbb{R}^+$ having drift $\tilde\gamma$ and infinitesimal variance $2x'$, with a zero-flux condition on the boundary $x'=0$. Initial condition (64) becomes $\lim_{t'\to 0^+} h'(x',t')=\delta(x')$. The proof thus proceeds similarly as Proposition 4.1 of Di Crescenzo and Nobile [11] assuming a zero initial state. □

In conclusion, from Eq. (65) we immediately obtain that a gamma-type stationary density exists when $\tilde\beta<0$.

Corollary 5.1 *If $\tilde\beta<0$, then*

$$\overline{h}(x):=\lim_{t\to+\infty} h(x,t)=\frac{1}{\Gamma(\tilde\gamma)}\left(\frac{|\tilde\beta|}{\tilde\mu}\right)^{\tilde\gamma} x^{\tilde\gamma-1}\,\exp\left(-\,x\,\frac{|\tilde\beta|}{\tilde\mu}\right), \qquad x\in\mathbb{R}^+. \tag{66}$$

A Proof of Theorem 2.1

In the appendix we provide the proof of Theorem 2.1 in 3 cases. Recall that the Laplace transform of any function $f(t)$ is denoted as in (51).

A.1 Case $\lambda = \mu$

From Eq. (17) in this case we obtain

$$\mathcal{L}_s[p(0,t)] = \mathcal{L}_s\left[\frac{1}{(1+\lambda t)^{\frac{\alpha}{\lambda}}}\right] - \alpha(d-1)\mathcal{L}_s[p(0,t)]\mathcal{L}_s\left[\frac{1}{(1+\lambda t)^{\frac{\alpha}{\lambda}+1}}\right], \tag{67}$$

where, for any $b \in \mathbb{R}$, $\lambda, s > 0$

$$\mathcal{L}_s\left[\frac{1}{(1+\lambda t)^b}\right] = \frac{\mathrm{e}^{s/\lambda}}{\lambda} E\left(b, \frac{s}{\lambda}\right),$$

and where $E(\nu, z)$ is defined in (53). Noting that

$$E(\nu, z) = \frac{1}{\nu - 1}[\mathrm{e}^{-z} - zE(\nu-1, z)], \quad \nu \in \mathbb{R}, \quad z > 0,$$

from Eq. (67) we obtain

$$\mathcal{L}_s[p(0,t)] = \frac{1}{\lambda d} \frac{\mathrm{e}^{s/\lambda} E\left(\frac{\alpha}{\lambda}, \frac{s}{\lambda}\right)}{1 - \frac{d-1}{d} \frac{s}{\lambda} \mathrm{e}^{s/\lambda} E\left(\frac{\alpha}{\lambda}, \frac{s}{\lambda}\right)}.$$

Hence, the above expression gives

$$\begin{aligned}\mathcal{L}_s[p(0,t)] &= \frac{1}{sd}\sum_{n=0}^{+\infty}\left(1-\frac{1}{d}\right)^n \left[\frac{s}{\lambda}\mathrm{e}^{s/\lambda} E\left(\frac{\alpha}{\lambda}, \frac{s}{\lambda}\right)\right]^{n+1} \\ &= \frac{1}{sd}\sum_{n=0}^{+\infty}\left(1-\frac{1}{d}\right)^n \sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j\left[1-\frac{s}{\lambda}\mathrm{e}^{s/\lambda} E\left(\frac{\alpha}{\lambda}, \frac{s}{\lambda}\right)\right]^j.\end{aligned} \tag{68}$$

Taking the inverse Laplace Transform, from Eq. (68) we obtain

$$p(0,t) = 1 + \frac{1}{d}\sum_{n=0}^{+\infty}\left(1-\frac{1}{d}\right)^n \sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^j F_Y^{(j)}(t), \tag{69}$$

where $F_Y^{(j)}(t)$, defined in (19), is the distribution function of the sum of j independent random variables having probability density

$$f_Y^{(1)}(t) = \frac{\alpha}{(1+\lambda t)^{\frac{\alpha}{\lambda}+1}}, \qquad t > 0.$$

Finally, by interchanging the order of summation in the right-hand side of (69), Eq. (20) immediately follows when $\lambda = \mu$.

