EXPONENTIAL AND INFINITARY DIVISORS

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Abstract. Our paper is devoted to several problems from the field of modified divisors: namely exponential and infinitary divisors. We study the behaviour of modified divisors, sum-of-divisors and totient functions. Main results concern with the asymptotic behaviour of mean values and explicit estimates of extremal orders.

1. Introduction

Let m be an *exponential divisor* (or *e-divisor*) of n (denote $m \mid^{(e)} n$) if $m \mid n$ and for each prime $p \mid n$ we have $a \mid b$, where $p^a \,||\, m$, $p^b \,||\, n$. This concept, introduced by Subbarao [18], leads us to the *e-divisor function* $\tau^{(e)}(n) = \sum_{m \mid^{(e)} n} 1$ (sequence A049419 in OEIS [20]) and *sum-of-e-divisors function* $\sigma^{(e)}(n) = \sum_{m \mid^{(e)} n} m$ (sequence A051377). These functions were studied by many authors, including among others Wu and Pétermann [17, 25].

Consider a set of arithmetic functions $\mathcal{A}$, a set of multiplicative prime-independent functions $\mathcal{M}_{PI}$ and an operator $E\colon \mathcal{A} \to \mathcal{M}_{PI}$ such that

$$(Ef)(p^a) = f(a).$$

One can check that $\tau^{(e)} = E\tau$, but $\sigma^{(e)} \neq E\sigma$. Section 3 is devoted to the latter new function $E\sigma$.

On contrary several authors, including Tóth [22, 24] and Pétermann [16], studied exponential analogue of the totient function, defining $\phi^{(e)} = Ef$. However $\phi^{(e)}$ lacks many significant properties of ϕ: it is prime-independent and $\phi^{(e)} \ll n^{\varepsilon}$. In Section 4 we construct more natural modification of the totient function, which will be denoted by $\mathfrak{f}^{(e)}$.

One can define *unitary divisors* as follows: $m \mid^{*} n$ if $m \mid n$ and $\gcd(m, n/m) = 1$. Further, define *bi-unitary divisors:* $m \mid^{**} n$ if $m \mid n$ and greatest common unitary divisor of m and n/m is 1; define *tri-unitary divisors:* $m \mid^{***} n$ if $m \mid n$ and greatest common bi-unitary divisor of m and n/m is 1; and so on. It appears that this process converges to the set of so-called *infinitary divisors* (or ∞-divisors): $m \mid^{\infty} n$ if $m \mid n$ and for each $p \mid n$, $p^a \,||\, m$, $p^b \,||\, n$, the binary digits of a have zeros in all places, where b's have. This notation immediately induces ∞-divisor function τ^{∞} (sequence A037445) and sum-of-∞-divisors function σ^{∞} (sequence A049417). See Cohen [1].

Recently Minculete and Tóth [15] defined and studied an exponential analogue of unitary divisors. We introduce e-∞-divisors: $m \mid^{(e)\infty} n$ if $m \mid n$ and for each $p \mid n$, $p^a \,||\, m$, $p^b \,||\, n$, we have $a \mid^{\infty} b$. In Section 5 we improve an estimate for $\sum_{n \leqslant x} \tau^{\infty}(n)$ by Cohen and Hagis [2] and briefly examine $\tau^{(e)\infty}$. Section 6 is devoted to e-∞-perfect numbers such that $\sigma^{(e)\infty}(n) = 2n$.

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2. Notations

Letter p with or without indexes denotes a prime number. Notation $p^a \,||\, n$ means that $p^a \mid n$, but $p^{a+1} \nmid n$.

We write $f \star g$ for Dirichlet convolution

$$(f \star g)(n) = \sum_{d|n} f(d) g(n/d).$$

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols O and o, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are given as an argument (usually x) tends to the infinity.

Letter γ denotes Euler—Mascheroni constant. Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

As usual $\zeta(s)$ is the Riemann zeta-function. Real and imaginary components of the complex s are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

We abbreviate $\operatorname{llog} x := \log\log x$, $\operatorname{lllog} x := \log\log\log x$, where $\log x$ is a natural logarithm.

Let τ be a divisor function, $\tau(n) = \sum_{d|n} 1$. Denote

$$\tau(a_1, \ldots, a_k; n) = \sum_{d_1^{a_1} \cdots d_k^{a_k} = n} 1$$

and $\tau_k = \tau(\underbrace{1, \ldots, 1}_{k \text{ times}}; \cdot)$. Then $\tau \equiv \tau_2 \equiv \tau(1,1;\cdot)$.

