Counting paths in corridors using circular Pascal arrays

Shaun V. Ault[a,*], Charles Kicey[a]

[a]*Department of Mathematics and Computer Science, Valdosta State University, 1500 N. Patterson St., Valdosta, GA, 31698*

Abstract

A circular Pascal array is a periodization of the familiar Pascal's triangle. Using simple operators defined on periodic sequences, we find a direct relationship between the ranges of the circular Pascal arrays and numbers of certain lattice paths within corridors, which are related to Dyck paths. This link provides new, short proofs of some nontrivial formulas found in the lattice-path literature.

Keywords: lattice path, corridor path, binomial coefficient, discrete linear operator

1. Circular Pascal Arrays and Corridor Paths

1.1. Circular Pascal Arrays

We begin by defining the *circular Pascal arrays* (one for each integer $d \geq 2$) and explore some of their amazing properties. By *Pascal array*, we mean something a little more general than the familiar Pascal's triangle. In what follows, we interpret the binomial coefficient $\binom{n}{k}$ as the coefficient of x^k in the expansion,

$$(1+x)^n = \sum_{k\in\mathbb{Z}} \binom{n}{k} x^k,$$

where we understand $\binom{n}{k} = 0$ if $k < 0$ or $k > n$. We also use the convention that $\mathbb{N}$ denotes the set of *non-negative integers*, $\{0, 1, 2, 3, \ldots\}$.

Definition 1. The **Pascal array** is the array whose row n, column k entry is equal to $\binom{n}{k}$ where $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

$n \setminus k$	...	−1	0	1	2	3	4	5	...
0	...	0	1	0	0	0	0	0	...
1	...	0	1	1	0	0	0	0	...
2	...	0	1	2	1	0	0	0	...
3	...	0	1	3	3	1	0	0	...
4	...	0	1	4	6	4	1	0	...
5	...	0	1	5	10	10	5	1	...
⋮		⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Recall that for any $n > 0$, entry (n, k) of the array can be found by the familiar formula,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{1}$$

Definition 2. Fix an integer $d \geq 2$. The **circular Pascal array of order** d is the array whose row n, column k entry, $\sigma^{(d)}_{n,k}$ (or just $\sigma_{n,k}$ when the context is clear), is the (finite) sum:

$$\sigma_{n,k} = \sigma^{(d)}_{n,k} = \sum_{j\in\mathbb{Z}} \binom{n}{k+dj}, \tag{2}$$

where $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

*Corresponding author

Email addresses: svault@valdosta.edu (Shaun V. Ault), ckicey@valdosta.edu (Charles Kicey)

In what follows, we will be interested in **periodic** sequences and arrays of numbers. To be precise, we say a sequence $(a_k)_{k\in\mathbb{Z}}$ is *periodic*, of period d, if $a_{k+dm} = a_k$ for every $m \in \mathbb{Z}$. Clearly, for fixed $n \geq 0$, the sequence $(\sigma^{(d)}_{n,k})_{k\in\mathbb{Z}}$ as defined in (2) is periodic of period d. Furthermore, it is easy to see that the entries of the circular Pascal array satisfy (1), in the sense that for $n > 0$,

$$\sigma_{n,k} = \sigma_{n-1,k-1} + \sigma_{n-1,k}. \tag{3}$$

For $d = 5$, our collection of periodic sequences is indicated below.

$n \setminus k$	$\cdots$	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	$\cdots$
0	$\cdots$	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	$\cdots$
1	$\cdots$	1	1	0	0	0	1	1	0	0	0	1	1	0	0	0	$\cdots$
2	$\cdots$	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0	$\cdots$
3	$\cdots$	1	3	3	1	0	1	3	3	1	0	1	3	3	1	0	$\cdots$
4	$\cdots$	1	4	6	4	1	1	4	6	4	1	1	4	6	4	1	$\cdots$
5	$\cdots$	2	5	10	10	5	2	5	10	10	5	2	5	10	10	5	$\cdots$
6	$\cdots$	7	7	15	20	15	7	7	15	20	15	7	7	15	20	15	$\cdots$
7	$\cdots$	22	14	22	35	35	22	14	22	35	35	22	14	22	35	35	$\cdots$
8	$\cdots$	57	36	36	57	70	57	36	36	57	70	57	36	36	57	70	$\cdots$
9	$\cdots$	127	93	72	93	127	127	93	72	93	127	127	93	72	93	127	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Henceforth, we will represent a circular Pascal array by showing only columns $0, 1, \ldots, d-1$.

While studying a related problem called the Sharing Problem, Charles Kicey, Katheryn Klimko, and Glen Whitehead[2] noticed that the circular Pascal array has surprising connections to well-known sequences for small values of d. Consider the case $d = 2$ (shown below, along with $d = 3$ and $d = 4$). Starting in row $n = 1$, $\sigma_{n,k} = 2^{n-1}$ for $k = 0, 1$. Of course, this reflects a well-known property of Pascal's triangle: $\sum_{j\in\mathbb{Z}} \binom{n}{2j} = \sum_{j\in\mathbb{Z}} \binom{n}{2j+1}$, if $n \geq 1$. Another way to state this result is to say that the *range* (difference between maximum and minimum values) of the n^{th} row is 0 for $n \geq 1$ in the circular Pascal array of order 2. The cases $d = 3, 4$ are interesting as well – for $d = 3$, the ranges are constantly 1, while for $d = 4$, the range of row n is $2^{\lfloor n/2 \rfloor}$ – but the most surprising case is, perhaps, $d = 5$. These ranges form the Fibonacci sequence, as was proved in [2]. Now Fibonacci numbers are no strangers to the Pascal's triangle; indeed the sequence of diagonal sums, $f_n = \sum_{j\in\mathbb{Z}} \binom{n-j}{j}$, is easily shown to be the Fibonacci sequence. However, this new manifestation of the Fibonacci numbers in the circular Pascal array of order 5 was quite unexpected.