A.2 Case $\lambda > \mu$

From Eqs. (17) and (18) in this case we obtain

$$\mathcal{L}_s[p(0,t)]\left\{1+(d-1)\mathcal{L}_s\left[\frac{\alpha(\lambda-\mu)^{\frac{\alpha}{\lambda}+1}\mathrm{e}^{(\lambda-\mu)t}}{(\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}+1}}\right]\right\}=\mathcal{L}_s\left[\frac{(\lambda-\mu)^{\frac{\alpha}{\lambda}}}{(\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}}}\right]. \tag{70}$$

We have (cf. Eq. (3.197.3) of Gradshteyn and Ryzhik [16])

$$\mathcal{L}_s\left[\frac{(\lambda-\mu)^{\frac{\alpha}{\lambda}}}{(\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}}}\right]=\frac{\lambda\left(\frac{\lambda-\mu}{\lambda}\right)^{\frac{\alpha}{\lambda}}}{\lambda s+(\lambda-\mu)\alpha}\,{}_2F_1\left(\frac{\alpha}{\lambda},\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};1+\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};\frac{\mu}{\lambda}\right),$$

where ${}_2F_1(a,b;c;z)$ is defined in Eq. (32). Moreover, since

$$\mathcal{L}_s\left[\frac{\alpha(\lambda-\mu)^{\frac{\alpha}{\lambda}+1}\mathrm{e}^{(\lambda-\mu)t}}{(\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}+1}}\right]=\frac{\lambda\alpha\left(\frac{\lambda-\mu}{\lambda}\right)^{\frac{\alpha}{\lambda}+1}}{\lambda s+(\lambda-\mu)\alpha}\,{}_2F_1\left(1+\frac{\alpha}{\lambda},\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};1+\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};\frac{\mu}{\lambda}\right)$$

and (cf., for instance, Eq. 15.2.14 of Abramowitz and Stegun [1]),

$$b\,{}_2F_1(a,b+1;c;z)-a\,{}_2F_1(a+1,b;c;z)+(a-b)\,{}_2F_1(a,b;c;z)=0, \tag{71}$$

from Eq. (70) we obtain

$$\mathcal{L}_s[p(0,t)]\left\{1+(d-1)\left[1-\frac{s\,(\lambda-\mu)^{\frac{\alpha}{\lambda}}}{\lambda^{\frac{\alpha}{\lambda}-1}(\lambda s+(\lambda-\mu)\alpha)}{}_2F_1\left(\frac{\alpha}{\lambda},\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};1+\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};\frac{\mu}{\lambda}\right)\right]\right\}$$
$$=\frac{(\lambda-\mu)^{\frac{\alpha}{\lambda}}}{\lambda^{\frac{\alpha}{\lambda}-1}(\lambda s+(\lambda-\mu)\alpha)}{}_2F_1\left(\frac{\alpha}{\lambda},\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};1+\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};\frac{\mu}{\lambda}\right).$$

Hence, after some calculations, the above equation gives

$$\mathcal{L}_s[p(0,t)]=\frac{1}{sd}\sum_{n=0}^{+\infty}\left(1-\frac{1}{d}\right)^n\left[\frac{s\,(\lambda-\mu)^{\frac{\alpha}{\lambda}}\,{}_2F_1\left(\frac{\alpha}{\lambda},\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};1+\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};\frac{\mu}{\lambda}\right)}{\lambda^{\frac{\alpha}{\lambda}-1}(\lambda s+(\lambda-\mu)\alpha)}\right]^{n+1}$$
$$=\frac{1}{sd}\sum_{n=0}^{+\infty}\left(1-\frac{1}{d}\right)^n\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j\left[1-\frac{s\,(\lambda-\mu)^{\frac{\alpha}{\lambda}}\,{}_2F_1\left(\frac{\alpha}{\lambda},\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};1+\frac{\alpha}{\lambda}+\frac{s}{\lambda-\mu};\frac{\mu}{\lambda}\right)}{\lambda^{\frac{\alpha}{\lambda}-1}(\lambda s+(\lambda-\mu)\alpha)}\right]^j. \tag{72}$$

