CONGRUENCES AND RELATIONS FOR r-FISHBURN NUMBERS

F. G. GARVAN

ABSTRACT. Recently Andrews and Sellers proved some amazing congruences for the Fishburn numbers. We extend their results to a more general sequence of numbers. As a result we prove a new congruence mod 23 for the Fishburn numbers and prove their conjectured mod 5 congruence for a related sequence. We also extend and prove some unpublished conjectures of Garthwaite and Rhoades.

1. INTRODUCTION

The Fishburn numbers $\xi(n)$ [2] are defined by the formal power series

$$\sum_{n=0}^{\infty} \xi(n) q^n = F(1-q), \tag{1.1}$$

where

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n, \tag{1.2}$$

and

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Zagier [10] showed that $\xi(n)$ is the number of linearized chord diagrams of degree n, and also the number of nonisomorphic interval orders on n unlabeled points. Andrews and Sellers [5] proved some amazing congruences for the Fishburn numbers. For example, for all $n \geq 0$,

$$\xi(5n+3) \equiv \xi(5n+4) \equiv 0 \pmod{5}, \tag{1.3}$$

$$\xi(7n+6) \equiv 0 \pmod{7}, \tag{1.4}$$

$$\xi(11n+8) \equiv \xi(11n+9) \equiv \xi(11n+10) \equiv 0 \pmod{11}, \tag{1.5}$$

$$\xi(17n+16) \equiv 0 \pmod{17}, \text{ and} \tag{1.6}$$

$$\xi(19n+17) \equiv \xi(19n+18) \equiv 0 \pmod{19}. \tag{1.7}$$

In fact, they prove that there are analogous congruences for all primes p that are quadratic nonresidues mod 23. For p prime they define

$$S(p) = \left\{ j \,:\, 0 \leq j \leq p-1 \text{ such that } \tfrac{1}{2}n(3n-1) \equiv j \pmod{p} \text{ for some } n \right\} \tag{1.8}$$

and

$$T(p) = \{ k \,:\, 0 \leq k \leq p-1 \text{ such that } k \text{ is larger than every element of } S(p) \}. \tag{1.9}$$

Date: June 24, 2014.

2010 *Mathematics Subject Classification.* 05A19, 11B65, 11P83.

Key words and phrases. Fishburn numbers, interval orders, congruence, Bernoulli polynomial, Stirling numbers, Glaisher T-numbers.

We state their main result.

Theorem 1.1 (Andrews and Sellers[5]). *If p is a prime and $i \in T(p)$ (as defined in (1.9)), then for all $n \geq 0$,*

$$\xi(pn+i) \equiv 0 \pmod{p}.$$

Remark 1.2. Congruences (1.3)–(1.7) are the cases $p = 5$, 7, 11, 17 and 19 of Theorem 1.1. Andrews and Sellers proved that $T(p)$ is nonempty whenever p is a quadratic nonresidue mod 23.

Recently Garthwaite and Rhoades [7] observed congruence pairs and triples such as

$$\xi(5n+2) - 2\,\xi(5n+1) \equiv 0 \pmod{5}, \tag{1.10}$$

$$\xi(11n+7) - 3\,\xi(11n+4) + 2\,\xi(11n+3) \equiv 0 \pmod{11}. \tag{1.11}$$

In Theorem 1.3 and Corollary 1.7 below we prove congruences relations mod p exist for all primes ≥ 5.

We extend the Andrews and Sellers result to what we call r-Fishburn numbers $\xi_r(n)$ and which we define by the formal power series

$$\sum_{n=0}^{\infty} \xi_r(n) q^n = F((1-q)^r), \tag{1.12}$$

where r is any nonzero integer. The case $r = 1$ corresponds to the ordinary Fishburn numbers. Even in the case $r = 1$ we are able to augment the set $T(p)$. For $p \geq 5$ prime, r relatively prime to p and nonzero and $0 \leq s \leq p-1$ we define

(1.13)

$$S^*(p,r,s) = \Big\{ j \,:\, 0 \leq j \leq p-1 \text{ such that } \tfrac{1}{2} rn(3n-1) \equiv j - s \pmod{p} \text{ for some } n \text{ and } 24(j-s) \not\equiv -r \pmod{p} \Big\}$$

and

(1.14)

$$T^*(p,r,s) = \{ k \,:\, 0 \leq k \leq p-1 \text{ such that } k \text{ is larger than every element of } S^*(p,r,s)\}.$$

We state our main

Theorem 1.3. *Suppose $p \geq 5$ is prime, r is a nonzero integer relatively prime to p and $0 \leq s \leq p-1$. If $m \in T^*(p,r,s)$ (as defined in (1.14)), then for all $n \geq 0$,*

$$\sum_{j=0}^{s} \binom{s}{j} (-1)^j \xi_r(pn+m-j) \equiv 0 \pmod{p}.$$

Remark 1.4. We make some remarks.

(i) The $(r,s) = (1,0)$ case of the theorem is slightly stronger than the Andrews-Sellers result. We observe that although $22 \in S(23)$, $22 \notin S^*(23,1,0)$ and we find $T^*(23,1,0) = \{18,19,20,21,22\}$ so that

(1.15)

$$\xi(23n+18) \equiv \xi(23n+19) \equiv \xi(23n+20) \equiv \xi(23n+21) \equiv \xi(23n+22) \equiv 0 \pmod{23}.$$

This is a congruence that Andrews and Sellers missed.

