m-DENDRIFORM ALGEBRAS

JEAN-CHRISTOPHE NOVELLI

ABSTRACT. The Fuss-Catalan numbers are a generalization of the Catalan numbers. They enumerate a large class of objects and in particular m-Dyck paths and $(m+1)$-ary trees. Recently, F. Bergeron defined an analogue for generic m of the Tamari order on classical Dyck words. The author and J.-Y. Thibon showed that the combinatorial Hopf algebras related to these m-Tamari orders are defined thanks to the same monoid, the sylvester monoid, as in the $m = 1$ case and that all related Hopf algebras also have m analogues.

We present here the m-generalization of another construction on Catalan sets: the dendriform algebras. These algebras are presented in two different ways: first by relations between the $m+1$ operations, relations that are very similar to the classical relations; and then by explicit operations splitting the classical dendriform operations defined on words into new operations. We then investigate their dual and show they are Koszul.

CONTENTS

1. Introduction

Among the so-called combinatorial Hopf algebras, the Loday-Ronco algebra of planar binary trees [9] plays a prominent role. Its original definition comes from uhe theory of operads, as this algebra is the free dendriform algebra on one generator. One can also define it in a combinatorial way by means of the so-called sylvester congruence [5] applied to the Malvenuto-Reutenauer algebra MR [10], an algebra defined on permutations. Since the product in this algebra consists in shuffling permutations, it is immediate to check that all products are intervals of the weak order on permutations. One very nice property of the Loday-Ronco algebra is that the congruence classes also are intervals of the weak order, so that products in this algebra also are intervals of the weak order. One can then consider a suborder by restricting to particular class representatives. As Loday and Ronco pointed out, this suborder appears to be a well-known order, the Tamari order initially defined as an order of parenthesized expressions.

Recently, the Tamari order has been generalized by Bergeron to an infinite series of lattices, the m-Tamari orders [1], defined on combinatorial objects counted by Fuss-Catalan numbers, for example $(m+1)$-ary trees. This raises the question of the existence of a generalization of the dendriform algebra for all values of m.

We shall give a positive answer to this question. Actually, in [11], the authors showed that there is a way to generalize permutations so that the number of sylvester classes of these m-permutations is the same as the number of $(m+1)$-ary trees. So, a way to look for an m-dendriform algebra is to start with the algebra on m-permutations and split the product into $m+1$ operations that preserve sylvester classes and such that all sylvester classes can be obtained by a suitable sequence of operations applied to one generator. We shall see that there is a very simple way to define these operations using the right-to-left minima on words. Note that Leroux also defined algebras that deserve the name of m-dendriform algebras in [7] very similar to ours but not isomorphic and of a different spirit: the author gave the relations without a realization. More details can be found in Note 3.2.

Finally, the initial study of dendriform algebras was deeply connected to the study of their dual, the $Dias$ algebras that are algebras satisfying monoidal relations between their operations. We shall compute here the dual of the m-dendriform algebras, hence generalizations of the $Dias$ algebras.

This article is structured as follows. We first fix our notations on words and their operations and recall the basic definitions and properties of the sylvester congruence. We then recall some facts about dendriform algebras and sketch our way to prove those. We then present in full detail the case $m = 2$ of the construction and show that our 2-dendriform algebra has as series of dimensions the generating series of ternary trees. We then present the case for general m which almost consists in copying

the $m = 2$ case. We end by presenting the structure of the dual algebras of our m-dendriform algebras and show that their operads are Koszul operads.

2. Background and notations

2.1. **Words.** In the sequel, we shall need a countable totally ordered alphabet A, usually labeled by the positive integers. We denote by A^* the free monoid generated by A.

All algebras will be taken over a field $\mathbb{K}$ of characteristic 0. The notation $\mathbb{K}\langle A \rangle$ means the free associative algebra over A when A is finite, and the inverse limit $\varprojlim_B \mathbb{K}\langle B \rangle$, where B runs over finite subsets of A, when A is infinite.

The *evaluation* $\mathrm{ev}(w)$ of a word w is the sequence whose i-th term is the number of occurrences of the letter a_i in w.

For a word w over the alphabet $\{1, 2, \ldots\}$, we denote by $w[k]$ the word obtained by replacing each letter i by the integer $i + k$. If u and v are two words, with u of length k, one defines the *shifted concatenation* $u \bullet v = u \cdot (v[k])$ and the *shifted shuffle* $u \uplus v = u \shuffle (v[k])$, where $w_1 \shuffle w_2$ is the usual shuffle product defined recursively by

- $w_1 \shuffle \epsilon = w_1, \quad \epsilon \shuffle w_2 = w_2,$
- $au \shuffle bv = a(u \shuffle bv) + b(au \shuffle v),$

where $w_1 = a \cdot u$ and $w_2 = b \cdot v$, and both a and b are letters and $\cdot$ means concatenation.
For example,

$$(1) \qquad 12 \uplus 21 = 12 \shuffle 43 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312 \, .$$

2.2. **The sylvester monoid.**

2.2.1. *Definition.* The sylvester monoid has been defined in [5] to show the parallel between the algebra of planar binary trees defined by Loday and Ronco in [9] and the algebra of standard Young tableaux defined in [4]. Indeed, the sylvester monoid has essentially the same properties as the plactic monoid and both presentations are indeed very close.

Let us recall that two words w_1 and w_2 are sylvester-adjacent if there exists three words u, v, w and three letters $a \leq b < c$ such that

$$(2) \qquad\qquad w_1 = u \; ac \; v \; b \; w \qquad \text{and} \qquad w_2 = u \; ca \; v \; b \; w.$$

The *sylvester congruence* is then the transitive closure of the sylvester adjacence and the *sylvester monoid* is the quotient of the free monoid A^* by the sylvester congruence, where A is any ordered alphabet.

There are many well-known facts about the sylvester monoid and we shall only recall here the ones we shall use. The reader can find more details and more results in [5, 11].

First, any sylvester class is in bijection with a binary search tree, the elements of the class then being its linear extensions. Moreover, the sylvester classes of permutations

form an interval of the weak order on permutations, and the greatest elements of the sylvester classes are the permutations avoiding the pattern 132.

In particular, the sylvester classes of permutations of size n are in bijection with unlabelled binary trees of size n since given such a tree, there is a unique labelling using each integer from 1 to n once, that provides a binary search tree. So sylvester classes of permutations of size n are enumerated by the Catalan numbers and are more precisely related to the well-known Tamari order since it is the order obtained by selecting in the weak order the permutations avoiding the pattern 132.