Now let $\Delta(a_1, \ldots, a_k; x)$ be an error term in the asymptotic estimate of the sum $\sum_{n \leqslant x} \tau(a_1, \ldots, a_k; n)$. (See [11] for the form of the main term.) For the sake of brevity denote $\Delta_k(x) = \Delta(\underbrace{1, \ldots, 1}_{k \text{ times}}; x)$.

Finally, $\theta(a_1, \ldots, a_k)$ denotes throughout our paper a real value such that

$$\Delta(a_1, \ldots, a_k; x) \ll x^{\theta(a_1, \ldots, a_k) + \varepsilon}$$

and we write θ_k for the exponent of x in $\Delta_k(x)$.

3. Values of $E\sigma$

Theorem 1.

$$\limsup_{n \to \infty} \frac{\log E\sigma(n) \operatorname{llog} n}{\log n} = \frac{\log 3}{2}. \tag{1}$$

Proof. Theorem of Suryanarayana and Sita Rama Chandra Rao [19] shows that

$$\limsup_{n \to \infty} \frac{\log E\sigma(n) \operatorname{llog} n}{\log n} = \sup_{n \geqslant 1} \frac{\log \sigma(n)}{n}.$$

The supremum can be split into two parts: we have

$$\max_{n \leqslant 6} \frac{\log \sigma(n)}{n} = \frac{\log \sigma(2)}{2} = \frac{\log 3}{2}$$

and for $n > 6$ we apply estimate of Ivić [7]

$$\sigma(n) < 2.59 n \operatorname{llog} n \tag{2}$$

to obtain

$$\frac{\log \sigma(n)}{n} < \frac{\log 2.59 + \log n + \operatorname{lllog} n}{n} := f(n),$$

where f is a decreasing function for $n > 6$ and $f(7) < (\log 3)/2$. Thus

$$\sup_{n \geqslant 1} \frac{\log \sigma(n)}{n} = \frac{\log 3}{2}.$$

□

Equation (1) shows that $E\sigma(n) \ll n^{\varepsilon}$.

Theorem 2.

$$\sum_{n\leqslant x} E\sigma(n) = C_1 x + (C_2 \log x + C_3)x^{1/2} + C_4 x^{1/3} + E(x), \tag{3}$$

where C_1, C_2, C_3, C_4 *are computable constants and*

$$x^{1/5} \ll E(x) \ll x^{1153/3613+\varepsilon}.$$

Proof. Let $F(s) = \sum_{n=1}^{\infty} E\sigma(n) n^{-s}$. We have utilizing (2)

$$F(s) = \prod_p \sum_{a=0}^{\infty} E\sigma(p^a) p^{-as} = \prod_p \left(1 + \sum_{a=1}^{\infty} \sigma(a) p^{-as}\right) =$$
$$= \prod_p \left(1 + p^{-s} + 3p^{-2s} + 4p^{-3s} + 7p^{-4s} + O(p^{\varepsilon-5s})\right) =$$
$$= \prod_p \frac{1 + O(p^{\varepsilon-5s})}{(1-p^{-s})(1-p^{-2s})^2(1-p^{-3s})},$$

so

$$F(s) = \zeta(s)\zeta^2(2s)\zeta(3s)H(s), \tag{4}$$

where series $H(s)$ converges absolutely for $\sigma > 1/5$.

Equation (4) shows that

$$E\sigma = \tau(1,2,2,3;\cdot) \star h,$$

where $\sum_{n\leqslant x} |h(n)| \ll x^{1/5+\varepsilon}$. We apply the result of Krätzel [12, Th. 3] together with Huxley's [6] exponent pair $k = 32/205 + \varepsilon$, $l = k + 1/2$ to obtain

$$\sum_{n\leqslant x} \tau(1,2,2,3;n) = B_1 x + (B_2 \log x + B_3)x^{1/2} + B_4 x^{1/3} + O(x^{1153/3613+\varepsilon})$$

for some computable constants B_1, B_2, B_3, B_4. Now convolution argument certifies (3) and the upper bound of $E(x)$. The lower bound for $E(x)$ follows from the theorem of Kühleitner and Nowak [14]. □

Theorem 3.