$d = 2$			
$n \setminus k$	0	1	Range
0	1	0	1
1	1	1	0
2	2	2	0
3	4	4	0
4	8	8	0
5	16	16	0
6	32	32	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$d = 3$				
$n \setminus k$	0	1	2	Range
0	1	0	0	1
1	1	1	0	1
2	1	2	1	1
3	2	3	3	1
4	5	5	6	1
5	11	10	11	1
6	22	21	21	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$d = 4$					
$n \setminus k$	0	1	2	3	Range
0	1	0	0	0	1
1	1	1	0	0	1
2	1	2	1	0	2
3	1	3	3	1	2
4	2	4	6	4	4
5	6	6	10	10	4
6	16	12	16	20	8
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$d=5$						
$n \setminus k$	0	1	2	3	4	Range
0	1	0	0	0	0	1
1	1	1	0	0	0	1
2	1	2	1	0	0	2
3	1	3	3	1	0	3
4	1	4	6	4	1	5
5	2	5	10	10	5	8
6	7	7	15	20	15	13
7	22	14	22	35	35	21
8	57	36	36	57	70	34
9	127	93	72	93	127	55
⋮	⋮	⋮	⋮	⋮	⋮	⋮

So the search began to find other well-known sequences in the d-circular Pascal array. Another of Charles Kicey's students, Jonathon Bryant, explored the sequences of ranges for larger values of d using the OEIS [7] and found that they were known but in different contexts. The OEIS entry for the $d=9$ case (A061551) gives a tantalizing clue that all of these number sequences are indeed related. The main description of A061551 is: "number of paths along a corridor width 8, starting from one side," and further down the page, the following note can be found.

> Narrower corridors effectively produce A000007, A000012, A016116, A000045, A038754, A028495, A030436. An infinitely wide corridor (i.e. just one wall) would produce A001405.

The main result of this paper is to prove that the corridor numbers are indeed the same as our sequences of ranges of circular Pascal arrays. First we define the corridor numbers precisely.

1.2. Corridor Paths

For a fixed number $m \geq 0$, the m**-corridor** is set of lattice points $(a,b) \in \mathbb{N} \times \{0,1,\ldots m\}$. We will show that the ranges of circular Pascal arrays of order $m+2$ coincide with the number of *m-corridor paths* beginning at the origin. However, we find it useful to consider paths within an m-corridor that may begin at an arbitrary point, since it makes the arguments no more difficult and provides a link to a more general formula found in the combinatorics literature (see §1.3 below). To be precise, a **corridor path** is a path in the m-corridor satisfying the following rules:

1. The initial point of the path is at $(0, y_0)$ for some chosen y_0 with $0 \leq y_0 \leq m$.
2. The path never leaves the corridor.
3. Each step in the path is either an up-and-right or down-and-right move.

An example corridor path is shown in Fig. 1.

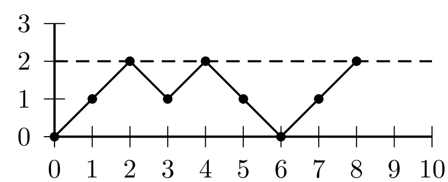

Figure 1: A path of length 8 in the 2-corridor with $y_0 = 0$.

Definition 3. For a fixed $m \geq 0$, the sequence of **corridor numbers of order** m, $(c_n^{(m)})_{n\in\mathbb{N}}$ (or just (c_n) when the context is clear), counts the number of paths of length n, starting at the origin, in the m-corridor. If we wish to count the number of corridor paths starting at $(0, y_0)$, then we may write $c_{n,y_0}^{(m)}$ (or c_{n,y_0}).

Remark 4. Corridor numbers are useful in graph theory as $c_n^{(m)}$ counts the number of length n paths in the path graph P_{m+1} that start at the initial node of the graph, while $c_{n,y_0}^{(m)}$ counts the number of such paths that start at node $y_0 + 1$.

1.3. Dyck Paths and K-M Paths

Corridor paths are certain types of *lattice paths*. Indeed, they may be identified with a variation of *Dyck paths*. Recall, a Dyck path of order m is a monotonic lattice path from the origin to the point (m, m) that does not cross the diagonal line $y = x$. Note that an order m Dyck path has length $2m$. For our purposes, we will assume the path is drawn above the line $y = x$, as in Fig. 2. It is well known that the number of Dyck paths of order $m \geq 0$ is equal to the m^{th} Catalan number, C_m [1]. We will consider the following variation of Dyck paths found in Krattenthaler-Mohanty [3]:

Definition 5. Let $s, t \in \mathbb{Z}$ such that $t \geq 0 \geq s$ and $a, b \in \mathbb{Z}$ such that $a + t \geq b \geq a + s$. A **K-M path** is a monotonic lattice path from the origin to (a, b) that does not cross either of the lines $y = x + s$ or $y = x + t$. The number of such K-M paths is denoted $D(a, b; s, t)$. If $b > a + t$ or $b < a + s$, then define $D(a, b; s, t) = 0$.

