# PARTITIONS WITH FIXED DIFFERENCES BETWEEN LARGEST AND SMALLEST PARTS

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Abstract. We study the number $p(n,t)$ of partitions of $n$ with difference $t$ between largest and smallest parts. Our main result is an explicit formula for the generating function $P_t(q) := \sum_{n\geq 1} p(n,t)\, q^n$. Somewhat surprisingly, $P_t(q)$ is a rational function for $t>1$; equivalently, $p(n,t)$ is a quasipolynomial in $n$ for fixed $t>1$. Our result generalizes to partitions with an arbitrary number of specified distances.

Enumeration results on integer partitions form a classic body of mathematics going back to at least Euler, including numerous applications throughout mathematics and some areas of physics; see, e.g., [2]. A *partition* of a positive integer $n$ is, as usual, an integer $k$-tuple $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, for some $k$, such that

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k \, .$$

The integers $\lambda_1, \lambda_2, \dots, \lambda_k$ are the *parts* of the partition. We are interested in the counting function

$$p(n,t) := \#\text{partitions of } n \text{ with difference } t \text{ between largest and smallest parts.}$$

It is immediate that

$$p(n,0) = d(n)$$

where $d(n)$ denotes the number of divisors of $n$. Charmingly, $p(n,1)$ equals the number of *non*divisors of $n$:

$$p(n,1) = n - d(n)\, ,$$

which can be explained bijectively by the fact that the partitions counted by $p(n,0)+p(n,1)$ contain exactly one sample with $k$ parts, for each $k = 1, 2, \dots, n$ [1, Sequence A049820], or by the generating function identity

$$\sum_{n\geq 1} p(n,1)\, q^n = \sum_{m\geq 1} \frac{q^m}{1-q^m} \frac{q^{m+1}}{1-q^{m+1}} = \frac{q}{(1-q)^2} - \sum_{m \geq 1} \frac{q^m}{1-q^m} \, .$$

(The last equation follows from a few elementary operations on rational functions). An even less obvious instance of our partition counting function is

$$p(n,2) = \binom{\lfloor \frac{n}{2} \rfloor}{2} \, , \tag{1}$$

as observed by Reinhard Zumkeller in 2004 [1, Sequence A008805]. (It is not clear to us where in the literature this formula first appeared, though specific values of $p(n,k)$ are well represented in

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[1], where Sequences A000005, A049820, A008805, A128508, and A218567–A218573 give the first values of $p(n,k)$ for fixed $k = 0, 1, \dots, 10$, and Sequence A097364 paints a general picture of $p(n,t)$.)

We remark that $p(n,2)$ is arithmetically quite different from $p(n,0)$ and $p(n,1)$: namely, $p(n,2)$ is a *quasipolynomial*, i.e., a function that evaluates to a polynomial when $n$ is restricted to a fixed residue class modulo some (minimal) positive integer, the *period* of the quasipolynomial. (For $p(n,2)$ this period is 2.) Equivalently, the accompanying generating function evaluates to a rational function all of whose poles are rational roots of unity. (See, e.g., [3, Chapter 4] for more on quasipolynomials and their rational generating functions.) Our goal is to prove closed formulas for these generating functions

$$P_t(q) := \sum_{n \ge 1} p(n,t)\, q^n .$$

**Theorem 1.** *For $t > 1$,*

$$P_t(q) = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}}{(1-q^t)^2(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q^2)} + \frac{q^t}{(1-q^t)(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q)} .$$

Written in terms of the usual shorthand $(q)_m := (1-q)(1-q^2)\cdots(1-q^m)$, Theorem 1 says

$$P_t(q) = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} + \frac{q^t}{(1-q^{t-1})(q)_t} .$$