$$\sum_{n\leqslant x} (E\sigma(n))^2 = Dx + P_7(\log x)x^{1/2} + E(x),$$

where D *is a computable constant,* P_7 *is a polynomial with* $\deg P = 7$ *and*

$$x^{4/17} \ll E(x) \ll x^{8/19+\varepsilon}.$$

Proof. We have

$$\sum_{n=1}^{\infty} (E\sigma(n))^2 n^{-s} = \prod_p \left(1 + \sum_{a=1}^{\infty} (\sigma(a))^2 p^{-as}\right) =$$
$$= \prod_p \left(1 + p^{-s} + 9p^{-2s} + O(p^{\varepsilon-3s})\right) =$$
$$= \prod_p \frac{1 + O(p^{\varepsilon-3s})}{(1-p^{-s})(1-p^{-2s})^8} = \zeta(s)\zeta^8(2s)G(s),$$

where series $G(s)$ converges absolutely for $\sigma > 1/3$.

Ω-estimate of the error term $E(x)$ follows again from [14]. To obtain $E(x) \ll \ll x^{8/19}$ we use [11, Th. 6.8], which implies

$$\theta(1,2,2,2,2,2,2,2,2) \leqslant \frac{1}{1+2-\theta_8} \leqslant \frac{8}{19}.$$

Here we used the estimate of Heath-Brown $\theta_8 \leqslant 5/8$ [21, p. 325]. □

4. Values of $\mathfrak{f}^{(e)}$

For the usual Möbius function μ, identity function id and unit function $\mathbf{1}$ we have

$$\begin{aligned} \tau &= \mathbf{1} \star \mathbf{1}, \\ \mathrm{id} &= \mathbf{1} \star \mu, \\ \sigma &= \mathbf{1} \star \mathrm{id}\,. \end{aligned}$$

Subbarao introduced in [18] the exponential convolution $\odot$ such that for multiplicative f and g their convolution $f \odot g$ is also multiplicative with

$$(f \odot g)(p^a) = \sum_{d|a} f(p^d) g(p^{a/d}). \tag{5}$$

For function $\mu^{(e)} = E\mu$ and defined in Section 1 functions $\tau^{(e)}$ and $\sigma^{(e)}$ we have

$$\begin{aligned} \tau^{(e)} &= \mathbf{1} \odot \mathbf{1}, \\ \mathrm{id} &= \mathbf{1} \odot \mu^{(e)}, \\ \sigma^{(e)} &= \mathbf{1} \odot \mathrm{id}\,. \end{aligned}$$

This leads us to the natural definition of $\mathfrak{f}^{(e)} = \mu^{(e)} \odot \mathrm{id}$ (similar to usual $\phi = = \mu \star \mathrm{id}$). Then by definition (5)

$$\mathfrak{f}^{(e)}(p^a) = \sum_{d|a} \mu(a/d) p^d.$$

Let us list a few first values of $\mathfrak{f}^{(e)}$ on prime powers:

a	1	2	3	4	5
$\mathfrak{f}^{(e)}(p^a)$	p	p^2-p	p^3-p	p^4-p^2	p^5-p

Note that $\mathfrak{f}^{(e)}(n)/n$ depends only on the square-full part of n. Trivially

$$\limsup_{n\to\infty} \frac{\mathfrak{f}^{(e)}(n)}{n} = 1$$

and one can show utilizing Mertens formula (cf. [5, Th. 328]) that

$$\liminf_{n\to\infty} \frac{\mathfrak{f}^{(e)}(n) \operatorname{llog} n}{n} = e^{-\gamma}.$$

Instead we prove an explicit result.

Theorem 4. *For any* $n > 44100$

$$\frac{\mathfrak{f}^{(e)}(n) \operatorname{llog} n}{n} \geqslant C e^{-\gamma}, \qquad C = 0.993957. \tag{6}$$

Proof. Denote for brevity $f(n) = \mathfrak{f}^{(e)}(n) \operatorname{llog} n / n$.

Let $s(n)$ count primes, squares of which divide n: $s(n) = \sum_{p^2 | n} 1$. Let p_k denote the k-th prime: $p_1 = 2$, $p_2 = 3$ and so on. One can check that

$$f(n) \leqslant f\left(\prod_{p \leqslant p_{s(n)}} p^2 \right).$$

Now we are going to prove that for every $x \geqslant 11$ inequality (6) holds for $n = \prod_{p \leqslant x} p^2$. On such numbers we have

$$f(n) = \prod_{p \leqslant x} (1 - p^{-1}) \cdot \log \left(2 \sum_{p \leqslant x} \log p \right)$$

and our goal is to estimate the right hand side from the bottom. By Dusart [3] we know that for $x \geqslant x_0 = 10\,544\,111$

$$\sum_{p \leqslant x} p^{-1} \leqslant \operatorname{llog} x + B + \frac{1}{10 \log^2 x} + \frac{4}{15 \log^3 x}, \tag{7}$$

$$\sum_{p \leqslant x} \log p \geqslant x \left(1 - \frac{0.006788}{\log x} \right) \tag{8}$$