There is an affine transformation taking K-M paths to corridor paths. With a, b, s, t as in Definition 5, map the point $(a, b) \mapsto (a + b, b - a - s)$. Then the line $y = x + s$ maps to the x-axis, the line $y = x + t$ maps to the line $y = t - s$, and the origin (the initial point for all K-M paths) maps to $(0, -s)$. Thus, with $m = t - s$ and $y_0 = -s$, the result is the m-corridor with initial point $(0, y_0)$. When $s = 0$ and $t = m$, we recover the corridor numbers of order m:

$$c_n = \sum_{a+b=n} D(a, b; 0, m). \tag{4}$$

That is, the number of length n paths in the m-corridor is equal to the number of K-M paths of length n, lying between $y = x$ and $y = x + m$ (Compare Fig. 3 and Fig. 1, for example).

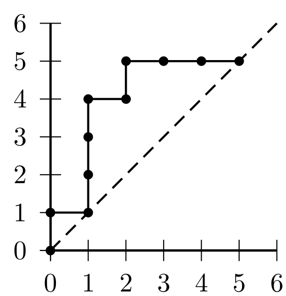

Figure 2: A Dyck path of order 5.

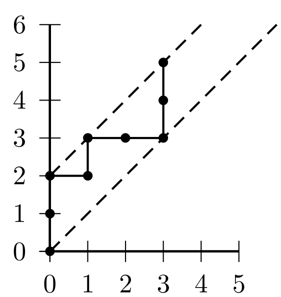

Figure 3: A K-M path of length 8 ending at (3,5)

These types of lattice paths and other variations have been studied extensively in the combinatorics literature (see, for example, [3, 5, 6]). Krattenthaler and Mohanty give a formula involving sums of binomial coefficients which looks quite related to our definition of the circular Pascal array (2).

$$D(a, b; s, t) = \sum_{k \in \mathbb{Z}} \left(\binom{a + b}{a - k(t - s + 2)} - \binom{a + b}{a - k(t - s + 2) + t + 1} \right). \tag{5}$$

Remark 6. This formula goes back to [5], §1.3, Thm. 2. Also see [9], and *Bertrand's ballot problem.*

In the next section, we prove the connection between circular Pascal arrays and corridor numbers, and this will lead to a simpler derivation of (5) than the proofs currently found in the literature, to our knowledge.

2. The Main Result

2.1. Shift and Difference Operators

Fix an integer $d \geq 2$. In what follows we work in the vector space $\mathbb{R}^\infty$ of real-valued sequences indexed by the set of integers $\mathbb{Z}$. All of our vectors will be periodic (in particular, with period d or $2d$). Let I be the identity operator on $\mathbb{R}^\infty$. For $\mathbf{x} = (x_k)_{k\in\mathbb{Z}} \in \mathbb{R}^\infty$, we will make use of the right shift operator defined by $R(\mathbf{x}) = (x_{k-1})_{k\in\mathbb{Z}}$. Let $L = R^{-1}$. Note that any powers or L and R would, of course, commute. Let us introduce a difference operator D on $\mathbb{R}^\infty$ by $D = I - L$ (a negative of a "discrete deriviative"). The role of D will become clear as we search for minimum and maximum values in our Pascal arrays. Define a periodic "unit vector," $\mathbf{e_0}$, by:

$$\mathbf{e}_0 = (\ldots, 0, 1, \underbrace{0, \ldots, 0}_{d-1 \text{ zeros}}, \mathbf{1}, \underbrace{0, \ldots, 0}_{d-1 \text{ zeros}}, 1, 0, \ldots). \tag{6}$$

From these building blocks alone, we can analyze the circular Pascal array as well as our corridor numbers. Note that the recursive formula (3) can be encoded by the operator $I + R$. In particular, if $\boldsymbol{\sigma}_n = (\sigma_{n,k})_{k\in\mathbb{Z}}$ is the n^{th} row of the circular Pascal array of order d, then for $n \geq 1$, we have:

$$\boldsymbol{\sigma}_n = (I + R)\boldsymbol{\sigma}_{n-1} = (I + R)^n \boldsymbol{\sigma}_0, \tag{7}$$

with the initial vector $\boldsymbol{\sigma}_0 = \mathbf{e}_0$. However, we will find it useful to allow more general initial vectors $\boldsymbol{\sigma}_0$ in the analysis below.

Remark 7. Here and throughout, as an aid to the reader, we will bold-face the entry corresponding to $k = 0$ for numerical vectors in $\mathbb{R}^\infty$ (unless it is clear from context).