Thus $P_t(q)$ is rational for $t > 1$, and so $p(n,t)$ is a quasipolynomial in $n$, of degree $t$ and period $\operatorname{lcm}(1, 2, \dots, t)$. For example, for $t = 2$, Theorem 1 gives

$$P_2(q) = \frac{q^4}{(1-q)^3(1+q)^2}$$

which confirms (1). The rational generating function given by Theorem 1 in the case $t = 3$ simplifies to

$$P_3(q) = \frac{q^5+q^6+q^7-q^8}{(1-q^2)^2(1-q^3)^2}$$

which (by way of a computer algebra system or a straightforward binomial expansion) translates to the partition counting function

$$p(n,3) = \frac{1}{108} \times \begin{cases} n^3 - 18n & \text{if } n \equiv 0 \bmod 6, \\ n^3 - 3n + 2 & \text{if } n \equiv 1 \bmod 6, \\ n^3 - 30n + 52 & \text{if } n \equiv 2 \bmod 6, \\ n^3 + 9n - 54 & \text{if } n \equiv 3 \bmod 6, \\ n^3 - 30n + 56 & \text{if } n \equiv 4 \bmod 6, \\ n^3 - 3n - 2 & \text{if } n \equiv 5 \bmod 6 \end{cases}$$

$$= \begin{cases} m(2m^2-1) & \text{if } n = 6m, \\ m(2m^2+1) & \text{if } n = 6m+1, \\ m(2m^2+2m-1) & \text{if } n = 6m+2, \\ m(2m^2+3m+2) & \text{if } n = 6m+3, \\ (m-1)(2m^2-1) & \text{if } n = 6m-2, \\ m^2(2m-1) & \text{if } n = 6m-1. \end{cases}$$

Using this explicit form of $p(n,3)$, one easily affirms a conjecture about the recursive structure of $p(n,3)$ given in [1, Sequence A128508] in the positive.

*Proof of Theorem 1.* We will use the usual shorthand

$$(A)_m := (1-A)(1-A\,q)\cdots(1-A\,q^{m-1})$$

as well as *Heine's transformation* (see, e.g., [2, p. 38])

$$\sum_{m\ge0} \frac{(a)_m (b)_m\, z^m}{(q)_m (c)_m} = \frac{(\frac{c}{b})_\infty (bz)_\infty}{(c)_\infty (z)_\infty} \sum_{j\ge0} \frac{(\frac{abz}{c})_j (b)_j (\frac{c}{b})^j}{(q)_j (bz)_j}\,. \tag{2}$$

Now we construct the generating function for $p(n,t)$. A partition of $n$ with difference $t$ between smallest and largest part starts with some part $m$, ends with the part $m+t$, and could include any of the numbers $m+1, m+2, \dots, m+t-1$ as parts. Translated into geometric series, this gives

$$\begin{aligned} P_t(q) &= \sum_{m\ge1} \frac{q^m}{1-q^m}\,\frac{1}{1-q^{m+1}}\cdots\frac{1}{1-q^{m+t-1}}\,\frac{q^{m+t}}{1-q^{m+t}} = q^t \sum_{m\ge1} \frac{q^{2m}(q)_{m-1}}{(q)_{m+t}} = q^{t+2} \sum_{m\ge0} \frac{q^{2m}(q)_m}{(q)_{m+t+1}} \\ &= \frac{q^{t+2}}{(q)_{t+1}} \sum_{m\ge0} \frac{(q)_m (q)_m\, q^{2m}}{(q)_m (q^{t+2})_m} \overset{(2)}{=} \frac{q^{t+2}(q^{t+1})_\infty (q^3)_\infty}{(q)_{t+1}(q^{t+2})_\infty (q^2)_\infty} \sum_{j\ge0} \frac{(q^{-t+2})_j (q)_j\, q^{j(t+1)}}{(q)_j (q^3)_j} \\ &= \frac{q^{t+2}}{(q)_t} \sum_{j=0}^{t-2} \frac{(q^{-t+2})_j\, q^{j(t+1)}}{(q^2)_{j+1}} = \frac{q^{t+2}}{(q)_t} \sum_{j=0}^{t-2} \frac{(1-q^{t-2})(1-q^{t-3})\cdots(1-q^{t-j-1})(-1)^j q^{2j+\binom{j+1}{2}}}{(q^2)_{j+1}} \\ &= \frac{q^{t+2}(1-q)}{(1-q^t)(1-q^{t-1})} \sum_{j=0}^{t-2} \frac{(-1)^j q^{2j+\binom{j+1}{2}}}{(q)_{j+2}(q)_{t-j-2}} = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} \sum_{j=0}^{t-2} \begin{bmatrix} t \\ j+2 \end{bmatrix} (-1)^j q^{\binom{j+3}{2}}\,. \end{aligned}$$

Thus, by the $q$-binomial theorem (see, e.g., [2, p. 36])

$$\begin{aligned} P_t(q) &= \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} \sum_{j=2}^{t} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} = \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} \left( (q)_t - 1 + q \begin{bmatrix} t \\ 1 \end{bmatrix} \right) \\ &= \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})} - \frac{q^{t-1}(1-q)}{(1-q^t)(1-q^{t-1})(q)_t} + \frac{q^t}{(1-q^{t-1})(q)_t}\,. \end{aligned}$$

□

A natural question concerns the growth behavior of $p(n,t)$. We see in the above example that the quasipolynomial $p(n,3)$ has a constant leading coefficient, which of course determines the asymptotic growth of $p(n,3)$. Something similar can be said in general.