Now since $\exp(-y - y^2/2 - cy^3) \leqslant 1 - y$ for $0 \leqslant y \leqslant 1/2$ and $c = 8 \log 2 - 5 = 0.545177$ we have

$$\prod_{p \leqslant x} (1 - p^{-1}) \geqslant \prod_{p \leqslant x} \exp(-p^{-1} - p^{-2}/2 - cp^{-3}) \geqslant$$
$$\geqslant \left(\prod_{p} \exp(-p^{-2}/2 - cp^{-3}) \right) \prod_{p \leqslant x} \exp(-p^{-1}) =: C_1 \prod_{p \leqslant x} \exp(-p^{-1}),$$

where $C_1 = 0.725132$. Further, by (7) for $x \geqslant x_0$

$$\prod_{p \leqslant x} \exp(-p^{-1}) = \exp \left(- \sum_{p \leqslant x} p^{-1} \right) \geqslant$$
$$\geqslant \log^{-1} x \cdot \exp \left(-B - \frac{1}{10 \log^2 x} - \frac{4}{15 \log^3 x} \right) \geqslant$$
$$\geqslant \log^{-1} x \cdot \exp \left(-B - \frac{1}{10 \log^2 x_0} - \frac{4}{15 \log^3 x_0} \right) =: C_2 \log^{-1} x,$$

where $C_2 = 0.769606$. And by (8) for $x \geqslant x_0$

$$\log \left(2 \sum_{p \leqslant x} \log p \right) \geqslant \log \left(2x \left(1 - \frac{0.006788}{\log x} \right) \right) \geqslant \log x.$$

Finally, we obtain that for $x \geqslant x_0$, $n = \prod_{p \leqslant x} p^2$ we have

$$f(n) \geqslant C_1 C_2 = 0.993957 e^{-\gamma}. \tag{9}$$

Numerical computations show that in fact (9) holds for $p_5 = 11 \leqslant x < x_0$ and $n = (2 \cdot 3 \cdot 5 \cdot 7)^2 = 44100$ is the largest exception of form $\prod_{p \leqslant x} p^2$.

To complete the proof we should show that the theorem is valid for each $n > 44100$ such that $s(n) \leqslant 4$. Firstly, one can validate that the only square-full numbers k for which $f(k) \geqslant Ce^{-\gamma}$ and $s(k) \leqslant 4$ are 4, 8, 9, 36, 900, 44100.

Secondly, let $n = kl$, where $k > 1$ stands for square-full part and l for square-free part, $\gcd(k, l) = 1$. Then

$$f(n) = \frac{\mathfrak{f}^{(e)}(k) \operatorname{llog} k}{k} \cdot \frac{\mathfrak{f}^{(e)}(l)}{l} \cdot \frac{\operatorname{llog} kl}{\operatorname{llog} k} \geqslant \frac{2 \operatorname{llog} 4}{4} \cdot \frac{\operatorname{llog} 4l}{\operatorname{llog} 4} = \frac{\operatorname{llog} 4l}{2}.$$

This inequality shows that if $f(n) \leqslant Ce^{-\gamma}$ then $\operatorname{llog} 4l \leqslant 2Ce^{-\gamma}$ or equivalently $l \leqslant \leqslant 5$.

Thus the complete set of suspicious numbers is

$$\bigl\{ kl \mid k \in (4, 8, 9, 36, 900, 44100), l \in \{1, 2, 3, 5\}, \gcd(k, l) = 1 \bigr\}$$

and fortunately all of them are less or equal to 44100. ☐

Theorem 5.

$$\sum_{n \leqslant x} \mathfrak{f}^{(e)}(n) = Cx^2 + O(x \log^{5/3} x),$$

where C is a computable constant.

