

Armstrong's Conjecture for $(k, mk + 1)$ -Core Partitions

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Abstract

A conjecture of Armstrong states that if $\gcd(a, b) = 1$, then the average size of an (a, b) -core partition is $(a-1)(b-1)(a+b+1)/24$. Recently, Stanley and Zanello used a recursive argument to verify this conjecture when $a = b - 1$. In this paper we use a variant of their method to establish Armstrong's conjecture in the more general setting where a divides $b - 1$.

1 Introduction

A *partition* is a finite, nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of positive integers. The sum $\sum_{i=1}^r \lambda_i$ is the *size* of λ and is denoted by $|\lambda|$. We may represent λ by a *Young diagram*, which is a collection of r left-justified rows of cells with λ_i cells in row i . The *hook length* of any cell C in the Young diagram is defined to be the number of cells to the right of, below, or equal to C . For instance, Figure 1 shows the Young diagram and hook lengths of the partition $(5, 3, 1, 1)$.

For any positive integers a and b , a partition is called an (a, b) -*core* if no cell in its Young diagram has hook length equal to a or b ; for instance, Figure 1 shows that $(5, 3, 1, 1)$ is a $(3, 7)$ -core. Simultaneous core partitions have been the topic of many articles during the past decade (see [3, 4, 5, 7, 8, 9, 10, 11, 14, 15]). They are particularly interesting when $\gcd(a, b) = 1$. In this case, there are only finitely many (a, b) -cores; in fact, a theorem of Anderson states that there are $\binom{a+b}{a}/(a+b)$ such cores [4].

The proof of Anderson's theorem is through a bijective correspondence between (a, b) -cores and order ideals of the poset $P_{a,b}$, whose elements are all positive integers not contained in the numerical semigroup generated by $\{a, b\}$ and whose partial order is fixed by requiring $p \in P_{a,b}$ to cover $q \in P_{a,b}$ if $p - q$ is either a or b (throughout the article, we will follow the poset terminology given in Chapter 3 of Stanley's text [12, 13]). Specifically, this correspondence sends an (a, b) -core partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ to the order ideal $I_\lambda = \{\lambda_1 + r - 1, \lambda_2 + r - 2, \dots, \lambda_r\} \in J(P_{a,b})$, where $J(P)$ denotes the set of order ideals of any poset P ; observe that I_λ consists of the hook lengths in the leftmost column of the Young diagram of λ . From this bijection, we deduce the identity

$$|\lambda| = \sigma(I_\lambda) - \binom{|I_\lambda|}{2}, \quad (1.1)$$

where $\sigma(I_\lambda) = \sum_{i \in I_\lambda} i$ is the sum of the elements in I_λ .

To see an example of this bijection, let $(a, b) = (3, 7)$; then, $P_{3,7} = \{1, 2, 4, 5, 8, 11\}$. In this poset, 11 covers 4 and 8; 8 covers 1 and 5; 5 covers 2; and 4 covers 1. The $(3, 7)$ -core $(5, 3, 1, 1)$

8	5	4	2	1
5	2	1		
2				
1				

Figure 1: The Young diagram of $(5, 3, 1, 1)$ is shown above; each cell contains its hook length.

corresponds to the order ideal $\{8, 5, 2, 1\} \subset P_{3,7}$; equation (1.1) may be verified since $(5, 3, 1, 1)$ has size 10 and $\sigma(\{8, 5, 2, 1\}) = 16$.

In 2011, Armstrong informally proposed the following conjecture that predicts the average size of an (a, b) -core; this conjecture was later published in [5].

Conjecture 1.1. *If $\gcd(a, b) = 1$, then*

$$\sum_{\lambda} |\lambda| = \frac{(a-1)(b-1)(a+b+1)}{24(a+b)} \binom{a+b}{a}, \quad (1.2)$$

where λ is summed over all (a, b) -cores. Equivalently, the average size of an (a, b) -core is $(a-1)(b-1)(a+b+1)/24$.

