COMBINATORIAL IDENTITIES FOR INCOMPLETE TRIBONACCI POLYNOMIALS

Mark Shattuck
Department of Mathematics, University of Tennessee, Knoxville, TN 37996
shattuck@math.utk.edu

ABSTRACT

The incomplete tribonacci polynomials, denoted by $T_n^{(s)}(x)$, generalize the usual tribonacci polynomials and were introduced in [10], where several algebraic identities were shown. In this paper, we provide a combinatorial interpretation for $T_n^{(s)}(x)$ in terms of weighted linear tilings involving three types of tiles. This allows one not only to supply combinatorial proofs of the identities for $T_n^{(s)}(x)$ appearing in [10] but also to derive additional identities. In the final section, we provide a formula for the ordinary generating function of the sequence $T_n^{(s)}(x)$ for a fixed s, which was requested in [10]. Our derivation is combinatorial in nature and makes use of an identity relating $T_n^{(s)}(x)$ to $T_n(x)$.

Keywords: tribonacci numbers, incomplete tribonacci polynomials, combinatorial proof

2010 Mathematics Subject Classification: 05A19, 05A15

1. Introduction

The *tribonacci numbers* t_n are defined by the recurrence relation $t_n = t_{n-1}+t_{n-2}+t_{n-3}$ if $n \geq 3$, with initial values $t_0 = 0$ and $t_1 = t_2 = 1$. See sequence A000073 in OEIS [11]. The tribonacci numbers are given equivalently by the explicit formula

$$t_{n+1} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} B(n-i,i), \qquad n \geq 0, \tag{1.1}$$

where $B(n,i) = \sum_{j=0}^{i} \binom{i}{j}\binom{n-j}{i}$, as shown in [2]. The number $B(n,i)$ is the n-th row, i-th column entry of the *tribonacci triangle* (see [1]).

The *tribonacci polynomials* $T_n(x)$ were introduced in [6] and are defined by the recurrence $T_n(x) = x^2T_{n-1}(x) + xT_{n-2}(x) + T_{n-3}(x)$ if $n \geq 3$, with initial values $T_0(x) = 0$, $T_1(x) = 1$, and $T_2(x) = x^2$. In analogy to (1.1), the tribonacci polynomials are given by the following explicit formula (see [10]):

$$T_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} \binom{i}{j}\binom{n-i-j}{i} x^{2n-3(i+j)}. \tag{1.2}$$

The *incomplete tribonacci polynomials* $T_n^{(s)}(x)$ were considered in [10] and are defined as

$$T_{n+1}^{(s)}(x) = \sum_{i=0}^{s}\sum_{j=0}^{i}\binom{i}{j}\binom{n-i-j}{i}x^{2n-3(i+j)}, \qquad 0 \le s \le \left\lfloor \frac{n}{2} \right\rfloor . \tag{1.3}$$

Note that the incomplete tribonacci polynomials generalize the ordinary ones and reduce to them when $s = \lfloor \frac{n}{2} \rfloor$. The *incomplete tribonacci number*, denoted by $t_n^{(s)}$, is defined as the value of $T_n^{(s)}(x)$ at $x = 1$. Incomplete Fibonacci numbers and polynomials have also been considered and are defined in a comparable way; see, e.g., [5, 8]. Some combinatorial identities for the incomplete Fibonacci numbers were given in [3] and a bi-periodic generalization was studied in [9].

In [10], several identities were derived for the incomplete tribonacci numbers and polynomials using various algebraic methods. In this paper, we supply combinatorial proofs of these identities using a weighted tiling interpretation of $T_n^{(s)}(x)$ (described in Theorem 2.1 below). In some cases, a further generalization of an identity can be given. In addition, using our interpretation, one also can find other relations not given in [10] that are satisfied by $T_n^{(s)}(x)$. In the final section, we provide an explicit formula for the generating function of $T_n^{(s)}(x)$, as requested in [10]. Our derivation is combinatorial in nature and makes use of some identities involving $T_n(x)$.