Proof. Let s be a complex number such that $\sigma > 4/5$. For $a \geqslant 4$ one have $\mathfrak{f}^{(e)}(p^a) = = p^a + O(p^{a/2})$ and

$$\sum_{a=4}^{\infty} p^{a/2 - 4a/5} = \sum_{a=4}^{\infty} p^{-3a/10} \ll p^{-12/10} \ll p^{-1}.$$

We have

$$\mathfrak{F}(p) := \sum_{a=0}^{\infty} \mathfrak{f}^{(e)}(p^a) p^{-as} =$$

$$= 1 + p^{1-s} + (p^{2-2s} - p^{1-2s}) + (p^{3-3s} - p^{1-3s}) + \sum_{a=4}^{\infty} p^{a-as} + O(p^{-1}).$$

Then

$$(1 - p^{1-s}) \mathfrak{F}(p) = 1 - p^{1-2s} + p^{2-3s} - p^{1-3s} + p^{2-4s} + O(p^{-1}) =$$

$$= 1 - p^{1-2s} + p^{2-3s} + O(p^{-1})$$

and

$$\frac{(1 - p^{1-s})(1 - p^{2-3s})}{1 - p^{1-2s}} \mathfrak{F}(p) = 1 + O(p^{-1}).$$

Taking product by p we obtain

$$\sum_{n=1}^{\infty} \mathfrak{f}^{(e)}(n) n^{-s} = \prod_p \mathfrak{F}(p) = \frac{\zeta(s-1)\zeta(3s-2)}{\zeta(2s-1)} G(s),$$

where $G(s)$ converges absolutely for $\sigma > 4/5$. This means that $\mathfrak{f}^{(e)} = z \star g$, where

$$z(n) = \sum_{n_1 n_2^2 n_3^3 = n} n_1 \mu(n_2) n_2 n_3^2$$

and $\sum_{n \leqslant x} |g(n)| \ll x^{4/5 + \varepsilon}$.

By [17, Th. 1] we have $\sum_{n_1 n_3^3 \leqslant y} n_1 n_3^2 = y^2 \zeta(4)/2 + O(y \log^{2/3} y)$, so

$$\sum_{n \leqslant x} z(n) = \sum_{n_2 \leqslant x^{1/2}} \mu(n_2) n_2 \left(\frac{\zeta(4)}{2} \frac{x^2}{n_2^4} + O\left(\frac{x}{n_2^2} \log^{2/3} x \right) \right) =$$

$$= \frac{\zeta(4)}{2\zeta(3)} x^2 + O(x \log^{5/3} x).$$

Standard convolution argument completes the proof. ☐

5. Values of τ^∞ and $\tau^{(e)\infty}$

Note that $\tau^\infty(p) = \tau^\infty(p^2) = \tau^\infty(p^4) = 2$, $\tau^\infty(p^3) = \tau^\infty(p^5) = 4$ and more generally

$$\tau^\infty(p^a) = 2^{u(a)}, \tag{10}$$

where $u(a)$ is equal to the number of units in binary representation of a. Thus $\tau^\infty(p^a) \leqslant \leqslant a+1$ and $\tau^\infty(n) \ll n^\varepsilon$.

Theorem 6.

$$\sum_{n\leqslant x} \tau^\infty(n) = (D_1 \log x + D_2)x + E(x),$$

where D_1, D_2 are computable constants. In unconditional case

$$E(x) \ll x^{1/2} \exp(-A \log^{3/5} x \operatorname{llog}^{-1/5} x), \qquad A > 0,$$

and under Riemann hypothesis $E(x) \ll x^{5/11+\varepsilon}$.

Proof. Let us transform Dirichlet series for τ^∞ into a product of zeta-functions:

$$\sum_{n=1}^\infty \tau^\infty(n) n^{-s} = \prod_p \sum_{a=0}^\infty \tau^\infty(p^a) p^{-as} =$$
$$= \prod_p \bigl(1 + 2p^{-s} + 2p^{-2s} + 4p^{-3s} + O(p^{\varepsilon-4s})\bigr) =$$
$$= \prod_p \frac{\bigl(1+O(p^{\varepsilon-4s})\bigr)(1-p^{-2s})}{(1-p^{-s})^2(1-p^{-3s})^2} = \frac{\zeta^2(s)\zeta^2(3s)}{\zeta(2s)} G(s),$$

where series $G(s)$ converges absolutely for $\sigma > 1/4$.

By [11, Th. 6.8] together with estimate $\theta_2 < 131/416 + \varepsilon$ from [6] we get

$$\sum_{n\leqslant x} \tau(1,1,3,3;n) = (C_1 \log x + C_2) + (C_3 \log x + C_4) x^{1/3} + O(x^{547/1664+\varepsilon}).$$

Now the statement of the theorem can be achieved by application of Ivić's [8, Th. 2]. Alas, term $(C_3 \log x + C_4)x^{1/3}$ will be absorbed by error term. ☐

Theorem 7.

$$\sum_{n\leqslant x} \bigl(\tau^\infty(n)\bigr)^2 = P_3(\log x)x + O(x^{1/2}\log^9 x), \tag{11}$$

where P_3 is a polynomial, $\deg P_3 = 3$.