(ii) We find that $S^*(5,-1,0)=\{0,3\}$ and $T^*(5,-1,0)=\{4\}$ so that

$$\xi_{-1}(5n+4) \equiv 0 \pmod 5. \tag{1.16}$$

This congruence was conjectured by Andrews and Sellers [5]. When $(r,s)=(-1,0)$ the Theorem only gives a congruence in the case $p=5$. This is because 1 is a pentagonal number so that when $p>5$ we have $p-1\in S^*(p,-1)$ and $T^*(p,-1,0)$ is empty. Using the facts that

$$T^*(5,-1,2)=\{3,4\},\quad T^*(5,-1,3)=\{4\},$$

we find that

$$\xi_{-1}(5n+3) \equiv 3\,\xi_{-1}(5n+2) \equiv 2\,\xi_{-1}(5n+1) \pmod 5. \tag{1.17}$$

Example 1.5.

$$S^*(43,-1,2)=\{0,1,2,5,7,10,13,14,16,18,19,23,29,30,31,33,37,38,39,40,41\},$$

so that

$$T^*(43,-1,2)=\{42\},$$

and

$$\xi_{-1}(43n+42)-2\,\xi_{-1}(43n+41)+\xi_{-1}(43n+40) \equiv 0 \pmod{43},$$

for all $n\ge 0$.

We highlight the $s=0$ case of the theorem.

Corollary 1.6. *Suppose $p\ge 5$ is prime and r is a nonzero integer relatively prime to p. If $m\in T^*(p,r,0)$ (as defined in (1.14)), then for all $n\ge 0$,*

$$\xi_r(pn+m)\equiv 0 \pmod p.$$

Corollary 1.7. *Suppose $p\ge 5$ is prime, and r is a nonzero integer relatively prime to p. Then there are at least $\frac{1}{2}(p+1)$ linearly independent congruence relations mod p of the form*

$$\sum_{j=0}^{p-1}\alpha_j\,\xi_r(pn+j)\equiv 0 \pmod p,$$

where n is any nonnnegative integer and $\vec{\alpha}\in\mathbb{F}_p^p$.

Remark 1.8. In Section 4 we prove Corollary 1.7 by showing that the relations

$$\sum_{j=0}^{s}\binom{s}{j}(-1)^j\xi_r(pn+p-1-j)\equiv 0 \pmod p,$$

where the Legendre symbol $\left(\frac{-24(1+s)\overline{r}+1}{p}\right)=-1$ or 0, form a set of $\frac{1}{2}(p+1)$ linearly independent congruence relations mod p. Here $r\overline{r}\equiv 1 \pmod p$. This also means that s can never equal $p-1$.

Example 1.9. When $p = 7$ and $r = 1$ there are 4 relations mod 7:

$$\begin{aligned}\xi(7n+6) &\equiv 0 \pmod{7},\\ \xi(7n+6) - 2\,\xi(7n+5) + \xi(7n+4) &\equiv 0 \pmod{7},\\ \xi(7n+6) - 3\,\xi(7n+5) + 3\,\xi(7n+4) - \xi(7n+3) &\equiv 0 \pmod{7},\\ \xi(7n+6) - 4\,\xi(7n+5) + 6\,\xi(7n+4) - 4\,\xi(7n+3) + \xi(7n+2) &\equiv 0 \pmod{7},\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\xi(7n+6) &\equiv 0 \pmod{7},\\ \xi(7n+5) + 5\,\xi(7n+2) &\equiv 0 \pmod{7},\\ \xi(7n+4) + 3\,\xi(7n+2) &\equiv 0 \pmod{7},\\ \xi(7n+3) + \xi(7n+2) &\equiv 0 \pmod{7}.\end{aligned}$$

Conjecture 1.10. *Suppose $p \ge 5$ is prime, and r is a nonzero integer relatively prime to p. Then there are exactly $\frac{1}{2}(p+1)$ linearly independent congruence relations mod p of the form*

$$\sum_{j=0}^{p-1} \alpha_j\, \xi_r(pn+j) \equiv 0 \pmod{p},$$

where n is any nonnnegative integer and $\vec{\alpha} \in \mathbb{F}_p^p$.

Following Andrews and Sellers [5], we define

$$F(q,N) = \sum_{n=0}^{N} (q;q)_n, \tag{1.18}$$

and the p-dissection

$$F(q,N) = \sum_{i=0}^{p-1} q^i A_p(N,i,q^p). \tag{1.19}$$

We consider the coefficients of the polynomials

$$A_p(pn-1,i,1-q) = \sum_{k\ge 0} \alpha(p,n,i,k) q^k. \tag{1.20}$$

The Andrews-Sellers Theorem 1.1 depends crucially on

Lemma 1.11 (Andrews and Sellers [5]). *If $i \notin S(p)$, then*

$$\alpha(p,n,i,k) = 0,$$

for $0 \le k \le n-1$.