In addition to having an intrinsic appeal, a proof of Conjecture 1.1 would yield implications about numerical semigroups generated by two elements. Yet, despite the ostensible simplicity of (1.2), it remains unproven. However, there have recently been several partial results towards Armstrong's conjecture. In 2013, Stanley and Zanello used a recursive method to prove Conjecture 1.1 when $a = b - 1$ [14]. In response to another conjecture in [4], Chen, Huang, and Wang later established an analog of Armstrong's conjecture for self-conjugate core partitions using the Ford-Mai-Sze bijection [7].

In this paper we use a variant of the recursive method given by Stanley and Zanello to verify a more general case of Conjecture 1.1. In particular, we prove the theorem below.

Theorem 1.2. *For any integers $k, m \geq 1$, Armstrong's conjecture holds for $(k, mk + 1)$ -cores. Specifically,*

$$\sum_{\lambda} |\lambda| = \frac{mk(k-1)((m+1)k+2)}{24(mk+1)} \binom{(m+1)k}{k},$$

where λ is summed over all $(k, mk + 1)$ -cores.

As Stanley and Zanello did in the case $m = 1$, we will prove the theorem above by using Anderson's bijection and the manageable behavior of the poset $P_{k, mk+1}$. Applying (1.1), we see that Theorem 1.2 is equivalent to the theorem below.

Theorem 1.3. *For any integers $m, k \geq 1$,*

$$\sum_{I \in J(P_{k, mk+1})} \left(\sigma(I) - \binom{|I|}{2} \right) = \frac{mk(k-1)((m+1)k+2)}{24(mk+1)} \binom{(m+1)k}{k}. \quad (1.3)$$

We will prove this theorem in Section 3.

2 Proof of Theorem 1.3 When $m = 2$

The proof of Theorem 1.3 is quite computational, so we will first verify Theorem 1.3 when $m = 2$. In particular, we will establish the following result.

Theorem 2.1. *For any integer $k \geq 1$,*

$$\sum_{I \in J(P_{k,2k+1})} \left(\sigma(I) - \binom{|I|}{2} \right) = \frac{k(k-1)(3k+2)}{12(2k+1)} \binom{3k}{k}. \quad (2.1)$$

Let us begin by introducing some notation. For each integer n , let $P_n = P_{n,2n+1}$. Let Q_n be the poset obtained by removing the minimal elements of P_{n+1} ; equivalently, $Q_n = P_{n+1} \setminus \{1, 2, \dots, n\}$. Figure 2 depicts the Hasse diagrams of P_4 and Q_3 . Let A_n denote the number of order ideals in P_n and let B_n denote the number of order ideals in Q_n . By a theorem of Bizley (see [6]),

$$A_n = \binom{3n+1}{n} / (3n+1). \quad (2.2)$$

Furthermore, one may check that there is a poset isomorphism $Q_n \simeq P_{n+1,2n+1}$ under the map sending $q \in Q_n$ to $(2n+2)\lfloor q/(n+1) \rfloor - q \in P_{n+1,2n+1}$; applying the theorem of Bizley again yields that $B_n = \binom{3n+2}{n+1} / (3n+2)$.

Define the generating functions $A(x) = \sum_{k=0}^{\infty} A_k x^k$ and $B(x) = \sum_{k=0}^{\infty} B_k x^k$, where x is a formal variable. It is known (see [1, 2]) that these generating functions have explicit forms given by

$$A(x) = \frac{2}{\sqrt{3x}} \sin \left(\frac{\arcsin(\sqrt{27x/4})}{3} \right)$$

and

$$\begin{aligned} B(x) &= \frac{4}{3x} \sin^2 \left(\frac{\arcsin(\sqrt{27x/4})}{3} \right) \\ &= A(x)^2. \end{aligned} \quad (2.3)$$

From [1] (or from (2.3)), we have that

$$xA(x)^3 - A(x) + 1 = 0. \quad (2.4)$$

For each $p \in P_n$, let $\rho_{P_n}(p) = \lfloor p/n \rfloor$; for each $q \in Q_n$, let $\rho_{Q_n}(q) = \rho_{P_{n+1}}(q)$. For each $S \in \{P, Q\}$, define the sums

$$T_n^{(S)} = \sum_{I \in J(S_n)} |I|; \quad R_n^{(S)} = \sum_{I \in J(S_n)} \sum_{i \in I} \rho_{S_n}(i); \quad G_n^{(S)} = \sum_{I \in J(S_n)} \left(\sigma(I) - \binom{|I|}{2} \right).$$