Since the Tamari order was generalized by Bergeron [1] where binary trees are replaced by $(m+1)$-ary trees, it was tempting to look for another set of words whose sylvester classes are enumerated by the number of such trees, which are the Fuss-Catalan numbers. It happens that one can enumerate in an efficient way the number of sylvester classes of all words of a given evaluation [11] but the final formula is not as simple as the Catalan numbers, except on the case we shall now detail.

2.2.2. *Sylvester classes of m-permutations and (m+1)-ary trees.* Indeed, there is one more general case where there is a nice formula for the number of sylvester classes of a given evaluation, that is, for evaluations of the form m^n. We shall call these elements the m-permutations since each integer from 1 to n appears exactly m times.

In that case, the binary search trees of this evaluation correspond to special binary trees called m-binary trees in [2] since they hide inside their structure a recursive m-structure. Châtel and Pons provided a bijection between these trees and the $(m + 1)$-ary trees hence showing in a direct combinatorial way that the number of sylvester classes of evaluation m^n is the Fuss-Catalan number

$$(3) \qquad C_n^{(m)} = \frac{1}{mn+1} \binom{mn+n}{n}.$$

Note that $m = 1$ gives back the Catalan numbers.

We shall not need the m-binary trees but we shall at some point make use of ternary trees and even $(m + 1)$-ary trees. Let us define a $(m + 1)$-*ary search tree* as the filling of such a tree by the integers from 1 to n each once in the following recursive way: label in increasing order first the m-th subtree, then the $(m-1)$-st subtree, and so on up to the first subtree, then the root and finally the $(m+1)$-st subtree. For example, Figure 1 presents the labelling of a ternary tree ($m = 2$ case).
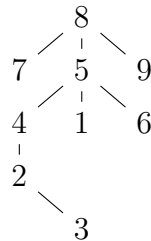

FIGURE 1. The unique labelling of a ternary tree.

Now, the bijection between the $(m + 1)$-ary search trees and the m-permutations avoiding the pattern 132 is the following: denote by r the root of the tree and read

the tree recursively by first reading its right subtree, then its left subtree, then put r, then the second subtree, then put r, and so on up to the m-th subtree and the root one last time.

For example, the tree represented Figure 1 corresponds to the word

(4)   997786643322451158.

One easily rebuilds the tree from the word since its root is labelled by the last letter, which gives back non ambiguously its subtrees since they are all separated by the other occurrences of the last letter except for the left and the right subtrees that can be dintinguished by the fact that the letters of the right subtree are the letters greater than the root.

2.3. **The Hopf algebra FQSym and its generalizations.** As Loday and Ronco made use in [9] of the algebra of permutations defined by Malvenuto and Reutenauer, we shall make use of other algebras defined in [11]. Recall that the Malvenuto-Reutenauer algebra was originally defined in an abstract way [10] and was later proved to be isomorphic to the algebra whose basis elements $\mathbf{F}_\sigma$ are given by sums of words having as result σ^{-1} by an algorithm called standardization. This last algebra, isomorphic to Malvenuto-Reutenauer is denoted by **FQSym** and is called its realization. Similarly, the Loday-Ronco algebra of planar binary trees is isomorphic to the algebra **PBT** obtained by summing the $\mathbf{F}_\sigma$ over sylvester classes. Loday and Ronco then proved that **PBT** is isomorphic to the free dendriform algebra on one generator.

The algebras we shall use are similar to **FQSym** and are indeed subalgebras of one of its generalization, **WQSym**. All details can be found in [11]. We shall only need here their definition and simplest properties: define ^{m}**FQSym** as the algebra on elements $\mathbf{F}_\sigma$ indexed by m-permutations whose product is given by

(5) $$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma \in \alpha \cupdot \beta} \mathbf{F}_\gamma,$$

where the shifted shuffle has been adapted in the following way: shift β not by the size of α but only by its maximal value. For example,

(6) $$\mathbf{F}_{11} \mathbf{F}_{11} = \mathbf{F}_{1122} + \mathbf{F}_{1212} + \mathbf{F}_{2112} + \mathbf{F}_{1221} + \mathbf{F}_{2121} + \mathbf{F}_{2211}.$$

It has been shown in [11] that there exists an analogue of **PBT** obtained by summing the $\mathbf{F}_\sigma$ again over sylvester classes, denoted by ^{m}**PBT**. We shall answer here the following question: is there a natural definition of a m-dendriform algebra so that ^{m}**PBT** is isomorphic to the free m-dendriform algebra on one generator?

2.4. **Dendriform algebras.** We shall first recall some highlights of the theory of dendriform algebras. A *dendriform algebra* A [8] is a vector space equipped with two binary operations $\prec$ and $\succ$ respectively called left and right, satisfying the three relations

(7) $$(u \prec v) \prec w = u \prec (v * w)$$

(8) $$(u \succ v) \prec w = u \succ (v \prec w)$$

(9) $$(u * v) \succ w = u \succ (v \succ w)$$

for any elements u, v, and w in A and where $* = \prec + \succ$.

It has been shown by Loday and Ronco in [9] that the free dendriform algebra on one generator $D^{(1)}$ has as Poincaré series the generating series of the Catalan numbers C_n. We shall sketch a combinatorial proof of both this result and the fact that **PBT** is isomorphic to $D^{(1)}$, since it is this proof that will be generalized later for general values of the parameter m.

The proof follows three steps:

- The dimension of the homogeneous component of $D^{(1)}$ of degree n is at most C_n.
- Splitting the shifted shuffle product of two permutations $\sigma \uplus \tau$ depending on its last letter being either σ_n or $|\sigma| + \tau_p$ endows **FQSym** with the structure of a dendriform algebra. Denote by $Dp^{(1)}$ the dendriform subalgebra of **FQSym** generated by $\mathbf{F}_1$.
- Given any sylvester class of permutations (represented by a binary tree), the sum of the $\mathbf{F}_\sigma$ over this class belongs to $Dp^{(1)}$.

The first step proves that $\dim(D_n^{(1)}) \leq C_n$ whereas the second step proves that $\dim(Dp_n^{(1)}) \leq \dim(D_n^{(1)})$ since $Dp^{(1)}$ is a dendriform algebra generated by one element, so has smaller dimension than the *free* dendriform algebra on one generator. The third step proves that $C_n \leq \dim(Dp_n^{(1)})$ since there are C_n sylvester classes of permutations of size n. All steps together prove

(10) $$C_n \leq \dim(Dp_n^{(1)}) \leq \dim(D_n^{(1)}) \leq C_n,$$

hence showing both $\dim(D_n^{(1)}) = C_n$ and $\dim(Dp_n^{(1)}) = \dim(D_n^{(1)})$, so that $Dp^{(1)}$ and $D^{(1)}$ are isomorphic, $Dp^{(1)}$ hence being an explicit realization of the abstract free dendriform algebra on one generator.

