APSIMON'S MINT PROBLEM WITH THREE OR MORE WEIGHINGS

RICHARD J. MATHAR

Abstract. ApSimon considered the problem of deciding by a process of two weighings on which of a known number of mints emit either coins of a known genuine weight or emit coins of a different secondary but unknown weight. The combinatorial problem consists of finding two sets of coin numbers to be loaded on the tray for each of the weighings, and then to minimize the total count of coins to be drawn from all mints for these two weighings.

This work yields numerical results for the generalized problem which allows three or more weighings to settle which of the mints produce either sort of coins.

1. Definitions

1.1. **Statement of the Problem.** Consider a set of M mints issuing a coin with a known nominal weight G. There are two suppliers for the coin material, each supplier producing the material for a fixed subset of the mints. Unfortunately one of the suppliers uses faulty material, so for some of the mints all of their coins weigh $G(1+\epsilon)$ characterized by some unknown nonzero excess ϵ. An investigator is equipped with an absolute scale, an allowance to draw any number C_m of coins from the mints numbered by m, $1 \le m \le M$, and ordered to find out which of the mints emit which of the two types of coins by weighing two times a subset of these coins.

1.2. **Algebra of the Search Space.** The investigator's art of solving this problem is in finding a vector of coin numbers $C_{1,m} \ge 0$ of the first weighing and another vector of coin numbers $C_{2,m} \ge 0$ of the second weighing such that for any outcome of two measured weights a unique correspondence exists to one of the 2^M variants of nominal and faulty coins of the mints.

ApSimon states the problem [2]: what is the minimum number of coins

$$C(W,M) = \sum_{r=m}^{M} C_m \tag{1}$$

involved in the $W = 2$ weighings that allows to put either a label $d_m = 0$ on mint m if it produces the correct coins or a label $d_m = 1$ on mint m if it produces the faulty coins?

To rephrase, consider the first weight measured, $\sum_{m=1}^{M} GC_{1,m}(1+d_m\epsilon)$, and the second weight measured, $\sum_{m=1}^{M} GC_{2,m}(1+d_m\epsilon)$ [4]. The investigator may subtract the known masses of the nominal coins, $\sum_m GC_{1,m}$ and $\sum_m GC_{2,m}$, to reduce the

Date: July 15, 2014.

2010 *Mathematics Subject Classification.* Primary 05A17, 90C27; Secondary 11P70, 49K35.

Key words and phrases. Combinatorics; Coin Problems; Weighing.

two measurements to their excess weights $\sum_{m=1}^{M} GC_{1,m}d_m\epsilon$ and $\sum_{m=1}^{M} GC_{2,m}d_m\epsilon$. The unknown excess ϵ can be eliminated by considering the measured known ratio of these two reduced weights because ϵ drops out,

$$X_1 \equiv \frac{\sum_{m=1}^{M} C_{2,m}d_m}{\sum_{m=1}^{M} C_{1,m}d_m}. \tag{2}$$

The problem is solved if two vectors $C_{1,m}$ and $C_{2,m}$ are found such that all these ratios differ for the 2^M different binary vectors d_m.

Remark 1. *This could also be rephrased as mixing the coin numbers such that no two of these excess vectors defined by plotting the points of the first weighing and second weighing in a two-dimensional coordinate system are collinear* [1, 7].

The coins of any mint may be re-used for the second weighing. If the total number $C = \sum_{m=1}^{M} C_m$ of the coins is sought to be minimal, it would be wasteful not to use the full set C_m of a mint with at least one of the two weighings. So the set of coin numbers to be searched for an optimum is evidently reduced to $0 \le C_{w,m} \le C_m$ for $w = 1, 2$. Another obvious constraint is that from each mint m at least one coin is to be put at the scale for at least one of the weighings—otherwise no information of that d_m would enter the weights. So the cases $C_{1,m} = C_{2,m} = 0$ do not need to be considered.

2. Known Solutions for Two Weighings

Guy and Nowakowski found upper bounds of $C(2, 6) \le 38$ for $M = 6$ mints and $C(2, 7) \le 74$ coins for $M = 7$ mints [4]. Li improved these upper bounds for two weighings to 31 coins for 6 mints and 63 coins for 7 mints [8]. Applegate settled the best value to 28 coins for 6 mints, 51 for 7 mints and 90 coins for 8 mints [9, A007673].

Example 1. *For $M = 6$ mints the full information on the d_m is extracted by loading $C_{1,m} = (0, 1, 2, 1, 8, 10)$ coins on the tray for the first weighing and $C_{2,m} = (1, 2, 2, 5, 5, 0)$ coins for the second. This needs $C(2, 6) = \sum_m C_m = \sum_m \max(C_{1,m}, C_{2,m}) = 1 + 2 + 2 + 5 + 8 + 10 = 28$ coins from all six mints for both weighings. These two vectors of $C_{w,r}$ are not unique, because one could as well combine $C_{1,m} = (0, 2, 1, 1, 8, 10)$ and $C_{2,m} = (1, 2, 2, 5, 5, 0)$ with the same total of $C(2, 6) = 28$ coins. There are two further solutions by just permuting the first and second weighing, and there are further solutions by permuting the enumeration of the M mints, but the two solutions shown above are the only two fundamentally different choices for the minimum of 28. Even these two solutions are degenerate because permutation of the subset of the $C_{\cdot,m}$ within a subset of constant C_m (here $C_2 = C_3 = 2$) does not cover different states of the d_m.*