2. Combinatorial interpretation for $T_n^{(s)}(x)$

We will use the following terminology. By a *square*, *domino*, or *tromino*, we will mean, respectively, a 1×1, 2×1, or 3×1 rectangular tile. A (linear) *tiling* of length n is a covering of the numbers $1, 2, \ldots, n$ written in a row by squares, dominos, and trominos, where tiles of the same kind are indistinguishable. Let $\mathcal{T}_n$ denote the set of all tilings of length n. It is well known that $\mathcal{T}_n$ has cardinality t_{n+1} (see, e.g., [4, p. 36]). We will often represent squares, dominos, and trominos by the letters r, d, and t, respectively. Thus, a member of $\mathcal{T}_n$ may be regarded as a word in the alphabet $\{r, d, t\}$ in which there are $n-2i-3j$, i, and j occurrences of the letters r, d, and t, respectively, for some i and j.

By a *longer piece* within a member of $\mathcal{T}_n$, we will mean one that is either a domino or a tromino. Given $0 \le s \le \lfloor \frac{n}{2} \rfloor$, let $\mathcal{T}_n^{(s)}$ denote the subset of $\mathcal{T}_n$ whose members contain *at most* s longer pieces. For example, if $n = 5$ and $s = 1$, then $\mathcal{T}_5^{(1)} = \{r^5, dr^3, rdr^2, r^2dr, r^3d, tr^2, rtr, r^2t\}$. Note that $\mathcal{T}_n^{(s)}$ is all of $\mathcal{T}_n$ when $s = \lfloor \frac{n}{2} \rfloor$. By a *square-and-domino* tiling, we will mean a member of $\mathcal{T}_n$ that contains no trominos.

Given $\pi \in \mathcal{T}_n^{(s)}$, let $\delta(\pi)$ and $\nu(\pi)$ record the number of squares and dominos, respectively, in π. We now provide a combinatorial interpretation of $T_{n+1}^{(s)}(x)$ in terms of linear tilings.

Theorem 2.1. *The polynomial $T_{n+1}^{(s)}(x)$ is the distribution for the statistic $2\delta + \mu$ on $\mathcal{T}_n^{(s)}$.*

Proof. First note that $T_{n+1}^{(s)}(x)$ may be written as

$$T_{n+1}^{(s)}(x) = \sum_{i=0}^{s} B(n-i, i)(x), \tag{2.1}$$

where $B(n,i)(x) = \sum_{j=0}^{i}\binom{i}{j}\binom{n-j}{i}x^{2n-i-3j}$. We next observe that when $x = 1$, the polynomial $B(n,i)(x)$ gives the cardinality of the set $\mathcal{B}_{n,i}$ consisting of square-and-domino tilings of length n in which the squares come in two colors, black and white, and containing i dominos and white squares combined. To see this, note that members of $\mathcal{B}_{n,i}$ containing exactly j dominos are in one-to-one correspondence with words in the alphabet $\{D, W, B\}$ containing j D's, $i-j$ W's, and $n-i-j$ B's and thus have cardinality

$$\binom{n-j}{j, i-j, n-i-j} = \frac{(n-j)!}{j!(i-j)!(n-i-j)!} = \binom{n-j}{i}\binom{i}{j}.$$

Summing over j gives

$$|\mathcal{B}_{n,i}| = \sum_{j=0}^{i}\binom{i}{j}\binom{n-j}{i}.$$

Given $\pi \in \mathcal{B}_{n,i}$, let $\delta_1(\pi)$ and $\delta_2(\pi)$ record the number of black and white squares, respectively. Thus, if $\pi \in \mathcal{B}_{n,i}$ has j dominos, then

$$2\delta_1(\pi) + \delta_2(\pi) = 2(n-i-j) + i - j = 2n - i - 3j.$$

Considering all j, this implies $B(n,i)(x)$ is the distribution on $\mathcal{B}_{n,i}$ for the statistic $2\delta_1(\pi) + \delta_2(\pi)$. Suppose now $\lambda \in \mathcal{B}_{n-i,i}$ is given and contains j dominos for some j, where $0 \leq i \leq s$. We replace each domino of λ with a tromino and each white square with a domino. The resulting tiling λ' belongs to $\mathcal{T}_n^{(s)}$ and has j trominos, $i-j$ dominos, and $n-2i-j$ squares. Thus we have

$$2\delta(\lambda') + \nu(\lambda') = 2\delta_1(\lambda) + \delta_2(\lambda)$$

for all $\lambda \in \mathcal{B}_{n-i,i}$. By (2.1), it follows that $T_{n+1}^{(s)}(x)$ is the distribution on $\cup_{i=0}^{s}\mathcal{B}_{n-i,i}$ for $2\delta_1 + \delta_2$, equivalently, for the distribution of $2\delta + \nu$ on $\mathcal{T}_n^{(s)}$. □

Remark: Taking $x = 1$ in the prior theorem shows that the cardinality of $\mathcal{T}_n^{(s)}$ is $t_{n+1}^{(s)}$. Taking $s = \lfloor \frac{n}{2} \rfloor$ shows that $T_{n+1}(x)$ is the distribution for $2\delta + \mu$ on all of $\mathcal{T}_n$.