Proof. We have

$$\bigl(\tau^\infty(p)\bigr)^2 = \bigl(\tau^\infty(p^2)\bigr)^2 = 4,$$

so

$$F(s) := \sum_{n=1}^\infty \bigl(\tau^\infty(n)\bigr)^2 n^{-s} = \frac{\zeta^4(s)}{\zeta^6(2s)} H(s),$$

where series $H(s)$ converges absolutely for $\sigma > 1/3$.

By Perron formula for $c := 1 + 1/\log x$, $\log T \asymp \log x$ we have

$$\sum_{n\leqslant x} \bigl(\tau^\infty(n)\bigr)^2 = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} F(s) x^s s^{-1} ds + O(x^{1+\varepsilon} T^{-1}).$$

Moving the contour of the integration till $[1/2 - iT, 1/2 + iT]$ we get

$$\sum_{n\leqslant x} \bigl(\tau^\infty(n)\bigr)^2 = \operatorname*{res}_{s=1} F(s) x^s s^{-1} + O(I_0 + I_- + I_+ + x^{1+\varepsilon} T^{-1}),$$

where

$$I_0 := \int_{1/2-iT}^{1/2+iT} F(s)x^s s^{-1} ds, \qquad I_\pm := \int_{1/2\pm iT}^{c\pm iT} F(s)x^s s^{-1} ds.$$

Function $F(s)x^s s^{-1}$ has a pole of fourth order at $s = 1$, so $\operatorname{res}_{s=1} F(s)x^s s^{-1}$ has form $P_3(\log x)x$. Let us estimate the error term.

Take $T = x^{3/4}$. Firstly,

$$I_+ \ll T^{-1} \int_{1/2}^{c} \frac{\zeta^4(\sigma + iT)}{\zeta^6(2\sigma + 2iT)} x^\sigma \, d\sigma$$

Using classic estimates $\zeta(\sigma + iT) \ll T^{(1-\sigma)/3}$ for $\sigma \in [1/2, 1)$ and $\zeta(\sigma + iT) \ll T^\varepsilon$ for $s \neq \sigma \geqslant 1$ we obtain $I_+ \ll x^{1/4+\varepsilon}$. The same can be proven for I_-.

Secondly, taking into account bounds $\zeta^{-1}(1 + it) \ll \log^{2/3} t$ and $\int_1^T \zeta(1/2 + it)^2 t^{-1} dt \ll \log^5 T$ (see [9] or [21]) we have

$$I_0 \ll x^{1/2} \int_1^T \frac{\zeta^4(1/2 + it)}{\zeta^6(1 + 2it)} \frac{dt}{t} \ll x^{1/2} \log^9 T, \tag{12}$$

which completes the proof. □

Recently Jia and Sankaranarayanan proved in the preprint [10] that

$$\int_1^T \frac{\zeta^4(1/2 + it)}{\zeta^k(1 + 2it)} dt \ll T \log^4 T,$$

so summing up integrals over intervals $[2^n, 2^{n+1}]$ for $n = 0, \ldots, \lfloor \log_2 T \rfloor$ leads to

$$\int_1^T \frac{\zeta^4(1/2 + it)}{\zeta^k(1 + 2it)} \frac{dt}{t} \ll \log^5 T.$$

Thus instead of (12) we get $I_0 \ll x^{1/2} \log^5 x$, which provides us with a better error term in (11).

Function $E\tau^\infty$ has Dirichlet series $\zeta(s)\zeta(2s)\zeta^{-1}(4s)H(s)$, where $H(s)$ converges absolutely for $\sigma > 1/5$, very similar to function $t^{(e)}$, studied by Tóth [23] and Pétermann [16]. The latter achieved error term $O(x^{1/4})$ in the asymptotic expansion of $\sum_{n \leqslant x} t^{(e)}$; the same result holds for $E\tau^\infty$.

Dirichlet series for $\big(E\tau^\infty(n)\big)^2$ is similar to $\big(\tau^{(e)}(n)\big)^2$: both of them are $\zeta(s) \times \zeta^3(2s)H(s)$, where $H(s)$ converges absolutely for $\sigma > 1/3$. Krätzel proved in [13] that the asymptotic expansion of $\sum_{n \leqslant x} \big(\tau^{(e)}(n)\big)^2$ has error term $O(x^{11/31})$; the same is true for $\big(E\tau^\infty(n)\big)^2$.