We consider the analog of this result when $24i \equiv -1 \pmod{p}$. For $p \ge 5$ prime we define $\bar{\xi}_p(n)$ by the formal power series

$$\sum_{n=0}^{\infty} \bar{\xi}_p(n) q^n = (1-q)^{\lfloor \frac{p}{24} \rfloor} F((1-q)^p), \tag{1.21}$$

where $F(q)$ is defined in (1.2). Observe that when $5 \le p \le 23$,

$$\bar{\xi}_p(n) = \xi_p(n),$$

for $n \ge 0$. We find the following new relation for Fishburn numbers.

Theorem 1.12. *Suppose $p \ge 5$ is prime and $24 i_0 \equiv -1 \pmod{p}$ where $1 \le i_0 \le p-1$. Then*

$$\alpha(p, n, i_0, k) = p \left(\frac{12}{p}\right) \bar{\xi}_p(k),$$

for $0 \le k \le n-1$. Here $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol.

Our main Theorem 1.3 will follow from Lemma 1.11 and Theorem 1.12 in a straightforward manner.

2. Preliminary results

Lemma 2.1. *Let p be prime and suppose $0 \le j \le p-1$ and $0 \le k \le M-1 \le N-1$. Then*

$$\alpha(p, N, j, k) = \alpha(p, M, j, k),$$

where $\alpha(p, n, i, k)$ is defined in (1.20).

Proof. Let $\zeta = \exp(2\pi i/p)$. Then from (1.19) we have

$$A_p(N, j, q) = \frac{1}{p} \sum_{k=0}^{p-1} \zeta^{-jk} q^{-\frac{j}{p}} F\left(\zeta^k q^{\frac{1}{p}}, N\right).$$

Next we suppose that $n \ge pM$. Then

$$\begin{aligned}
\left(\zeta^k (1-q)^{\frac{1}{p}}; \zeta^k (1-q)^{\frac{1}{p}}\right)_n &= \prod_{j=1}^{n} \left(1 - \left(\zeta^k (1-q)^{\frac{1}{p}}\right)^j\right) \\
&= \prod_{j=1}^{\lfloor n/p \rfloor} (1 - (1-q)^j) \prod_{\substack{j=1 \\ j \not\equiv 0 \pmod{p}}}^{n} \left(1 - \left(\zeta^k (1-q)^{\frac{1}{p}}\right)^j\right) \\
&= \prod_{j=1}^{M} (jq + O(q^2)) \prod_{\substack{j=1 \\ j \not\equiv 0 \pmod{p}}}^{n} \left(1 - \left(\zeta^k (1-q)^{\frac{1}{p}}\right)^j\right) \\
&= O\left(q^M\right).
\end{aligned}$$

Thus

$$\begin{aligned}
A_p(pN-1, j, 1-q) &= \frac{1}{p} \sum_{k=0}^{p-1} \zeta^{-jk} q^{-\frac{j}{p}} \sum_{n=0}^{pN-1} \left(\zeta^k (1-q)^{\frac{1}{p}}; \zeta^k (1-q)^{\frac{1}{p}}\right)_n \\
&= A_p(pM-1, j, 1-q) + O(q^M).
\end{aligned}$$

The result follows. ∎

Andrews and Sellers define a Stirling like array of numbers $C(n,i,j,p)$ for $n \geq 0$, $0 \leq i \leq p-1$, and $0 \leq j \leq n$, which are defined by the recursion

$$C(n+1,i,j,p) = (i+jp)C(n,i,j,p) + pC(n,i,j-1,p), \tag{2.1}$$

and the initial value

$$C(0,i,0,p) = 1. \tag{2.2}$$

It is understood that if either of the conditions $n \geq 0$ or $0 \leq j \leq n$ are not satisfied, then $C(n,i,j,p) = 0$. We note that

$$C(n,i,0,p) = i^n.$$

We need a generalization of the signless Stirling numbers of the first kind. We define the numbers $s_1(n,j,m)$ for $0 \leq j \leq n$ by

$$\sum_{j=0}^{n} s_1(n,j,m)x^j = (x-m)(x-m+1)\cdots(x-m+n-1). \tag{2.3}$$

We note case $m = 0$ correspond to the signless Stirling numbers of the first kind $s_1(n,j)$. We define

$$f(x,n,k,m) = (-1)^n \sum_{j=k}^{n} \binom{j}{k} s_1(n,j,m)x^j, \tag{2.4}$$

for $0 \leq k \leq n$, otherwise define $f(x,n,k,m) = 0$.

Lemma 2.2. *For* $0 \leq k \leq n+1$ *we have*

$$f(x,n+1,k,m) = -((x+n-m)f(x,n,k,m) + xf(x,n,k-1,m)). \tag{2.5}$$

Proof. First we observe that

$$f(x,n,k,m) = \frac{x^k}{k!}\left(\frac{\partial}{\partial x}\right)^k f(x,n,0,m),$$

where

$$f(x,n,0,m) = (-1)^n(x-m)(x-m+1)\cdots(x-m+n-1) \quad \text{(by (2.3))},$$

for $k \geq 0$. Now we let

$$\tilde{f}(x,n,k,m) = \left(\frac{\partial}{\partial x}\right)^k (-1)^n(x-m)(x-m+1)\cdots(x-m+n-1),$$