Using our interpretation for $T_n^{(s)}(x)$, one obtains the following recurrence formula from [10] as a corollary.

Corollary 2.2. *If $n \geq 2s+1$, then*

$$T_{n+3}^{(s)}(x) = x^2T_{n+2}^{(s)}(x) + xT_{n+1}^{(s)}(x) + T_n^{(s)}(x) - (xB(n-s,s)(x) + B(n-1-s,s)(x)). \tag{2.2}$$

Proof. We will show that the right-hand side of (2.2) gives the weighted sum of all the members of $\mathcal{T}_{n+2}^{(s)}$ with respect to the statistic $2\delta+\nu$ by considering the final piece. The first term clearly accounts for all tilings ending in a square. On the other hand, if a member of $\mathcal{T}_{n+2}^{(s)}$ ends in a longer piece, then there can be at most $s-1$ additional longer pieces. From the proof of Theorem 2.1 above, we have for each m that $B(m-s,s)(x)$ gives the weight of all members of $\mathcal{T}_m^{(s)}$ containing exactly s longer pieces. Note that addition of a longer piece to the end of a tiling already containing s longer pieces is not allowed. Thus, by subtraction, the total weight of all members of $\mathcal{T}_{n+2}^{(s)}$ ending in a domino is given by $x(T_{n+1}^{(s)}(x) - B(n-s,s)(x))$ and the weight of those ending in a tromino by $T_n^{(s)}(x) - B(n-1-s,s)(x)$, which completes the proof. □

3. Some identities of $T_n^{(s)}(x)$

In this section, we provide combinatorial proofs of some identities involving the incomplete tribonacci polynomials that generalize those shown in [10] using algebraic methods. We also consider some further identities that can be obtained from the combinatorial interpretation given in Theorem 2.1.

In this section and the next, by the *weight* of a subset S of $\mathcal{T}_n$ or $\mathcal{T}_n^{(s)}$, we will mean the sum $\sum_{\lambda\in S} x^{2\delta(\lambda)+\nu(\lambda)}$.

The $x = 1$ case of the following identity was shown in [10] by an inductive argument.

Identity 3.1. *If $h \geq 1$ and $n \geq 2s + 2$, then*

$$\sum_{i=0}^{h-1} x^{2(h-i-1)} T_{n+i}^{(s)}(x) = \frac{1}{1+x^3}\left(T_{n+h+2}^{(s+1)}(x) - x^{2h}T_{n+2}^{(s+1)}(x) + x^{2h+1}T_n^{(s)}(x) - xT_{n+h}^{(s)}(x)\right). \tag{3.1}$$

Proof. We show equivalently

$$T_{n+h+2}^{(s+1)}(x) = (1+x^3)\sum_{i=0}^{h-1} x^{2(h-i-1)} T_{n+i}^{(s)}(x) + x^{2h}T_{n+2}^{(s+1)}(x) + xT_{n+h}^{(s)}(x) - x^{2h+1}T_n^{(s)}(x).$$

For this, we'll argue that the right-hand side gives the total weight of all the members of $\mathcal{T}_{n+h+1}^{(s+1)}$. First note that $x^{2h}T_{n+2}^{(s+1)}(x)$ gives the weight of the members of $\mathcal{T}_{n+h+1}^{(s+1)}$ in which positions $n+2$ through $n+h+1$ are covered by squares (i.e., the right-most longer piece ends at position $n+1$ or before). On the other hand, the weight of all members of $\mathcal{T}_{n+h+1}^{(s+1)}$ whose right-most longer piece starts at position $n+i-1$ for some $0 \leq i \leq h-1$ is given by $\left(x^{2(h-i)+1} + x^{2(h-i-1)}\right)T_{n+i}^{(s)}(x)$ since such tilings λ are of the form $\lambda = \lambda' dr^{h-i}$ or $\lambda = \lambda' tr^{h-i-1}$ for some tiling λ' of length $n+i-1$ where r^m denotes a sequence of m squares. Note that $\lambda' \in \mathcal{T}_{n+i-1}^{(s)}$ since the number of longer pieces in λ' is limited to s. Summing over $0 \leq i \leq h-1$ gives the indexed sum on the right-hand side. Next, the term $xT_{n+h}^{(s)}(x)$ accounts for all members of $\mathcal{T}_{n+h+1}^{(s+1)}$ whose final piece is a domino which were missed in the sum. Finally, members of $\mathcal{T}_{n+h+1}^{(s+1)}$ of the form $\lambda' dr^h$, where λ' has length $n-1$, were accounted for by both the $x^{2h}T_{n+2}^{(s+1)}(x)$ term and by the $i = 0$ term of the indexed sum; hence, we must subtract their weight, $x^{2h+1}T_n^{(s)}(x)$, to correct for this double count. Combining all of the cases above completes the proof. □

