

TILINGS OF RECTANGULAR REGIONS BY RECTANGULAR TILES: COUNTS DERIVED FROM TRANSFER MATRICES.

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ABSTRACT. Step by step completion of a left-to-right tiling of a rectangular floor with tiles of a single shape starts from one edge of the floor, considers the possible ways of inserting a tile at the leftmost uncovered square, passes through a sequence of rugged shapes of the front line between covered and uncovered regions of the floor, and finishes with a straight front line at the opposite edge. We count the tilings by mapping the front shapes to nodes in a digraph, then counting closed walks on that digraph with the transfer matrix method.

Generating functions are detailed for tiles of shape 1×3 , 1×4 and 2×3 and modestly wide floors. Equivalent results are shown for the 3-dimensional analog of filling bricks of shape $1 \times 1 \times 2$, $1 \times 1 \times 3$, $1 \times 1 \times 4$, $1 \times 2 \times 2$ or $1 \times 2 \times 3$ into rectangular containers of small cross sections.

1. DEFINITIONS (DIMENSIONS, STATE VECTORS)

Given a floor of width m and length n and a prototile of width t_m and length t_n , $t_m \leq t_n$, we consider the number of ways of covering the floor by

$$(1) \quad N = \frac{mn}{t_n t_m}$$

non-overlapping tiles. Tiles may be placed on the floor mixing both orientations. Only the case of coprime t_n and t_m is of interest, because the geometry could otherwise be shrunk by the common factor without changing the number of tilings.

The symmetry of the $m \times n$ rectangle will not be taken into account; tilings which are equivalent to other tilings through reflections or rotations of the stack are counted including their multiplicity due to their (missing) symmetry.

The following counting technique reflects the act of paving the floor starting from the (left) short edge. The rule of placing the next tile is to cover the leftmost not yet covered unit square of the floor, and if there are more than one of them to cover, the one closest to the front edge of the floor. Each intermediate tiling shall be characterized by a *state vector* $(h_1 h_2 \dots h_m)$ of heights $0 \leq h_i \leq t_n$ that encode how far the tiles at vertical position $1 \leq i \leq m$ penetrate into the uncovered territory beyond the rightmost occupied square [9, 10]. Figure 1 illustrates these integers h_i of a state vector (0231102) with a floor width of $m = 7$.

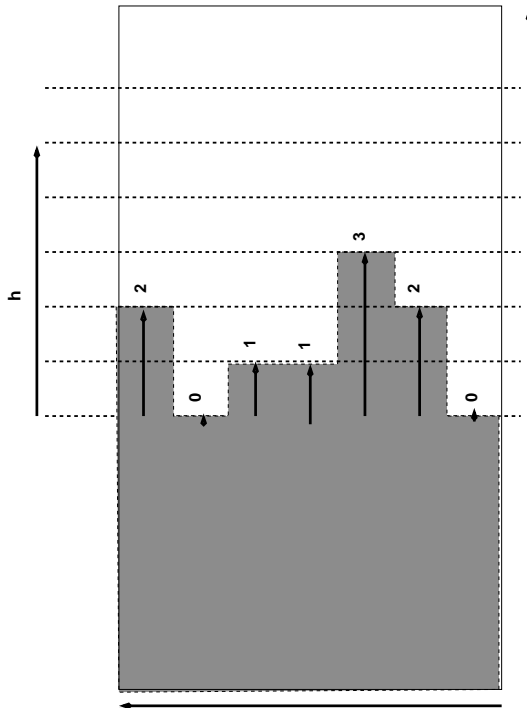
A directed graph is defined where the different state vectors are the nodes; edges point from each state vector to the state vectors that can be reached by placing the next tile, so walking along a single edge represents adding a tile. The possibility

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FIGURE 1. Illustration of the vector of heights, the profile of the covered region into the uncovered part of the floor.



of reaching a state where no further tile can be placed in conjunction with the two orientations of the tile lead to outdegrees in the range 0 up to 2.

The number of state vectors (of nodes in the graph) is finite. An upper limit is $(1 + t_n)^m$, because the rule of the next placement puts a limit of t_n , the tile's long edge, to each profile. In addition, the maximum height $h_i = t_n$ may only appear for the first consecutive vector elements starting at $i = 1$, because otherwise at least one tile would have been placed prematurely at an advanced position reserved for later placements. Further reduction of the upper limit stems from the requirement that at least one h_i must equal zero.

2. COUNTING CLOSED WALKS ON THE DIGRAPH

2.1. Generating Function. We wish to count the number $T(N)$ of different closed walks of N steps starting from and returning to the state vector $(00 \dots 00)$, which represents the straight empty front edge at the start of the paving and represents also the straight front edge when the opposite edge is reached. The aforementioned rule to cover the "lowest" unoccupied unit square (employing some sequential enumeration of the nm unit squares of the floor) ensures that there is a 1-to-1 map of the different tilings to the walks on the graph.

The number of walks are registered by construction of their ordinary generating function (GF)

$$(2) \quad T_{t_m \times t_n}(m, z) \equiv \sum_{N \geq 0} T_{t_m \times t_n}(m, N) z^N.$$

Remark 1. *This definition uses the tile count N to manage the results, whereas my earlier notation uses the floor length n to tabulate them [6]. For the cases studied here, the shape $t_m \times t_n$ of the tile and the width m of the floor are kept fixed, so N and n are easily translated using (1).*

Some relation between the GF and the graph topology can easily be established akin to the Kirchoff rules of impedances of circuit networks [8]:

- (Disconnected) Subgraphs not connected to $(00 \dots 0)$ are discarded. Their contribution to the generating function is multiplication by the factor 1.
- (Edge weights) A step along a single edge multiplies the generating function by z .
- (Serial Paths) A non-splitting chain of walks multiplies the generating functions along the sub-paths.
- (Parallel Paths) At a node with outdegree larger than one, where diverting alternative walks lead across disconnected subgraphs and rejoin later, so exactly one of the paths is to be taken to reach the join, the sum of the generating functions of the dispatched subgraphs is build.

These basics may be combined to formulate rules for simply weaved networks:

- A circuit from a node back to itself, which allows to walk that circuit any number of times or not at all, contributes $1/(1 - l(z))$ to the generating function, where $l(z)$ is the GF of the circuit. The formula sums a geometric series of the two rules for chains and parallel paths.
- A node with several disconnected walks returning to the node, where the generating functions are $T^{(1)}(z)$, $T^{(2)}(z)$ etc. and where walks may be executed in any order, represents

$$(3) \quad \frac{1}{1 - \sum_i T^{(i)}(z)} = 1 + \sum_i T^{(i)}(z) + \sum_{ij} T^{(i)}(z) T^{(j)}(z) + \dots$$

The terms on the right hand side accumulate (as products) the number of walks choosing first circuit i , then circuit j etc, each as many times as wished.

Figure 2 shows the fundamental example of placing 1×2 tiles on $2 \times n$ floors. Orienting the tile parallel to n leads from (00) to (20) or (in the second lane) from (20) to (00) , whereas orienting it parallel to the short edge loops from (00) back to (00) . The disconnected subgraph with the circuit $(01) \rightarrow (10) \rightarrow (01)$ is not plotted. Starting at (00) there is one loop with a single step, with GF z . The circuit passing by the intermediate (20) represents a walk with two steps, with GF z^2 . The rule (3) for multiple circuits attached to a node combines these two GF's to $T_{1 \times 2}(2, z) = 1/(1 - z - z^2)$, the Fibonacci series [12, A000045].

A second example of application is the tiling of $3 \times n$ floors with 1×2 tiles, Figure 3. One option is to walk $(000) \rightarrow (200) \rightarrow (220) \rightarrow (000)$ which contributes $T^{(1)} = z^3$ by the chain rule. There is a subwalk $(100) \rightarrow (120) \rightarrow (011) \rightarrow (100)$ in a circuit which represents $1/(1 - z^3)$; since it is reached from (000) via (200) with

FIGURE 2. State diagram while tiling $2 \times n$ floors with dominoes, generating the Fibonacci sequence.