for $0 \leq k \leq n$, otherwise define $\tilde{f}(x,n,k,m) = 0$. We show that

$$\tilde{f}(x,n+1,k,m) = -\Big((x-m+n)\tilde{f}(x,n,k,m) + k\tilde{f}(x,n,k-1,m)\Big) \tag{2.6}$$

where $0 \leq k \leq n+1$. Since

$$\tilde{f}(x,n+1,0,m) = -(x-m+n)\tilde{f}(x,n,0,m),$$

we have

$$\frac{\partial}{\partial x}\tilde{f}(x,n+1,0,m) = -(x-m+n)\frac{\partial}{\partial x}\tilde{f}(x,n,0,m) - \tilde{f}(x,n,0,m),$$

so that

$$\tilde{f}(x,n+1,1,m) = -\Big((x-m+n)\tilde{f}(x,n,1,m) + \tilde{f}(x,n,0,m)\Big)$$

and (2.6) holds for $k=1$. We assume (2.6) holds for $k \le K$.

$$\begin{aligned}\tilde{f}(x,n+1,K+1,m) &= \frac{\partial}{\partial x}\tilde{f}(x,n+1,K,m)\\ &= -\frac{\partial}{\partial x}\Big[(x-m+n)\tilde{f}(x,n,K,m) + K\tilde{f}(x,n,k-1,m)\Big]\\ &= -\Big[(x-m+n)\tilde{f}(x,n,K+1,m) + \tilde{f}(x,n,K,m) + K\tilde{f}(x,n,K,m)\Big]\\ &= -\Big[(x-m+n)\tilde{f}(x,n,K+1,m) + (K+1)\tilde{f}(x,n,K,m)\Big],\end{aligned}$$

and (2.6) holds for $k = K+1$. Hence (2.6) holds for all k by induction. Since

$$f(x,n,k,m) = \frac{x^k}{k!}\tilde{f}(x,n,k,m),$$

the result (2.5) follows easily. □

Theorem 2.3. *Let* $i_0 = (p^2-1)z - mp$. *Suppose that*

$$\sum_{\ell=0}^{n} C(n,i_0,\ell,p)A_1(\ell,m) = (-1)^n z^n \sum_{k=0}^{n}\binom{n}{k}X(k), \tag{2.7}$$

for $n \ge 0$. *Then*

$$A_1(n,m) = (-1)^n \sum_{k=0}^{n}\sum_{j=k}^{n}\binom{j}{k} s_1(n,j,m)p^{j-2k}X(k)z^j, \tag{2.8}$$

for $n \ge 0$.

Proof. Since for $n \ge 0$ (2.7) forms a triangular system of equations in the unknowns $A_1(\ell,m)$ (for fixed m) and each diagonal coefficient

$$C(n,i_0,n,p) = p^n \ne 0,$$

it suffices to show that $A_1(n,m)$ given by (2.8) satisfies (2.7). By considering the coefficient of $X(L)$ it suffices to show that

$$\sum_{\ell=0}^{n} C(n,i_0,\ell,p)f(pz,\ell,L,m) = G(n,L), \tag{2.9}$$

where

$$G(n,L) = (-1)^n\binom{n}{L}z^n p^{2L},$$

for $0 \le L \le n$ and $m \ge 0$. We proceed by induction on n. The result is clearly true for $n = 0$. We assume (2.9) holds for $n = N$. Now

$$\sum_{\ell=0}^{N+1} C(N+1,i_0,\ell,p)f(pz,\ell,L,m)$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{N+1} \left(((p^2-1)z - mp + \ell p) C(N,i_0,\ell,p) + pC(N,i_0,\ell-1,p) \right) f(pz,\ell,L,m) \quad \text{(by (2.1))} \\
&= \sum_{\ell=0}^{N} ((p^2-1)z - mp + \ell p) C(N,i_0,\ell,p) f(pz,\ell,L,m) \\
&\quad + \sum_{\ell=0}^{N} pC(N,i_0,\ell,p) f(pz,\ell+1,L,m) \\
&= \sum_{\ell=0}^{N} ((p^2-1)z - mp + \ell p) C(N,i_0,\ell,p) f(pz,\ell,L,m) \\
&\quad - \sum_{\ell=0}^{N} pC(N,i_0,\ell,p) \Big((pz+\ell-m) f(pz,\ell,L,m) + pz f(pz,\ell,L-1,m) \Big) \\
&= -z \sum_{\ell=0}^{N} C(N,i_0,\ell,p) f(pz,\ell,L,m) - p^2 z \sum_{\ell=0}^{N} C(N,i_0,\ell,p) f(pz,\ell,L-1,m) \\
&\qquad \text{(by (2.5))} \\
&= -zG(N,L) - p^2 z G(N,L-1) \\
&= (-1)^{N+1} \left(\binom{N}{L} z^{N+1} p^{2L} + \binom{N}{L-1} z^{N+1} p^{2L} \right) \\
&= (-1)^{N+1} \binom{N+1}{L} z^{N+1} p^{2L} = G(N+1,L),
\end{aligned}
$$

and (2.9) holds for $n = N+1$, thus completing our induction proof. □

We will also need some results of Zagier [10] on the Fishburn numbers. We need the following formal power series identity [10, Eqn.(4),p.946]

$$
e^{t/24} \sum_{n=0}^{\infty} (1-e^t) \cdots (1-e^{nt}) = \sum_{n=0}^{\infty} \frac{T_n}{n!} \left(\frac{-t}{24} \right)^n, \tag{2.10}
$$