The following identity from [10] gives a formula for the sum of all the incomplete tribonacci polynomials of a fixed order.

Identity 3.2. *If $n \geq 1$, then*

$$\sum_{s=0}^{\ell} T_{n+1}^{(s)}(x) = (\ell+1)T_{n+1}(x) - \sum_{i=0}^{\ell}\sum_{j=0}^{i} i\binom{i}{j}\binom{n-i-j}{i} x^{2n-3(i+j)}, \tag{3.2}$$

where $\ell = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let $\lambda \in \mathcal{T}_n$ and suppose that it contains exactly k longer pieces, where $0 \leq k \leq \ell$. Then the weight of λ is counted by each summand of $\sum_{s=0}^{\ell} T_{n+1}^{(s)}(x)$ such that $s \geq k$. That is, the tiling λ is counted $\ell+1-k$ times by this sum. The proof of Theorem 2.1 above shows that the total weight of all members of $\mathcal{T}_n$ containing exactly k longer pieces is given by

$$\sum_{j=0}^{k}\binom{k}{j}\binom{n-k-j}{k}x^{2n-3(k+j)},$$

upon considering the number j of dominos. Thus, the only inner sum in the double sum on the right-hand side of (3.2) in which λ is counted occurs when $i=k$ and here it is counted k times (due to the extra factor of $i=k$). Since λ is clearly counted $\ell+1$ times by the term $(\ell+1)T_{n+1}(x)$, we have by subtraction that λ is counted $\ell+1-k$ times by the right-hand side of (3.2) as well. Since λ was arbitrary, the identity follows. □

The $x=1$ case of the following identity was conjectured in [10] and follows from the generating function proof given in [7].

Identity 3.3. *If $n \geq 1$, then*

$$\sum_{s=0}^{\ell}T_{n+1}^{(s)}(x)=(\ell+1)T_{n+1}(x)-\sum_{j=1}^{n-1}(xT_j(x)+T_{j-1}(x))T_{n-j}(x), \tag{3.3}$$

where $\ell=\lfloor \frac{n}{2}\rfloor$.

Proof. Suppose $\lambda \in \mathcal{T}_n$ has exactly k longer pieces. By the proof of the preceding identity, we need only show that the weight of λ is counted k times by the sum on the right-hand side of (3.3). Note that $xT_j(x)T_{n-j}(x)$ gives the weight of all members of $\mathcal{T}_n$ in which a domino covers positions j and $j+1$, while $T_{j-1}(x)T_{n-j}(x)$ gives the weight of those in which a tromino covers positions $j-1$, j, and $j+1$. Thus, for each longer piece of λ, there is a term in the sum that counts the weight of λ, which implies that λ is counted k times by the sum, as desired. □

Remark: Comparing the $x=1$ cases of the preceding two identities, it follows that

$$\sum_{n\geq 1}a_nz^n=\frac{z^2+z^3}{(1-z-z^2-z^3)^2},$$

where

$$a_n=\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{j=0}^{i}i\binom{i}{j}\binom{n-i-j}{i},$$

which can also be shown directly using the methods of [12, Section 4.3] (see [7]).

The next three identities follow from the combinatorial interpretation of $T_n^{(s)}(x)$ given in Theorem 2.1 and do not occur in [10].