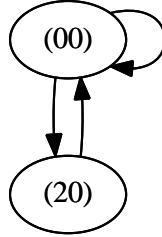
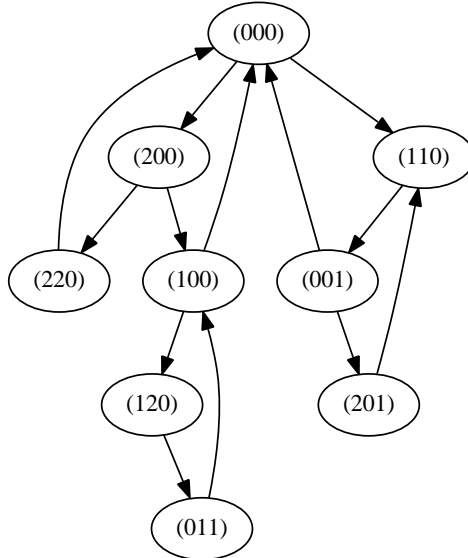


FIGURE 3. State diagram while tiling $3 \times n$ floors with dominoes.



two steps and returns to (000) in one step, it actually represents $T^{(2)} = z^3/(1-z^3)$. Finally there is a walk to (110) \rightarrow (001) with an option to circle any number of times via (201) \rightarrow (110) \rightarrow (001) before moving back to (000), $T^{(3)} = [z^2/(1-z^3)]z$. The effective total of the three circuits originating from (000) is [12, A001835]

$$(4) \quad T_{1 \times 2}(3, z) = \frac{1}{1 - T^{(1)} - T^{(2)} - T^{(3)}} = \frac{(1-z)(1+z+z^2)}{1-4z^3+z^6} = 1+3z^3+11z^6+41z^9+\dots$$

2.2. Irreducible GF. The function $T(m, N)$ counts tilings including any number of blocks of side-by-side smaller tilings $N = N' + N'' + \dots$ of the common width m . In the speak of the digraph, $T(m, z)$ counts walks that pass through (00...0) any number of times. The *irreducible* tilings may be defined as those that cannot be cut into smaller tilings by cuts of length m parallel to the short side of the floor. Their walk returns to (00...0) just once. Their GF shall be denoted by $\hat{T}(m, z)$:

$$(5) \quad T(m, z) = \frac{1}{1 - \hat{T}(m, z)}$$

The reduction of (4) to these irreducible cases yields for example

$$(6) \quad \hat{T}_{1 \times 2}(3, z) = \frac{z^3(3 - z^3)}{1 - z^3} = 3z^3 + 2z^6 + 2z^9 + 2z^{15} + 2z^{18} + \dots$$

2.3. Inverting Transfer Matrices. An iterative procedure to compute the Taylor Series of the generating function according to the rules of the previous section is to “load” the node $(00 \dots 0)$ with $T = 1$ and all other nodes with $T = 0$, and to multiply the vector of loads with increasingly higher powers of the Transfer Matrix of the digraph. The Transfer Matrix is defined to contain the weight z of the edge at the column associated with the start of the step and the row associated with the end of the step along the edge, i.e., essentially the incidence matrix pairing direct predecessors and successors in the graph. The number of z in the columns equals the outdegree of the node; the number of z in the columns equals the indegree of the node.

The 9×9 Transfer Matrix of Figure 3 with 9 nodes and a vector of 9 states sorted as (000) , (001) , (011) , (100) , (110) , (120) , (200) , (201) , and (220) becomes

$$(7) \quad X = \begin{pmatrix} 0 & z & 0 & z & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 & z & 0 & 0 \\ z & 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \end{pmatrix}$$

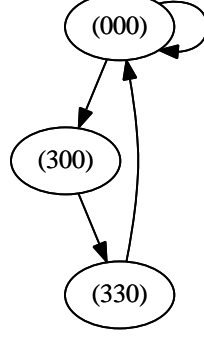
for example. The two z in its first column instantiate the walks $(000) \rightarrow (110)$ and $(000) \rightarrow (200)$.

The full GF follows by (i) summation of the geometric series of the matrix powers, which ultimately is the inversion $T(z) = \sum_{i \geq 0} [X(z)]^i = \text{inv}(I - X(z))$ with I the identity matrix, (ii) multiplying that inverse from the right by the initial load $(1, 0, 0, \dots)$ and (iii) taking the element of that vector associated with $(00 \dots 0)$ [14].

Remark 2. *Our application does not rely on the full inverse of the Transfer Matrix but only its first column; the effort reduces to solving a linear system of equations. As the number of states is finite, solving the equation by Cramer’s rule proves that the GF’s are rational functions of z .*

3. FLOOR TILINGS

3.1. Dominoes. Tiling with dominoes is much better studied in the literature than tiling with other polyominoes because thermodynamic properties of dimer coverings are of interest to the physics of surfaces [3, 2, 13, 15, 11, 4]. We skip this special subject because no new results arise in the present context. For increasing width m , the sequences $T_{1 \times 2}(m, n)$ are entries A000075, A005178, A003775, A028469–A028474 in the Encyclopedia of Integer Sequences [12].

FIGURE 4. State diagram while tiling $3 \times n$ floors with 1×3 3-ominoes.

3.2. 1×3 Straight Trominoes. For tiles of 1×3 shape we find

$$(8) \quad T_{1 \times 3}(2, z) = \frac{1}{1 - z^2},$$

which essentially says that there is one tiling whenever (1) is an integer, because the long side of the tile needs to be aligned with the long side of the floor. We find via Figure 4 [12, A000930]

$$(9) \quad T_{1 \times 3}(3, z) = \frac{1}{1 - z - z^3} = 1 + z + z^2 + 2z^3 + 3z^4 + 4z^5 + 6z^6 + 9z^7 + 13z^8 + 19z^9 + 28z^{10} + \dots$$

For floors of width $m = 4$ the digraph of Figure 5 reduces to sequence [12, A049086] and basically the even numbers:

$$(10) \quad T_{1 \times 3}(4, z) = \frac{(-1 + z)^2 (z + 1)^2 (z^2 + 1)^2}{-5z^4 + 1 + 3z^8 - z^{12}} = 1 + 3z^4 + 13z^8 + 57z^{12} + 249z^{16} + 1087z^{20} + \dots;$$

(11)

$$\hat{T}_{1 \times 3}(4, z) = \frac{z^4 (3 - 2z^4 + z^8)}{1 - 2z^4 + z^8} = 3z^4 + 4z^8 + 6z^{12} + 8z^{16} + 10z^{20} + 12z^{24} + 14z^{28} + 16z^{32} + \dots$$

Figure 6 gives [12, A236576]

(12)

$$T_{1 \times 3}(5, z) = \frac{(1 - z^5)^2}{1 - 6z^5 + 3z^{10} - z^{15}} = 1 + 4z^5 + 22z^{10} + 121z^{15} + 664z^{20} + 3643z^{25} + 19987z^{30} + \dots$$

and basically multiples of three:

(13)

$$\hat{T}_{1 \times 3}(5, z) = \frac{z^5 (4 - 2z^5 + z^{10})}{(1 - z^5)^2} = 4z^5 + 6z^{10} + 9z^{15} + 12z^{20} + 15z^{25} + 18z^{30} + 21z^{35} + 24z^{40} + 27z^{45} + \dots$$

Figure 7 reduces to [12, A236577]

(14)

$$T_{1 \times 3}(6, z) = \frac{(1 - z^6)^2 (-z^4 + 1 - z^6)}{-z^{20} + z^{24} + z^{22} + 10z^{12} - 5z^{18} - 3z^{16} + z^{14} + z^8 - 7z^6 + 5z^{10} - z^4 - z^2 + 1} = 1 + z^2 + z^4 + 6z^6 + 13z^8 + 22z^{10} + 64z^{12} + 155z^{14} + 321z^{16} + 783z^{18} + 1888z^{20} + 4233z^{22} + \dots$$

FIGURE 5. State diagram while tiling $4 \times n$ floors with 1×3 3-ominoes.