6. E-∞-perfect numbers

Let $\sigma^{(e)\infty}$ denote *sum-of-e-∞-divisor function,* where e-∞-divisors were defined in Section 1. We call n e-∞-perfect if $\sigma^{(e)\infty}(n) = 2n$. As far as $\sigma^{(e)\infty}(n)/n$ depends only on square-full part of n, we consider only square-full n below. We found following examples of e-∞-perfect numbers:

$$36, 2700, 1800, 4769\,856, 357\,739\,200, 238\,492\,800, 54\,531\,590\,400,$$
$$1307\,484\,087\,615\,221\,689\,700\,651\,798\,824\,550\,400\,000.$$

All of them are e-perfect also: $\sigma^{(e)}(n) = 2n$. We do not know if there are any e-∞-perfect numbers, which are not e-perfect.

Equation (10) implies that $\tau^\infty(n)$ is even for $n \neq 1$. Then for $p > 2$, $a > 1$ the value of $\sigma^{(e)\infty}(p^a)$ is a sum of even number of odd summands and is even. Thus if n is odd and $\sigma^{(e)\infty}(n) = 2n$ then $n = p^a$, $p > 2$. But definitely $\sigma^{(e)\infty}(p^a) \leqslant \sigma(p^a) < 2p^a$. We conclude that all e-∞-perfect numbers are even.

Are there e-∞-perfect numbers, which are not divisible by 3? For e-perfect numbers Fabrykowski and Subbarao [4] have obtained that if $\sigma^{(e)}(n) = 2n$ and $3 \nmid n$ then $n > 10^{664}$. We are going to show that in the case of e-∞-perfect even better estimate can be given.

Lemma 1.

$$\frac{\sigma^{(e)\infty}(p)}{p} = 1,$$

$$\frac{\sigma^{(e)\infty}(p^2)}{p^2} = 1 + p^{-1},$$

$$\frac{\sigma^{(e)\infty}(p^a)}{p^a} \leqslant 1 + 2p^{-a/2} \textit{ for } a \geqslant 6, \tag{13}$$

$$\frac{\sigma^{(e)\infty}(p^a)}{p^a} \leqslant 1 + p^{-2} \textit{ for } a \geqslant 3. \tag{14}$$

Proof. Two first identities are trivial. For $a \geqslant 6$ all non-proper divisors of a are less or equal to $a/2$, so

$$\sigma^{(e)\infty}(p^a) \leqslant p^a + \sum_{b=1}^{a/2} p^b \leqslant p^a + \frac{p(p^{a/2}-1)}{p-1} \leqslant p^a + 2p^{a/2}.$$

This provides (13). Inequality (14) can be directly verified for $a = 3, 4, 5$ and follows from (13) for $a \geqslant 6$. □

Lemma 2. *Let* $b(t) = \max_{\tau \geqslant t} \sigma^{(e)\infty}(2^\tau)2^{-\tau}$. *Then*

$$b(t) \leqslant \begin{cases} 5/4, & t \leqslant 3, \\ 39/32, & 2 < t \leqslant 6, \\ 1 + 2^{1-t/2}, & t > 6. \end{cases} \tag{15}$$

Proof. Follows from (13) and direct computations for small τ:

$$\sigma^{(e)\infty}(2^3) = 10, \qquad \sigma^{(e)\infty}(2^6) = 78.$$

□

Theorem 8. *If n is e-∞-perfect and $3 \nmid n$ then $n > 1.35 \cdot 10^{816}$.*

Proof. In fact we will give a lower estimate for square-full n such that for $u/v = \sigma^{(e)\infty}(n)/n$, $\gcd(u, v) = 1$, we have

$$3 \nmid u, \quad 3 \nmid v, \tag{16}$$

$$2 \mid u, \quad 4 \nmid u, \quad 2 \nmid v, \tag{17}$$

$$u/v \geqslant 2. \tag{18}$$

If this conditions are not satisfied then n is not e-∞-perfect or $3 \mid n$.

