

# Generalising Tuenter's binomial sums

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### Abstract

Tuenter [*Fibonacci Quarterly* 40 (2002), 175-180] and other authors have considered centred binomial sums of the form

$$S_r(n) = \sum_k \binom{2n}{k} |n - k|^r,$$

where  $r$  and  $n$  are non-negative integers. We consider sums of the form

$$U_r(n) = \sum_k \binom{n}{k} |n/2 - k|^r$$

which are a generalisation of Tuenter's sums as  $S_r(n) = U_r(2n)$  but  $U_r(n)$  is also well-defined for odd arguments  $n$ .  $U_r(n)$  may be interpreted as a moment of a symmetric Bernoulli random walk with  $n$  steps. The form of  $U_r(n)$  depends on the parities of both  $r$  and  $n$ . In fact,  $U_r(n)$  is the product of a polynomial (depending on the parities of  $r$  and  $n$ ) times a power of two or a binomial coefficient. In all cases the polynomials can be expressed in terms of Dumont-Foata polynomials. We give recurrence relations, generating functions and explicit formulas for the functions  $U_r(n)$  and/or the associated polynomials.

*Keywords:* Bernoulli random walks, binomial sum identities, Catalan numbers, Dumont-Foata polynomials, explicit formulas, generating functions, Genocchi numbers, moments, polynomial interpolation, secant numbers, tangent numbers

*MSC classes:* 05A10, 11B65 (Primary); 05A15, 05A19, 44A60, 60G50 (Secondary)

# 1 Introduction

We consider centred binomial sums of the form

$$U_r(n) = \sum_k \binom{n}{k} \left| \frac{n}{2} - k \right|^r, \quad (1)$$

where  $r \in \mathbb{N}$  and  $n \in \mathbb{Z}$ . These generalise the binomial sums

$$S_r(n) = \sum_k \binom{2n}{k} |n - k|^r \quad (2)$$

previously considered by Tuentler [12] and other authors [1, 4, 5, 9, 10], since  $S_r(n) = U_r(2n)$  but  $U_r(n)$  is well-defined for both even and odd values of  $n$ . The generalisation arises naturally in the study of certain two-fold centred binomial sums [3] of the form  $\sum_j \sum_k \binom{2n}{n+j} \binom{2n}{n+k} P(j, k)$ .

In definitions such as (1) and (2) we always interpret  $0^0$  as 1. Thus  $U_0(n) = 2^n$  and  $S_0(n) = 2^{2n}$  for all  $n \in \mathbb{N}$ . By our summation convention (see §1.1), we have  $U_r(n) = S_r(n) = 0$  if  $n < 0$ . Thus, in the following we assume that  $n \geq 0$ .

For  $r > 0$  we can avoid the absolute value function in (1) by writing

$$U_r(n) = 2 \sum_{k < n/2} \binom{n}{k} \left( \frac{n}{2} - k \right)^r.$$

Tuentler [12] showed in a direct manner that, for  $r \geq 0$  and  $n > 0$ ,  $S_r(n)$  satisfies the recurrence

$$S_{r+2}(n) = n^2 S_r(n) - 2n(2n - 1) S_r(n - 1). \quad (3)$$

Observe that this recurrence splits into two separate recurrences, one involving odd values of  $r$  and the other involving even values of  $r$ . Also,  $S_0(n) = 2^{2n}$  and  $S_1(n) = n \binom{2n}{n}$  (see for example [4, 10]). It follows from (3) that

$$S_{2r}(n) = Q_r(n) 2^{2n-r}, \quad S_{2r+1}(n) = P_r(n) n \binom{2n}{n}, \quad (4)$$

where  $P_r(n)$  and  $Q_r(n)$  are polynomials of degree  $r$  with integer coefficients, satisfying the recurrences

$$P_{r+1}(n) = n^2 P_r(n) - n(n - 1) P_r(n - 1), \quad (5)$$

$$Q_{r+1}(n) = 2n^2Q_r(n) - n(2n - 1)Q_r(n - 1) \quad (6)$$

for  $r \geq 0$ , with initial conditions  $P_0(n) = Q_0(n) = 1$ . The polynomials  $P_r, Q_r$  for  $0 \leq r \leq 5$  are given in Appendix 1.

The Dumont-Foata polynomials  $F_r(x, y, z)$  are 3-variable polynomials satisfying the recurrence relation

$$F_{r+1}(x, y, z) = (x + z)(y + z)F_r(x, y, z + 1) - z^2F_r(x, y, z) \quad (7)$$

for  $r \geq 1$ , with  $F_1(x, y, z) = 1$ . Dumont and Foata [8] gave a combinatorial interpretation for the coefficients of  $F_r(x, y, z)$  and showed that  $F_r(x, y, z)$  is symmetric in the three variables  $x, y, z$ .