Since we shall copy these steps in the general m case, let us recall how one proves each step separately. The first step is generally proven by orienting the dendriform relations, then showing that any expression in a dendriform algebra is equivalent to at least a linear combination of expressions that avoid some patterns as subexpressions, and that the total number of expressions with $n-1$ symbols avoiding these patterns is given by C_n. It is here that we differ from the usual proof that consists in showing that the linear combination of expressions avoiding the patterns does not depend on the sequence of oriented relations applied to an element. In our case, the second step consists in checking the dendriform relations on the splitting of the associative product of **FQSym**. The third step amounts to see that any binary tree with respective left and right subtrees T_1 and T_2 can be written as $T_1 \succ \mathbf{F}_1 \prec T_2$ if one understands T_i as its expression as an element of $Dp^{(1)}$.

3. THE CASE $m = 2$, WITH THREE OPERATORS

Having recalled our proof in the case $m = 1$, we shall detail the whole procedure in the case $m = 2$ where all ideas are already needed and where notations and computations are easier than in the general case.

3.1. 2-dendriform algebras.

A 2-dendriform algebra A is a vector space equipped with three binary operations $\prec$, $\circ$, $\succ$ respectively called left, middle, and right, satisfying the six relations

$$(11) \qquad (u \prec v) \prec w = u \prec (v * w)$$

$$(12) \qquad (u \circ v) \prec w = u \circ (v \prec w)$$

$$(13) \qquad (u \succ v) \prec w = u \succ (v \prec w)$$

$$(14) \qquad (u \prec v) \circ w = u \circ (v \circ w + v \succ w)$$

$$(15) \qquad (u \circ v + u \succ v) \circ w = u \succ (v \circ w)$$

$$(16) \qquad (u * v) \succ w = u \succ (v \succ w)$$

for any elements u, v, w in A and where $* = \prec + \circ + \succ$.

Note 3.1. The operations $\prec$ and $\circ + \succ$ satisfy the three usual dendriform relations, so that a 2-dendriform algebra is also a dendriform algebra: (11) is (7), adding (12) and (13) gives back (8) while adding (14), (15), and (16) gives back (9).

Note 3.2. Let us compare our definition with the definition of Leroux in [7], p.13, Sec. 5.1. First, it is easy to check that up to a scalar, both algebras have a unique associative product: consider a generic element, write the associativity axiom, and check the system involved has only one solution up to a constant. Moreover, again in both algebras, given the operation $*$, there are only two operations satisfying the first axiom of Leroux (which is also Equation (11)), one of those being unsatisfactory since it is $*$ itself. So if one takes an operad morphism from the algebra of Leroux to ours, it must, up to a scalar that can be chosen as 1, send $*$ to $*$, send $\prec$ to $\prec$, and send $\succ$ to $\circ + \succ$. Now, concerning $\bullet$, if one considers rule 4 of Leroux, it implies that $\bullet$, has to be sent to our $\circ$. By now, the operad morphism is well-defined and the image of his rule 5 reads in our 3-dendriform algebras as

$$(17) \qquad (x \circ y) \circ z + (x \succ y) \circ z = x \circ (y \circ z) + x \succ (y \circ z).$$

which is no linear combination of our relations.

3.2. First upper bound.

Let us now define $D^{(2)}$ as the free 2-dendriform algebra on one generator. Let us also recall that the Fuss-Catalan numbers with $m = 2$ are

$$(18) \qquad C_n^{(2)} = \frac{1}{2n+1} \binom{3n}{n}.$$

3.2.1. *Writing the relations as trees.* Note that any monomial in the 2-dendriform algebra can be represented as a complete binary tree with internal nodes labelled by one the three binary operators and leaves by elements of the algebra. So here follow the six 2-dendriform relations written as trees where we have omitted the leaves since they always are x, y, and z in that order.

(19)

Note that the trees with the symbol $*$ represent the sum of all trees where $*$ is replaced by the three symbols $\prec$, $\circ$, and $\succ$.

Let us now prove

Theorem 3.3. *Let n be a positive integer. Then*

$$(20) \qquad\qquad \dim(D_n^{(2)}) \leq C_n^{(2)}.$$

3.2.2. *Orienting the relations.* Let us orient the 2-dendriform relations as follows:

(21)

This amounts to forbid the following six tree patterns:

(22)

To get to the conclusion that $\dim(D^{(2)})_n \leq C_n^{(2)}$, we need to prove two facts: first, orienting the 2-dendriform relations as before brings ultimately linear combinations of trees avoiding as subtrees the six trees presented in (22); and the number of binary trees avoiding these six tree patterns is enumerated by $C_n^{(2)}$. We say here that a tree T avoids a subtree of size 2 if there is no edge oriented in the same way and with the same parent and child as in the subtree.

3.2.3. *Rewriting trees with forbidden patterns.*

Proposition 3.4. *Any tree can be expressed as a linear combination of trees avoiding the six forbidden tree patterns.*

Proof – Let us consider a tree with n operators. If $n \leq 2$, the statement holds.

Otherwise, we shall prove the statement by induction. Assume that any tree with at most $n - 1$ operators satisfies the statement. Hence, given a tree T with subtrees T_1 and T_2, one can assume that both T_1 at T_2 avoid the six forbidden trees since one

can rewrite separately the left and right subtrees of T. Let us now split the cases depending on the root of T.

First, if the root of T is $\prec$, then T avoids the six patterns if the root of T_1 is not $\prec$, so we can assume that the root of T_1 is $\prec$ and denote by T_1' and T_1'' its own subtrees. We can then apply the rewriting

$$(23) \qquad T = \underset{T_1 \quad T_2}{\overset{\prec}{\diagup \diagdown}} = \underset{\underset{T_1' \quad T_1''}{\diagup \diagdown}}{\overset{\prec}{\diagup}} \overset{\displaystyle \overset{\prec}{\diagup \diagdown}}{T_2} = \underset{T_1''}{\overset{\displaystyle \overset{\prec}{\diagup \diagdown}}{T_1'}} \underset{T_1'' \quad T_2}{\overset{*}{\diagup \diagdown}} \; .$$

Note that since T_1 avoids all patterns, the root of T_1' is different from $\prec$, so the patterns can only be found in the right subtree with root label $*$. But, again by induction, this tree is a linear combination of trees avoiding the patterns, and so is the linear combination obtained for T.