2.2. Up-Sampling

We also introduce a variation of the *up-sample* operator of digital signal processing [8]. For $\mathbf{x} = (x_k)_{k\in\mathbb{Z}}$, we will define our up-sample operator U as follows:

$$U(\mathbf{x}) = (x_{\lfloor k/2 \rfloor})_{k\in\mathbb{Z}} = (\ldots, x_{-1}, x_{-1}, x_0, x_0, x_1, x_1, \ldots)\ .$$

Definition 8. The **up-sampled circular Pascal array of order** d, denoted $(\mathbf{p}_n)_{n\geq 0}$ is defined by $\mathbf{p}_n = U(\boldsymbol{\sigma}_n)$ for all $n \geq 0$.

Note that up-sampled array of order d repeats in blocks of size $2d$. Observe that $U(I + R) = (I + R^2)U$ on $\mathbb{R}^\infty$, so it follows that the up-sampled array can be expressed inductively as

$$\mathbf{p}_n = (I + R^2)^n \mathbf{p}_0, \tag{8}$$

where $\mathbf{p}_0$ is the initial vector. In our applications, we typically use $\mathbf{p}_0 = U(\boldsymbol{\sigma}_0)$, for our chosen initial vector $\boldsymbol{\sigma}_0$, and so in the prototypical case (*i.e.*, $\boldsymbol{\sigma}_0 = \mathbf{e}_0$), we have

$$\mathbf{p}_0 = (\ldots, \mathbf{1}, 1, \underbrace{0, 0, \ldots, 0, 0}_{2d-2 \text{ zeros}}, 1, 1, 0, \ldots). \tag{9}$$

Now define the differences: $(\mathbf{q}_n)_{n\geq 0}$, by $\mathbf{q}_n = D(\mathbf{p}_n)$ for all $n \geq 0$. Each row in this collection of differences is of course $2d$-periodic. It turns out that $\mathbf{q}_{n+1}$ can be obtained from $\mathbf{q}_n$ the same way as $\mathbf{p}_{n+1}$ is obtained from $\mathbf{p}_n$:

Lemma 9. *For all* $n \geq 0$, $\mathbf{q}_n = (I + R^2)^n \mathbf{q}_0$.

Proof. Because $I + R^2$ and $D = I - L$ commute, we have, for $n \geq 1$,

$$\mathbf{q}_n = D(\mathbf{p}_n) = D(I + R^2)(\mathbf{p}_{n-1}) = (I + R^2)D(\mathbf{p}_{n-1}) = (I + R^2)\mathbf{q}_{n-1}.$$

A simple induction gives the result. □

Note that in the prototypical case, when $\mathbf{p}_0$ is given by (9), we have

$$\mathbf{q}_0 = (\dots, 0, -1, \mathbf{0}, 1, \underbrace{0, 0, \dots, 0}_{2d-3 \text{ zeros}}, -1, 0, 1, 0, \dots).$$

However, let us analyze a more general case that will become useful in later sections. Let $0 \leq y_0 \leq d - 2$ and define the initial vector $\boldsymbol{\sigma}_0$ via

$$\boldsymbol{\sigma}_0 = \left(I + R + R^2 + \cdots + R^{y_0}\right)\mathbf{e}_0 = (\dots, \underbrace{\mathbf{1}, 1, \dots, 1}_{y_0 + 1 \text{ ones}}, \underbrace{0, 0, \dots, 0}_{d-(y_0+1) \text{ zeros}}, \dots), \tag{10}$$

and define $\boldsymbol{\sigma}_n$ inductively by (7). For the up-sampled version we set $\mathbf{p}_n = U(\boldsymbol{\sigma}_n)$ for all $n \geq 0$. In particular, for $n = 0$,

$$\mathbf{p}_0 = (I + R + R^2 + \cdots + R^{2y_0+1})\mathbf{e}'_0 = (\dots, \underbrace{\mathbf{1}, 1, \dots 1, 1}_{2y_0 + 2 \text{ ones}}, \underbrace{0, 0, \dots, 0, 0}_{2d - 2y_0 - 2 \text{ zeros}}, \dots), \tag{11}$$

where $\mathbf{e}'_0$ is the $2d$-periodic analog of $\mathbf{e}_0$. The relationship between up-sampled Pascal arrays and corridor number rests on the following key fact.

Lemma 10. *With the general initial vector* $\mathbf{p}_0$ *defined by (11),* $L^{y_0}\mathbf{q}_0 = (-L^{y_0+1} + R^{y_0+1})\mathbf{e}'_0$.

Proof.

$$\begin{aligned} \mathbf{q}_0 &= D(\mathbf{p}_0) \\ &= (I - R^{-1})(I + R + R^2 + \cdots + R^{2y_0+1})\mathbf{e}'_0 \\ &= (-R^{-1} + R^{2y_0+1})\mathbf{e}'_0 \\ &= (-L + R^{2y_0+1})\mathbf{e}'_0 \\ L^{y_0}\mathbf{q}_0 &= (-L^{y_0+1} + R^{y_0+1})\mathbf{e}'_0. \end{aligned}$$