Also define the generating functions

$$T_S(x) = \sum_{k=0}^{\infty} T_k^{(S)} x^k; \quad R_S(x) = \sum_{k=0}^{\infty} R_k^{(S)} x^k; \quad G_S(x) = \sum_{k=0}^{\infty} G_k^{(S)} x^k.$$

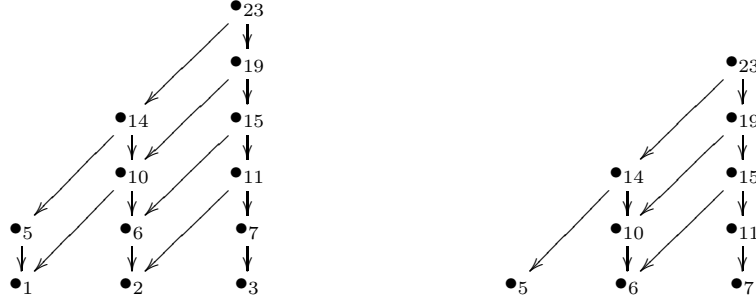


Figure 2: The Hasse diagrams of the posets P_4 and Q_3 are shown to the left and right, respectively.

The equality (2.1) is equivalent to

$$G_n^{(P)} = \frac{n(n-1)(3n+2)}{12(2n+1)} \binom{3n}{n}.$$

Therefore, by (2.2), it suffices to show that

$$G_P(x) = \frac{3x^3 A'''(x) + 8x^2 A''(x)}{12} \quad (2.5)$$

in order to establish Theorem 2.1.

In order to prove (2.5), we will derive several recursions that yield algebraic relations between the generating functions A , T , R , and G . These relations will allow us to solve for G as a rational function in x and A and thereby deduce the above equality.

To obtain these recursions, we partition the sets $J(P_n)$ and $J(Q_n)$ in a way similar to that done by Stanley and Zanello in [14]. For each integer $i \in [1, n]$, let $J_i(P_n) \subset J(P_n)$ denote the set of order ideals of P_n that contain $\{1, 2, \dots, i-1\}$ but not i . Similarly, for each integer $i \in [1, n+1]$, let $J_i(Q_n)$ denote the set of order ideals of Q_n that contain $\{n+2, n+3, \dots, n+i\}$ but not $n+i+1$. One may refer to Figure 3 for examples. On the left is the Hasse diagram of P_6 ; any ideal in $J_3(P_6)$ must contain the elements labelled by squares, must avoid the elements labelled by white circles, and may contain some of the elements labelled by black circles; the analogous figure for $J_4(Q_5)$ is shown on the right. We have the decompositions $J(P_n) = \bigcup_{i=1}^n J_i(P_n)$ and $J(Q_n) = \bigcup_{i=1}^{n+1} J_i(Q_n)$.

For any integer $i \in [1, n]$, let $P_n(i) \subset P_n$ denote the poset of elements in P_n that are greater than some integer in $[1, i-1]$ and incomparable to each integer in $[i, n-1]$, where the ordering is with respect to the poset P_n ; for instance, $P_6(3) = \{7, 8, 14, 20\}$, as seen from Figure 3. Also let $P_n(-i)$ denote the poset of elements $p \in P_n$ that are incomparable to each integer in $[1, i]$; for instance, $P_6(-3) = \{4, 5, 10, 11, 17, 23\}$. Observe that there is a poset isomorphism $Q_{i-1} \simeq P_n(i)$ under the map sending a $q \in Q_{i-1}$ to $q + (n-i)\rho_{Q_{i-1}}(q) \in P_n(i)$. Similarly, there is a poset isomorphism $P_{n-i} \simeq P_n(-i)$ that maps $p \in P_{n-i}$ to $p+i+i\rho_{P_{n-i}}(p) \in P_n(-i)$.