Now, if the root of T is $\circ$, then either the root of T_1 is $\prec$ or is not. If the root of T_1 is not $\prec$, then T avoids the six patterns if the root of T_2 is not $\prec$, so we can assume that the root of T_2 is $\prec$ and denote by T_2' and T_2'' its subtrees. We can then apply the rewriting

$$(24) \qquad T = \underset{T_1 \quad T_2}{\overset{\circ}{\diagup \diagdown}} = T_1 \underset{\underset{T_2' \quad T_2''}{\diagup \diagdown}}{\overset{\displaystyle \overset{\circ}{\diagup \diagdown}}{\prec}} = \underset{\underset{T_1 \quad T_2'}{\diagup \diagdown}}{\overset{\circ}{}} \overset{\displaystyle \overset{\prec}{\diagup \diagdown}}{T_2''} \; .$$

Since T_2 avoids all patterns, the root of T_2' is not $\prec$, and this last tree avoids all patterns.

Now, assume that the root of T_1 is $\prec$. In that case, we have the following picture:

$$(25) \qquad T = \underset{T_1 \quad T_2}{\overset{\circ}{\diagup \diagdown}} = \underset{\underset{T_1' \quad T_1''}{\diagup \diagdown}}{\overset{\prec}{\diagup}} \overset{\displaystyle \overset{\circ}{\diagup \diagdown}}{T_2} = \underset{T_1''}{\overset{\displaystyle \overset{\circ}{\diagup \diagdown}}{T_1'}} \underset{T_1'' \quad T_2}{\overset{\circ}{\diagup \diagdown}} + \underset{T_1''}{\overset{\displaystyle \overset{\circ}{\diagup \diagdown}}{T_1'}} \underset{T_1'' \quad T_2}{\overset{\succ}{\diagup \diagdown}} \; .$$

Since T_1 avoids all patterns, the root of T_1' is different from $\prec$, and the two trees on the right belong to a case seen previously where the root is $\circ$ and the root of its left subtree is not $\prec$. So by induction, all these trees rewrite as linear combinations of trees avoiding the six patterns.

The case where the root is a $\succ$ is analogous to the previous cases and is treated without difficulties. ∎

3.2.4. *Enumerating the trees avoiding the patterns.* The technique to enumerate a set of trees with forbidden patterns is always the same. Define by U the generating series of all these trees enumerated thanks to their number of operators, and by $U_\prec$, $U_\circ$, and $U_\succ$ the subseries of these trees where the root respectively is $\prec$, $\circ$, and $\succ$.

We then have the following system of equations directly derived from the forbidden patterns:

(26)
$$\begin{cases} U = 1 + U_{\prec} + U_{\circ} + U_{\succ} \\ U_{\prec} = x\,(U - U_{\prec})\,U \\ U_{\circ} = x\,(U - U_{\prec})^2 \\ U_{\succ} = x\,U \end{cases}$$

One then easily eliminates the three partial series and gets

(27)
$$U = (1 + xU)^3,$$

which is the generating series of non-empty ternary trees enumerated by their number of nodes minus 1. Since $\dim(D_n^{(2)})$ is smaller than or equal to the coefficient of x^{n-1} in U (all operators are binary so there is one more operand than operators), we conclude to the theorem: $\dim(D_n^{(2)}) \leq C_n^{(2)}$.

Note 3.5. One needs to carefully choose the forbidden patterns since, choosing *e.g.*, to forbid the other pattern of (12), one gets a non-optimal hence useless upper bound.

3.3. **An explicit 2-dendriform algebra.** This research has of course been done in a different order than the presentation given here. To get the correct rewriting rules with no explicit object would have been pointless and it has indeed been done the other way round: first guess a way to split the shuffle product of 2-permutations into three operations that separate their sylvester classes. And then look for the relations these operations satisfy. We shall now present how we adressed this question.

When one computes $\mathbf{F}_{11}^2$, one finds three sylvester classes, and more precisely, splitting according to the dendriform operations $\prec$ and $\succ$, one has

(28)
$$\mathbf{F}_{11} \prec \mathbf{F}_{11} = \mathbf{F}_{1221} + \mathbf{F}_{2121} + \mathbf{F}_{2211},$$

(29)
$$\mathbf{F}_{11} \succ \mathbf{F}_{11} = \mathbf{F}_{1122} + \mathbf{F}_{1212} + \mathbf{F}_{2112}.$$

The first set of 2-permutations constitutes a single sylvester class, whereas the second one can be split in two, hence suggesting that one must split $\succ$ into two operations, and preserve $\prec$. Hence justifying Note 3.1.

3.3.1. *Splitting $*$ into three operations.* Let us now get to the construction itself. We shall use the right-to-left minima of 2-permutations $\tau = \tau_1 \ldots \tau_p$ that is the set of values $i \in [1, p]$ such that $\tau_j \geq \tau_i$ for all $j > i$. For example, the right-to-left minima of 212313 are in decreasing order $\{6, 5, 2\}$ whereas the right-to-left minima of 4121235453 are $\{10, 6, 5, 4, 2\}$. We shall write $m_1(\tau)$ for the first right-to-left minimum, that is always p, and $m_2(\tau)$ for the second one.

Write τ as $\tau = \tau' \cdot \tau'' \cdot \tau_p$ where the first letter of τ'' is $m_2(\tau)$. Note that any 2-permutation has at least *two* right-to-left minima thanks to the definition, so this way of splitting τ always makes sense.

For example,

(30) $212313 = 2123.1.3$ and $4121235453 = 41212.3545.3.$

Then define the three operations on $\sigma = \sigma_1 \ldots \sigma_n$ and $\tau[n] = \tau' \cdot \tau'' \cdot \tau_p$ as

(31)
$$
\begin{cases}
\sigma \prec \tau = (\sigma_1 \ldots \sigma_{n-1} \, \shuffle \, \tau[n]).\sigma_n \\
\sigma \circ \tau = (\sigma \, \shuffle \, (\tau' \cdot \tau'')).\tau_p - \sigma \succ \tau \\
\sigma \succ \tau = (\sigma \, \shuffle \, \tau').\tau''.\tau_p
\end{cases}
$$

In other words, $\sigma \prec \tau$ (respectively $\sigma \circ \tau$ and $\sigma \succ \tau$) is the subset of $\sigma \, \uplus \, \tau$ where the right-most letter of σ ends up after $m_1(\tau)$ (resp. between $m_2(\tau)$ and $m_1(\tau)$, resp. before $m_2(\tau)$).