□

2.3. Dual Corridors

Let us shift our attention now to corridors and corridor numbers. Again fix $d \geq 2$. Recall that the corridor numbers of order $m = d - 2$ count corridor paths starting at $(0, 0)$ in $\mathbb{N} \times \{0, 1, \dots, d - 2\}$. Instead, let us shift up a unit and consider corridor paths starting at $(0, 1)$ in $\mathbb{N} \times \{1, 2, \dots, d - 1\}$. In fact, we introduce a dual corridor structure, of *positive* corridor paths starting at $(0, 1)$ in $\mathbb{N} \times \{1, 2, \dots, d - 1\}$ together with *negative* corridor paths starting at $(0, -1)$ in $\mathbb{N} \times \{-1, -2, \dots, -(d - 1)\}$. Embed the dual corridors into the lattice $\mathbb{N} \times \mathbb{Z}$ and extend by $2d$-periodicity in the second component. Let us denote the signed number of paths incoming to vertex (n, k) by $v_{n,k}$. Here the signs are chosen according to $v_{n,k} > 0$ if $k \equiv 1, 2, \dots, d - 1$ (mod $2d$) and $v_{n,k} < 0$ if $k \equiv -1, -2, \dots, -(d - 1)$ (mod $2d$). Let $\mathbf{v}_n = (v_{n,k})_{k\in\mathbb{Z}}$, which one may say is the **state** of our periodic corridor at step n. More generally, consider the initial state, $\mathbf{v}_0$, defined so that it will correspond to a corridor path starting at vertex $(0, y_0 + 1)$, *i.e.*,

$$\mathbf{v}_0 = (\dots, 0, -1, \underbrace{0, \dots, 0, \mathbf{0}, 0, \dots, 0}_{2y_0 + 1 \text{ zeros}}, 1, 0, \dots). \tag{12}$$

The key observation, using Lemma 10, is that there is a link between differences of circular Pascal array entries and the vertex state numbers:

$$\mathbf{v}_0 = (-L^{y_0+1} + R^{y_0+1})\mathbf{e}'_0 = L^{y_0}\mathbf{q}_0. \tag{13}$$

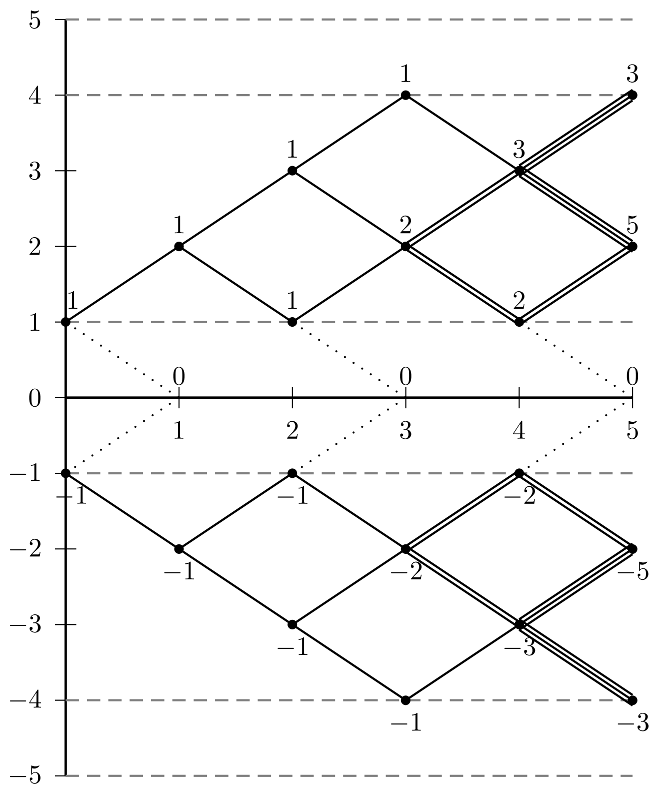

Figure 4: The dual corridor structure for $d = 5$ and $y_0 = 0$ For example, the initial state is $\mathbf{v}_0 = (\ldots, 0, -1, \mathbf{0}, 1, 0, \ldots)$, and we have $\mathbf{v}_5 = (\ldots, 0, -3, 0, -5, 0, \mathbf{0}, 0, 5, 0, 3, 0, \ldots)$.

By the very definition of the $\mathbf{v}_n$ we have

$$v_{n,0} = v_{n,\pm d} = v_{n,\pm 2d} = \cdots = 0. \tag{14}$$

Moreover, $v_{n,k}$ are antisymmetric about $k = 0$, *i.e.*,

$$v_{n,-k} = -v_{n,k} \text{ for all } k \in \mathbb{Z}. \tag{15}$$

Now consider the state of our corridor at step $n + 1$. Recall that the upper and lower corridor states are represented by $v_{n+1,k}$ for $k = 1, 2, \ldots, d - 1$ and $k = -1, -2, \ldots, -(d - 1)$ respectively. An interior vertex (n, k), $k = 2, \ldots, d - 2$ receives paths from both $(n, k - 1)$ and $(n, k + 1)$, and so

$$v_{n+1,k} = v_{n,k-1} + v_{n,k+1}. \tag{16}$$

At the boundaries, however, $v_{n+1,1} = v_{n,2}$ and $v_{n+1,d-1} = v_{n,d-2}$. But observe that (14) and (15) imply that (16) does hold for all $k = -(d-1), -(d-2), \ldots, 0, 1, \ldots d$. Then by periodicity, (16) holds for all $k \in \mathbb{Z}$. In our operator notation, this gives

$$\mathbf{v}_{n+1} = (L + R)\mathbf{v}_n. \tag{17}$$

2.4. Proof of the Main Theorem

We are now in position to prove the main result of this paper.