For any integer $i \in [1, n]$, let $Q_n(i) \subset Q_n$ denote the poset of elements in Q_n that are greater than some integer in $[n+2, n+i]$ and incomparable to each integer in $[n+i+1, 2n+1]$, where the ordering is with respect to the poset Q_n ; for instance, $Q_5(4) = \{14, 15, 20, 21, 27, 33\}$, as seen from Figure 3. Also let $Q_n(-i)$ denote the poset of elements in Q_n that are incomparable to each integer

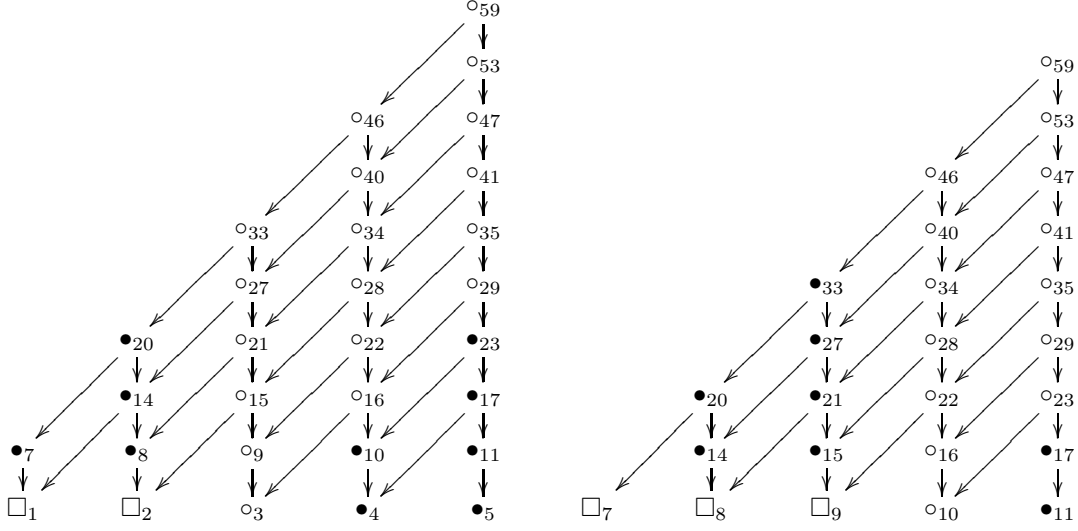


Figure 3: The Hasse diagram of P_6 is on the left. Any order ideal in $J_3(P_6)$ must avoid the elements labelled by white circles, must contain elements labelled by squares, and might contain some of the elements labelled by black circles. A similar diagram for $J_4(Q_5)$ is on the right.

in $[n+2, n+i+1]$; for instance, $Q_5(-4) = \{11, 17\}$. Observe that there is a poset isomorphism $P_{i-1} \simeq Q_n(i)$ under the map that sends $p \in P_{i-1}$ to $p+2n+3+(n+2-i)\rho_{P_{i-1}}(p) \in Q_n(i)$. Similarly, there is a poset isomorphism $P_{n-i+1} \simeq Q_n(-i)$ that maps $p \in P_{n-i+1}$ to $p+n+1+i\rho_{P_{n-i+1}}(p) \in Q_n(-i)$.

We will now deduce the following recursive identities.

Proposition 2.2. *For each integer $n \geq 0$,*

$$T_n^{(P)} = \sum_{i=0}^{n-1} (A_{n-i-1}T_i^{(Q)} + iB_iA_{n-i-1} + B_iT_{n-i-1}^{(P)}) \quad (2.6)$$

and

$$T_n^{(Q)} = \sum_{i=0}^n (A_{n-i}T_i^{(P)} + iA_iA_{n-i} + A_iT_{n-i}^{(P)}). \quad (2.7)$$