Tuenter [12] showed that  $P_r(n)$  and  $Q_r(n)$  may be expressed in terms of Dumont-Foata polynomials. In fact, for  $r \geq 1$ ,  $P_r(n) = (-1)^{r-1}nF_r(1, 1, -n)$  and  $Q_r(n) = (-2)^{r-1}nF_r(\frac{1}{2}, 1, -n)$ . Thus, we can obtain explicit formulas and generating functions for the polynomials  $P_r(n)$  and  $Q_r(n)$  as special cases of the results of Carlitz [6] on Dumont-Foata polynomials.

We can obtain explicit formulas for  $S_{2r}(n)$  and  $S_{2r-1}(n)$  by using Carlitz's results for Dumont-Foata polynomials – see Theorem 3 in §2<sup>1</sup>. We note that these formulas are different from the explicit formulas (36)–(37) of Guo and Zeng [9], which are discussed in Remark 3.

We show that all the above results for  $S_r(n)$  can be generalised to cover  $U_r(n)$ . In particular, Theorem 1 shows that  $U_r(n)$  satisfies a recurrence (8) similar to the recurrence (3) satisfied by  $S_r(n)$ . Theorem 2 shows that  $U_r(n)$  is the product of a polynomial in  $n$  times a power of two or a binomial coefficient, depending on the parity of  $r$ , as in (4). These polynomials can be expressed in terms of Dumont-Foata polynomials, so the results of Carlitz allow us to obtain explicit formulas for  $U_r(n)$  such as those given in Theorem 3, and to obtain new exponential generating functions (*egfs*) such as (39) and (43)–(45) in §4. We give some additional explicit formulas in §3, and consider the asymptotic behaviour of  $U_r(n)$  as  $n \rightarrow \infty$  in §5.

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<sup>1</sup>In fact, our Theorem 3 is more general, since it covers  $U_r(n)$  for both even and odd  $n$ .

## 1.1 Notation

The set of non-negative integers is denoted by  $\mathbb{N}$ , and the set of positive integers by  $\mathbb{N}^*$ .

For  $k \in \mathbb{N}$  and  $x \in \mathbb{R}$  we denote the *Pochhammer symbol* or *rising factorial* by

$$(x)_k := x(x+1) \cdots (x+k-1),$$

with the special case  $(x)_0 = 1$ . The falling factorial may be written as  $(x+1-k)_k$  or  $(-)^k(-x)_k$ , where we use  $(-)^k$  as an abbreviation for  $(-1)^k$ .

The binomial coefficient  $\binom{n}{k}$  is defined<sup>2</sup> for all  $k \in \mathbb{Z}$  and  $n \in \mathbb{N}$  by

$$\binom{n}{k} := \begin{cases} 0 & \text{if } k < 0 \text{ or } k > n; \\ \frac{n!}{(n-k)!k!} & \text{otherwise.} \end{cases}$$

Thus we can often write sums over all  $k \in \mathbb{Z}$  without explicitly giving upper and lower limits on  $k$ .

## 2 Main results

Our main results on  $U_r(n)$  are summarised in the following Theorems 1–3. The recurrence (8) in Theorem 1 implies the recurrence (3) satisfied by  $S_r(n)$ , since (3) follows on replacing  $n$  by  $2n$  in (8).

**Theorem 1.** *For all  $r, n \in \mathbb{N}$ ,  $U_r(n)$  satisfies the recurrence*

$$4U_{r+2}(n) = n^2U_r(n) - 4n(n-1)U_r(n-2), \quad (8)$$

*and may be computed from the recurrence using the initial conditions*

$$U_0(n) = 2^n, \quad U_1(2n) = n \binom{2n}{n}, \quad U_1(2n+1) = (2n+1) \binom{2n}{n} \quad \text{for all } n \in \mathbb{N}.$$

*Proof.* We have

$$4U_{r+2}(n) = \sum_k 4 \binom{n}{k} \left| \frac{n}{2} - k \right|^{r+2} = \sum_k \binom{n}{k} \left| \frac{n}{2} - k \right|^r (n-2k)^2, \quad (9)$$

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<sup>2</sup>Guo and Zeng [9] implicitly define the binomial coefficient to be zero if  $-k \in \mathbb{N}^*$ ,  $n \in \mathbb{Z}$ , and  $n(n-1) \cdots (n-k+1)/k!$  if  $k \in \mathbb{N}$ ,  $n \in \mathbb{Z}$ . We do not use this definition because it is incompatible with our convention of summing over all  $k \in \mathbb{Z}$ .

$$n^2 U_r(n) = \sum_k \binom{n}{k} \left| \frac{n}{2} - k \right|^r n^2, \quad (10)$$

and

$$\begin{aligned} 4n(n-1)U_r(n-2) &= \sum_k 4n(n-1) \binom{n-2}{k} \left| \frac{n-2}{2} - k \right|^r \\ &= \sum_k 4n(n-1) \binom{n-2}{k-1} \left| \frac{n}{2} - k \right|^r \\ &= \sum_k 4k(n-k) \binom{n}{k} \left| \frac{n}{2} - k \right|^r. \end{aligned} \quad (11)$$

Since  $(n-2k)^2 - n^2 - 4k(n-k) = 0$ , the recurrence (8) follows from (9)–(11).