Lemma 11. *Fix $d \geq 2$. Let $0 \leq y_0 \leq d - 2$. For each $n \geq 0$, $\mathbf{v}_n = L^{n+y_0}\mathbf{q}_n$, i.e., $v_{n,k} = q_{n,k+n+y_0}$, for all $k \in \mathbb{Z}$.*

Proof. By (17) and induction, $\mathbf{v}_n = (L + R)^n\mathbf{v}_0$ for all $n \geq 0$. But $(L + R)^n = [L(I + R^2)]^n = L^n(I + R^2)^n$. Using $\mathbf{v}_0 = L^{y_0}\mathbf{q}_0$ and Lemma 9, we have

$$\mathbf{v}_n = L^n(I + R^2)^n(L^{y_0}\mathbf{q}_0) = L^{n+y_0}(I + R^2)^n\mathbf{q}_0 = L^{n+y_0}\mathbf{q}_n.$$

□

$d=8$									
$n \setminus k$	0	1	2	3	4	5	6	7	Range
0	1	1	1	0	0	0	0	0	1
1	1	2	2	1	0	0	0	0	2
2	1	3	4	3	1	0	0	0	4
3	1	4	7	7	4	1	0	0	7
4	1	5	11	14	11	5	1	0	14
5	1	6	16	25	25	16	6	1	24
6	2	7	22	41	50	41	22	7	48
7	9	9	29	63	91	91	63	29	82
8	38	18	38	92	154	182	154	92	164
9	130	56	56	130	246	336	336	246	280
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Figure 5: Circular Pascal numbers corresponding to $d = 8$ and $y_0 = 2$.

Theorem 12. *For fixed $d \geq 2$, and $0 \leq y_0 \leq d-2$, the n^{th} corridor number of order $d-2$, with paths beginning at $(0, y_0)$ is given by $c_{n,y_0}^{(d-2)} = p_{n,n+y_0}^{(d)} - p_{n,n+y_0+d}^{(d)}$. Moreover, $p_{n,n+y_0}^{(d)}$ and $p_{n,n+y_0+d}^{(d)}$ are respectively, the maximum and minimum values found in the n^{th} row of the circular Pascal array of order d whose initial vector, $\boldsymbol{\sigma}_0$ is given by (10).*

Proof. Let $p_{n,k}$, *resp.* $q_{n,k}$, be the k^{th} entry of $\mathbf{p}_n$, *resp.* $\mathbf{q}_n$. By definition, $c_{n,y_0} = \sum_{k=0}^{d-1} v_{n,k}$. So by Lemma 11,

$$c_{n,y_0} = \sum_{k=0}^{d-1} q_{n,k+n+y_0} = \sum_{k=n+y_0}^{n+y_0+d-1} q_{n,k} = \sum_{k=n+y_0}^{n+y_0+d-1} (p_{n,k} - p_{n,k+1}) = p_{n,n+y_0} - p_{n,n+y_0+d}.$$

Moreover, by the definition of the dual corridor states, $\mathbf{v}_n$, it is clear that the differences $q_{n,k}$ satisfy:

$$q_{n,n+y_0+j} = \begin{cases} v_{n,0} = 0, & \text{if } j \equiv 0 \pmod{2d}, \\ v_{n,j} \geq 0, & \text{if } j \equiv 1, 2, \ldots, d-1 \pmod{2d}, \\ v_{n,d} = 0, & \text{if } j \equiv d \pmod{2d}, \\ v_{n,j} \leq 0, & \text{if } j \equiv d+1, d+2, \ldots, 2d-1 \pmod{2d}. \end{cases}$$

Now since $q_{n,n+y_0+j} = p_{n,n+y_0+j} - p_{n,n+y_0+j+1}$ is essentially a negative discrete derivative, it follows that $p_{n,n+y_0}$ is a maximum value of $\mathbf{p}_n$, and $p_{n,n+y_0+d}$ is a minimum value of $\mathbf{p}_n$. Of course the maximum and minimum values of the up-sampled array correspond to those in the original $(\boldsymbol{\sigma}_n)_{n \in \mathbb{N}}$, which completes the proof. □

Example. Let $d = 8$ and $y_0 = 2$. To find the corridor numbers of order $d - 2 = 6$ in $\mathbb{N} \times \{0, 1, \ldots, 6\}$, but with corridor paths starting at $(0, 2)$, we take the range of the n^{th} of the Pascal array mod 8, but using $\boldsymbol{\sigma}_0 = (1, 1, 1, 0, 0, 0, 0, 0) \in \mathbb{R}^8$ (extended periodically) as our initial row (See Fig. 5).

