# There are $\frac{1}{30}(\mathbf{r}+1)(\mathbf{r}+2)(2\mathbf{r}+3)(\mathbf{r}^2+3\mathbf{r}+5)$ Ways For the Four Teams of a World Cup Group to Each Have $r$ Goals For and $r$ Goals Against [Thanks to the Soccer Analog of Prop. 4.6.19 of Richard Stanley's (Classic!) EC1]

*By Shalosh B. EKHAD and Doron ZEILBERGER*

*Dedicated to Richard Peter Stanley who just turned "number of ways for a simple 1D Drunkard to return home after 8 steps"-years-old*



**Proof of the Statement in the Title**: Calling this quantity $S_4(r)$, and using the case $n = 4$ of the Soccer analog of Prop. 4.6.19 of [EC1] (see Comment 1 below), we see that

- $S_4(r)$ is a polynomial of degree 5 .
- $S_4(-1) = 0$ , $S_4(-2) = 0$ .
- $S_4(-3-r) = -S_4(r)$ .

Hence, it suffices to check the statement at the **two** values $r = 0$ and $r = 1$. Obviously $S_4(0) = 1$ (there is just one all-zero $4 \times 4$ matrix), and almost obviously, $S_4(1) = 9$ (the number of derangements of length 4 equals 9). QED!

**Comments**

**1**. The same method of proof that guru Richard Stanley used to prove Prop. 4.6.19 of [EC1] (first edition; it is Prop. 4.6.2 in the second edition.) [the first edition is missing $(-1)^{n-1}$ in front of $H_n(r)$] yields:

**The Soccer analog of Prop. 4.6.19 of EC1**

Let $S_n(r)$ be the number of ways $n$ Soccer teams, playing a round-robin tournament, each scored a total of $r$ "Goals For" (GF), and a total of $r$ "Goals Against" (GA), (in other words the number of $n \times n$ magic squares whose diagonal entries are all 0 (no team plays against itself!)).

For each fixed integer $n \geq 3$, the function $S_n(r)$ is a polynomial in $r$ of degree $n^2 - 3n + 1$. Since it is a polynomial, it can be evaluated at *any* (not necessarily positive) integer, and we have

$$S_n(-1) = S_n(-2) = \ldots = S_n(-n+2) = 0 \quad ,$$

$$S_n(-(n-1)-r) = -S_n(r) \quad .$$

**2.** We got the idea for this tribute to our beloved guru when we attended the Stanley@70 conference, held at MIT, June 23-27, 2014, at the same time as the preliminary Group stage of the World Cup 2014.

**3.** While the proof of the statement of the title did not require any computer, the analogous result for $n = 5$ is already beyond the scope of mere humans (or the human would have to be very stupid to spend time on it). We have

$$S_5(r) = \frac{1}{241920}\,(r+1)\,(r+2)\,(r+3)\cdot$$

$$\left(43\,r^8 + 688\,r^7 + 4934\,r^6 + 20680\,r^5 + 55907\,r^4 + 101272\,r^3 + 123436\,r^2 + 96240\,r + 40320\right) \quad .$$

Of course $S_3(r) = r + 1$ (why?).

To see $S_6(r)$, please go to: `http://www.math.rutgers.edu/~zeilberg/tokhniot/oGOALS1` .

One can get this, and (potentially!) infinitely more results, using the Maple package **GOALS** available directly from: `http://www.math.rutgers.edu/~zeilberg/tokhniot/GOALS` , or via the front of this article: `http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/worldcup.html` .

In particular, procedure `MagicPolAG(n,r,Sr,Sc,B,A)` can find polynomial expressions for the number of $n \times n$ matrices with non-negative integer entries, whose row-sums are

$$(r + Sr[1], \dots, r + Sr[n]) \quad ,$$

and whose column sums are

$$(r + Sc[1], \dots, r + Sc[n]) \quad ,$$

for *any* fixed numerical vectors `Sr`, `Sc`, of length $n$ (of course they have to add-up to the same number), and any assignment $A$ where some entries are fixed beforehand, where $B$ denotes "wild-card". See the on-line help there. See the above-mentioned front of this article for numerous sample input and output files.

**4.** The sequence for $S_4(r)$ is already in Neil Sloane's OEIS, (see [OEIS1]), but for **different** reasons! Can you find a bijection? On the other hand, the sequence for $S_5(r)$ is not (yet!) there:

$$1, 44, 870, 9480, 68290, 365936, 1573374, 5709120, 18107760, 51488800, 133748186, 321979164, \dots \quad ,$$

but we are sure that very soon it will!

**5.** Another, even more useful, Maple package accompanying this article is `WorldCup`, one of whose many procedures is '`Ptor`', that finds all the possible scenarios that lead to a given *score board*, consisting of the "Goals For", "Goals Against", and "PTS", vectors. (Recall that a win yields 3 points, a draw, 1 point, and a loss, 0 points.).

The number of possible scenarios for groups A-H were as follows

**2014**: $8, 32, 7, 3, 13, 3, 12, 3$ .

**2010**: $2, 3, 2, 2, 3, 3, 1, 2$ (in particular, the score-board for Group $G$ *uniquely* determined the individual game scores).

**2006**: $11, 1, 5, 3, 2, 6, 1, 8$ (in particular, the score-boards for Groups $B$ and $G$ *uniquely* determined the individual game scores).

**2002**: $1, 12, 5, 5, 1, 2, 3, 1$ (in particular, the score-boards for Groups $A, E, H$ *uniquely* determined the individual game scores).

**1998**: $9, 6, 4, 3, 3, 3, 1, 1$ (in particular, the score-boards for Groups $G, H$ *uniquely* determined the individual game scores).

**6.** The Israeli daily *Yedioth Ahronoth* published, at the start of the 2014 World Cup, a "soccer puzzle" where the above-mentioned vectors, GF, GA, and PTS, are given, and the solver has to reconstruct the scores of the individual matches. The above-mentioned Maple package `WorldCup` has a procedure, `Khida`, that makes up such puzzles, and another procedure, `Sefer1`, that creates challenging puzzle books. For some samples, see

`http://www.math.rutgers.edu/~zeilberg/tokhniot/oWorldCupi` , for $1 \leq i \leq 4$. Enjoy!

**7.** Happy birthday Richard, and keep up the good work! (and continue to practice what Phil Hanlon calls the "the three H's").

**References**

[OEIS1] Neil J.A. Sloane, *Sequence A061927*, `http://oeis.org/A061927`.

[EC1] Richard P. Stanley, "*Enumerative Combinatorics, volume I*", first edition: Wadsworth & Brooks/Cole, 1986; second edition: Cambridge University Press, 2011.
Available on-line (viewed July 7, 2014): `http://math.mit.edu/~rstan/ec/ec1.pdf` .

---

Shalosh B. Ekhad, c/o D. Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
url: `http://www.math.rutgers.edu/~zeilberg/` .
Email: `zeilberg at math dot rutgers dot edu` .

---

**EXCLUSIVELY PUBLISHED IN THE PERSONAL JOURNAL OF SHALOSH B. EKHAD and DORON ZEILBERGER** `http://www.math.rutgers.edu/~zeilberg/pj.html` and `arxiv.org` .

---

July 7, 2014