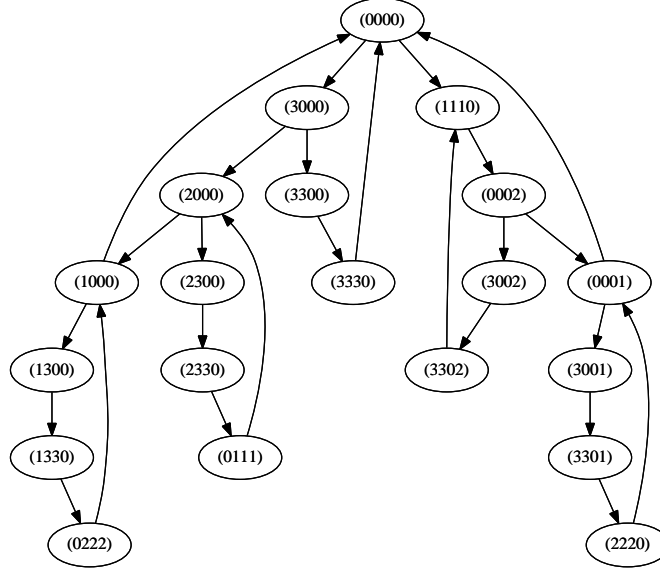
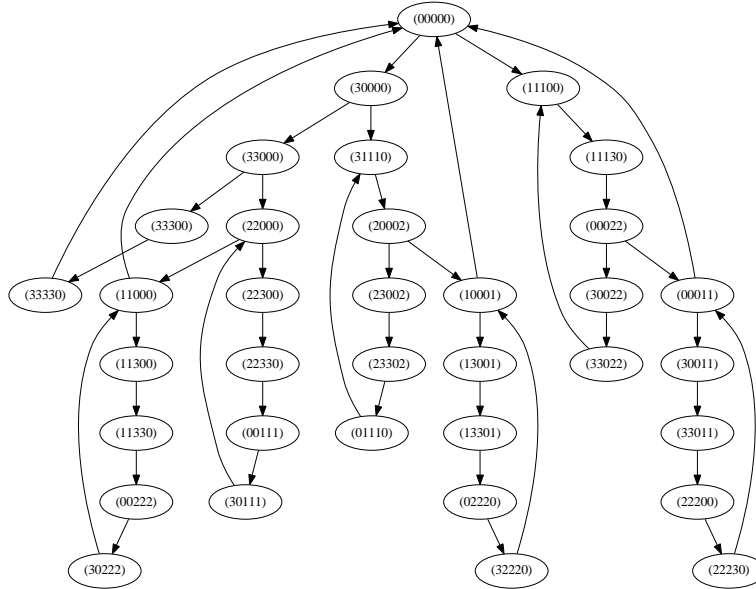


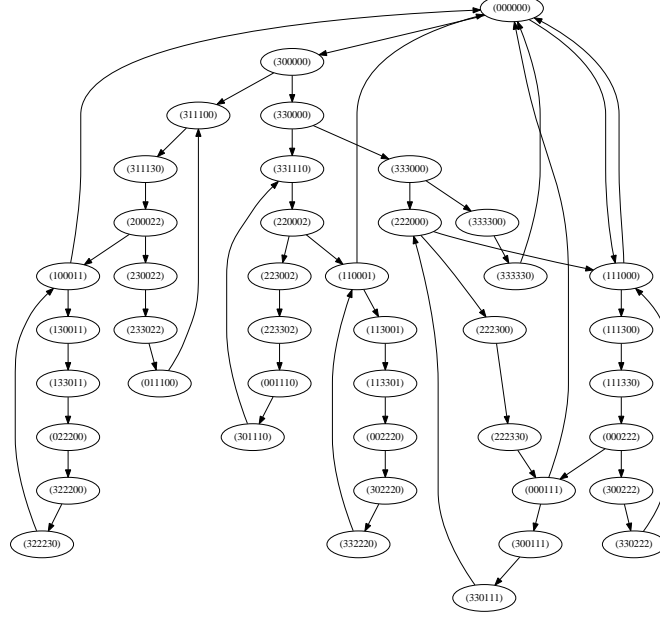
FIGURE 6. State diagram while tiling $5 \times n$ floors with 1×3 3-ominoes.



The Transfer Matrix for filling $7 \times n$ floors with 1×3 3-ominoes has dimension 273×273 [12, A236578],

(15)

$$T_{1 \times 3}(7, z) = \frac{p_{1 \times 3}(7, z)}{q_{1 \times 3}(7, z)} = 1 + 9z^7 + 155z^{14} + 2861z^{21} + 52817z^{28} + 972557z^{35} + \dots$$

FIGURE 7. State diagram while tiling $6 \times n$ floors with 1×3 3-ominoes.

with numerator

(16)

$$p_{1 \times 3}(7, z) \equiv (z^7 - 1)^2 (-z^{105} + 14z^{98} - 104z^{91} + 527z^{84} - 1971z^{77} + 5573z^{70} - 11973z^{63} + 19465z^{56} - 23695z^{49} + 21166z^{42} - 13512z^{35} + 5915z^{28} - 1685z^{21} + 291z^{14} - 27z^7 + 1);$$

and denominator

(17)

$$q_{1 \times 3}(7, z) \equiv -17z^{119} + 293180z^{56} - 236178z^{49} + 142400z^{42} - 62621z^{35} + 19420z^{28} - 4062z^{21} + 533z^{14} - 38z^7 + z^{126} + 1 + 151z^{112} - 946z^{105} + 4558z^{98} - 17135z^{91} + 50164z^{84} - 114198z^{77} + 202080z^{70} - 277277z^{63}.$$

(18)

$$\hat{T}_{1 \times 3}(7, z) = 9z^7 + 74z^{14} + 800z^{21} + 8398z^{28} + 85908z^{35} + 867148z^{42} + 8697028z^{49} + 86962830z^{56} + \dots$$

Filling $8 \times n$ floors with 1×3 3-ominoes is counted by

(19)

$$T_{1 \times 3}(8, z) = \frac{p_{1 \times 3}(8, z)}{q_{1 \times 3}(8, z)} = 1 + 13z^8 + 321z^{16} + 8133z^{24} + 204975z^{32} + 5158223z^{40} \dots,$$

with

(20)

$$\begin{aligned} p_{1 \times 3}(8, z) \equiv & 133678z^{32} + 1 - 190305075z^{88} + 2914z^{192} - 19827z^{184} - 326z^{200} + 223054092z^{96} \\ & + 108161z^{176} - 486843z^{168} + 25z^{208} + 35967130z^{64} - 51z^8 - z^{216} + 3806952z^{48} - 13264117z^{56} \\ & - 15217z^{24} - 5879363z^{152} + 179445867z^{112} - 218171819z^{104} + 1838302z^{160} + 16010861z^{144} \\ & + 134225178z^{80} - 77378505z^{72} + 73831771z^{128} - 124861824z^{120} + 1148z^{16} - 37212426z^{136} - 830622z^{40}, \end{aligned}$$

(21)

$$\begin{aligned} q_{1 \times 3}(8, z) \equiv & 233525z^{32} + 1 + z^{224} - 609762885z^{88} + 24393z^{192} - 146966z^{184} - 3312z^{200} \\ & + 779625485z^{96} + 738848z^{176} - 3126151z^{168} + 351z^{208} + 88153581z^{64} - 64z^8 - 26z^{216} \\ & + 7843386z^{48} - 29769135z^{56} - 24373z^{24} - 33883064z^{152} + 739898086z^{112} - 829644305z^{104} \\ & + 11180105z^{160} + 87159919z^{144} + 393536359z^{80} - 207405453z^{72} + 353345037z^{128} \\ & - 555959103z^{120} + 1659z^{16} - 190440779z^{136} - 1575184z^{40}. \end{aligned}$$

3.3. 1×4 **Straight 4-ominoes.** Similar to (8)–(9), widths m smaller than the long edge of the polyomino lead to simple results [12, A003269]:

$$(22) \quad T_{1 \times 4}(3, z) = \frac{1}{1 - z^3};$$

(23)

$$T_{1 \times 4}(4, z) = \frac{1}{1 - z - z^4} = 1 + z + z^2 + z^3 + 2z^4 + 3z^5 + 4z^6 + 5z^7 + 7z^8 + 10z^9 + 14z^{10} + 19z^{11} + \dots.$$