Corollary 13. *The $(d-2)$-corridor number c_n (i.e., when $y_0 = 0$) equals the range of the n^{th} row of the (standard) circular Pascal array or order d. More explicitly, $c_n^{(d-2)} = \sigma_{n,\lfloor n/2 \rfloor}^{(d)} - \sigma_{n,\lfloor (n+d)/2 \rfloor}^{(d)}$.*

3. Further Results

3.1. Deriving the K-M Formula

In this section we prove the K-M formula (5) in full generality. Let $s, t \in \mathbb{Z}$ such that $t \geq 0 \geq s$, and $a, b \in \mathbb{Z}$ such that $a + t \geq b \geq a + s$. Then the number of K-M paths, $D(a, b; s, t)$ starting at $(0, 0)$ and ending at (a, b) is equivalent to the number of m-corridor paths in $\mathbb{N} \times \{1, 2, \ldots, d-1\}$ beginning at $(0, y_0 + 1)$ and ending at $(a+b, b-a+y_0+1)$, where $m = d-2 = t-s$ and $y_0 = -s$. That is, $D(a,b;s,t) = v_{a+b,\,b-a+y_0+1}$. Using Lemma 11 and the definitions of $\mathbf{q}_n$, $\mathbf{p}_n$, and $\boldsymbol{\sigma}_n$, this leads to

$$\begin{aligned} D(a,b;s,t) &= q_{a+b,\,(b-a+y_0+1)+(a+b)+y_0} \\ &= q_{a+b,\,2b-2s+1} \\ &= p_{a+b,\,2b-2s+1} - p_{a+b,\,2b-2s+2} \\ &= \sigma_{a+b,\,b-s} - \sigma_{a+b,\,b-s+1}. \end{aligned}$$

Using the linearity of $(I+R)^n$ and the fact that R commutes with $(I+R)^n$, then it is easy to see that

$$\sigma_{n,k} = \sum_{j\in\mathbb{Z}} \left[\binom{n}{k+dj} + \binom{n}{(k-1)+dj} + \cdots + \binom{n}{(k-y_0)+dj} \right]. \tag{18}$$

Then by (18), we have:

$$\begin{aligned} D(a,b;s,t) &= \sigma_{a+b,\,b-s} - \sigma_{a+b,b-s+1} \\ &= \sum_{k\in\mathbb{Z}} \left[\binom{a+b}{b-s+dk} + \binom{a+b}{b-s-1+dk} + \cdots + \binom{a+b}{b-s-(-s)+dk} \right] \\ &\quad - \sum_{k\in\mathbb{Z}} \left[\binom{a+b}{b-s+1+dk} + \binom{a+b}{b-s+dk} + \cdots + \binom{a+b}{b-s+1-(-s)+dk} \right] \\ &= \sum_{k\in\mathbb{Z}} \left[\binom{a+b}{b+dk} - \binom{a+b}{b-s+1+dk} \right] \\ &= \sum_{k\in\mathbb{Z}} \left[\binom{a+b}{a-dk} - \binom{a+b}{a+s-1-dk} \right] \\ &= \sum_{k\in\mathbb{Z}} \left[\binom{a+b}{a-k(t-s+2)} - \binom{a+b}{a-k(t-s+2)+s-1} \right]. \end{aligned}$$

The final step is to re-index the second terms via $k \mapsto k+1$, and obtain (5).

3.2. Infinite Width Corridors

Consider corridor paths beginning at $(0, y_0)$, but in the infinite corridor $\mathbb{N}\times\mathbb{N}$; we denote the number of such paths of length n by $c^{(\infty)}_{n,y_0}$. It is easy to see that $c^{(\infty)}_{n,y_0}$ is the number of n-tuples $(r_1, r_2, \ldots, r_n)$ satisfying $r_k \in \{-1, 1\}$ and $\sum_{j=1}^{k} r_j \geq -y_0$ for all $k = 1, 2, \ldots, n$. We pass on the challenge found in [4].

> The most basic case of the enumerative coincidences that we shall study is the fact that there are $\binom{2n}{n}$ positive walks of length $2n$, a number that also (and more obviously) counts the recurrent walks of that length. This result appears to be well known, at least in the lattice path community, but in view of its simplicity it is somewhat surprising that it does not receive prominent mention in the enumerative combinatorics literature. We do not know whether any nice bijective proofs for this result are known, but it would at least seem that none are "well known."

Although we do not have a *bijective* proof, we now provide a simple proof based on our structure already in place. Fix $n \geq 0$ and $y_0 \geq 0$. For $m = n + y_0$, *i.e.*, $d = n + y_0 + 2$, there is no difference between $c^{(\infty)}_{n,y_0}$ and $c^{(m)}_{n,y_0}$, since the paths of length n in $\mathbb{N}\times\{0, 1, \ldots, m\}$ are not yet restricted by the upper wall.

Thus by Theorem 12,

$$\begin{aligned} c^{(\infty)}_{n,y_0} = c^{(n+y_0)}_{n,y_0} &= p^{(n+y_0+2)}_{n,\,n+y_0} - p^{(n+y_0+2)}_{n,\,n+y_0+(n+y_0+2)} \\ &= p^{(n+y_0+2)}_{n,\,n+y_0} - p^{(n+y_0+2)}_{n,\,2n+2y_0+2} \\ &= \sigma^{(n+y_0+2)}_{n,\,\lfloor (n+y_0)/2 \rfloor} - \sigma^{(n+y_0+2)}_{n,\,n+y_0+1}. \end{aligned}$$

By (18) we see that the second term must be zero and so $c^{(\infty)}_{n,y_0} = \sigma_{n,\lfloor (n+y_0)/2\rfloor}$. Finally, in the case $y_0 = 0$, we have $c^{(\infty)}_{n,0} = c^{(n)}_n = \sigma^{(n+2)}_{n,\lfloor n/2\rfloor} - \sigma^{(n+2)}_{n,n+1} = \binom{n}{\lfloor n/2\rfloor} - 0$, the central binomial coefficient as expected.