3. More than Two Weighings

3.1. **Excess Weight Ratios.** Naturally the total number of coins needed becomes smaller if the investigator may use a larger number W of weighings, each with its own set $C_{w,m}$ of coins, $1 \le w \le W$. There is no new methodology to the analysis but to require that the sets of $(W - 1)$ potentially measured excess ratios

$$X_w \equiv \frac{\sum_{m=1}^{M} C_{w+1,m}d_m}{\sum_{m=1}^{M} C_{w,m}d_m}, \quad 1 \le w < W, \tag{3}$$

are 2^M different vectors of rational numbers as a function of the binary state vectors d_m [3, 6, 5].

Remark 2. *Other definitions of the ratios may serve the same purpose. One might for example use a constant reference value for w in all the denominators, or invert all ratios.*

Still 0-vectors of the form $C_{1,m} = C_{2,m} = \ldots = C_{w,m} = 0$ do not need to be considered because such an input cannot reveal information on d_m. As for the case of two weighings, minimization of the total number of coins requires

- to use the full number C_m in at least one weighing to avoid waste,

$$C_m = \max(C_{1,m}, C_{2,m}, \ldots, C_{w,m}), \tag{4}$$

- to use at least one coin of each mint in at least one weighing,

$$C_m \geq 1, \tag{5}$$

- and to search for the minimum sum of coins purchased from all the mints,

$$C(W, M) = \min_{\{C_{w,m}\}} C = \min_{\{C_{w,m}\}} \sum_{m=1}^{M} C_m. \tag{6}$$

For the purpose of testing whether the ratios (3) differ for different sets of d_m, two different types of numbers are assigned to the X_w if the denominator is zero: If the numerator is positive, $X_w = \infty$ as usual; if the numerator is also zero, a different quantity $X_w = 0/0$ is placed. Two different symbols for this case obviously helps to reduce the number of coins needed, because a larger variation of the components in the vectors X_w helps to cover the d_m-space.

To illustrate this managing of zeros, consider the solution

$$C_{w,m} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix} \tag{7}$$

to the problem with $W = 3$ weighings and $M = 4$ mints. Table 1 shows the 2^M different states, their W excess weights and ratios. For this distribution of coins in the three weighings the states of $d_m = (1, 0, 0, 0)$ and $d_m = (1, 0, 0, 1)$ can be distinguished, because the vector of the ratios is $(0, 0/0)$ in the former case and differs from the vector of ratios $(0, \infty)$ in the latter case. If $0/0$ and ∞ were considered the *same* ratio, (7) would not be flagged as a solution to the problem.

3.2. **Mints Equal to Weighings.** If the weighing number W equals the number M of mints, one could use a single coin from a different mint m for each of the weighings and find individually one d_m per weighing. Therefore a diagonal C-matrix with column maximum 1,

$$C_{w,m} = \delta_{w,m}, \quad C_m = 1 \tag{8}$$

suffices if $W \geq M$, and

$$C(W, M) = M, \quad W \geq M \tag{9}$$

is an upper bound and also the optimum.

d_1	d_2	d_3	d_4	$\sum_m C_{1,m}d_m$	$\sum_m C_{2,m}d_m$	$\sum_m C_{3,m}d_m$	X_1	X_2
0	0	0	0	0	0	0	0/0	0/0
0	0	0	1	0	0	2	0/0	∞
0	0	1	0	0	1	1	∞	1
0	0	1	1	0	1	3	∞	3
0	1	0	0	1	1	0	1	0
0	1	0	1	1	1	2	1	2
0	1	1	0	1	2	1	2	1/2
0	1	1	1	1	2	3	2	3/2
1	0	0	0	1	0	0	0	0/0
1	0	0	1	1	0	2	0	∞
1	0	1	0	1	1	1	1	1
1	0	1	1	1	1	3	1	3
1	1	0	0	2	1	0	1/2	0
1	1	0	1	2	1	2	1/2	2
1	1	1	0	2	2	1	1	1/2
1	1	1	1	2	2	3	1	3/2

TABLE 1. Decision table for 4 mints and 3 weighings with their reduced weights, assuming coin counts specified by (7).

3.3. **Simple Bounds.** It is obvious that the number of coins needed is monotonous in both variables:

- $$C(W, M) \ge C(W + 1, M) \tag{10}$$

  because increasing the number of weighings does not require to increase the number of coins to find the d_m. This is demonstrated by weighing two times with the same assembly of coins, i.e., by duplicating a row in the matrix $C_{w,m}$ of coins.