Figure 8 is condensed to [12, A236579]

(24)

$$T_{1 \times 4}(5, z) = \frac{(1 - z^5)^3}{-6z^5 + 1 + 6z^{10} - 4z^{15} + z^{20}} = 1 + 3z^5 + 15z^{10} + 75z^{15} + 371z^{20} + 1833z^{25} + \dots$$

and [12, A002378]

(25)

$$\hat{T}_{1 \times 4}(5, z) = \frac{z^5(3 - 3z^5 + 3z^{10} - z^{15})}{(1 - z^5)^3} = 3z^5 + 6z^{10} + 12z^{15} + 20z^{20} + 30z^{25} + 42z^{30} + 56z^{35} + \dots$$

Figure 9 represents [12, A236580]

(26)

$$T_{1 \times 4}(6, z) = \frac{(1 - z^6)^3}{-7z^6 + 1 + 6z^{12} - 4z^{18} + z^{24}} = 1 + 4z^6 + 25z^{12} + 154z^{18} + 943z^{24} + 5773z^{30} + \dots$$

Finally [12, A236581]

(27)

$$T_{1 \times 4}(7, z) = \frac{(1 - z^7)^3}{-8z^7 + 1 + 6z^{14} - 4z^{21} + z^{28}} = 1 + 5z^7 + 37z^{14} + 269z^{21} + 1949z^{28} + 14121z^{35} + \dots$$

and [12, A236582]

(28)

$$T_{1 \times 4}(8, z) = \frac{p_{1 \times 4}(8, z)}{q_{1 \times 4}(8, z)} = 1 + z^2 + z^4 + z^6 + 7z^8 + 15z^{10} + 25z^{12} + 37z^{14} + 100z^{16} + 229z^{18} + \dots,$$

FIGURE 8. State diagram while tiling $5 \times n$ floors with 1×4 4-ominoes.

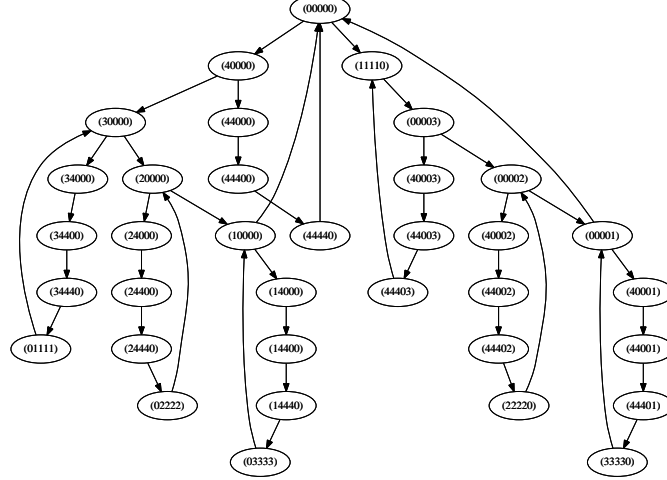
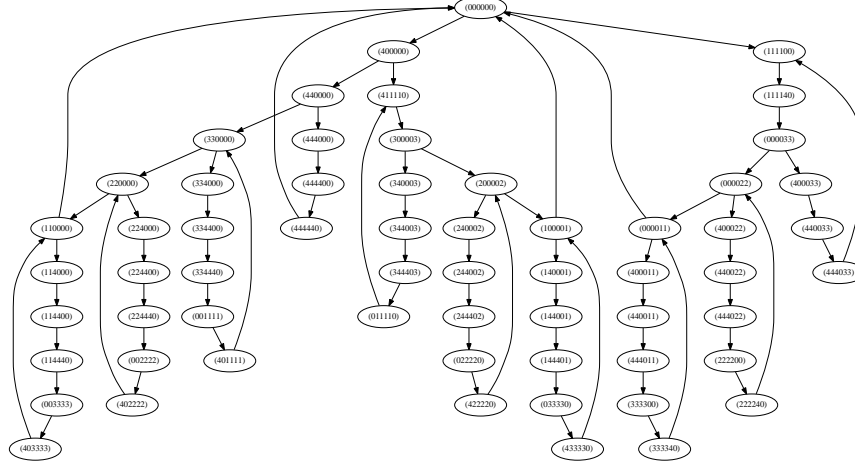


FIGURE 9. State diagram while tiling $6 \times n$ floors with 1×4 4-ominoes.



where

$$(29) \quad p_{1 \times 4}(8, z) = (1 - z^2)^3 (z^2 + 1)^3 (z^4 + 1)^3 (z^{12} - z^8 - z^6 - z^4 + 1);$$

$$(30) \quad q_{1 \times 4}(8, z) = -z^4 - 13z^{20} - 5z^{36} + 8z^{12} - z^2 - z^{40} - 9z^8 + 16z^{16} - 13z^{24} - 2z^{38} + 1 \\ + 10z^{28} + 5z^{14} + 6z^{30} - 6z^{22} + z^{44} + 6z^{32} + z^{34} + 2z^{10} - 2z^{26}.$$

3.4. **2×3 Hexominoes.** The state diagram in Figure 10 is a first example with “dangling” nodes with zero outdegree. The shortest side $t_m = 2$ of the tile is larger than the slit of unit width if the height vector has the pattern $(\dots a0b \dots)$ for some $a, b \geq 1$. After pruning these dead ends (reducing the graph to the strongly connected subgraph), the graph reduces to two isolated circuits of 5 edges attached

FIGURE 10. State diagram while tiling $5 \times n$ floors with 2×3 6-ominoes.

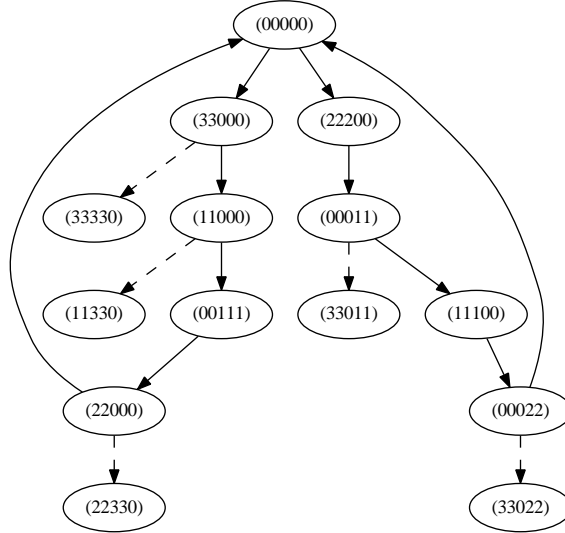
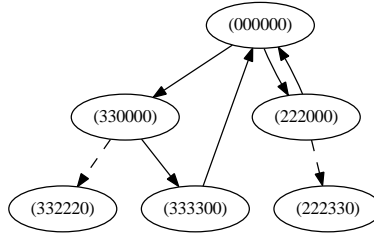


FIGURE 11. State diagram while tiling $6 \times n$ floors with 2×3 6-ominoes.



to the start position:

$$(31) \quad T_{2 \times 3}(5, z) = \frac{1}{1 - 2z^5} = 1 + 2z^5 + 4z^{10} + 8z^{15} + 16z^{20} + 32z^{25} + \dots$$

The state diagram in Figure 11 contains two nodes with zero outdegree, (332220) and (222330). After pruning these, the topology is a circuit of 3 steps and a circuit of 2 steps [12, A182097]:

$$(32) \quad T_{2 \times 3}(6, z) = \frac{1}{1 - z^2 - z^3} = 1 + z^2 + z^3 + z^4 + 2z^5 + 2z^6 + 3z^7 + 4z^8 + 5z^9 + 7z^{10} + 9z^{11} + \dots$$

After pruning the dead ends of Figure 12, we are left with a skeleton of 3 circuits of 7 steps each, so [12, A000244]

$$(33) \quad T_{2 \times 3}(7, z) = \frac{1}{1 - 3z^7} = 1 + 3z^7 + 9z^{14} + \dots$$

FIGURE 12. State diagram while tiling $7 \times n$ floors with 2×3 6-ominoes.