For the initial values, we easily verify that  $U_0(n) = 2^n$ . Also, the “official” solution [10] to the Putnam problem 35-A4 gives

$$\begin{aligned} U_1(n) &= \sum_k \binom{n}{k} \left| \frac{n}{2} - k \right| = \sum_{k < n/2} \binom{n}{k} (n - 2k) \\ &= \sum_{k < n/2} \left\{ \binom{n}{k} (n - k) - \binom{n}{k} k \right\} \\ &= \sum_{k < n/2} \left\{ \binom{n-1}{k} n - \binom{n-1}{k-1} n \right\} \\ &= n \sum_{k < n/2} \left\{ \binom{n-1}{k} - \binom{n-1}{k-1} \right\} \\ &= n \binom{n-1}{\lfloor n/2 \rfloor}. \end{aligned}$$

Thus,

$$U_1(2n) = 2n \binom{2n-1}{n} = n \binom{2n}{n}$$

and

$$U_1(2n+1) = (2n+1) \binom{2n}{\lfloor (2n+1)/2 \rfloor} = (2n+1) \binom{2n}{n}.$$

□

Theorem 2 shows that  $U_r(n)$  can be expressed as the product of a polynomial in  $n$  multiplied by a simple non-polynomial function of  $r$  and  $n$ . There are four cases, depending on the parities of  $r$  and  $n$ , although only three of the cases are essentially different.

**Theorem 2.** *For  $r \in \mathbb{N}$  there exist polynomials  $P_r(n), \overline{P}_r(n), Q_r(n), \overline{Q}_r(n)$  of degree  $r$  over  $\mathbb{Z}$ , such that, for all  $n \in \mathbb{N}^*$ ,*

$$U_{2r+1}(2n) = nP_r(n) \binom{2n}{n}, \quad (12)$$

$$U_{2r+1}(2n-1) = 2^{-(2r+1)} n \overline{P}_r(n) \binom{2n}{n}, \quad (13)$$

$$U_{2r}(2n) = 2^{2n-r} Q_r(n), \quad (14)$$

$$U_{2r}(2n+1) = 2^{2n+1-2r} \overline{Q}_r(n). \quad (15)$$

The polynomials satisfy the following recurrence relations:

$$P_{r+1}(n) = n^2 P_r(n) - n(n-1) P_r(n-1), \quad (16)$$

$$\overline{P}_{r+1}(n) = (2n-1)^2 \overline{P}_r(n) - 4(n-1)^2 \overline{P}_r(n-1), \quad (17)$$

$$Q_{r+1}(n) = 2n^2 Q_r(n) - n(2n-1) Q_r(n-1), \quad (18)$$

$$\overline{Q}_{r+1}(n) = (2n+1)^2 \overline{Q}_r(n) - 2n(2n+1) \overline{Q}_r(n-1), \quad (19)$$

with initial conditions  $P_0(n) = \overline{P}_0(n) = Q_0(n) = \overline{Q}_0(n) = 1$ .

*Proof.* For  $r \in \mathbb{N}$  and  $n \in \mathbb{N}^*$  we define functions  $P_r(n), \overline{P}_r(n), Q_r(n), \overline{Q}_r(n)$  by (12)–(15) respectively. Using the second half of Theorem 1, it is easy to see that  $P_0(n) = \overline{P}_0(n) = Q_0(n) = \overline{Q}_0(n) = 1$ . Thus, it only remains to show that  $P_r(n), \overline{P}_r(n), Q_r(n)$  and  $\overline{Q}_r(n)$  satisfy the claimed recurrences (16)–(19), since these recurrences enable us to show by induction on  $r$  that the functions  $P_r(n), \overline{P}_r(n), Q_r(n)$  and  $\overline{Q}_r(n)$  are polynomials over  $\mathbb{Z}$ .