Identity 3.4. *If $n \geq 2s+1$, then*

$$T_{n+1}^{(s)}(x)=\sum_{i=0}^{s}(x^{i+2}T_{n-2i}^{(s-i)}(x)+x^{i}T_{n-2i-2}^{(s-i-1)}(x)). \tag{3.4}$$

Proof. Suppose a member of $\mathcal{T}_n^{(s)}$ ends in exactly i dominos, where $0 \le i \le s$. If the right-most piece that is not a domino is a square, then the tiles coming to the left of this square constitute a member of $\mathcal{T}_{n-2i-1}^{(s-i)}$ and thus the weight of the corresponding subset of $\mathcal{T}_n^{(s)}$ is $x^{i+2}T_{n-2i}^{(s-i)}(x)$. On the other hand, if the right-most non-domino piece is a tromino, then the tiles to the left of this tromino form a member of $\mathcal{T}_{n-2i-3}^{(s-i-1)}$ and thus the weight of the corresponding subset is $x^iT_{n-2i-2}^{(s-i-1)}(x)$. Considering all possible i gives (3.4). □

Our next formula relates the incomplete tribonacci polynomials to the trinomial coefficients.

Identity 3.5. *If $n \ge 3s+1$, then*

$$T_n^{(s)}(x) = \sum_{i=0}^{s}\sum_{j=0}^{s-i}\binom{s}{i,j,s-i-j}x^{2s-i-2j}T_{n-s-i-2j}^{(s-i-j)}(x). \tag{3.5}$$

Proof. Suppose that there are i dominos and j trominos among the final s tiles within a member of $\mathcal{T}_{n-1}^{(s)}$, where $n \ge 3s+1$. Then there are $\binom{s}{i,j,s-i-j}$ ways to arrange these tiles, which contribute $x^{2(s-i-j)+i}$ towards the weight, with the remaining tiles forming a member of $\mathcal{T}_{n-s-i-2j-1}^{(s-i-j)}$. Considering all possible i and j gives (3.5). □

The incomplete Fibonacci polynomials introduced in [8] are given as

$$F_n^{(s)}(x) = \sum_{r=0}^{s}\binom{n-r-1}{r}x^{n-2r-1}, \qquad 0 \le s \le \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Our next identity relates the incomplete Fibonacci and tribonacci polynomials.

Identity 3.6. *If $n \ge 2s$, then*

$$T_{n+1}^{(s)}(x) = x^{n/2}F_{n+1}^{(s)}(x^{3/2}) + \sum_{i=1}^{n-2}x^{(i-1)/2}\sum_{j=0}^{s-1}(T_{n-i-1}^{(j)}(x) - T_{n-i-1}^{(j-1)}(x))F_i^{(s-j-1)}(x^{3/2}). \tag{3.6}$$

Proof. First note that the weight of all members of $\mathcal{T}_n^{(s)}$ that contain no trominos is given by

$$\sum_{r=0}^{s}\binom{n-r}{r}x^{2n-3r} = x^{n/2}F_{n+1}^{(s)}(x^{3/2}).$$

So assume a member of $\mathcal{T}_n^{(s)}$ contains at least one tromino and that the left-most tromino covers positions i through $i+2$. Suppose further that there are exactly r dominos to the left of the left-most tromino. Then the weight of all such members of $\mathcal{T}_n^{(s)}$ is given by $\binom{i-r-1}{r}x^{2i-3r-2}T_{n-i-1}^{(s-r-1)}(x)$. Summing over the possible i and r implies that the total weight of all the members of $\mathcal{T}_n^{(s)}$ containing at least one tromino is

$$\sum_{i=1}^{n-2}\sum_{r=0}^{s-1}\binom{i-r-1}{r}x^{2i-3r-2}T_{n-i-1}^{(s-r-1)}(x).$$

To obtain the expression in (3.6), we write $T^{(s-r-1)}_{n-i-1}$ as $\sum_{j=0}^{s-r-1}(T^{(j)}_{n-i-1}-T^{(j-1)}_{n-i-1})$, where $T^{(-1)}_{n-i-1}=0$. We then obtain a total weight formula of

$$\begin{aligned}&\sum_{i=1}^{n-2}\sum_{r=0}^{s-1}\binom{i-r-1}{r}x^{2i-3r-2}\sum_{j=0}^{s-r-1}(T^{(j)}_{n-i-1}-T^{(j-1)}_{n-i-1})\\&=\sum_{i=1}^{n-2}\sum_{j=0}^{s-1}(T^{(j)}_{n-i-1}-T^{(j-1)}_{n-i-1})\sum_{r=0}^{s-j-1}\binom{i-r-1}{r}x^{2i-3r-2}\\&=\sum_{i=1}^{n-2}\sum_{j=0}^{s-1}(T^{(j)}_{n-i-1}-T^{(j-1)}_{n-i-1})x^{(i-1)/2}F_i^{(s-j-1)}(x^{3/2}),\end{aligned}$$