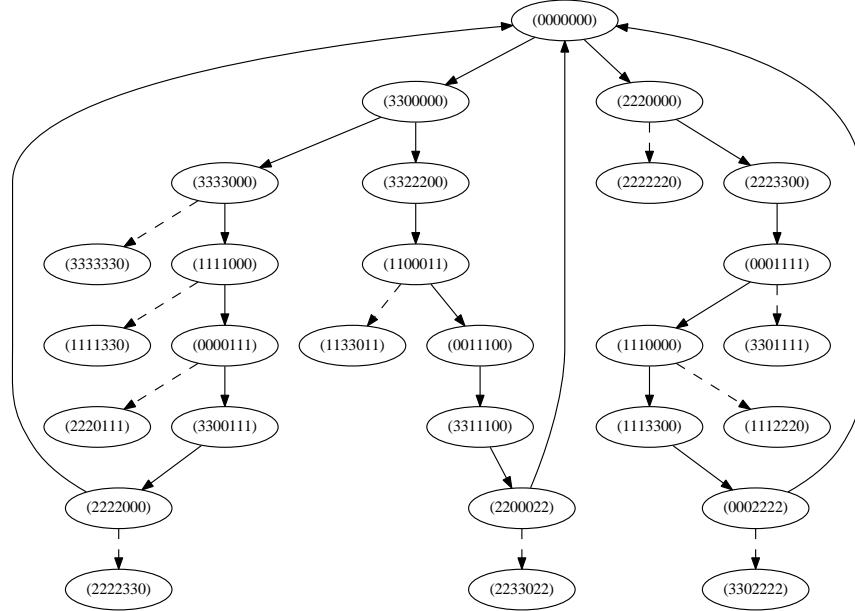


FIGURE 13. State diagram while tiling $8 \times n$ floors with 2×3 6-ominoes.

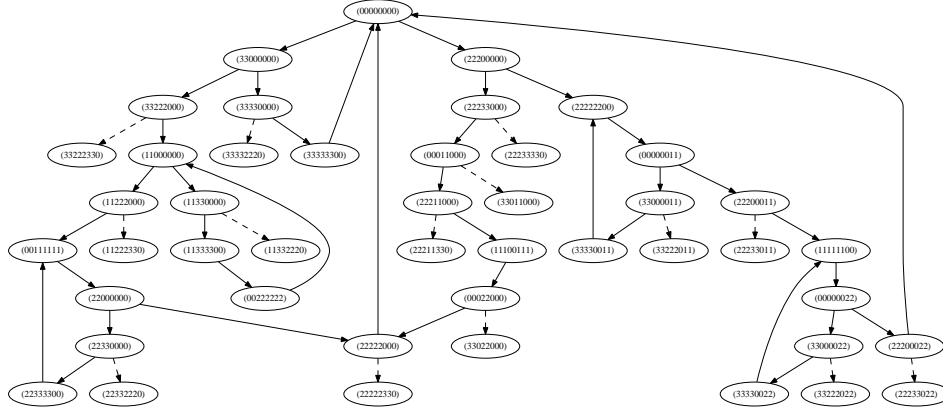


Figure 13 reduces to [12, A236583]

(34)

$$T_{2 \times 3}(8, z) = \frac{(-1 + z^4)^2}{z^{12} - z^{16} + 1 - 3z^4} = 1 + z^4 + 4z^8 + 11z^{12} + 33z^{16} + 96z^{20} + 281z^{24} + 821z^{28} + \dots,$$

and the case of width $m = 9$ 14 to [12, A236584]

(35)

$$T_{2 \times 3}(9, z) = \frac{1 - z^3}{-4z^9 + 1 - 2z^3 + z^6 + 2z^{12}} = 1 + z^3 + z^6 + 5z^9 + 11z^{12} + 19z^{15} + 45z^{18} + 105z^{21} + \dots.$$

FIGURE 14. State diagram while tiling $9 \times n$ floors with 2×3 6-ominoes.

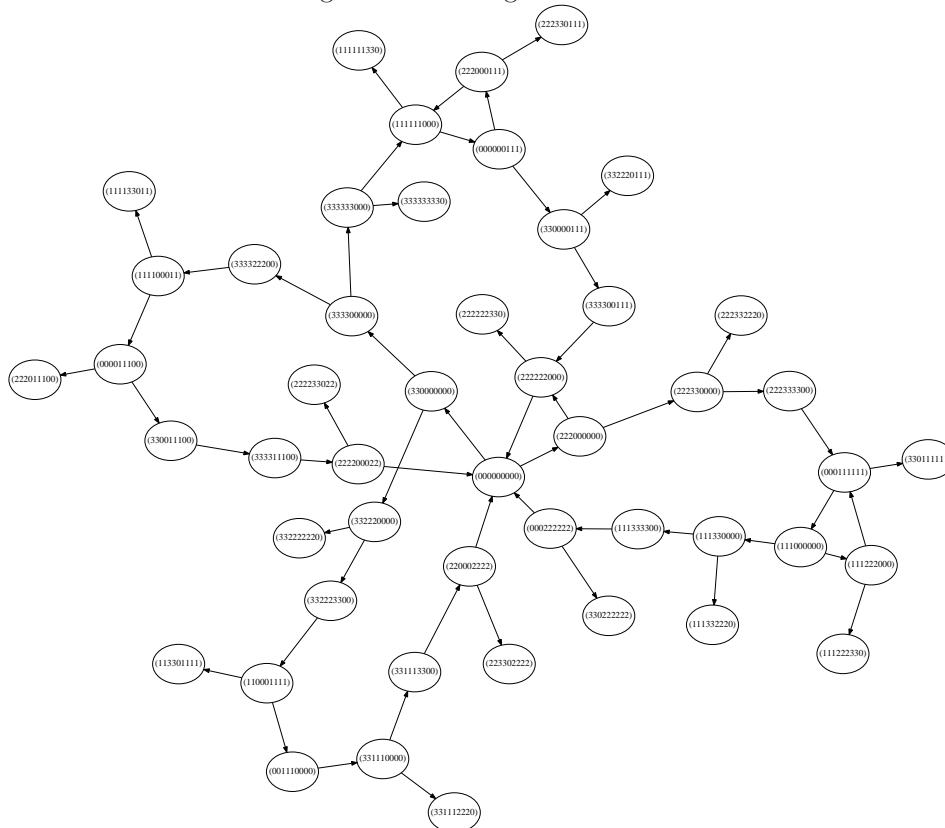
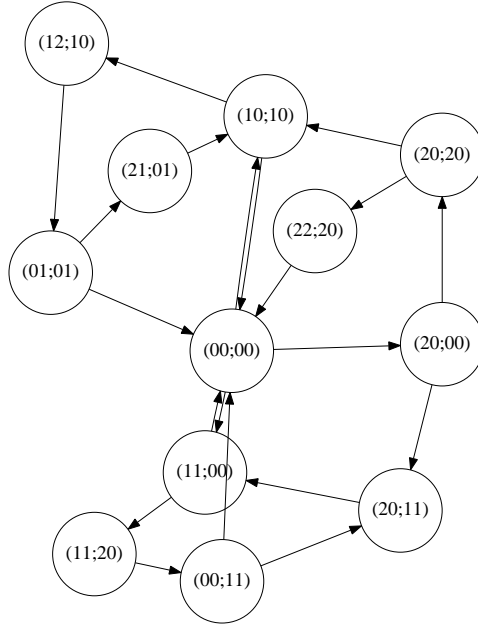


FIGURE 15. State diagram while tiling $2 \times 2 \times n$ rooms with $1 \times 1 \times 2$ dominoes.

4. STACKING BRICKS IN ROOMS—THE 3D ANALOG

4.1. Digraphs. The three-dimensional equivalent to the tilings packs impenetrable bricks of dimension $t_k \times t_m \times t_n$ into containers of shape $k \times m \times n$, leaving no hole [7]. The volume ratio $kmn/(t_k t_m t_n)$ is an integer.