First consider the recurrence (16) for  $P_r(n)$ . Replacing  $n$  by  $2n$  and  $r$  by  $2r+1$  in the recurrence (8), we obtain

$$U_{2r+3}(2n) = n^2 U_{2r+1}(2n) - 2n(2n-1) U_{2r+1}(2n-2),$$

and in view of (12) this implies

$$nP_{r+1}(n) \binom{2n}{n} = n^2 P_r(n) \binom{2n}{n} - 2n(2n-1)(n-1) P_r(n-1) \binom{2n-2}{n-1}. \quad (20)$$

Now, dividing each side of (20) by  $n\binom{2n}{n}$  and using  $\binom{2n-2}{n-1} = \frac{n}{2(2n-1)}\binom{2n}{n}$ , we obtain (16). The other three cases are similar.  $\square$

Lemma 1 expresses the four families of polynomials  $P_r(n), \dots, \overline{Q}_r(n)$  in terms of Dumont-Foata polynomials, and incidentally shows that only three of the four cases are essentially different, since  $\overline{Q}_r(n)$  is just a shifted and scaled version of  $Q_r(n)$ .

**Lemma 1.** *For  $r \in \mathbb{N}^*$ , the polynomials of Theorem 2 can be expressed in terms of Dumont-Foata polynomials, as follows:*

$$P_r(n) = (-)^{r-1} n F_r(-n, 1, 1), \quad (21)$$

$$\overline{P}_r(n) = (-4)^r F_{r+1}\left(\frac{1}{2} - n, \frac{1}{2}, \frac{1}{2}\right), \quad (22)$$

$$Q_r(n) = (-2)^{r-1} n F_r\left(-n, \frac{1}{2}, 1\right), \quad (23)$$

$$\overline{Q}_r(n) = (-)^{r-1} 2^{2r-1} \left(n + \frac{1}{2}\right) F_r\left(-n - \frac{1}{2}, \frac{1}{2}, 1\right) = 2^r Q_r\left(n + \frac{1}{2}\right). \quad (24)$$

*Proof.* Since  $F_r(x, y, z)$  is undefined for  $r \leq 0$ , we assume that  $r \in \mathbb{N}^*$ . From the defining recurrence (7) it is easy to verify that the function  $f_r(n) := (-)^{r-1} n F_r(-n, 1, 1)$  satisfies the recurrence (16) that is satisfied by  $P_r(n)$ . Also,  $f_1(n) = n F_1(-n, 1, 1) = n$ , so  $f_1(n) = P_1(n)$ . The recurrence (16) uniquely defines the polynomial  $P_r(n)$ , so  $P_r(n) = f_r(n)$  and (21) holds. The other three cases are similar.  $\square$

Using Lemma 1, we obtain  $U_r(n)$  in terms of Dumont-Foata polynomials. There are three cases, depending on the parities of  $r$  and  $n$ : (even, any), (odd, even) and (odd, odd).

**Corollary 1.**

$$U_{2r}(n) = 2^{n-2} n (-)^{r-1} F_r\left(-\frac{n}{2}, \frac{1}{2}, 1\right), \quad (25)$$

$$U_{2r+1}(2n) = n^2 (-)^{r-1} F_r(-n, 1, 1) \binom{2n}{n}, \quad (26)$$

$$U_{2r+1}(2n-1) = \frac{1}{2} n (-)^r F_{r+1}\left(\frac{1}{2} - n, \frac{1}{2}, \frac{1}{2}\right) \binom{2n}{n}. \quad (27)$$

*The expressions (25)–(26) are valid for  $r \geq 1$ , and (27) is valid for  $r \geq 0$ .*

*Proof.* This is immediate from (12)–(15) of Theorem 2 and (21)–(24) of Lemma 1.  $\square$

Proposition 1, due to Carlitz [6], gives an explicit formula for the Dumont-Foata polynomials.

**Proposition 1.** *For  $r \in \mathbb{N}^*$  we have*

$$F_r(x, y, z) = 2(-)^{r-1} \sum_{0 \leq j \leq k < r} (-)^j \frac{(x+z)_k (y+z)_k (z+j)^{2r-1}}{j!(k-j)!(2z+j)_{k+1}}, \quad (28)$$

*provided that the Pochhammer symbol in the denominator does not vanish, which is equivalent to saying that  $(2z)_{2r-1} \neq 0$ . In particular, (28) is valid for all positive  $z$ .*

*Proof.* Modulo a small change of notation, this is Carlitz [6, Thm. 1]<sup>3</sup>. □

Theorem 3 gives explicit formulas for  $U_r(n)$ . The definition (1) can be used to evaluate  $U_r(n)$  for small  $n$ , but this is infeasible if  $n$  is large, as there are  $n+1$  terms in the defining sum. Hence it is preferable to use the appropriate explicit formula of Theorem 3 when  $r$  is small but  $n$  is large<sup>4</sup>.