- $$C(W, M) \le C(W, M + 1), \tag{11}$$

  because increasing the number of mints requires no less coins to find the d_m. This is proven by considering some minimizing solution $C_{w,m}$ with ratios X_w for $M+1$ mints, chopping off the component d_{M+1} of the binary vector and removing the associated ratios X_w related to $d_{M+1} = 1$, and observing that in the reduced decision table all remaining X_w vectors are still pairwise different. (Example: delete all rows where $d_4 = 1$ and then the column d_4 in Table 1, which ends up in a decision table for 3 mints.)

3.4. **Solutions.** A list of one example of a matrix $C_{w,m}$ for the numerical solutions that are found by exhaustive search follows. They have been computed with a dedicated JAVA program reproduced in the `anc` directory.

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \therefore \quad C(3,4) = 1 + 1 + 1 + 2 = 5. \tag{12}$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \therefore C(3,5) = 1+1+1+2+2 = 7. \tag{13}$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \therefore C(3,6) = 1+1+1+2+2+4 = 11. \tag{14}$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 3 & 0 & 2 & 4 \\ 1 & 1 & 1 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 3 & 3 & 4 \end{pmatrix} \tag{15}$$
$$\therefore C(3,7) = 1+1+1+3+3+3+4 = 16.$$

$$C_{w,m} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 1 & 2 & 4 & 1 & 2 & 5 \\ 0 & 1 & 1 & 2 & 2 & 5 & 5 & 0 \end{pmatrix} \tag{16}$$
$$\therefore C(3,8) = 1+1+1+2+4+5+5+5 = 24.$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \therefore C(4,5) = 1+1+1+1+1 = 5. \tag{17}$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \therefore C(4,6) = 1+1+1+1+1+2 = 7. \tag{18}$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 3 \end{pmatrix} \tag{19}$$
$$\therefore C(4,7) = 1+1+1+1+1+2+3 = 10.$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 0 \end{pmatrix} \tag{20}$$
$$\therefore C(4,8) = 1+1+1+1+2+2+2+2 = 12.$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{21}$$
$$\therefore C(5,6) = 1+1+1+1+1+1 = 6.$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{22}$$

$$\therefore C(5,7) = 1+1+1+1+1+1+1 = 7.$$

$$C_{w,m} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{23}$$

$$\therefore C(5,8) = 1+1+1+1+1+1+1+1+2 = 9.$$

$$C_{w,m} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \tag{24}$$

$$\therefore C(5,9) = 1+1+1+1+1+1+1+2+3 = 12.$$

$$C_{w,m} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 \end{pmatrix} \tag{25}$$

$$\therefore C(5,10) = 1+1+1+1+1+1+2+2+2+2 = 14.$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{26}$$

$$\therefore C(6,9) = 1+1+1+1+1+1+1+1+1 = 9.$$

$$C_{w,m} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \tag{27}$$

$$\therefore C(7,10) = 1+1+1+1+1+1+1+1+1+1 = 10.$$

W\ M	1	2	3	4	5	6	7	8	9	10
2	1	2	4	8	15	28	51	90		
3			3	5	7	11	16	24	≤ 37	≤ 59
4				4	5	7	10	12	$> 15, \leq 18$	≤ 28
5					5	6	7	9	12	14
6						6	7	8	9	
7							7	8	9	10

TABLE 2. The minimum number of coins $C(W, M)$ as a function of weighings W and number of mints M.

4. Summary

Table 2 shows the array $C(W, M)$ of the minimum number of individual coins required with W weighings for M mints by collecting result from equations (12)-(27). Entries below the diagonal are constant down the columns according to (9), and are not shown. Entries with upper or lower bounds indicate that the space of the $C_{w,m}$-matrices has not been scanned in full.

References

1. Noga Alon, Dmitry N. Kozlov, and Van H. Vu, *The geometry of coin-weighing problems*, Symp. on Found. Comput. Science, vol. 37, IEEE, 1996, pp. 524–532. MR 1450650
2. Hugh ApSimon, *Mathematical byways in ayling, beeling and ceiling*, Oxford University Press, Oxford, 1991. MR 0771115 (86c:00008)
3. Nader H. Bshouty, *Optimal algorithms for the coin weighing problem with a spring scale*, COLT 2009 (Montreal, Quebec), Conference on Learning Theory, 18–21 June 2009, p. #4.
4. Richard K. Guy and Richard J. Nowakowsky, *ApSimon's mints problem*, Amer. Math. Monthly **101** (1994), no. 4, 358–359. MR 1542518
5. _____, *Coin-weighing problems*, Amer. Math. Monthly **102** (1995), no. 2, 164–167.
6. Michael Hendy, *The retrial of the lower slobbovian counterfeiters*, Amer. Math. Monthly **87** (1980), no. 3, 200–201. MR 1539313
7. Tanya Khovanova, *Attacking ApSimon's mints*, arXiv:1406.3012 [math.HO] (2014).
8. Xue-Wu Li, *A new algorithm for ApSimon's mints problem*, J. Tianjin Normal University **23** (2003), no. 2, 39–42.
9. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://oeis.org/. MR 1992789 (2004f:11151)

E-mail address: mathar@mpia.de
URL: http://www.mpia.de/~mathar

MAX-PLANCK INSTITUTE OF ASTRONOMY, KÖNIGSTUHL 17, 69117 HEIDELBERG, GERMANY