For example,

(32) $11 \circ 11 = 1212 + 2112$ and $11 \succ 11 = 1122.$

So this splits the right hand-side of (29) into its two sylvester classes. As a bigger example, one also has

$$11 \circ 2112 = 132213 + 312213 + 321213 + 322113,$$

(33) $11 \succ 2112 = (11 \, \shuffle \, 32).23$

$$= 113223 + 131223 + 132123 + 311223 + 312123 + 321123.$$

3.4. **The 2-dendriform structure.** Let us now check that the three operations defined in the last paragraph indeed define a 2-dendriform algebra, that is, satisfy Relations (11) up to (16).

Relation (11) is automatic since it is a known dendriform relation. Relations (12) and (13) sum up to another known dendriform relation, so we only have to check one relation to get the other. Relation (13) is indeed true since the values of the right-to-left minima of v are equal to the values of the right-to-left minima of $v \prec w$: in particular, $m_1(v) = m_1(v \prec w)$ and $m_2(v) = m_2(v \prec w)$. So u is shuffled with the same prefix of v on both sides of the relation regardless of w.

The last three relations all sum up to the last dendriform relation, so we only need to prove two of those to get the last one. Relation (16) is direct since it amounts in both terms to compute $(u \, \uplus \, v \, \uplus \, w').w''$ where $w = w'.w''$ is the splitting of w such that $w_1'' = m_2(\tau)$. Relation (14) is also true since in both expressions the sum consists in the elements such that the last letter of u is to the right of v and between the first two right-to-left minima of w. Indeed, the second right-to-left minimum of $v \circ w + v \succ w$ is always to the right of the last letter of v and hence is the second right-to-left minimum of w.

We can then conclude:

Theorem 3.6. *The three operations defined previously endow* $^2\mathbf{FQSym}$ *with the structure of a 2-dendriform algebra.*

We shall denote by $Dp^{(2)}$ the 2-dendriform subalgebra of $^2\mathbf{FQSym}$ generated by the element $\mathbf{F}_{11}$.

Note 3.7. The previous theorem shows in particular that the dimension of the homogeneous component of size n of $Dp^{(2)}$ is smaller than or equal to the dimension of the same component of $\mathcal{D}^{(2)}$.

3.5. 2-dendriform products and sylvester classes. Let us now prove that the three 2-dendriform products satisfy the property they were designed for, that is, send a pair of sylvester classes to an union of sylvester classes. First, if the product is $\prec$, it comes from known results on the dendriform structure of ${}^2\mathbf{FQSym}$, see [11]. The same argument holds if one considers $\circ+ \succ$ so we only need to prove that a product, say $\succ$, of two sylvester classes is indeed an union of sylvester class.

Let us then consider two words w and w' that are equivalent up to one sylvester rewriting: up to exchanging the roles of w and w', we have $w = \ldots ac \ldots b \ldots$ and $w' = \ldots ca \ldots b \ldots$ with $a \leq b < c$. Assume now that w belongs to a product $u \succ v$.

If the three letters a, b, c belong to u, then w' belongs to $u' \succ v$ where u and u' are in the same sylvester class. The same holds if the three letters belong to v. Now, if the three letters do not belong to the same word, then a belongs to u and c belongs to v, and w' also belongs to $u \succ v$: if b belongs to u, a is not the right-most letter of u and hence is swapped with c in $u \succ v$ and if b belongs to v, then c cannot be $m_2(v)$ since $c > b$ so a can again be swapped with c in $u \succ v$.

So since $Dp^{(2)}$ is generated by $\mathbf{F}_{11}$ which is a sylvester class all by itself, $Dp^{(2)}$ has a basis given by linear combinations of sylvester classes of 2-permutations which is coherent with the combination of Note 3.7 and Theorem 3.3 that together show that $\dim Dp_n^{(2)} \leq C_n^{(2)}$. We shall see in the next paragraph that any sylvester class belongs to $Dp^{(2)}$.

3.6. Sylvester classes of 2-permutations as 2-dendriform products. Let us summarize what we have proved up to now. We showed in Theorem 3.3 that the free 2-dendriform algebra on one generator has dimension at most $C_n^{(2)}$ in size n. We showed in Theorem 3.6 that our way of cutting the shifted shuffle product of permutations in three has indeed the structure of a 2-dendriform algebra, so that (Note 3.7) the dimension of the 2-dendriform algebra $Dp^{(2)}$ generated by $\mathbf{F}_{11}$ is at most $C_n^{(2)}$ in size n too. Finally, we showed in Section 3.5 that $Dp^{(2)}$ has a basis given by linear combinations of sylvester classes of 2-permutations.

Since sylvester classes of 2-permutations are enumerated by the Fuss-Catalan numbers, we only have to prove that any sylvester class of 2-permutations belong to $Dp^{(2)}$ to conclude. Indeed, it would prove that $\dim(Dp^{(2)})_n$ is of dimension at least $C_n^{(2)}$, hence of dimension $C_n^{(2)}$ exactly, so that $\dim(D_n^{(2)})$ also is the same.

This last result is indeed correct and comes from the following three relations we shall prove in the next section. The relations are written in terms of ternary trees since sylvester classes of 2-permutations are indeed encoded by ternary trees (see Section 2.2.2).

$$(34) \qquad \begin{array}{c} \bullet \\ T_1 \quad T_2 \quad T_3 \end{array} = \begin{array}{c} \bullet \\ T_1 \quad T_2 \end{array} \prec T_3.$$

$$(35) \qquad \begin{array}{c} \bullet \\ T'_1 \quad \bullet \\ T_1 \quad T_2 \quad T'_2 \end{array} = \begin{array}{c} \bullet \\ T_1 \quad T_2 \end{array} \circ \begin{array}{c} \bullet \\ T'_1 \quad T'_2 \end{array}.$$

(36)
$$T \nearrow^{\bullet} = T \succ \bullet.$$

3.7. **Proving the three reduction relations.** Let us first show that all these equations are correct. We shall see that the products on the right hand-side always consist in exactly one tree and that it is the tree given on the left hand-side.

Indeed, let us translate the trees as their canonical words obtained by reading the unique labelling of their tree as ternary search tree (see Section 2.2.2 for details about this construction). Then, since canonical words of sylvester classes are the words avoiding the pattern 132, a product of two sylvester classes, being an union of sylvester classes, contains as many terms as there are words avoiding 132 in the shifted shuffle of their canonical elements. So one can restrict to the corresponding product of canonical words.

For example, the product in (34) reads as $T_1 r T_2 r \prec T_3$ where r is the value of the root. The product is $(T_1 r T_2 \,\text{ш}\, T_3[n])r$ if n denotes the number of nodes of the tree on the left. Since all letters of $T_3[n]$ are greater than both T_1 and r, and that all letters of T_1 are smaller than r, there is only one word avoiding 132 in this set. It is $T_3[n]T_1 r T_2 r$, which is exactly the canonical word of the tree on the left hand-side of the equation.