**Theorem 3.** *For  $r, n \in \mathbb{N}^*$  we have the following explicit formulas:*

$$U_{2r}(n) = 2^{n+1} \sum_{1 \leq j \leq k \leq r} (-)^j \frac{\left(-\frac{n}{2}\right)_k \left(\frac{1}{2}\right)_k}{(k-j)!(k+j)!} j^{2r}, \quad (29)$$

$$U_{2r+1}(2n) = 2n \binom{2n}{n} \sum_{1 \leq j \leq k \leq r} (-)^j \frac{(-n)_k}{(k-j)!(k+1)_j} j^{2r}, \quad (30)$$

$$U_{2r-1}(2n-1) = \binom{2n}{n} \sum_{1 \leq j \leq k \leq r} (-)^j \frac{(-n)_k}{(k-j)!(k)_j} \left(j - \frac{1}{2}\right)^{2r-1}. \quad (31)$$

*Proof.* We can substitute the identity (28) into each of (25)–(27) to obtain explicit formulas for the different cases of  $U_r(n)$ . For example, (25) gives (29). Similarly, (26) gives (30), and (27) gives (31) on replacing  $r$  by  $r-1$ . □

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<sup>3</sup>Carlitz does not state the condition  $(2z)_{2r-1} \neq 0$ . In fact, if we evaluate the RHS of (28) symbolically and cancel any factors of the form  $2z+j$  that occur in both the numerator and the denominator, then the result is always valid.

<sup>4</sup>The recurrence relations (8) or (16)–(19) may also be used in this case.

**Remark 1.** Replacing  $F_r(-\frac{n}{2}, \frac{1}{2}, 1)$  by  $F_r(-\frac{n}{2}, 1, \frac{1}{2})$  before applying (28) gives

$$U_{2r}(n) = 2^n n \sum_{1 \leq j \leq k \leq r} (-)^{j-1} \frac{\left(\frac{1-n}{2}\right)_{k-1} \left(\frac{1}{2}\right)_k}{(k+j-1)!(k-j)!} \left(j - \frac{1}{2}\right)^{2r-1}. \quad (32)$$

We note that formulas (29) and (32) are different.

**Remark 2.** Other explicit expressions for  $U_r(n)$  may be obtained by applying the above process after interchanging  $x$  and  $z$  in (28), but these expressions require the condition  $n \geq r$  to avoid division by zero due to the vanishing of a Pochhammer symbol in the denominator. Hence, we omit the details.

### 3 Explicit formulas via Lagrange interpolation

We can obtain explicit formulas via polynomial interpolation whenever the function being interpolated is a polynomial of known degree. For example, from (14)–(15) and (24) we have  $U_{2r}(n) = 2^{n-r} Q_r(n/2)$ , where  $Q_r(x)$  is a polynomial of degree  $r$ . Thus, we can obtain an explicit formula for  $Q_r(x)$ , and hence for  $U_{2r}(n)$ , by evaluating  $Q_r(x)$  at  $r + 1$  distinct points  $x_k$  and using Lagrange's polynomial interpolation formula

$$Q_r(x) = \sum_{k=0}^r Q_r(x_k) L_k(x),$$

where

$$L_k(x) = \prod_{\substack{0 \leq j \leq r \\ j \neq k}} \frac{x - x_j}{x_k - x_j}.$$

For example, taking  $x_k = k$  for  $0 \leq k \leq r$  gives the explicit formula

$$U_{2r}(n) = (-)^r 2^{n+1} \sum_{1 \leq j \leq k \leq r} \frac{\left(-\frac{n}{2}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{n}{2} - r\right)_{r-k}}{(k+j)!(k-j)!(r-k)!} j^{2r}, \quad (33)$$

which is valid for  $n \in \mathbb{N}^*$ , and may be compared with (29) and (32).

In the same way, by interpolating  $P_r(n)$  at  $n = 0, 1, \dots, r$ , using  $P_r(0) = 0$  for  $r \geq 1$ , we obtain the explicit formula

$$U_{2r+1}(2n) = 2n \binom{2n}{n} (-)^r \sum_{1 \leq j \leq k \leq r} \frac{(-n)_k (n-r)_{r-k}}{(r-k)!(k-j)!(k)_{j+1}} j^{2r+1}, \quad (34)$$

which is valid for  $r \in \mathbb{N}^*$  and  $n \in \mathbb{N}$ , and may be compared with (30).

Similarly, by interpolating  $n\overline{P}_{r-1}(n)$  at  $n = 0, 1, \dots, r$ , we obtain

$$U_{2r-1}(2n-1) = (-)^r \binom{2n}{n} \sum_{0 \leq j \leq k \leq r} \frac{(1-n)_k (n-1-r)_{r-k}}{(r-k)!(k-j)!(k+2)_j} \left(j + \frac{1}{2}\right)^{2r-1} \quad (35)$$

for  $r \geq 1$  and  $n \geq 1$ . This may be compared with (31).