No generic new aspects relative to the handling of the two dimensions arise. Because the side lengths of the brick may be any (coprime) triple of positive integers, the maximum number of orientations of the placement of each further brick, and also the maximum outdegree of each node of the state graph, is six. (Consider three choices fixing the direction of the long edge t_n either forwards, sideways or up, followed by another two choices for an orientation around the long axis. If two edge lengths of the brick are equal, this set reduces to three choices.) The encoding of the two-dimensional front of the brick insertions needs a matrix of integers; we write one vector for each layer, separated by semicolons.

4.2. $1 \times 1 \times 2$ Bricks. To illustrate the notation, consider the simplest nontrivial layout with rooms of 2×2 cross sections filled with bricks of size $1 \times 1 \times 2$. Figure 15 evaluates to [12, A006253][1]

$$\begin{aligned}
 (36) \quad T_{1 \times 1 \times 2}(2, 2, z) &= \frac{1 - z^2}{(1 + z^2)(z^4 - 4z^2 + 1)} \\
 &= 1 + 2z^2 + 9z^4 + 32z^6 + 121z^8 + 450z^{10} + 1681z^{12} + 6272z^{14} + 23409z^{16} + 87362z^{18} + \dots; \\
 \hat{T}_{1 \times 1 \times 2}(2, 2, z) &= \frac{z^2(2 + 3z^2 - z^4)}{1 - z^2} = 2z^2 + 5z^4 + 4z^6 + 4z^8 + 4z^{10} + 4z^{12} + 4z^{14} + 4z^{16} + \dots
 \end{aligned}$$

Filling $2 \times 3 \times n$ rooms with $1 \times 1 \times 2$ bricks generates a state diagram with 60

nodes and yields [12, A028447]

$$(37) \quad T_{1 \times 1 \times 2}(2, 3, z) = -\frac{14z^{12} - 7z^6 + 1 + z^{24} - 7z^{18} + 3z^{21} + 16z^9 - 3z^3 - 16z^{15}}{(-7z^3 + 7z^{21} + 48z^9 + 1 - 13z^6 + 28z^{12} - 48z^{15} - 13z^{18} + z^{24})(z^6 - z^3 - 1)}$$

$$= 1 + 3z^3 + 32z^6 + 229z^9 + 1845z^{12} + 14320z^{15} + 112485z^{18} + 880163z^{21} + 6895792z^{24} + 54003765z^{27} + \dots$$

Filling $3 \times 3 \times n$ rooms with $1 \times 1 \times 2$ bricks delivers [12, A028452]

$$(38) \quad T_{1 \times 1 \times 2}(3, 3, z) = 1 + 229z^9 + 117805z^{18} + 64647289z^{27} + 35669566217z^{36} + \dots$$

Filling $3 \times 4 \times n$ rooms with $1 \times 1 \times 2$ bricks generates a graph with 5544 nodes [12, A028453], and Lundow has also tabulated the case of $4 \times 4 \times n$ rooms [12, A028454] [5].

4.3. $1 \times 1 \times 3$ Bricks. Figure 16 describes placements of $1 \times 1 \times 3$ bricks into $2 \times 3 \times n$ rooms [12, A233247]:

$$(39) \quad T_{1 \times 1 \times 3}(2, 3, z) = \frac{1 - z^6 - z^4}{(z^4 + 1 - z^6)(1 - z^2 - 2z^4 - z^6)} = 1 + z^2 + z^4 + 4z^6 + 9z^8 + 16z^{10} + 36z^{12} + \dots$$

The expansion coefficients are $T_{1 \times 1 \times 3}(2, 3, N) = T_{1 \times 3}^2(3, N)$, the squares of those in (9), because all solutions are classified as stacks of two independent solutions of the 2-dimensional problem [10].

Tiling $3 \times 3 \times n$ rooms with $1 \times 1 \times 3$ bricks yields [12, A233289]

$$(40) \quad T_{1 \times 1 \times 3}(3, 3, z) = \frac{z^{12} - z^{15} - 4z^9 - z^{21} - 2z^6 + z^{18} - z^3 + 1}{(-z^{27} - 3z^{21} + z^{18} - 6z^{15} - 17z^{12} - 15z^9 - 2z^6 - 2z^3 + 1)(1 - z^3)} =$$

$$1 + 2z^3 + 4z^6 + 21z^9 + 92z^{12} + 320z^{15} + 1213z^{18} + 4822z^{21} + 18556z^{24} + \dots$$

Tiling $3 \times 4 \times n$ rooms with $1 \times 1 \times 3$ bricks yields [12, A237355]

$$(41) \quad T_{1 \times 1 \times 3}(3, 4, z) = 1 + 3z^4 + 9z^8 + 92z^{12} + 749z^{16} + 4430z^{20} + 30076z^{24} + 217579z^{28} + \dots$$

The full GF is not noted because the numerator is a polynomial of order 87 in z^4 , and the denominator a polynomial of order 88 in z^4 .

4.4. $1 \times 2 \times 2$ Bricks. Figure 17 shows one loop of 1 step and two circuits of 2 steps [12, A001045]:

$$(42) \quad T_{1 \times 2 \times 2}(2, 2, z) = \frac{1}{1 - z - 2z^2} = 1 + z + 3z^2 + 5z^3 + 11z^4 + 21z^5 + 43z^6 + 85z^7 + 171z^8 + \dots$$

Figure 18 is [12, A083066]

$$(43) \quad T_{1 \times 2 \times 2}(2, 3, z) = \frac{1 - 2z^3}{(1 - z^3)(1 - 6z^3)} = 1 + 5z^3 + 29z^6 + 173z^9 + 1037z^{12} + 6221z^{15} + \dots$$

Figure 19 is an example of a state diagram without any of the required closed walks, $T_{1 \times 2 \times 2}(3, 3, z) = 1$. This becomes plausible if one considers the subproblem of filling the first layer of cross section 3×3 with any combination of the 2×2 or 1×2 cross sections of the brick—which cannot succeed because the area of 9 cannot

FIGURE 16. State diagram while tiling $2 \times 3 \times n$ rooms with $1 \times 1 \times 3$ 3-ominoes.

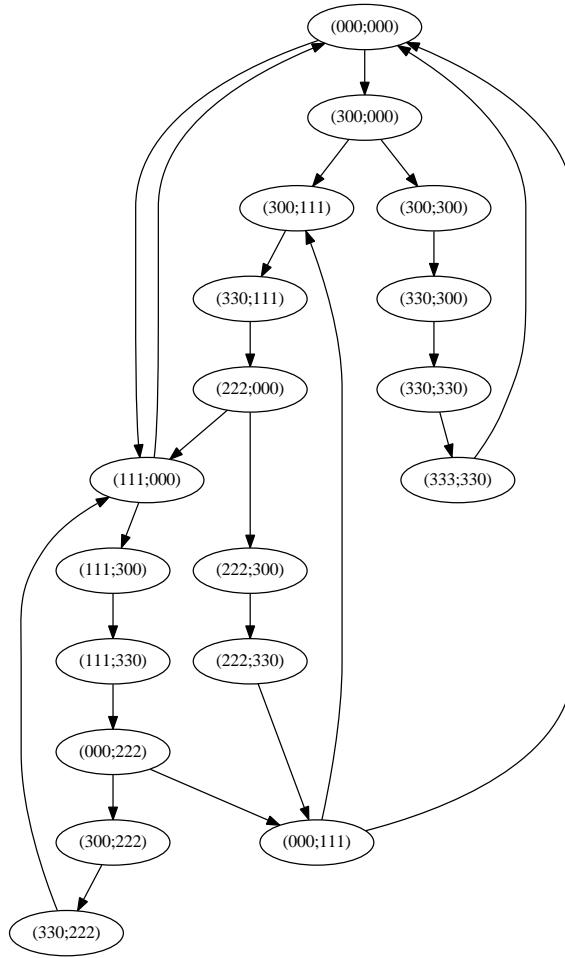
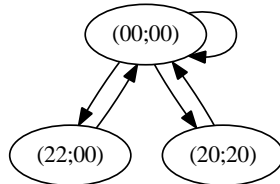


FIGURE 17. State diagram while tiling $2 \times 2 \times n$ rooms with $1 \times 2 \times 2$ 4-ominoes.



be partitioned into parts of 4 and 2. This result is of a more general nature: if the cross section $m \times k$ of the room has an odd area km , at least one of the three cross sections $t_m t_n$, $t_m t_k$ or $t_k t_n$ of the brick needs to be odd to allow a complete filling.