which gives (3.6). □

4. Generating function formula for $T_n^{(s)}(x)$

The generating function formula for the incomplete tribonacci numbers was found in [10] and a formula was requested for the tribonacci polynomials. The next result provides such a formula. We remark that our method is more combinatorial than that used in [10] in the case $x=1$ and thus supplies an alternate proof in that case.

Theorem 4.1. *Let $Q_s(z)$ be the generating function for the incomplete tribonacci polynomials $T_n^{(s)}(x)$, where $n\geq 2s+1$. Then*

$$\frac{Q_s(z)}{z^{2s+1}}=\frac{T_{2s+1}(x)+(T_{2s-1}(x)+xT_{2s}(x))z+T_{2s}(x)z^2-z^2\left(\frac{x+z}{1-x^2z}\right)^{s+1}}{1-x^2z-xz^2-z^3}. \tag{4.1}$$

Proof. Let $r_n=r_n(x)$ be given by

$$r_n=\sum_{j=0}^{s}\binom{s}{j}\binom{n+s-j-2}{s}x^{2n+s-3j-3}+\sum_{j=0}^{s}\binom{s}{j}\binom{n+s-j-3}{s}x^{2n+s-3j-6},\qquad n\geq 3,$$

with $r_0=r_1=0$ and $r_2=x^{s+1}$.

We claim that $r_i(x)$ gives the total weight with respect to the statistic $2\delta+\nu$ of all the members of $\mathcal{T}_{i+2s}$ containing exactly $s+1$ longer pieces and ending in a longer piece, the subset of which we will denote by $\mathcal{A}$. To show this, first note that $r_i(x)$ evaluated at $x=1$ is seen to give the number of square-and-domino tilings of length $i+s-2$ or $i+s-3$ in which squares are black or white and having exactly s white squares and dominos combined. We then increase the length of each white square and each domino by one and add a domino to the end if the original tiling had length $i+s-2$ and add a tromino to the end if it had length $i+s-3$. This yields all members of $\mathcal{A}$ in a bijective manner and thus implies $r_i(x)$ at $x=1$ gives the cardinality of $\mathcal{A}$. Note that members of $\mathcal{A}$ ending in a domino contain $i-j-2$ squares, $s-j+1$ dominos, and j trominos for some $0\leq j\leq s$, while members of $\mathcal{A}$ ending is a tromino contain $i-j-3$ squares, $s-j$ dominos, and $j+1$ trominos for some j. Summing over j then implies that $r_i(x)$ is the distribution for the statistic $2\delta+\nu$ on $\mathcal{A}$, as claimed.

By the interpretation for $r_i(x)$ just described, the product $r_i(x)T_{n-2s-i}(x)$ gives the total weight of all members of $\mathcal{T}_{n-1}$ containing at least $s+1$ longer pieces in which the $(s+1)$-st longer piece ends at position $i+2s$ since the final $n-2s-i-1$ positions of such a member of $\mathcal{T}_{n-1}$ may be covered by any tiling of the same length. Summing over all possible i then gives the total weight of all members of $\mathcal{T}_{n-1}$ containing *strictly more* than s longer pieces. Subtracting from $T_n(x)$ thus gives the weight of all members of $\mathcal{T}_{n-1}$ containing *at most* s longer pieces and implies the following identity:

$$T_n^{(s)}(x) = T_n(x) - \sum_{i=0}^{n-2s-1} r_i(x) T_{n-2s-i}(x), \qquad n \geq 2s+1. \tag{4.2}$$

In order to find a closed form expression for $Q_s(z)$ using (4.2), we express $T_n = T_n(x)$ as follows:

$$T_n = T_{n-2s}T_{2s+1} + T_{n-2s-1}(T_{2s-1} + xT_{2s}) + T_{n-2s-2}T_{2s}, \qquad n \geq 2s+1. \tag{4.3}$$