where T_n are the Glaisher T-numbers [1] and which are given explicitly by

$$
T_n = 6 \frac{(-144)^n}{n+1} \left[B_{2n+2}\left(\frac{1}{12}\right) - B_{2n+2}\left(\frac{5}{12}\right) \right], \tag{2.11}
$$

where $B_n(x)$ denotes the n-th Bernoulli polynomial. We remark that letting $t = \log(1-q)$ in (2.10) and using (3.10) below we find that

$$
\xi(n) = \sum_{m=0}^{n} \sum_{k=0}^{m} (-1)^{n+k} \binom{-1/24}{n-m} \frac{s_1(m,k)}{m! 24^k} T_k, \tag{2.12}
$$

which is useful for calculation. Zagier [10] also determined the behaviour of $F(q)$ when q is near a root of unity. In particular, if $\zeta = \zeta_p$ is a p-th root of unity and $N = 12p$ then

$$e^{t/24}F(\zeta e^t) = \sum_{n=0}^{\infty} \frac{c_n(\zeta)}{n!}\left(\frac{-t}{24}\right)^n, \tag{2.13}$$

where

$$c_n(\zeta) = \frac{(-1)^n N^{2n+1}}{2n+2} \sum_{m=1}^{N/2} \chi(m)\zeta^{\frac{1}{24}(m^2-1)} B_{2n+2}\left(\frac{m}{N}\right), \tag{2.14}$$

and where χ is the character mod 12 given by

$$\chi(n) = \left(\frac{12}{n}\right) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.15}$$

This character occurs in the statement of Theorem 1.12.

3. Proof of Theorem 1.12

In this section we assume $p > 3$ is prime, $N = 12p$ and ζ is any p-th root of unity. Following [10] we define the sequence $(b_n(\zeta))$ formally by

$$F(\zeta e^t) = \sum_{n=0}^{\infty} \frac{b_n(\zeta)}{n!} t^n. \tag{3.1}$$

From (2.10), (2.13), (2.14) we have for $n \geq 0$

$$b_n(\zeta) = \frac{(-1)^n}{24^n} \sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{p-1} \gamma(j,i)\zeta^i, \tag{3.2}$$

where

$$\gamma(j,i) = \frac{(-1)^j N^{2j+1}}{2j+2} \sum_{\substack{m=1 \\ (m^2-1)/24 \equiv i \pmod{p}}}^{N/2} \chi(m) B_{2j+2}\left(\frac{m}{N}\right). \tag{3.3}$$

As in [5] we have for $n \geq 0$

$$b_n(\zeta) = \left(\frac{d}{dt}\right)^n F(\zeta e^t)\bigg|_{t=0} = \left(q\frac{d}{dq}\right)^n F(q)\bigg|_{q=\zeta} = \left(q\frac{d}{dq}\right)^n F(q,m)\bigg|_{q=\zeta}$$

for $m \geq (n+1)p - 1$. Proceeding as in the proof of [5, lemma 2.5] we have from (2.14), (3.2) and Lemma 2.1 that

$$\begin{aligned} b_n(\zeta) &= \left(q\frac{d}{dq}\right)^n F(q,(n+1)p-1)\bigg|_{q=\zeta} \\ &= \sum_{j=0}^{n}\sum_{i=0}^{p-1} C(n,i,j,p)\zeta^i A_p^{(j)}(p(n+1)-1,i,1) \end{aligned}$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{p-1} C(n,i,j,p)\zeta^i A_p^{(j)}(p(j+1)-1,i,1)$$
$$= \frac{(-1)^n}{24^n} \sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{p-1} \gamma(j,i)\zeta^i.$$

Since this identity holds for all p-th roots of unity ζ (including $\zeta = 1$) and all the coefficients involved are all rational numbers we may equate coefficients of ζ^i on both sides to obtain

$$\sum_{j=0}^{n} C(n,i,j,p) A_p^{(j)}(p(j+1)-1,i,1) = \frac{(-1)^n}{24^n} \sum_{j=0}^{n} \binom{n}{j} \gamma(j,i), \tag{3.4}$$

for $0 \le i \le p-1$. Now we let i_0 be the least nonnegative integer satisfying $24 i_0 \equiv -1 \pmod{p}$. We find that

$$i_0 = \frac{p^2-1}{24} - \left\lfloor \frac{p}{24} \right\rfloor p. \tag{3.5}$$

We now calculate $\gamma(j,i_0)$. We see that

$$\frac{m^2-1}{24} \equiv i_0 \pmod{p} \text{ if and only if } m \equiv 0 \pmod{p}.$$

In the sum (3.3) (with ($N = 12p$) we only consider the terms with $m = p$, $5p$ to find that

$$\gamma(j,i_0) = \chi(p) \frac{(-1)^j 12^{2j+1} p^{2j+1}}{2j+2} \left(B_{2j+2}\left(\frac{1}{12}\right) - B_{2j+2}\left(\frac{5}{12}\right)\right). \tag{3.6}$$