Let

$$n = 2^t \prod_{p \in P} p^2 \prod_{q \in Q} q^{a_q}, \qquad t \geqslant 1, \qquad a_q \geqslant 3,$$

sets P and Q contain primes $\geqslant 5$ and $P \cap Q = \varnothing$. Then

$$\frac{u}{v} = \frac{\sigma^{(e)\infty}(n)}{n} = \frac{\sigma^{(e)\infty}(2^t)}{2^t} \prod_{p \in P} \frac{p+1}{p} \prod_{q \in Q} \frac{\sigma^{(e)\infty}(q^{a_q})}{q^{a_q}}.$$

Condition (16) implies that all $p \in P$ are of form $p = 6k+1$. Split P into three disjoint sets:

$$\begin{aligned} P_8 &= \{p \in P \mid p+1 \equiv 0 \pmod 8\}, \\ P_4 &= \{p \in P \mid p+1 \equiv 4 \pmod 8\}, \\ P_2 &= P \setminus P_4 \setminus P_8. \end{aligned}$$

Let $t_2 = |P_2|$, $t_4 = |P_4|$, $t_8 = |P_8|$. Then condition (17) implies

$$t \geqslant t_2 + 2t_4 + 3t_8 + |Q| + 1.$$

Now we utilize (18) to get

$$\begin{aligned} 2 \leqslant \frac{u}{v} &\leqslant b(t_2 + 2t_4 + 3t_8 + |Q| + 1) \prod_{p\in P} (1+p^{-1}) \prod_{p \in Q} (1+q^{-2}) = \\ &= b(t_2 + 2t_4 + 3t_8 + |Q| + 1) \prod_{p\in P} \frac{1+p^{-1}}{1+p^{-2}} \prod_{q \in P\cup Q} (1+q^{-2}). \end{aligned}$$

But

$$\prod_{q\in P\cup Q} (1+q^{-2}) \leqslant \frac{\prod_q (1+q^{-2})}{(1+2^{-2})(1+3^{-2})} = \frac{\zeta(2)/\zeta(4)}{25/18} = \frac{54}{5\pi^2}, \tag{19}$$

so we obtain

$$\frac{10\pi^2}{54} \leqslant b(t_2 + 2t_4 + 3t_8 + 1) \prod_{p\in P} \frac{1+p^{-1}}{1+p^{-2}}.$$

Denote $f(p) = (1+p^{-1})/(1+p^{-2})$. As soon as f is decreasing we can estimate

$$\prod_{p\in P} f(p) = \prod_{\substack{j\in\{2,4,8\} \\ p\in P_j}} f(p) \leqslant \prod_{k=1}^{t_2} f(p_{2,k}) \prod_{k=1}^{t_4} f(p_{4,k}) \prod_{k=1}^{t_8} f(p_{8,k}),$$

where p_2 is a sequence of consecutive primes such that $p_{2,k} \equiv 1 \pmod 6$ and $p_{2,k} + 1 \not\equiv 0 \pmod 4$; p_4 is a sequence of consecutive primes such that again $p_{4,k} \equiv 1 \pmod 6$, but $p_{4,k}+1 \equiv 4 \pmod 8$; and p_8 is such that $p_{8,k} \equiv 1 \pmod 6$, $p_{8,k}+1 \equiv 0 \pmod 8$.

Now conditions (16), (17), (18) can be rewritten as

$$n \geqslant \min_{t_2,t_4,t_8} \left\{ 2^{t_2+2t_4+3t_8+1} \prod_{j\in\{2,4,8\}} \prod_{k=1}^{t_j} p_{j,k}^2 \;\middle|\; \frac{10\pi^2}{54} \leqslant b(t_2+2t_4+3t_8+1) \prod_{j\in\{2,4,8\}} \prod_{k=1}^{t_j} \frac{1+p_{j,k}^{-1}}{1+p_{j,k}^{-2}} \right\}.$$

This optimization problem can be solved numerically utilizing (15):

$$t_2 = 70, \quad t_4 = 32, \quad t_8 = 31, \quad n > 8.49 \cdot 10^{801}.$$

We can use n's factor $\prod_{q\in Q} q^{a_q}$ to improve obtained bound. Suppose that any of primes 5, 11, 17, 23 (all of form $6k-1$) is not in Q. Then instead of (19) we derive

$$\prod_{q\in P\cup Q} (1+q^{-2}) \leqslant \frac{54}{5\pi^2(1+23^{-2})}.$$

Same arguments as above shows that in this case $n > 3 \cdot 10^{823}$. Otherwise, if 5, 11, 17 and 23 are present in Q we get

$$n > (8.49 \cdot 10^{801}) \cdot (5 \cdot 11 \cdot 17 \cdot 23)^3 \cdot 2^4 > 1.35 \cdot 10^{816}.$$

□

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