Taking the inverse Laplace Transform, in Eq. (72) we get

$$p(0,t)=1+\frac{1}{d}\sum_{n=0}^{+\infty}\left(1-\frac{1}{d}\right)^n\sum_{j=1}^{n+1}\binom{n+1}{j}(-1)^jF_Y^{(j)}(t), \tag{73}$$

where $F_Y^{(j)}(t)$ is the distribution function of the sum of j independent random variables having probability density

$$f_Y^{(1)}(t)=\frac{\alpha(\lambda-\mu)^{\frac{\alpha}{\lambda}+1}\mathrm{e}^{(\lambda-\mu)t}}{\left[\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu\right]^{\frac{\alpha}{\lambda}+1}},\qquad t>0.$$

By interchanging the order of summation in Eq. (73) we immediately obtain Eq. (20) when $\lambda > \mu$.

A.3 Case $\lambda < \mu$

When $\lambda < \mu$, Eq. (70) can be rewritten as

$$\mathcal{L}_s[p(0,t)]\left\{1+(d-1)\left[1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right]\mathcal{L}_s\left[\frac{\alpha(\lambda-\mu)^{\frac{\alpha}{\lambda}+1}\mathrm{e}^{(\lambda-\mu)t}}{\left[1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right](\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}+1}}\right]\right\}$$
$$=\mathcal{L}_s\left[\frac{(\lambda-\mu)^{\frac{\alpha}{\lambda}}}{(\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}}}\right], \tag{74}$$

with (cf. Eq. (3.197.3) of Gradshteyn and Ryzhik [16])

$$\mathcal{L}_s\left[\frac{(\lambda-\mu)^{\frac{\alpha}{\lambda}}}{(\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}}}\right]=\frac{1}{s}\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}{}_2F_1\left(\frac{\alpha}{\lambda},\frac{s}{\mu-\lambda};1+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right),$$

and

$$\mathcal{L}_s\left[\frac{\alpha(\lambda-\mu)^{\frac{\alpha}{\lambda}+1}\mathrm{e}^{(\lambda-\mu)t}}{\left[1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right](\lambda\mathrm{e}^{(\lambda-\mu)t}-\mu)^{\frac{\alpha}{\lambda}+1}}\right]=\frac{\alpha}{\left[\left(\frac{\mu}{\mu-\lambda}\right)^{\frac{\alpha}{\lambda}}-1\right]}\frac{{}_2F_1\left(1+\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)}{\mu[1+\frac{s}{\mu-\lambda}]}.$$
$$=\frac{\lambda\left(\frac{\mu}{\mu-\lambda}\right)^{\frac{\alpha}{\lambda}}}{\mu\left[\left(\frac{\mu}{\mu-\lambda}\right)^{\frac{\alpha}{\lambda}}-1\right]}\left\{1+\frac{\left[\frac{\alpha}{\lambda}-1-\frac{s}{\mu-\lambda}\right]}{[1+\frac{s}{\mu-\lambda}]\left(\frac{\mu}{\mu-\lambda}\right)^{\frac{\alpha}{\lambda}}}{}_2F_1\left(\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\right\},$$

where use of Eq. (71) has been made. Hence, performing some calculations, Eq. (74) becomes

$$\mathcal{L}_s[p(0,t)]\left\{1-\frac{\lambda(\mu-\lambda)^{\frac{\alpha}{\lambda}}(d-1)}{\mu^{\frac{\alpha}{\lambda}}(\mu-\lambda+\lambda d)}\left[1-\frac{\alpha(\mu-\lambda)}{\lambda(\mu-\lambda+s)}\right]{}_2F_1\left(\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\right\}$$
$$=\frac{\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}}{s\left[1+\frac{\lambda}{\mu}(d-1)\right]}{}_2F_1\left(\frac{\alpha}{\lambda},\frac{s}{\mu-\lambda};1+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)$$

so that

$$\mathcal{L}_s[p(0,t)]=\frac{\mu}{s[\mu+\lambda(d-1)]}\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}{}_2F_1\left(\frac{\alpha}{\lambda},\frac{s}{\mu-\lambda};1+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\sum_{n=0}^{+\infty}\left[\frac{\lambda(d-1)}{\mu+\lambda(d-1)}\right]^n$$
$$\times\left[\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\left[1-\frac{\alpha(\mu-\lambda)}{\lambda(\mu-\lambda+s)}\right]{}_2F_1\left(\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\right]^n. \tag{75}$$