We can obtain an infinite number of explicit formulas by evaluating the relevant polynomials at different sets of  $r+1$  distinct points. The examples given above seem the most natural.

**Remark 3.** Recently, Guo and Zeng [9] considered  $S_r(n)$ . They obtained explicit formulas which may be written, for  $r \geq 1$ , as

$$S_{2r}(n) = \sum_{0 \leq j \leq k < r} 2^{2n-2k-1} (-)^{k-j} \binom{2n}{j} \frac{(2n-2k)_{k-j}}{(k-j)!} (n-j)^{2r-1} \quad (36)$$

and

$$S_{2r-1}(n) = \sum_{0 \leq j \leq k \leq \min(r-1, n)} (-)^{k-j} \binom{2n-2k}{n-k} \binom{2n}{j} \frac{(2n-2k)_{k-j}}{(k-j)!} (n-j)^{2r-1}. \quad (37)$$

Assuming that  $n \geq r$ , the explicit formulas (36)–(37) may be obtained by interpolation of the polynomials  $Q_r(n)/n$  and  $P_{r-1}(n)$  respectively, using the  $r$  points  $n, n-1, \dots, n-r+1$ . Similar remarks apply to the results of Chen and Chu [7, Thm. 1 – Cor. 4].

## 4 Generating functions

Theorem 4 gives an egf (39) for  $U_{2r}(n)$  and any fixed  $n \in \mathbb{N}$ . It generalises the egf

$$\sum_{r \geq 0} S_{2r}(n) \frac{z^{2r}}{(2r)!} = 2^{2n} \cosh^{2n}(z/2) \quad (38)$$

given by Tuentler [12, §5], since replacing  $n$  by  $2n$  in (39) gives (38). The proof is straightforward, and does not require the results of Carlitz.

**Theorem 4.** For  $n \in \mathbb{N}$  we have the exponential generating function

$$\sum_{r \geq 0} U_{2r}(n) \frac{z^{2r}}{(2r)!} = 2^n \cosh^n(z/2). \quad (39)$$

*Proof.* From the definition (1) with  $r$  replaced by  $2r$  (so the absolute value signs can be omitted), we have

$$\begin{aligned}
\sum_{r \geq 0} U_{2r}(n) \frac{z^{2r}}{(2r)!} &= \sum_{r \geq 0} \sum_k \binom{n}{k} \left(\frac{n}{2} - k\right)^{2r} \frac{z^{2r}}{(2r)!} \\
&= \sum_k \binom{n}{k} \sum_{r \geq 0} \left(\frac{n}{2} - k\right)^{2r} \frac{z^{2r}}{(2r)!} \\
&= \sum_k \binom{n}{k} \cosh\left(\left(\frac{n}{2} - k\right)z\right) \\
&= \frac{1}{2} \sum_k \binom{n}{k} (e^{(n/2-k)z} + e^{(k-n/2)z}) \\
&= \sum_k \binom{n}{k} e^{(k-n/2)z} \\
&= e^{-nz/2} (1 + e^z)^n \\
&= (e^{-z/2} + e^{z/2})^n = 2^n \cosh^n(z/2),
\end{aligned}$$

as required. The series converges absolutely for all  $z$ .  $\square$

We can obtain other egfs from the results of Carlitz. First, we note that Carlitz [6, eqn. (4.2)] gives the egf

$$\sum_{r \geq 1} (-)^r F_r(x, y, 1) \frac{z^{2r}}{(2r)!} = \frac{1}{xy} \sum_{k \geq 1} (-)^k \frac{(x)_k (y)_k}{(2k)!} \left(2 \sinh \frac{z}{2}\right)^{2k}. \quad (40)$$

In view of (25) and (26), this allows us to obtain egfs for  $U_{2r}(n)$  and  $U_{2r+1}(2n)$ . From (25) and (40) we obtain

$$\sum_{r \geq 0} U_{2r}(n) \frac{z^{2r}}{(2r)!} = 2^n \sum_{k \geq 0} (-)^k \left(\frac{1}{2}\right)_k \left(-\frac{n}{2}\right)_k \frac{(2 \sinh(z/2))^{2k}}{(2k)!}. \quad (41)$$

Comparing this with (39), we must have

$$\cosh^n(z) = \sum_{k \geq 0} (-)^k \left(\frac{1}{2}\right)_k \left(-\frac{n}{2}\right)_k \frac{(2 \sinh(z))^{2k}}{(2k)!}. \quad (42)$$

Indeed, if we define  $s := \sinh^2 z$ , so  $1 + s = \cosh^2 z$ , then the left side of (42) is  $(1 + s)^{n/2}$ , and the right side is a disguised form of the binomial expansion  $(1 + s)^{n/2} = 1 + (n/2)s + (n/2)(n/2 - 1)s^2/2! + \dots$ . Thus, (25) and (40) give nothing new; they merely confirm (39).