**Corollary 2.** *If $t > 1$ then $p(n,t) = \dfrac{n^t}{t\,(t!)^2} + O(n^{t-1})$ as $n \to \infty$.*

*Proof.* It is well known that the first-order asymptotics of a quasipolynomial stems from the highest-order poles of its rational generating function. (This follows from first principles, essentially partial-fraction decomposition; see [**?**] for far-reaching generalizations.) In our case, $P_t(q)$ has a unique highest-order pole at $q = 1$ of order $t$. Thus the leading coefficient of $p(n,t)$ equals $\frac{1}{t!}$ times the lowest coefficient of the Laurent series of $P_t(q)$ at $q = 1$ which is

$$\lim_{q\to 1} \frac{(1-q)^{t+1}(2q^t - q^{2t} - q^{t-1})}{(1-q^t)^2(1-q^{t-1})^2(1-q^{t-2})\cdots(1-q)} = \frac{1}{t\cdot t!}\,. \qquad \square$$

Next we shall generalize Theorem 1 by considering *partitions with specified distances.* Let $p(n,t_1,t_2,\dots,t_k)$ be the number of partitions of $n$ such that, if $\sigma$ is the smallest part then $\sigma + t_1 + t_2 + \cdots + t_k$ is the largest part and each of $\sigma + t_1$, $\sigma + t_1 + t_2$, $\dots$, $\sigma + t_1 + t_2 + \cdots + t_{k-1}$ appear as parts. We consider the related generating function

$$P_{t_1,\dots,t_k}(q) := \sum_{n\ge 1} p(n,t_1,t_2,\dots,t_k)\, q^n.$$

We note that when $k = 1$ this is simply $P_t(q)$ from above.

**Theorem 3.** *For $t := t_1 + t_2 + \cdots + t_k > k$,*

$$P_{t_1,\dots,t_k}(q) = \frac{(-1)^k q^{T-\binom{k+1}{2}} \left(\sum_{j=0}^{k} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} - (q)_t\right)}{\begin{bmatrix} t-1 \\ k \end{bmatrix} (1-q^t)(q)_t}\,,$$

*where $T := kt_1 + (k-1)t_2 + \cdots + 2t_{k-1} + t_k$ and $\begin{bmatrix} A \\ B \end{bmatrix} := \frac{(q)_A}{(q)_B(q)_{A-B}}$.*

For example, for $k = 2$ and $t_1 = t_2 = 2$, we have $p(11,2,2) = 2$ since $1+1+1+3+5$ and $1+2+3+5$ are the unique two partitions of 11 that contain three parts whose consecutive distances are 2. Theorem 3 says in this case

$$P_{2,2}(q) = \frac{q^9 + q^{10} + q^{11} + q^{12} - q^{13}}{(1-q^2)(1-q^3)^2(1-q^4)^2}$$

which translates to

$$p(n,2,2) = \frac{1}{6912}\begin{cases} 3n^4 - 20n^3 - 24n^2 + 288n & \text{if } n \equiv 0 \bmod 12,\\ 3n^4 - 20n^3 - 78n^2 + 492n - 397 & \text{if } n \equiv 1 \bmod 12,\\ 3n^4 - 20n^3 - 24n^2 - 48n + 304 & \text{if } n \equiv 2 \bmod 12,\\ 3n^4 - 20n^3 - 78n^2 + 1260n - 2781 & \text{if } n \equiv 3 \bmod 12,\\ 3n^4 - 20n^3 - 24n^2 - 480n + 2816 & \text{if } n \equiv 4 \bmod 12,\\ 3n^4 - 20n^3 - 78n^2 + 492n + 155 & \text{if } n \equiv 5 \bmod 12,\\ 3n^4 - 20n^3 - 24n^2 + 720n - 3024 & \text{if } n \equiv 6 \bmod 12,\\ 3n^4 - 20n^3 - 78n^2 + 492n + 35 & \text{if } n \equiv 7 \bmod 12,\\ 3n^4 - 20n^3 - 24n^2 - 480n + 3328 & \text{if } n \equiv 8 \bmod 12,\\ 3n^4 - 20n^3 - 78n^2 + 1260n - 3213 & \text{if } n \equiv 9 \bmod 12,\\ 3n^4 - 20n^3 - 24n^2 - 48n - 208 & \text{if } n \equiv 10 \bmod 12,\\ 3n^4 - 20n^3 - 78n^2 + 492n + 547 & \text{if } n \equiv 11 \bmod 12. \end{cases}$$