We provide a combinatorial proof of (4.3) as follows. Note that (4.3) is clearly true if $s = 0$ or if $n = 2s+1$ since $T_0 = T_{-1} = 0$, so we may assume $s \geq 1$ and $n \geq 2s+2$. Observe first that the $T_{n-2s}T_{2s+1}$ term gives the weight of all members of $\mathcal{T}_{n-1}$ in which there is no piece covering the boundary between positions $2s$ and $2s+1$. On the other hand, the total weight of the members of $\mathcal{T}_{n-1}$ in which a domino covers this boundary is given by $xT_{n-2s-1}T_{2s}$. Finally, if a tromino covers the boundary between positions $2s$ and $2s+1$, then that tromino covers either positions $2s-1$, $2s$, and $2s+1$ or positions $2s$, $2s+1$, and $2s+2$. In the former case, the weight of the corresponding members of $\mathcal{T}_{n-1}$ is $T_{n-2s-1}T_{2s-1}$, while in the latter it would be $T_{n-2s-2}T_{2s}$. Combining all of the cases above gives (4.3).

Multiplying both sides of the equation

$$T_n^{(s)} = T_{n-2s}T_{2s+1} + T_{n-2s-1}(T_{2s-1} + xT_{2s}) + T_{n-2s-2}T_{2s} - \sum_{i=0}^{n-2s-1} r_i T_{n-2s-i}$$

by z^n and summing over $n \geq 2s+1$ yields

$$\frac{Q_s(z)}{z^{2s}} = \left(T_{2s+1} + (T_{2s-1} + xT_{2s})z + T_{2s}z^2 - \sum_{i\geq 0} r_i z^i \right) \cdot \sum_{n\geq 1} T_n z^n.$$

The proof is completed by noting

$$\sum_{i\geq 0} r_i z^i = z^2 \left(\frac{x+z}{1-x^2z} \right)^{s+1}$$

and

$$\sum_{n\geq 1} T_n z^n = \frac{z}{1 - x^2z - xz^2 - z^3},$$

the former being computed by the methods given in [12, Section 4.3]. □

Taking $x = 1$ in the prior theorem yields the following result.

Corollary 4.2. *Let $q_s(z)$ be the generating function for the incomplete tribonacci numbers $t_n^{(s)}$. Then*

$$\frac{q_s(z)}{z^{2s+1}} = \frac{t_{2s+1} + (t_{2s-1} + t_{2s})z + t_{2s}z^2 - z^2 \left(\frac{1+z}{1-z}\right)^{s+1}}{1 - z - z^2 - z^3}. \tag{4.4}$$

Remark: Equation (4.4) appears as Theorem 8 of [10]. We note however that there was a slight misstatement of this theorem; in particular, there should be no -2 in the factor multiplying z^2 in the numerator on the right-hand side.

References

[1] K. Alladi and V. E. Hoggatt Jr., On tribonacci numbers and related functions, *Fibonacci Quart.* **15** (1977), 42–45.

[2] P. Barry, On integer-sequence-based constructions of generalized Pascal triangles, *J. Integer Seq.* **9** (2006), Article 06.2.4.

[3] H. Belbachir and A. Belkhir, Combinatorial expressions involving Fibonacci, hyperfibonacci, and incomplete Fibonacci numbers, *J. Integer Seq.* **17** (2014), Article 14.4.3.

[4] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, 2003.

[5] P. Filipponi, Incomplete Fibonacci and Lucas numbers, *Rend. Circ. Mat. Palermo* **45** (1996), 37–56.

[6] V. E. Hoggatt Jr. and M. Bicknell, Generalized Fibonacci polynomials, *Fibonacci Quart.* **11** (1973), 457–465.

[7] E. Kilic and H. Prodinger, A note on the conjecture of Ramirez and Sirvent, *J. Integer Seq.* **17** (2014), Article 14.5.8.

[8] J. L. Ramírez, Incomplete generalized Fibonacci and Lucas polynomials, pre-print, http://arxiv.org/abs/1308.4192.

[9] J. L. Ramírez, Bi-periodic incomplete Fibonacci sequences, *Ann. Math. Inform.* **42** (2013), 83–92.

[10] J. L. Ramírez and V. F. Sirvent, Incomplete tribonacci numbers and polynomials, *J. Integer Seq.* **17** (2014), Article 14.4.2.

[11] N. J. A. Sloane, On-line Encyclopedia of Integer Sequences, http://oeis.org, 2010.

[12] H. Wilf, *generatingfunctionology*, CRC Press, third edition, 2005.