We apply Theorem 2.3 with $z = \frac{1}{24}$ and $m = \left\lfloor \frac{p}{24} \right\rfloor$ to equation (3.4) with $i = i_0$ to obtain

$$\begin{aligned} A_p^{(n)}(p(n+1)-1,i_0,1) &= (-1)^n \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} s_1(n,j,\lfloor p/24 \rfloor) \frac{p^{j+1}}{24^j} \\ &\qquad \chi(p) \frac{(-1)^k 12^{2k+1}}{2k+2} \left(B_{2k+2}\left(\frac{1}{12}\right) - B_{2k+2}\left(\frac{5}{12}\right)\right) \\ &= (-1)^n \chi(p) \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} s_1(n,j,\lfloor p/24 \rfloor) \frac{p^{j+1}}{24^j} T_k, \end{aligned}$$

by (2.11). From (1.20) we see that

$$\alpha(p,n+1,i_0,n) = \frac{(-1)^n}{n!} A_p^{(n)}(p(n+1)-1,i_0,1).$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha(p,n+1,i_0,n) x^n &= p\chi(p) \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k=0}^{n} \sum_{j=k}^{n} \binom{j}{k} s_1(n,j,\lfloor p/24 \rfloor) \frac{p^j}{24^j} T_k, \\ &= p\chi(p) \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left(\sum_{n=j}^{\infty} s_1(n,j,\lfloor p/24 \rfloor) \frac{x^n}{n!} \right) \frac{p^j}{24^j} \binom{j}{k} T_k \qquad (3.7) \end{aligned}$$

It is well-known that the signless Stirling numbers of the first kind have generating function

$$G(x,u)=\exp\left(-u\log 1-x\right)=(1-x)^{-u}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}s_1(n,j)u^j\,\frac{x^n}{n!}. \tag{3.8}$$

Our generalized signless Stirling numbers have generating function

$$G_k(x,u)=(1-x)^k\exp\left(-u\log 1-x\right)=(1-x)^{k-u}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}s_1(n,j,k)u^j\,\frac{x^n}{n!}. \tag{3.9}$$

Thus

$$\sum_{n=j}^{\infty}s_1(n,j,k)\frac{x^n}{n!}=(1-x)^k\frac{(-\log(1-x))^j}{j!}. \tag{3.10}$$

Hence from (3.10) and (3.7) we have

$$\sum_{n=0}^{\infty}\alpha(p,n+1,i_0,n)q^n=p\chi(p)(1-q)^{\lfloor p/24\rfloor}\sum_{j=0}^{\infty}\left(\sum_{k=0}^{j}\binom{j}{k}T_k\right)\frac{(-p\log(1-q))^j}{24^j j!} \tag{3.11}$$

From Zagier's result (2.10) we have

$$F(\exp(t))=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}T_k\right)\frac{(-t)^n}{24^n n!}. \tag{3.12}$$

Thus using this result in (3.11) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\alpha(p,n+1,i_0,n)q^n&=p\chi(p)(1-q)^{\lfloor p/24\rfloor}F(\exp(p\log(1-q))\\&=p\chi(p)(1-q)^{\lfloor p/24\rfloor}F((1-q)^p).\end{aligned} \tag{3.13}$$

Theorem 1.12 follows from (1.21) and (3.13) since

$$\alpha(p,n,i_0,k)=\alpha(p,k+1,i_0,k),$$

for $0\le k\le n-1$ by Lemma 2.1.

4. Proof of Theorem 1.3 and Corollary 1.7

We assume $p\ge 5$ is prime and $24i_0\equiv -1\pmod{p}$ where $1\le i_0\le p-1$. For two formal power series

$$A=\sum_{n=0}^{\infty}a_nq^n,\qquad B=\sum_{n=0}^{\infty}b_nq^n\in\mathbb{Z}[[q]],$$

we write

$$A\equiv B\pmod{p}\quad\text{if}\quad a_n\equiv b_n\pmod{p},$$

for all $n\ge 0$. If p is prime and m is a nonnegative integer

$$m=jp+r,\quad\text{where } 0\le r<p \text{ and } j,r\in\mathbb{N},$$

then

$$(1-q)^m=(1-q)^r(1-q)^{pj}\equiv(1-q)^r(1-q^p)^j\pmod{p}. \tag{4.1}$$

Now suppose r is a nonzero integer relatively prime to p and $0 \le s \le p-1$. We consider two cases.

Case I. $r > 0$. We proceed as in [5, Section 3]. We note that

$$\sum_{n=0}^{\infty} \xi(n) q^n = F(1-q, N) + O(q^{N+1}). \tag{4.2}$$

Now from (1.19) we have

$$\begin{aligned} F(q, pn-1) &= \sum_{i=0}^{p-1} q^i A_p(pn-1, i, q^p) \\ &= \sum_{i \in S(p)\setminus\{i_0\}} q^i A_p(pn-1, i, q^p) + q^{i_0} A_p(pn-1, i_0, q^p) \\ &\qquad + \sum_{i \notin S(p)} q^i A_p(pn-1, i, q^p). \end{aligned}$$

Hence

$$\begin{aligned} F((1-q)^r, pn-1) = &\sum_{i \in S(p)\setminus\{i_0\}} (1-q)^{ri} A_p(pn-1, i, (1-q)^{rp}) \\ &+ (1-q)^{ri_0} A_p(pn-1, i_0, (1-q)^{rp}) \\ &+ \sum_{i \notin S(p)} (1-q)^{ri} A_p(pn-1, i, (1-q)^{rp}). \end{aligned}$$