Proof. Let us first verify (2.6). Suppose that $i \in [1, n]$ is some integer; let $I \in J_i(P_n)$ be an order ideal. Then I can be partitioned as the disjoint union $\{1, 2, \dots, i-1\} \cup I_1 \cup I_2$, where $I_1 = I \cap P_n(i)$

and $I_2 = I \cap P_n(-i)$. Since $P_n(i) \simeq Q_{i-1}$ and $P_n(-i) \simeq P_{n-i}$, we have that

$$\begin{aligned} T_n^{(P)} &= \sum_{i=1}^n \sum_{I \in J_i(P_n)} |I| \\ &= \sum_{i=1}^n \sum_{I_1 \in J(Q_{i-1})} \sum_{I_2 \in J(P_{n-i})} (i-1 + |I_1| + |I_2|) \\ &= \sum_{i=1}^n (T_{i-1}^{(Q)} |J(P_{n-i})| + (i-1) |J(Q_{i-1})| |J(P_{n-i})| + T_{n-i}^{(P)} |J(Q_{i-1})|), \end{aligned}$$

which implies (2.6) since $|J(P_{n-i})| = A_{n-i}$ and $|J(Q_{i-1})| = B_{i-1}$.

The proof of (2.7) is analogous. Suppose that $i \in [1, n+1]$ is an integer and let $I \in J_i(Q_n)$ be an order ideal. Then I may be partitioned as the disjoint union $\{n+2, n+3, \dots, n+i\} \cup I_1 \cup I_2$, where $I_1 = I \cap Q_n(i)$ and $I_2 = I \cap Q_n(-i)$. Since $Q_n(i) \simeq P_{i-1}$ and $Q_n(-i) \simeq P_{n-i+1}$, we have that

$$\begin{aligned} T_n^{(Q)} &= \sum_{i=1}^{n+1} \sum_{I \in J_i(Q_n)} |I| \\ &= \sum_{i=1}^n \sum_{I_1 \in J(P_{i-1})} \sum_{I_2 \in J(P_{n-i+1})} (i-1 + |I_1| + |I_2|) \\ &= \sum_{i=1}^n (T_{i-1}^{(P)} |J(P_{n-i+1})| + (i-1) |J(P_{i-1})| |J(P_{n-i+1})| + T_{n-i+1}^{(P)} |J(P_{i-1})|), \end{aligned}$$

which implies (2.7). □

This yields a linear system of equations for the generating functions $T_P(x)$ and $T_Q(x)$ that can be solved explicitly.

Corollary 2.3. *We have that*

$$T_P(x) = \frac{3x^2 A'(x)^2}{A(x)} \tag{2.8}$$

and

$$T_Q(x) = 2A(x)T_P(x) + xA'(x)A(x). \tag{2.9}$$

Moreover,

$$T_Q'(x) = 2A'(x)T_P(x) + 2A(x)T_P'(x) + A'(x)A(x) + xA''(x)A(x) + xA'(x)^2. \tag{2.10}$$

Proof. The relation (2.9) follows from (2.7). Differentiating (2.9) yields (2.10). From (2.6), we deduce that

$$T_P(x) = xA(x)T_Q(x) + x^2 B'(x)A(x) + xB(x)T_P(x). \tag{2.11}$$

By (2.3), $B'(x) = 2A'(x)A(x)$; thus, inserting (2.9) into (2.11) yields

$$T_P(x) = \frac{3x^2 A'(x)A(x)^2}{1 - 3xA(x)^2}.$$

Applying (2.22) to the above equality yields (2.8). \square

We may use a similar method to evaluate $R_P(x)$ and $R_Q(x)$.

Proposition 2.4. *For each integer $n \geq 0$,*

$$R_n^{(P)} = \sum_{i=0}^{n-1} (A_{n-i-1}R_i^{(Q)} + B_i R_{n-i-1}^{(P)}) \quad (2.12)$$

and

$$R_n^{(Q)} = \sum_{i=0}^n (A_{n-i}(R_i^{(P)} + 2T_i^{(P)}) + iA_i A_{n-i} + A_i(R_{n-i}^{(P)} + T_{n-i}^{(P)})). \quad (2.13)$$