The product in (35) is proven in the same way: one has to compute $T_1 r T_2 r \circ T_1' r' T_2' r'$. This product is equal to $(T_1 r T_2 \,\text{ш}\, (T_1' r' T_2')[n])rr'[n]$. Again all letters of $(T_1' r' T_2')[n]$ are greater than r whereas all letters of $T_1 r T_2$ are smaller than or equal to r, so that there is only one word avoiding 132 in this shuffle product, which is $T_1' r' T_2' T_1 r T_2 rr'$. Equation (36) is immediate.

3.8. **Concluding remarks.** Thanks to the three relations given before, one now sees that any sylvester class is obtained as a suitable product of the generator $\mathbf{F}_{11}$ by induction on the sizes of the trees. So the 2-dendriform algebra generated by $\mathbf{F}_{11}$ has a basis indexed by sylvester classes of 2-permutations.

For example,

$$(37)$$

$$
\cdots = \cdots \prec \bullet = \left(\cdots \right) \prec \bullet
$$

$$
= \left(\left(\bullet \circ \cdots \right) \circ \cdots \right) \prec \bullet
$$

$$
= \left(\left(\bullet \circ \left(\cdots \succ \bullet \right) \right) \circ \cdots \right) \prec \bullet
$$

$$
= \left(\left(\bullet \circ \left(\left(\bullet \circ \cdots \right) \succ \bullet \right) \right) \circ \left(\bullet \circ \cdots \right) \right) \prec \bullet
$$

$$
= ((\bullet \circ ((\bullet \circ (\bullet \circ \bullet)) \succ \bullet)) \circ (\bullet \circ (\bullet \succ \bullet))) \prec \bullet
$$

We can summarize all the previous results as

Theorem 3.8. *The free 2-dendriform algebra $D^{(2)}$ has as series of dimensions the series of Fuss-Catalan numbers $C_n^{(2)}$ and the 2-dendriform algebra $Dp^{(2)}$ that is a 2-dendriform subalgebra of $^2\mathbf{FQSym}$ is free and isomorphic to $D^{(2)}$.*

4. The general case

We shall now consider the case where m is any integer and not only 1 or 2. Those two cases were discussed before and the general case is just an adaptation of those for general m.

The strategy is once again the same:

- define an abstract m-dendriform algebra by providing relations between $m+1$ operations,
- prove that the Fuss-Catalan numbers provide an upper bound of the dimension of the homogeneous component of size n of the free m-dendriform algebra on one generator $D^{(m)}$,
- define a way to split the shifted shuffle of m-permutations into $m+1$ operations and check that these operations satisfy the m-dendriform relations,
- show that these $m+1$ operations send pairs of sylvester classes into unions of sylvester classes,
- show conversely that each sylvester class can be obtained as a suitable product from the generator $\mathbf{F}_{1^m}$.

Let us now show how each step presents itself in the generic case.

4.1. m-dendriform algebras.

An m-*dendriform algebra* is an algebra with $m + 1$ operations labelled $\prec, \circ_1, \ldots, \circ_{m-1}, \succ$ subject to the relations

$$(38) \qquad (u \prec v) \prec w = u \prec (v * w),$$

$$(39) \qquad (u \circ_i v) \prec w = u \circ_i (v \prec w), \qquad \text{for all } i,$$

$$(40) \qquad (u \succ v) \prec w = u \succ (v \prec w),$$

$$(41) \qquad (u \prec v) \circ_i w = u \circ_i (v \succ w + \sum_{j \geq i} v \circ_j w), \qquad \text{for all } i,$$

$$(42) \qquad (u * v) \succ w = u \succ (v \succ w),$$

$$(43) \qquad (u \succ v + \sum_{j \geq m-i} u \circ_j v) \circ_i w = u \succ (v \circ_i w), \qquad \text{for all } i,$$

$$(44) \qquad (u \circ_k v) \circ_i w = u \circ_{k+i} (v \circ_i w), \qquad \text{if } k + i < m,$$

for all u, v, w, and where $* = \prec + \circ_1 + \cdots + \circ_{m-1} + \prec$.

Note that all expressions with two rules are represented in the products so that adding all these relations together shows that $*$ is associative.

Note 4.1. There are respectively $1, m-1, 1, m-1, 1, m-1$ and $\binom{m-1}{2}$ relations of the corresponding types hence $\binom{m+2}{2}$ relations in total, so that, starting with $2(m+1)^2$ expressions involving two operators, there only remains $2(m+1)^2 - \binom{m+2}{2}$ different expressions, which is the pentagonal number $(m+1)(3m+2)/2 = \binom{3m+3}{2}/3$, which is exactly the third Fuss-Catalan number $C_3^{(m)} = \frac{1}{3m+1}\binom{3m+3}{3}$.

Note 4.2. Thanks to this definition, any m-dendriform algebra is also an $(m-1)$-dendriform algebra with the operations $\prec, \circ_1, \ldots, \circ_{m-2}$, and $\circ_{m-1} + \succ$. Indeed, the relations concerning this last operation are obtained by summing together relations involving both $\circ_{m-1}$ and $\succ$.

Note 4.3. As it was already the case with $m = 2$, our algebras are not isomorphic to the algebras of Leroux defined in [7]. Indeed, the same technique as in Note 3.2 proves the fact.

4.2. The upper bound.

Once again, we orient the relations so that, the forbidden tree patterns are

$$(45)$$

for all i and all trees

$$(46)$$

with $1 \leq i < j \leq m - 1$.

4.2.1. *Removing the forbidden tree patterns.*

Proposition 4.4. *Any tree can be expressed as a linear combination of trees avoiding the general forbidden tree patterns.*

The proof is the exact copy of the proof in the case $m = 2$: prove it by induction on trees, and split the cases according to the operator on the root. Then solve the cases starting with $\prec$, then all $\circ_i$ in increasing order then $\succ$. This order guarantees that given a tree that has a forbidden pattern, either its rewriting does not have any pattern anymore, or the pattern has already been dealt with before. Indeed, this works because the forbidden patterns were not chosen at random. They all are the patterns in their corresponding equation that are maximal in a certain sense: they are the patterns whose root is maximal according to the natural order of the operators and whose other vertex is extremal (maximal if the root is $\succ$ and minimal otherwise). So their rewritings are smaller in a certain sense so that the algorithm necessarily ends.