*Proof of Theorem 3.* Again we start with the natural generating function

$$\begin{aligned}
P_{t_1,\dots,t_k}(q) &= \sum_{m\geq 1} \frac{q^m\, q^{m+t_1}\, q^{m+t_1+t_2}\cdots q^{m+t_1+t_2+\cdots+t_k}}{(1-q^m)(1-q^{m+1})\cdots(1-q^{m+t_1+t_2+\cdots+t_k})} = \sum_{m\geq 1} \frac{q^{(k+1)m+T}}{(q^m)_{t+1}} \\
&= \sum_{m\geq 1} \frac{q^{(k+1)m+T}(q)_{m-1}}{(q)_{m+t}} = q^{T+k+1} \sum_{m\geq 0} \frac{q^{(k+1)m}(q)_m}{(q)_{m+t+1}} = \frac{q^{T+k+1}}{(q)_{t+1}} \sum_{m\geq 0} \frac{(q)_m (q)_m\, q^{(k+1)m}}{(q)_m (q^{t+2})_m} \\
&\overset{(2)}{=} \frac{q^{T+k+1}(q^{t+1})_\infty (q^{k+2})_\infty}{(q)_{t+1}(q^{k+1})_\infty (q^{t+2})_\infty} \sum_{j\geq 0} \frac{(q^{k+1-t})_j (q)_j q^{(t+1)j}}{(q)_j (q^{k+2})_j} \\
&= \frac{q^{T+k+1}(q)_k}{(q)_t} \sum_{j=0}^{t-k-1} \frac{(q^{-(t-k+1)})_j q^{(t+1)j}}{(q)_{j+k+1}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{q^{T+k+1}(q)_k}{(q)_t} \sum_{j=0}^{t-k-1} \frac{(1-q^{t-k-1})(1-q^{t-k-2})\cdots(1-q^{t-k-j})(-1)^j q^{\binom{j}{2}-j(t-k-1)+(t+1)j}}{(q)_{j+k+1}} \\
&= \frac{q^{T+k+1}(q)_k}{(q)_t} \sum_{j=0}^{t-k-1} \frac{(q)_{t-k-1}(-1)^j q^{\binom{j+1}{2}+j(k+1)}}{(q)_{j+k+1}(q)_{j-k-j-1}} \\
&= \frac{q^{T+k+1}(q)_k (q)_{t-k-1}}{(q)_t (q)_t} \sum_{j=0}^{t-k-1} \begin{bmatrix} t \\ j+k+1 \end{bmatrix} (-1)^j q^{\binom{j+k+2}{2}-\binom{k+2}{2}} \\
&= \frac{q^{T+k+1}(q)_k}{\begin{bmatrix} t-1 \\ k \end{bmatrix}(1-q^t)(q)_t} \sum_{j=0}^{t-k-1} \begin{bmatrix} t \\ j+k+1 \end{bmatrix} (-1)^j q^{\binom{j+k+2}{2}-\binom{k+2}{2}} \\
&= \frac{q^{T-\binom{k+1}{2}}(-1)^{k+1}}{\begin{bmatrix} t-1 \\ k \end{bmatrix}(1-q^t)(q)_t} \sum_{j=k+1}^{t} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} \\
&= \frac{q^{T-\binom{k+1}{2}}(-1)^{k}}{\begin{bmatrix} t-1 \\ k \end{bmatrix}(1-q^t)(q)_t} \left( \sum_{j=0}^{k} \begin{bmatrix} t \\ j \end{bmatrix} (-1)^j q^{\binom{j+1}{2}} - (q)_t \right). \qquad \square
\end{aligned}$$

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