Proof. The proof is similar to that of Proposition 2.2. Let us verify (2.13) because the proof of (2.12) is similar. Suppose that $i \in [1, n+1]$ is an integer and let $I \in J_i(Q_n)$ be an order ideal. Then I may be partitioned as the disjoint union $\{n+2, n+3, \dots, n+i\} \cup I_1 \cup I_2$, where $I_1 = I \cap Q_n(i)$ and $I_2 = I \cap Q_n(-i)$. Then, since $Q_n(i) \simeq P_{i-1}$ and $Q_n(-i) \simeq P_{n-i+1}$, we have that

$$\begin{aligned} R_n^{(Q)} &= \sum_{i=1}^{n+1} \sum_{I \in J_i(Q_n)} \sum_{q \in I} \rho_{Q_n}(q) \\ &= \sum_{i=1}^{n+1} \sum_{I_1 \in J(P_{i-1})} \sum_{I_2 \in J(P_{n-i+1})} \left(i-1 + \sum_{p_1 \in I_1} (\rho_{P_{i-1}}(p_1) + 2) + \sum_{p_2 \in I_2} (\rho_{P_{n-i+1}}(p_2) + 1) \right) \\ &= \sum_{i=1}^{n+1} (A_{n-i+1}(R_{i-1}^{(P)} + 2T_{i-1}^{(P)}) + (i-1)A_{i-1}A_{n-i+1} + A_{i-1}(R_{n-i+1}^{(P)} + T_{n-i+1}^{(P)})), \end{aligned}$$

which implies (2.13). \square

Corollary 2.5. *We have that*

$$R_P(x) = \frac{3xA'(x)T_P(x) + x^2A'(x)^2}{A(x)} \quad (2.14)$$

and

$$R_Q(x) = 2A(x)R_P(x) + 3A(x)T_P(x) + xA'(x)A(x). \quad (2.15)$$

Proof. The relation (2.15) follows from (2.13). From (2.12), we deduce that

$$R_P(x) = xA(x)R_Q(x) + xB(x)R_P(x). \quad (2.16)$$

Inserting (2.15) into (2.16) and using (2.3) yields that

$$R_P(x) = \frac{3xA(x)^2T_P(x) + x^2A'(x)A(x)^2}{1 - 3xA(x)^2}.$$

Applying (2.22) to the above gives (2.14). \square

We will now express $G_P(x)$ in terms of $A(x)$, $T_P(x)$, and $R_P(x)$.

Proposition 2.6. *For each integer $n \geq 0$,*

$$\begin{aligned} G_n^{(P)} &= \sum_{i=0}^{n-1} \left(A_{n-i-1}(G_i^{(Q)} + (n-i-1)R_i^{(Q)} - iT_i^{(Q)}) \right. \\ &\quad \left. + B_i(G_{n-i-1}^{(P)} + (i+1)R_{n-i-1}^{(P)} + T_{n-i-1}^{(P)}) \right. \\ &\quad \left. + iB_iA_{n-i-1} - T_i^{(Q)}T_{n-i-1}^{(P)} \right) \end{aligned} \quad (2.17)$$

and

$$\begin{aligned} G_n^{(Q)} &= \sum_{i=0}^n \left(A_{n-i}(G_i^{(P)} + (n-i+1)R_i^{(P)} + (2n+3-i)T_i^{(P)}) \right. \\ &\quad \left. + A_i(G_{n-i}^{(P)} + (i+1)R_{n-i}^{(P)} + (n+2)T_{n-i}^{(P)}) \right. \\ &\quad \left. + (n+2)iA_iA_{n-i} - T_i^{(P)}T_{n-i}^{(P)} \right). \end{aligned} \quad (2.18)$$

Proof. Again the proof is similar to the proofs of Proposition 2.2 and Proposition 2.4. We will verify (2.18) since the proof of (2.17) is similar. Let $i \in [1, n+1]$ be an integer and $I \in J_i(Q_n)$ be an order ideal. Using the decomposition $I = \{n+2, n+3, \dots, n+i\} \cup I_1 \cup I_2$ as before (and the isomorphisms $Q_n(i) \simeq P_{i-1}$ and $Q_n(-i) \simeq P_{n-i+1}$), one may check that

$$\begin{aligned} \sum_{I \in J(Q_n)} \sigma(I) &= \sum_{i=1}^{n+1} \sum_{I_1 \in J(P_{i-1})} \sum_{I_2 \in J(P_{n-i+1})} \left(\sum_{j=n+2}^{n+i} j + \sum_{p_1 \in I_1} (p_1 + (n+2-i)\rho_{P_{i-1}}(p_1) + 2n+3) \right. \\ &\quad \left. + \sum_{p_2 \in I_2} (p_2 + i\rho_{P_{n-i+1}}(p_2) + n+i+1) \right). \end{aligned} \quad (2.19)$$