4.2.2. *Enumerating the trees avoiding the patterns.* Let us again denote by U the generating series of all the trees avoiding the forbidden patterns enumerated thanks to their number of operators, and by $U_\prec$, $U_{\circ_i}$, $U_\succ$ the subseries of these trees where the root respectively is $\prec$, $\circ_i$, or $\succ$. We then have the following system of equations directly derived from the forbidden patterns:

(47)
$$
\begin{cases}
U = 1 + U_\prec + \sum_i U_{\circ_i} + U_\succ \\[2mm]
U_\prec = x\,(U - U_\prec)\,U \\[1mm]
U_{\circ_1} = x\,(U - U_\prec)^2 \\[1mm]
U_{\circ_2} = x\,(U - U_\prec)(U - U_\prec - \sum_{i<2} U_{\circ_i}) \\[1mm]
\quad\vdots \quad = \qquad \vdots \\[1mm]
U_{\circ_{m-1}} = x\,(U - U_\prec)(U - U_\prec - \sum_{i<m-1} U_{\circ_i}) \\[1mm]
U_\succ = x\,U.1
\end{cases}
$$

Then one easily checks that this rewrites as $U_\prec = U\frac{xU}{1+xU}$, and $U_{\circ_i} = U\frac{xU}{(1+xU)^{i+1}}$ for all i, and $U_\succ = xU$, so that one directly obtains

(48)
$$
\begin{aligned}
U &= 1 + xU + U\left(\sum_{i=0}^{m-1} \frac{xU}{(1+xU)^{i-1}}\right) \\
&= 1 + xU + U\left(1 - \frac{1}{(1+xU)^m}\right)
\end{aligned}
$$

which implies

(49)
$$
U = (1 + xU)^{m+1},
$$

hence the same conclusion as in the case $m = 2$ since this series enumerates the $(m + 1)$-ary trees by their number of nodes minus 1.

Note that the general computation of U gives away how to choose the forbidden patterns.

4.3. An explicit m-dendriform algebra: Splitting the shifted shuffle into $m + 1$ parts.

The recipe is the same as in the $m = 2$ case: given an m-permutation, compute its first m right-to-left minima. Then define $u \circ_i v$ as the subset of $u \uplus v$ where the last letter of u is between $m_{i+1}(v)$ and $m_i(v)$. In particular, $\circ_1$ is equal to the operator $\circ$ of the 2-dendriform algebra and $\succ$ just selects the words where the last letter of u is to the left of $m_m(v)$.

One then easily checks that this way of splitting the shifted shuffle indeed endows $^m\textbf{FQSym}$ of a m-dendriform algebra structure since all relations hold. Relation (38) holds since it is a dendriform relation. Relations (39) and (40) hold for the same reason Relations (12) and (13) hold in the 2-dendriform algebra: the left-to-right minima of $v \prec w$ are the same as the left-to-right minima of v. Relation (41) holds since the two conditions: the last letter of u is to the right of the last letter of v and the last letter of u is between $m_{i+1}(w)$ and $m_i(w)$ are equivalent to the two conditions: the last letter of v is to the left of $m_i(w)$ and the last letter of u is between the same letters $m_{i+1}(w)$ and $m_i(w)$ and to the right of the last letter of v.

Relation (42) holds for the same reason as in the 2-dendriform algebra. Relation (43) holds since the two conditions: the last letter of u is to the left of $m_{m-i}(v)$ and the last letter of v is to the left of $m_i(z)$ are equivalent to the two conditions: the last letter of u is to the left of $m_m(v \circ_i w)$ and the last letter of v is to the left of $m_i(z)$. Finally, Relation (44) holds again by equivalence of the two pairs of conditions.

4.4. Multiplying sylvester classes.

Proving that any product of sylvester classes is an union of sylvester classes is done exactly as in the $m = 2$ case: either both letters a and c come from the same word and it is obvious, either they do not but in that case, either b was with a and one can exchange a and c, or b was with c but then c cannot be a right-to-left minimum, hence can also be exchanged with a.

4.5. Obtaining a particular sylvester class.

As in the $m = 2$ case, there exists $m + 1$ relations that allow one to strip a tree off its $m + 1$ subtrees, in order. The first and last step of this prodedure are as easy as in the $m = 2$ case so we shall concentrate on the case where we want to obtain a tree whose k subtrees from the right are empty. It just amounts to the following formula:

$$
(50) \qquad T'_1 \ldots T'_{m-k} \underset{T_1 \ldots T_m\, T'_{m+1-k}}{\overset{\bullet}{\diagup | \diagdown}} = T_1 \ldots T_m \;\circ_k\; T'_1 \ldots T'_{m+1-k}.
$$

This formula holds since it amounts to compute $T_1 r \ldots r T_m r \circ_k T'_1 r' \ldots T'_{m+1-k} r'^k$ which is equal to

$$
(51) \qquad (T_1 r \ldots r T_m \sqcup\!\sqcup (T'_1 r' \ldots T'_{m+1-k})[n]) r r'^k,
$$

where the left (respectively right) part of the shuffle consists in letters smaller (resp. greater) than r, so that there is only one word in this shuffle product that avoids the pattern 132 and it is

$$(52) \qquad (T_1' r' \dots T_{m+1-k}')[n]) \ T_1 r \dots r T_m \ rr'^k,$$

which is the corresponding 132-avoiding m-permutation of the tree on the left of the formula.

Piecing all the results of this Section together, we get

Theorem 4.5. *The free m-dendriform algebra $D^{(m)}$ has as series of dimensions the series of Fuss-Catalan numbers $C_n^{(m)}$ and the m-dendriform subalgebra $Dp^{(m)}$ of $^m\mathbf{FQSym}$ generated by $\mathbf{F}_{1^m}$ is free and isomorphic to $D^{(m)}$.*

5. THE DUAL OF THE m-DENDRIFORM ALGEBRAS

The dual of the dendriform algebras is well-known and is the *Dias* algebras, some monoidal algebras. Since our relations are very similar to the dendriform relations, it shall not come as a surprise that the m-dendriform algebras also have duals and that these duals generalize *Dias*.

Recall that the dual of a quadratic algebra A is defined as an algebra on the same number of generators whose relations are orthogonal to the relations of A. Here, since the relations defining the m-dendriform algebras do not share any 2-vertices tree, the computation of the orthogonal is as easy as in the dendriform case.