To obtain something new, we consider (26) and (40). Proceeding as above, we obtain the interesting egf:

$$\sum_{r \geq 0} U_{2r+1}(2n) \frac{z^{2r}}{(2r)!} = n \binom{2n}{n} \sum_{k=0}^n 2^{2k} \binom{n}{k} \binom{2k}{k}^{-1} \sinh^{2k} \left( \frac{z}{2} \right). \quad (43)$$

Observe that, in order to calculate  $U_{2r+1}(2n)$  from (43), it is only necessary to sum the terms on the right-hand side for  $k \leq \min(r, n)$ .

The final case  $U_{2r+1}(2n - 1)$  is more difficult because (40) does not apply to (27), as the last argument of  $F_{r+1}(x, y, z)$  in (27) is  $\frac{1}{2}$ , not 1. However, we can use the egf

$$\begin{aligned} & \sum_{r \geq 0} (-)^r F_{r+1}(x, y, z) \frac{u^{2r+1}}{(2r+1)!} \\ &= 2 \sum_{k=0}^{\infty} \sum_{j=0}^k \frac{(-)^j (x+z)_k (y+z)_k (2z)_j}{j! (k-j)! (2z)_{k+1} (2z+k+1)_j} \sinh((z+j)u), \end{aligned} \quad (44)$$

which follows<sup>5</sup> from the discussion in Carlitz [6, pp. 221–222]. Using (27) and (44), after some simplification followed by a change of variables ( $u \mapsto z$ ), we obtain the following egf:

$$\begin{aligned} & \sum_{r \geq 0} U_{2r+1}(2n-1) \frac{z^{2r+1}}{(2r+1)!} \\ &= n \binom{2n}{n} \sum_{0 \leq j \leq k < n} \frac{(-)^{k-j} \binom{n-1}{k} \binom{2k}{k-j}}{\binom{2k}{k}} \frac{\sinh \left( \left( j + \frac{1}{2} \right) z \right)}{j+k+1}, \end{aligned} \quad (45)$$

which is valid for  $n \in \mathbb{N}^*$ .

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<sup>5</sup>Carlitz does not give (44) explicitly. He gives a generating function [6, eqn. (4.6)] involving the hypergeometric function. However, there is a problem with convergence of the Maclaurin series involved, because the hypergeometric function occurs with arguments  $e^u$  and  $e^{-u}$ , one of which lies outside the unit circle. Thus, Carlitz's generating function is only valid (if at all) in the context of formal power series. We prefer to use (44), for which there is no problem with convergence.

## 5 Asymptotics

Using Stirling's formula to approximate the binomial coefficient in (1), we see that  $U_r(n)$  can be regarded as a Riemann sum approximating a suitably scaled integral of the form

$$\int_{-\infty}^{\infty} e^{-x^2} |x|^r dx.$$

This gives an asymptotic approximation to  $U_r(n)$  as  $n \rightarrow \infty$  with  $r$  fixed. More precisely, if the notation  $O_r(\dots)$  means that the implied constant depends on  $r$ , we have

$$U_r(n) = \pi^{-1/2} 2^n \left(\frac{n}{2}\right)^{r/2} \Gamma\left(\frac{r+1}{2}\right) \left(1 + O_r\left(\frac{1}{n}\right)\right) \quad (46)$$

as  $n \rightarrow \infty$ . The same asymptotic approximation can be obtained by ignoring all but the leading coefficient in the polynomials  $P_r(n), \dots, \overline{Q}_r(n)$ . (The leading coefficients are given in Table 1 of Appendix 2.) For example,

$$P_r(n) = r!n^r + O_r(n^{r-1}),$$

so (12) and the approximation

$$\binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

give

$$U_{2r+1}(2n) = \pi^{-1/2} 2^{2n} n^{r+1/2} r! \left(1 + O_r\left(\frac{1}{n}\right)\right),$$

which is a special case of (46). Other special cases of (46) can be obtained from (13)–(15).