FIGURE 18. State diagram while tiling $2 \times 3 \times n$ rooms with $1 \times 2 \times 2$ 4-ominoes.

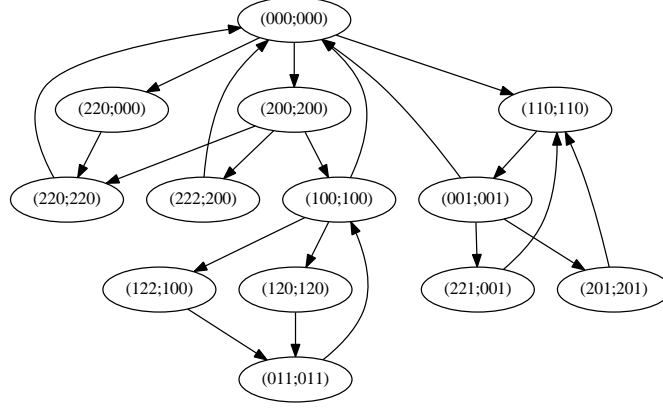
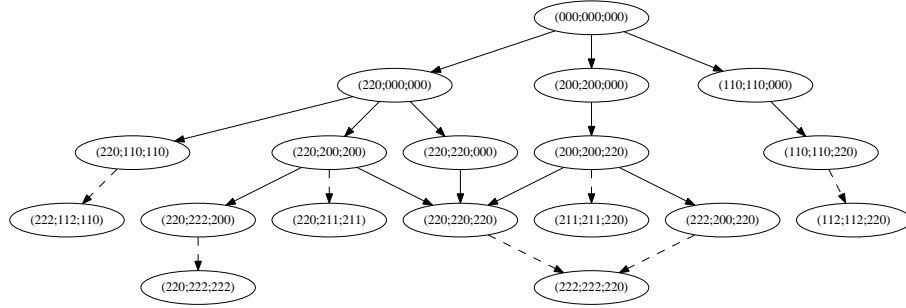


FIGURE 19. State diagram while tiling $3 \times 3 \times n$ rooms with $1 \times 2 \times 2$ 4-ominoes.



Filling $3 \times 4 \times n$ rooms with $1 \times 2 \times 2$ bricks is described by [12, A237356]

$$(44) \quad T_{1 \times 2 \times 2}(3, 4, z) = \frac{(1 - 2z^6)(-120z^{18} + 122z^{12} - 24z^6 + 1)}{(1 - z^6)(2640z^{24} - 2540z^{18} + 646z^{12} - 54z^6 + 1)}$$

$$= 1 + 29z^6 + 1065z^{12} + 41097z^{18} + 1602289z^{24} + 62603505z^{30} + 2447085377z^{36} + \dots$$

Filling $4 \times 4 \times n$ rooms with $1 \times 2 \times 2$ bricks is described by

$$(45) \quad T_{1 \times 2 \times 2}(4, 4, z) = \frac{p_{1 \times 2 \times 2}(4, 4, z)}{q_{1 \times 2 \times 2}(4, 4, z)} = 1 + z^4 + 165z^8 + 1065z^{12} + 44913z^{16} + 561689z^{20} + \dots,$$

where

$$(46) \quad p_{1 \times 2 \times 2}(4, 4, z) = (1 - 2z^4)(1 + 2z^4)(16896000z^{68} - 21811200z^{64} - 27278080z^{60} + 43889536z^{56} + 11612256z^{52} - 32759456z^{48} + 1863960z^{44} + 11174296z^{40} - 2373860z^{36} - 1742780z^{32} + 538060z^{28} + 111540z^{24} - 46358z^{20} - 1940z^{16} + 1612z^{12} - 44z^8 - 18z^4 + 1);$$

(47)

$$\begin{aligned}
q_{1 \times 2 \times 2}(4, 4, z) = & -207578z^{20} + 3942932672z^{56} - 46651584z^{36} + 1620z^{16} - 597079616z^{60} \\
& + 1 - 5117931520z^{76} + 229966968z^{44} - 3208396800z^{80} - 194z^8 - 418425808z^{52} - 19z^4 \\
& + 2433024000z^{84} + 3948z^{12} + 351702z^{24} + 4524176z^{28} + 7975271424z^{72} - 7861379200z^{64} \\
& - 11992040z^{32} - 1077496088z^{48} + 3593770496z^{68} + 160007540z^{40}.
\end{aligned}$$

4.5. $1 \times 1 \times 4$ Bricks. Figure 20 constructs

$$\begin{aligned}
(48) \quad T_{1 \times 1 \times 4}(2, 4, z) &= \frac{1 - z^4 - z^8 + z^{12} - z^6}{(1 - z^2 - 2z^4 + z^8)(-z^{12} + z^4 + 1 - z^8 + z^6)} \\
&= 1 + z^2 + z^4 + z^6 + 4z^8 + 9z^{10} + 16z^{12} + 25z^{14} + 49z^{16} + 100z^{18} + 196z^{20} + \dots
\end{aligned}$$

The coefficients are the squares $T_{1 \times 1 \times 4}(2, 4, N) = T_{1 \times 4}^2(4, N)$ of the coefficients of (23) for the reason discussed in conjunction with Eq. (39).

Similarly the tiling $3 \times 4 \times n$ rooms with $1 \times 1 \times 4$ 4-ominoes is counted by the cubes $T_{1 \times 1 \times 4}(3, 4, N) = T_{1 \times 4}^3(4, N)$:

$$(49) \quad T_{1 \times 1 \times 4}(3, 4, z) = \frac{p_{1 \times 1 \times 4}(3, 4, z)}{q_{1 \times 1 \times 4}(3, 4, z)} = 1 + z^3 + z^6 + z^9 + 8z^{12} + 27z^{15} + 64z^{18} + 125z^{21} + 343z^{24} + \dots,$$

where

$$\begin{aligned}
(50) \quad p_{1 \times 1 \times 4}(3, 4, z) = & z^{48} + z^{45} + 2z^{39} - 2z^{36} - 3z^{33} - z^{30} + 5z^{27} + 6z^{24} + 3z^{21} \\
& + 5z^{18} - 6z^{15} - 8z^{12} - 3z^9 - 2z^6 + 1;
\end{aligned}$$

$$\begin{aligned}
(51) \quad q_{1 \times 1 \times 4}(3, 4, z) = & (1 - z^3 - 3z^6 - 3z^9 - z^{12})(-z^{12} - z^9 + 1) \\
& \times (-z^{36} + 3z^{33} - 6z^{30} + 7z^{27} - 3z^{24} - 3z^{21} + 4z^{18} - 2z^{15} - 4z^{12} + 4z^9 + z^6 + 1).
\end{aligned}$$

Tiling $4 \times 4 \times n$ rooms with $1 \times 1 \times 4$ 4-ominoes is [12, A233291]