Example. Let $n = 4$ and $y_0 = 2$. To find the infinite corridor number, we have $c^{(\infty)}_{4,2} = c^{(6)}_{4,2} = p_{4,6} - p_{4,14} = \sigma^{(8)}_{4,3} - \sigma^{(8)}_{4,7}$, which may be computed using (18), but actually appear in Fig. 5. Indeed, we have $c^{(\infty)}_{4,2} = \sigma_{4,3} - \sigma_{4,7} = 14 - 0 = 14$.

3.3. Three-choice Corridors

When the allowable moves in a corridor include remaining at the same level, the paths are often called "Motzkin." The structure we have already set up extends easily to such Motzkin paths. Fix $d \geq 2$. We define an (up-sampled) circular Pascal-type array $(\mathbf{p}_n)_{n\in\mathbb{N}}$ still using $\mathbf{p}_0 = \mathbf{e}'_0 + R(\mathbf{e}'_0)$ defined above, but now using $T = I + R + R^2$ to transition from $\mathbf{p}_n$ to $\mathbf{p}_{n+1}$. Define the difference array $(\mathbf{q}_n)_{n\in\mathbb{N}}$ as above, $\mathbf{q}_n = D(\mathbf{p}_n)$ where $D = I - L$.

Remark 14. If we had begun with $\mathbf{p}_0 = \mathbf{e}'_0$ instead, then the resulting array whose n^{th} row is $\mathbf{p}_n = T^n\mathbf{p}_0$ is a periodization of what has been called the *trinomial triangle* [10].

Now in the corridor $\mathbb{N}\times\{1, 2, \ldots, d-1\}$ let c'_n be the number of paths of length n (for simplicity, beginning at $(0, 1)$), but now allowing three choices in movement, up-and-right, down-and-right and right. With $\mathbf{v}_n$ defined in the periodic dual corridor, exactly as above, we have $\mathbf{v}_{n+1} = (L + I + R)\mathbf{v}_n$. With no extra effort, one can conclude that maximum and minimum values on $\mathbf{p}_n$ occur on the diagonals $p_{n,n}$ and $p_{n,n+d}$ respectively, and that the difference is our new corridor number c'_n. To this point, the proofs are exactly the same as above; there is just no precursor array of period d. Now let us use this fact to find these corridor numbers within Pascal's triangle itself, *i.e.* to obtain a binomial coefficient based formula for c'_n.

The key is to understand the effect of T^n on $\mathbf{e}'_0 = (\ldots, 0, \mathbf{1}, 0, \ldots, 1, 0, \ldots)$, the $2d$-periodic "unit vector", then to use the linearity of T^n. Using the commutativity of R and $I+R$, we may apply the binomial theorem to write

$$T^n = [I + R(I + R)]^n = \sum_{j=0}^{n}\binom{n}{j}\sum_{\ell=0}^{j}\binom{j}{\ell}R^{j+\ell}. \tag{19}$$

Theorem 15. *Let $d \geq 2$. Then $p_{n,k} = \sum_{j=0}^{n}\binom{n}{j}\sum_{m\in\mathbb{Z}}\binom{j+1}{2dm-j+k}$.*

Proof. Let us represent $T^n(\mathbf{e}'_0)$ and $T^n(R\mathbf{e}'_0)$ by $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ respectively; of course these sequences must also be $2d$-periodic. The nonzero contributions of $\mathbf{e}'_0$ to a_k come from the 1's in $\mathbf{e}'_0$ occurring at the integer multiples of $2d$ that are shifted by R^{j+l} to position k (mod $2d$), *i.e.* if $j + \ell = 2dm + k$ for $m \in \mathbb{Z}$. Thus, using (19),

$$a_k = \sum_{j=0}^{n}\binom{n}{j}\sum_{m\in\mathbb{Z}}\binom{j}{2dm - j + k}, \quad \text{and} \quad b_k = \sum_{j=0}^{n}\binom{n}{j}\sum_{m\in\mathbb{Z}}\binom{j}{2dm - j + k - 1}.$$

Since $p_{n,k} = T^n\mathbf{e}'_0 + T^nR\mathbf{e}'_0 = a_k + b_k$ we immediately obtain the result. □

As a consequence of Theorem 15, we find

$$c'_n = p_{n,n} - p_{n,n+d} = \sum_{j=0}^{n}\binom{n}{j}\sum_{m\in\mathbb{Z}}\left[\binom{j+1}{2dm - j + n} - \binom{j+1}{2dm - j + n + d}\right].$$

Other counts associated with the three-way corridor paths may also be analyzed along these lines.

4. Conclusion

Early on in this project we used MAPLE to animate plots of a damped version of $\boldsymbol{\sigma}_n$, for $n = 0, 1, \ldots N$, to observe a "wave" determined by the maximum and minimum values moving along the successive rows of the Pascal arrays. Upon the up-sampling, the extreme values fell nicely on diagonals of the circular Pascal arrays. Finally, with the introduction of the dual corridor, we found the strong connection between the two structures. Using the most basic properties of a few simple operators, we arrived at our main result, which easily led to a few nontrivial lattice path results.

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