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## Appendix 1: Small cases of the polynomials $P, Q, \overline{P}, \overline{Q}$

$$\begin{aligned}
P_0(n) &= 1, \\
P_1(n) &= n, \\
P_2(n) &= n(2n - 1), \\
P_3(n) &= n(6n^2 - 8n + 3), \\
P_4(n) &= n(24n^3 - 60n^2 + 54n - 17), \\
P_5(n) &= n(120n^4 - 480n^3 + 762n^2 - 556n + 155), \\
Q_0(n) &= 1, \\
Q_1(n) &= n, \\
Q_2(n) &= n(3n - 1), \\
Q_3(n) &= n(15n^2 - 15n + 4), \\
Q_4(n) &= n(105n^3 - 210n^2 + 147n - 34), \\
Q_5(n) &= n(945n^4 - 3150n^3 + 4095n^2 - 2370n + 496), \\
\overline{P}_0(n) &= 1, \\
\overline{P}_1(n) &= 4n - 3, \\
\overline{P}_2(n) &= 32n^2 - 56n + 25, \\
\overline{P}_3(n) &= 384n^3 - 1184n^2 + 1228n - 427, \\
\overline{P}_4(n) &= 6144n^4 - 29184n^3 + 52416n^2 - 41840n + 12465, \\
\overline{P}_5(n) &= 122880n^5 - 829440n^4 + \\
&\quad 2258688n^3 - 3076288n^2 + 2079892n - 555731, \\
\overline{Q}_0(n) &= 1, \\
\overline{Q}_1(n) &= 2n + 1, \\
\overline{Q}_2(n) &= 12n^2 + 8n + 1, \\
\overline{Q}_3(n) &= 120n^3 + 60n^2 + 2n + 1, \\
\overline{Q}_4(n) &= 1680n^4 - 168n^2 + 128n + 1, \\
\overline{Q}_5(n) &= 30240n^5 - 25200n^4 + 5040n^3 + 7320n^2 - 2638n + 1.
\end{aligned}$$

The triangles of coefficients of  $-P_r(-n)/n$ ,  $-Q_r(-n)/n$  and  $\overline{Q}_r(-n)$  for  $r \geq 1$  are OEIS [11] sequences A036970, A083061 and A160485 respectively. We have contributed the coefficients of  $\overline{P}_r(n)$  as sequence A245244. The values  $(-)^r \overline{P}_r(0)$  are sequence A009843 (see Appendix 2 for details).

The bijection (24) between A083061 and A160485 (by a shift of  $\pm\frac{1}{2}$  and scaling by a power of 2) was not mentioned in the relevant OEIS entries as at July 14, 2014; we have now contributed comments to this effect.

## Appendix 2: Special values of the polynomials $P, Q, \bar{P}, \bar{Q}$

Let  $\mathcal{P}_r$  denote any of the polynomials  $P_r, Q_r, \bar{P}_r, \bar{Q}_r$ . In Table 1 we give the special values  $\mathcal{P}_r(0)$ ,  $\mathcal{P}_r(1)$ , and  $\mathcal{P}_r(\infty)$ , where the latter denotes the leading coefficient of  $\mathcal{P}_r$ .

$\mathcal{P}_r$	$P_r$	$Q_r$	$\bar{P}_r$	$\bar{Q}_r$
$\mathcal{P}_r(0)$	$\delta_{0,r}$	$\delta_{0,r}$	$(-)^r(2r+1)\mathcal{S}_r$	1
$\mathcal{P}_r(1)$	1	$\max(1, 2^{r-1})$	1	$(3^{2r} + 3)/4$
$\mathcal{P}_r(\infty)$	$r!$	$(2r)!/(2^r r!)$	$2^{2r} r!$	$(2r)!/r!$

Table 1: Special values of the polynomials

In Table 1,  $\delta_{i,j}$  is 1 if  $i = j$  and 0 otherwise;  $\mathcal{S}_r$  is the  $r$ -th Secant number [2], defined by

$$\sec x = \frac{1}{\cos x} = \sum_{r \geq 0} \mathcal{S}_r \frac{x^{2r}}{(2r)!}.$$

The values  $(-)^r \bar{P}_r(0)$  are OEIS sequence A009843, and are given by the egf

$$\sum_{r=0}^{\infty} \bar{P}_r(0) \frac{x^{2r+1}}{(2r+1)!} = \frac{x}{\cosh x}. \quad (47)$$

They may be expressed in terms of the Secant numbers  $\mathcal{S}_r$ , which comprise OEIS sequence A000364. In view of (22), we obtain a special value of the Dumont-Foata polynomials:

$$F_{r+1} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) = 2^{-2r} (2r+1) \mathcal{S}_r. \quad (48)$$

The values  $\bar{Q}_r(1)$  comprise OEIS sequence A054879. The values in the last row of Table 1 may also be found in OEIS: they are sequences A000142, A001147, A047053, and A001813.

Tuenter [12] observed that, for  $r \geq 1$ , the constant terms of  $-P_r(n)/n$  are the Genocchi numbers (A001469), and the constant terms of  $(-)^{r-1} Q_r(n)/n$  are the reduced tangent numbers (A002105).