For $i \notin S(p)$ we have

$$A_p(pn-1, i, q) = \sum_{k \ge 0} \alpha(p, n, i, k)(1-q)^k,$$

and

$$\begin{aligned} A_p(pn-1, i, (1-q)^{rp}) &= \sum_{k \ge 0} \alpha(p, n, i, k)(1-(1-q)^{rp})^k \\ &\equiv \sum_{k \ge 0} \alpha(p, n, i, k)(1-(1-q^p)^r)^k \pmod{p} \\ &\equiv O(q^{pn}) \pmod{p}, \end{aligned}$$

by Lemma 1.11. In a similar fashion we have

$$A_p(pn-1, i_0, (1-q)^{rp}) \equiv O(q^{pn}) \pmod{p},$$

using Theorem 1.12. Thus

$$(1-q)^s F((1-q)^r, pn-1) \equiv \sum_{i \in S(p)\setminus\{i_0\}} (1-q)^{ri+s} A_p(pn-1, i, (1-q^p)^r) + O(q^{pn}) \pmod{p}.$$

By (4.1) we see that the only terms $q^{j'}$ that occur in $(1-q)^{ri+s}$ (where $i \in S(p) \setminus \{i_0\}$) satisfy

$$j' \equiv j \pmod{p} \quad \text{where } 0 \le j \le m \text{ and } m \in S^*(p, r, s).$$

This is because $i \in S(p) \setminus \{i_0\}$ if and only if $ri + s$ is congruent to an element of $S^*(p, r, s)$. Since $A_p(pn-1, i, (1-q^p)^r)$ is a polynomial in q^p the result follows by letting $n \to \infty$; i.e. every term $q^{j'}$ in

$$(1-q)^s F((1-q)^r) = \sum_{n=0}^{\infty} \sum_{j=0}^{s} \binom{s}{j} (-1)^j \xi_r(n-j) q^n$$

where j' is congruent to an element of $T^*(p, r, s)$ must have a coefficient that is congruent to $0 \bmod p$.

Case II. $r < 0$. This time we choose integers β and m such that

$$r = mp + \beta,$$

where $0 < \beta \le p - 1$ and $m < 0$. We find that

$$(1-q)^r = (1-q)^\beta \Phi(q^p) + p\Psi(q) \equiv (1-q)^\beta \Phi(q^p) \pmod p,$$

where $\Phi(q)$, $\Psi(q) \in \mathbb{Z}[[q]]$ and $\Phi(q)$ has constant term 1 so that it is a unit in the ring of formal power series. In fact,

$$\Phi(q) = (1-q)^m = 1 + \sum_{k=1}^{\infty} \binom{k-m-1}{k} q^k = 1 - mq + \frac{m(m-1)}{2} q^2 + \dots .$$

We basically proceed as in Case I. We have

$$\begin{aligned}
&F((1-q)^r, pn-1) = F((1-q)^\beta \Phi(q^p) + p\Psi(q), pn-1) \\
&= \sum_{i=0}^{p-1} ((1-q)^\beta \Phi(q^p) + p\Psi(q))^i A_p(pn-1, i, ((1-q)^\beta \Phi(q^p) + p\Psi(q))^p) \\
&\equiv \sum_{i=0}^{p-1} (1-q)^{\beta i} \left[\Phi(q^p)\right]^i A_p(pn-1, i, ((1-q^p)^\beta \left[\Phi(q^p)\right]^p) \pmod p \\
&\equiv \sum_{i \in S(p)\setminus\{i_0\}} (1-q)^{\beta i} \left[\Phi(q^p)\right]^p A_p(pn-1, i, ((1-q^p)^\beta \left[\Phi(q^p)\right]^p) + O(q^{pn}) \pmod p,
\end{aligned}$$

and

$$\begin{aligned}
&(1-q)^s F((1-q)^r, pn-1) \\
&\equiv \sum_{i \in S(p)\setminus\{i_0\}} (1-q)^{\beta i + s} \left[\Phi(q^p)\right]^p A_p(pn-1, i, ((1-q^p)^\beta \left[\Phi(q^p)\right]^p) + O(q^{pn}) \pmod p,
\end{aligned}$$

arguing as before. This time instead of the term

$$A_p(pn-1, i, (1-q^p)^r),$$

which is a polynomial in q^p with integer coefficients we have the term

$$\left[\Phi(q^p)\right]^p A_p(pn-1, i, ((1-q^p)^\beta \left[\Phi(q^p)\right]^p),$$

which is a formal power series in q^p with integer coefficients. The result follows as before by letting $n \to \infty$. This completes the proof of our main theorem.