Recalling Eq. (33), from Eq. (75), after some calculations, we obtain

$$
\begin{aligned}
\mathcal{L}_s[p(0,t)] &= \frac{\lambda}{s[\mu+\lambda(d-1)]}\sum_{n=0}^{+\infty}\left[\frac{\lambda(d-1)}{\mu+\lambda(d-1)}\right]^n \\
&\times\left\{\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\left[1-\frac{\alpha(\mu-\lambda)}{\lambda(\mu-\lambda+s)}\right]{}_2F_1\left(\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\right\}^{n+1} \\
&+\frac{\mu-\lambda}{s[\mu+\lambda(d-1)]}\sum_{n=0}^{+\infty}\left[\frac{\lambda(d-1)}{\mu+\lambda(d-1)}\right]^n \\
&\times\left\{\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\left[1-\frac{\alpha(\mu-\lambda)}{\lambda(\mu-\lambda+s)}\right]{}_2F_1\left(\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\right\}^{n} \\
&=\frac{\lambda}{s[\mu+\lambda(d-1)]}\sum_{n=0}^{+\infty}\left[\frac{\lambda(d-1)}{\mu+\lambda(d-1)}\right]^n\sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^j \\
&\times\left\{1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\left[1-\frac{\alpha(\mu-\lambda)}{\lambda(\mu-\lambda+s)}\right]{}_2F_1\left(\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\right\}^{j} \\
&+\frac{\mu-\lambda}{s[\mu+\lambda(d-1)]}\sum_{n=0}^{+\infty}\left[\frac{\lambda(d-1)}{\mu+\lambda(d-1)}\right]^n\sum_{j=0}^{n}\binom{n}{j}(-1)^j \\
&\times\left\{1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\left[1-\frac{\alpha(\mu-\lambda)}{\lambda(\mu-\lambda+s)}\right]{}_2F_1\left(\frac{\alpha}{\lambda},1+\frac{s}{\mu-\lambda};2+\frac{s}{\mu-\lambda};\frac{\lambda}{\mu}\right)\right\}^{j}. \qquad (76)
\end{aligned}
$$

Taking the inverse Laplace Transform, from Eq. (76) we get

$$
\begin{aligned}
p(0,t) &= 1+\frac{\lambda}{\mu+\lambda(d-1)}\sum_{n=0}^{+\infty}\left[\frac{\lambda(d-1)}{\mu+\lambda(d-1)}\right]^n \\
&\times\sum_{j=1}^{n+1}\binom{n+1}{j}\left(-\frac{\mu}{\lambda}\right)^j\left[1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right]^j F_Y^{(j)}(t) \\
&+\frac{\mu-\lambda}{\mu+\lambda(d-1)}\sum_{n=0}^{+\infty}\left[\frac{\lambda(d-1)}{\mu+\lambda(d-1)}\right]^n\sum_{j=1}^{n}\binom{n}{j}\left(-\frac{\mu}{\lambda}\right)^j\left[1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right]^j F_Y^{(j)}(t), \qquad (77)
\end{aligned}
$$

where $F_Y^{(j)}(t)$ is the distribution function of the sum of j independent random variables having probability density

$$
f_Y^{(1)}(t)=\frac{\alpha(\mu-\lambda)^{\frac{\alpha}{\lambda}+1}\mathrm{e}^{-(\mu-\lambda)t}}{\left[1-\left(\frac{\mu-\lambda}{\mu}\right)^{\frac{\alpha}{\lambda}}\right]\left[\mu-\lambda\mathrm{e}^{-(\mu-\lambda)t}\right]^{\frac{\alpha}{\lambda}+1}}, \qquad t>0.
$$

Finally, by interchanging the order of summation in Eq. (77), we come to Eq. (20) when $\lambda<\mu$.

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