5.1. **Presentation of m-Dias.** The dual of the m-dendriform algebras defined as m-Dias algebras are defined as algebras on $m+1$ binary operations labelled $\dashv$, $\perp_1$, ..., $\perp_{m-1}$, $\vdash$ satisfying the set of relations

$$(53) \qquad (u \dashv v) \dashv w = u \dashv (v \dashv w) = u \dashv (v \perp_i w) = u \dashv (v \vdash w), \qquad \text{for all } i,$$

$$(54) \qquad (u \perp_i v) \dashv w = u \perp_i (v \dashv w), \qquad \text{for all } i,$$

$$(55) \qquad (u \vdash v) \dashv w = u \vdash (v \dashv w),$$

$$(56) \qquad (u \dashv v) \perp_i w = u \perp_i (v \vdash w) = u \perp_i (v \perp_j w), \qquad \text{for all } j \geq i,$$

$$(57) \qquad (u \dashv v) \vdash w = (u \perp_i v) \vdash w = (u \vdash v) \vdash w = u \vdash (v \vdash w), \qquad \text{for all } i,$$

$$(58) \qquad (u \perp_j v) \perp_i w = (u \vdash v) \perp_i w = u \vdash (v \perp_i w), \qquad \text{for all } i+j \geq m,$$

$$(59) \qquad (u \perp_k v) \perp_i w = u \perp_{k+i} (v \perp_i w), \qquad \text{for all } k+i < m,$$

for all words u, v, and w.

5.2. **The free m-Dias algebra on one generator.** As is customary for the dual of quadratic algebras, one obtains the elements of the dual by considering the trees formed only by the forbidden tree patterns of the original algebra. It amounts here to consider the patterns

for all i and all trees

$$\begin{array}{c} \perp_j \\ {}^{\diagdown} \\ \perp_i \end{array}$$

(61)

with $1 \le i < j \le m-1$. We shall call these patterns the *valid* patterns in the sequel.

As before, we shall prove that the free m-Dias algebra on one generator, denoted here by $Di^{(m)}$ satisfies

(62)
$$\dim Di_n^{(m)} = \binom{n+m-1}{n-1}.$$

Note that this number is equal to the number of trees having only valid tree patterns. Indeed, denoting again by U and $U_\bullet$ the total number of trees or the number of trees with a given root, we have

(63)
$$\begin{cases} U = 1 + U_{\dashv} + \sum_i U_{\perp_i} + U_{\vdash} \\[2mm] U_{\dashv} = \dfrac{x}{1-x} \\[2mm] U_{\perp_1} = x\,(1+U_{\dashv})^2 \\[2mm] U_{\perp_2} = x\,(1+U_{\dashv})(1+U_{\dashv}+\sum_{i<2} U_{\perp_i}) \\[2mm] \vdots \quad = \qquad \vdots \\[2mm] U_{\perp_{m-1}} = x\,(1+U_{\dashv})(1+U_{\dashv}+\sum_{i<m-1} U_{\perp_i}) \\[2mm] U_{\vdash} = x\,U \end{cases}$$

Then one easily checks that this rewrites as $U_{\perp_i} = \frac{x}{(1-x)^{i+1}}$ for all i, and $U_{\succ} = xU$, so that one directly obtains

(64)
$$U = 1 + xU + \sum_{i=1}^{m} \frac{x}{(1-x)^i}$$

so that $U = \frac{1}{(1-x)^{m+1}}$.

Let us observe that the m-Dias algebras are monoidal (all relations involve one tree equal to another). So we have to prove that any tree is equivalent up to the relations to exactly one tree that has only the valid tree patterns.

5.2.1. *Upper bound.* First, let us see why any tree is equivalent to a tree with only valid tree patterns. The method is the same as for m-dendriform algebras: do it by induction on trees, splitting the cases according to the operator on the root. Indeed, the same argument as presented in the m-dendriform case (see Section 4.2.1) works since the valid patterns, being the same as the forbidden patterns in the m-dendriform case are *maximal* in a certain sense among their rewritings. So all trees rewrite to smaller trees in a certain sense, hence get equal to a tree with only valid patterns by induction on that particular order.

5.2.2. *Lower bound.* We now want to show that two trees with only valid patterns cannot be rewritten into one another, that is, do not belong to the same class of trees. Let us explain how this proof works using rewriting graphs and a confluence property. So let us consider the graph G of all rewritings between 2-edges subtrees and their corresponding non-forbidden 2-edges subtree in the m-Dias algebras. The trees in the connected components of G are all trees that are equivalent in the sense of the previous section. Now consider G' the graph obtained from G by orienting all edges of G by rewriting towards the valid subtrees. Note that in our case G' is acyclic since our valid subtrees, as already noted in Section 4.2.1 are maximal in a certain sense. We shall assume here that G' is acyclic.

Let us now assume that there is a connected component of G containing strictly more that one tree with only valid patterns. Then inside G', consider the subcomponents going to each of the trees with only valid patterns. These subcomponents cannot be nonintersecting: since G is connected, there has to be two trees T and T' that are related by an edge of G so that T and T' belong to different subcomponents. Then, depending on the orientation of the rewriting, either T or T' belongs to both subcomponents. So, there exists a tree that can go in G' to two different trees with only valid patterns. Then consider such a tree T minimal in the sense that it has two successors going towards different valid trees that do not share the previous property. Such a tree T exists since G' is acyclic. Then these two successors have disjoint sets of images in G'.

So we shall here show that given a tree T and any two edges in G' starting from T defining two trees T' and T'', both trees have images in common in G' hence proving that all connected components of G have only one tree with only valid patterns by this *confluence* property. Note that if the two rewritings do not share a vertex, the property is obvious. So we only have to prove the property on rewritings having a common vertex, hence on trees with three vertices. Since all rewritings concerning $\perp_i$ look the same for all i, we can restrict ourselves to $m = 2$ and check the property by hand or by computer. Since it holds in all that cases, it holds in general.

5.2.3. *Conclusion.* Piecing all the results of this Section together, we get

Theorem 5.1. *The free m-Dias algebra $Di^{(m)}$ has as series of dimensions the series of binomial numbers $\binom{n+m-1}{n-1}$.*

Concerning the operad associated with m-Dias algebras, the presentation given here is quadratic and confluent. So by a known result [3, 6], this implies

Theorem 5.2. *The m-Dias operad and the m-dendriform operad are both Koszul operads and dual to each other.*

One can easily check the first consequence of this fact since the Poincaré series of the m-dendriform algebras and their duals are indeed inverses of each other for the composition of functions: the free m-dendriform algebra satisfies $g(-x) = -x(1 + g)^{m+1}$ whereas its dual satisfies $h(-x) = \frac{-x}{(1+x)^{m+1}}$.

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Laboratoire d'informatique Gaspard-Monge, Université Paris-Est Marne-la-Vallée, 5, Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France
E-mail address, Jean-Christophe Novelli: `novelli@univ-mlv.fr`