We now prove Corollary 1.7. As before suppose $p \geq 5$ is prime and let r be a fixed nonzero integer relatively prime to p. Suppose $\overline{r}$ is the multiplicative inverse of r mod p. We see that

$$r\tfrac{1}{2}n(3n-1) \equiv -1-s \pmod{p} \quad \text{if and only if} \quad (6n-1)^2 \equiv -24(1+s)\overline{r}+1 \pmod{p}.$$

Thus $p-1 \not\in S^*(p,r,s)$ if $-24(1+s)\overline{r}+1$ is either a quadratic nonresidue mod p or congruent to zero mod p. There are $\frac{1}{2}(p+1)$ such values of s for $0 \leq s \leq p-1$. Thus

$$\sum_{j=0}^{s} \binom{s}{j} (-1)^j \xi_r(pn+p-1-j) \equiv 0 \pmod{p}, \tag{4.3}$$

for all $n \geq 0$ provided $\left(\frac{-24(1+s)\overline{r}+1}{p}\right) = -1$ or 0. It is clear that this set of $\frac{1}{2}(p+1)$ congruence relations mod p is linearly independent. This completes the proof of Corollary 1.7.

We illustrate (4.3) with some examples.

Example 4.1. $s=0$ We have

$$\xi(pn+p-1) \equiv 0 \pmod{p}$$

for all $n \geq 0$ provided $p \geq 5$ is prime and $\left(\frac{-23}{p}\right) = 0$ or -1; i.e. $p \equiv 0, 5, 7, 10, 11, 14, 15, 17,$ $19, 20, 21,$ or $22 \pmod{23}$.

Example 4.2. $s=1$ We have

$$\xi(pn+p-1) \equiv \xi(pn+p-2) \pmod{p}$$

for all $n \geq 0$ provided $p \geq 5$ is prime and $\left(\frac{-47}{p}\right) = 0$ or -1; i.e. $p \equiv 0, 5, 10, 11, 13, 15, 19, 20,$ $22, 23, 26, 29, 30, 31, 33, 35, 38, 39, 40, 41, 43, 44, 45,$ or $46 \pmod{47}$.

Example 4.3. $s=2$ We have

$$\xi(pn+p-1) - 2\,\xi(pn+p-2) + \xi(pn+p-3) \equiv 0 \pmod{p}$$

for all $n \geq 0$ provided $p \geq 5$ is prime and $\left(\frac{-71}{p}\right) = 0$ or -1; i.e. $p = 71$ or $\left(\frac{71}{p}\right) = -1$. We have

$$\xi_{-1}(pn+p-1) - 2\,\xi_{-1}(pn+p-2) + \xi_{-1}(pn+p-3) \equiv 0 \pmod{p}$$

for all $n \geq 0$ provided $p \geq 5$ is prime and $\left(\frac{73}{p}\right) = 0$ or -1; i.e. $p = 73$ or $\left(\frac{73}{p}\right) = -1$.

5. Conclusion

We pose the following problems.

(i) The numbers $\xi(n)$ and $\xi_{-1}(n)$ have many combinatorial interpretations [6], [8], [9]. Use one of the interpretations to find a rank or crank-type function [3] to explain combinatorially the simplest congruences

$$\xi(5n+3) \equiv \xi(5n+4) \equiv 0 \pmod{5},$$
$$\xi_{-1}(5n+4) \equiv 0 \pmod{5}.$$

(ii) Zagier [10] showed that $q^{1/24}F(q)$ is a so-called quantum modular form [11]. Find and prove congruences for the coefficients of other quantum modular forms. In particular look at the functions considered by Andrews, Jiménez-Urroz and Ono [4].

Acknowledgements
I would like to thank George Andrews, Sharon Garthwaite, Ira Gessel, Ken Ono and Rob Rhoades for their comments and suggestions. In particular I thank Ira Gessel for reminding me of the exponential generating function for the signless Stirling numbers of the first kind, and I thank Sharon Garthwaite and Rob Rhoades for sharing their observations on congruences for pairs and triples for the sequences $\xi(n)$ and $\xi_{-1}(n)$.

REFERENCES

1. *Sequence* `A002439` *(Glaisher's T numbers)*, in "The On-Line Encyclopedia of Integer Sequences", published electronically at `http://oeis.org/A002439`, 2010.
2. *Sequence* `A022493` *(The Fishburn numbers)*, in "The On-Line Encyclopedia of Integer Sequences", published electronically at `http://oeis.org/A022493`, 2010.
3. G.E. Andrews and F. G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. (N.S.) **18** (1988), 167–171.
4. G. E. Andrews, J. Jiménez-Urroz, and K. Ono, *q-series identities and values of certain L-functions*, Duke Math. J. **108** (2001), 395–419.
5. G. E. Andrews and J. A. Sellers, *Congruences for the Fishburn numbers*, preprint (arXiv:1401.5345).
6. K. Bringmann, Y. Li and R. Rhoades, *Asymptotics for the number of row Fishburn matrices*, J. Combin. Theory Ser. A, to appear.
7. S. A. Garthwaite and R. C. Rhoades, Private communication, March 4, 2014.
8. S. M. Khamis, *Exact counting of unlabeled rigid interval posets regarding or disregarding height*, Order **29** (2012), 443–461.
9. P. Levande, *Fishburn diagrams, Fishburn numbers and their refined generating functions*, J. Combin. Theory Ser. A **120** (2013), 194–217.
10. D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function*, Topology **40** (2001), 945–960.
11. D. Zagier, *Quantum modular forms*, in "Quanta of Maths," Clay Math. Proc. **11**, Amer. Math. Soc., Providence, RI, (2010) 659–675.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FL 32611-8105
E-mail address: `fgarvan@ufl.edu`