(52)

$$T_{1 \times 1 \times 4}(4, 4, z) = \frac{p_{1 \times 1 \times 4}(4, 4, z)}{q_{1 \times 1 \times 4}(4, 4, z)} = 1 + 2z^4 + 4z^8 + 8z^{12} + 45z^{16} + 248z^{20} + 1032z^{24} + \dots,$$

with

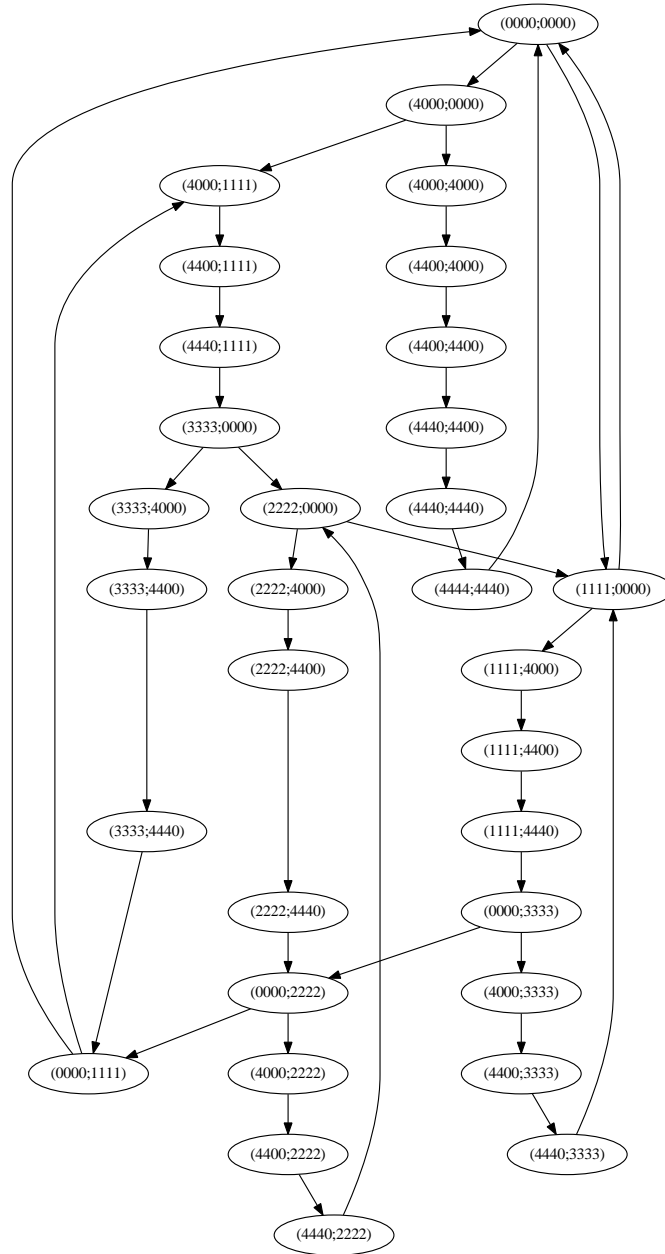
(53)

$$\begin{aligned}
p_{1 \times 1 \times 4}(4, 4, z) = & 1 + 30z^{28} - 20z^{20} - 112z^{80} + z^{116} - 171z^{40} - 151z^{56} + 90z^{32} \\
& + 34z^{72} + 69z^{36} + 174z^{60} + 5z^{108} + 61z^{24} + 3z^{112} - 4z^{12} - z^4 + 188z^{64} - 8z^{104} + 63z^{76} \\
& - 57z^{48} - 166z^{68} - 34z^{88} + 251z^{52} + z^{120} - 3z^{100} - 48z^{92} - 104z^{44} - 4z^8 - z^{124} + 11z^{84} + 39z^{96} - 31z^{16};
\end{aligned}$$

(54)

$$\begin{aligned}
q_{1 \times 1 \times 4}(4, 4, z) = & (1 + z^4)(1 + 184z^{28} + 25z^{20} + 89z^{80} + 9z^{116} + z^{136} - 758z^{40} \\
& - 715z^{56} + 49z^{32} + 436z^{72} + 120z^{36} + 1435z^{60} - 55z^{108} + 72z^{24} + z^{112} + 2z^{12} - 4z^4 \\
& - 1830z^{64} + 120z^{104} - 2z^{132} - 883z^{76} - 1557z^{48} + 545z^{68} - 306z^{88} + 917z^{52} + 2z^{120} - 72z^{100} \\
& + z^{128} + 75z^{92} + 990z^{44} + 2z^8 - 6z^{124} + 384z^{84} + 120z^{96} - 54z^{16}).
\end{aligned}$$

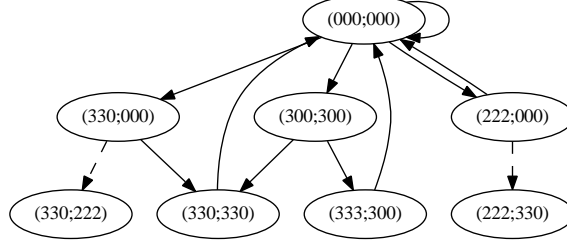
FIGURE 20. State diagram while tiling $2 \times 4 \times n$ rooms with $1 \times 1 \times 4$ 4-ominoes.



4.6. $1 \times 2 \times 3$ Bricks. Tiling $2 \times 2 \times n$ rooms with $1 \times 2 \times 3$ 6-ominoes gives the powers of 2 [12, A000079],

$$(55) \quad T_{1 \times 2 \times 3}(2, 2, z) = \frac{1}{1 - 2z^2} = 1 + 2z^2 + 4z^4 + 8z^6 + 16z^8 + 32z^{10} + \dots$$

FIGURE 21. State diagram while tiling $2 \times 3 \times n$ rooms with $1 \times 2 \times 3$ 6-ominoes.



Results of that type are generally understood by the constraint that the 2×2 cross section of the room permits only two different orientations of the 1×2 cross section of the brick, and that after each such placement the next placement is forced to return to the straight profile $(00 \dots 0)$.

Tiling $2 \times 3 \times n$ rooms with $1 \times 2 \times 3$ 6-ominoes is analyzed in Figure 21 [12, A103143]:

(56)

$$T_{1 \times 2 \times 3}(2, 3, z) = \frac{1}{1 - z - z^2 - 3z^3} = 1 + z + 2z^2 + 6z^3 + 11z^4 + 23z^5 + 52z^6 + 108z^7 + \dots$$

After pruning $(330; 222)$ and $(222; 330)$, the single loop, the circuit with 2 steps passing through $(222; 000)$ and the 3 circuits with 3 steps that contribute to the denominator of this generating function are easily recognized in the figure.

Tiling $3 \times 3 \times n$ rooms with $1 \times 2 \times 3$ 6-ominoes is counted by [12, A237357]

(57)

$$T_{1 \times 2 \times 3}(3, 3, z) = \frac{1 - z^3}{1 - 22z^6 - 7z^3 - 36z^9} = 1 + 6z^3 + 64z^6 + 616z^9 + 5936z^{12} + 57408z^{15} + \dots;$$

$$\hat{T}_{1 \times 2 \times 3}(3, 3, z) = \frac{2z^3(3 + 11z^3 + 18z^6)}{1 - z^3} = 6z^3 + 28z^6 + 64z^9 + 64z^{12} + 64z^{15} + 64z^{18} + \dots$$

Tiling $3 \times 4 \times n$ rooms with $1 \times 2 \times 3$ 6-ominoes is summarized by the GF [12, A237358]

(58)

$$T_{1 \times 2 \times 3}(3, 4, z) = \frac{p_{1 \times 2 \times 3}(3, 4, z)}{q_{1 \times 2 \times 3}(3, 4, z)} = 1 + z^2 + 11z^4 + 64z^6 + 296z^8 + 1716z^{10} + 9123z^{12} + \dots$$

with numerator

(59)

$$p_{1 \times 2 \times 3}(3, 4, z) = (1 - z^2)(1 + z^2)(1 - 3z^2)(3z^4 + 2z^2 + 1)(1 - z^4 - 7z^6 + 9z^{12})$$

and denominator

(60)

$$q_{1 \times 2 \times 3}(3, 4, z) = 504z^{12} + 306z^{22} + 1 - 1012z^{18} + 103z^{14} - 2z^2 + 54z^{32} - 162z^{34} - 450z^{28} + 74z^{24} - 14z^4 - 487z^{16} - 42z^6 - 448z^{20} + 915z^{26} + 237z^{10} + 873z^{30} + 42z^8.$$

5. SUMMARY

We have transformed Read's profile vectors of incomplete tilings into Transfer Matrices of associated digraphs, and obtained generating functions for some tilings of rectangular floors and rooms by 2- and 3-dimensional rectangular tiles.

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