Furthermore,

$$\begin{aligned} \sum_{I \in J(Q_n)} \binom{|I|}{2} &= \sum_{i=1}^{n+1} \sum_{I_1 \in J(P_{i-1})} \sum_{I_2 \in J(P_{n-i+1})} \binom{|I_1| + |I_2| + i - 1}{2} \\ &= \sum_{i=1}^{n+1} \sum_{I_1 \in J(P_{i-1})} \sum_{I_2 \in J(P_{n-i+1})} \left(\binom{|I_1|}{2} + \binom{|I_2|}{2} + |I_1||I_2| + (i-1)|I_1| \right. \\ &\quad \left. + (i-1)|I_2| + \binom{i-1}{2} \right). \end{aligned} \quad (2.20)$$

Subtracting (2.20) from (2.19) yields

$$G_n^{(Q)} = \sum_{i=1}^{n+1} \sum_{I_1 \in J(P_{i-1})} \sum_{I_2 \in J(P_{n-i+1})} \left((i-1)(n+2) + \sum_{p_1 \in I_1} ((n+2-i)\rho_{P_{i-1}}(p_1) + 2n+4-i) + \sigma(I_1) \right. \\ \left. - \binom{|I_1|}{2} + \sum_{p_2 \in I_2} (i\rho_{P_{n-i+1}}(p_2) + n+2) + \sigma(I_2) - \binom{|I_2|}{2} - |I_1||I_2| \right),$$

which implies (2.18). \square

Corollary 2.7. *We have that*

$$G_P(x) = (3xA(x)^2R_P(x) + 6x^2A'(x)A(x)R_P(x) + 6xA(x)^2T_P(x) \\ + 3x^2A'(x)A(x)T_P(x) + 4x^2A'(x)A(x)^2 + x^3A'(x)^2A(x) \\ - 3xA(x)T_P(x)^2)(1 - 3xA(x)^2)^{-1}. \quad (2.21)$$

Proof. From (2.18) and (2.17), we deduce that

$$G_Q(x) = 2A(x)G_P(x) + 2A(x)R_P(x) + 2xA'(x)R_P(x) + 5A(x)T_P(x) + 3xA'(x)T_P(x) \\ + 2xA(x)T_P'(x) + 3xA'(x)A(x) + x^2A'(x)^2 + x^2A''(x)A(x) - T_P(x)^2$$

and

$$G_P(x) = xA(x)G_Q(x) + x^2A'(x)R_Q(x) - x^2A(x)T_Q'(x) + xB(x)G_P(x) + xB(x)R_P(x) \\ + x^2B'(x)R_P(x) + xB(x)T_P(x) + x^2B'(x)A(x) - xT_P(x)T_Q(x),$$

respectively. Inserting the first equality above into the second and using (2.3) gives

$$G_P(x) = 3xA(x)^2G_P(x) + 3xA(x)^2R_P(x) + 4x^2A'(x)A(x)R_P(x) + x^2A'(x)R_Q(x) \\ + 6xA(x)^2T_P(x) + 3x^2A'(x)A(x)T_P(x) + 2x^2A(x)^2T_P'(x) - x^2A(x)T_Q'(x) \\ + 5x^2A'(x)A(x)^2 + x^3A'(x)^2A(x) + x^3A''(x)A(x)^2 - xA(x)T_P(x)^2 - xT_P(x)T_Q(x).$$

Applying (2.9), (2.10), and (2.15) to the above yields (2.21). \square

We may now prove Theorem 2.1.

Proof of Theorem 2.1. As noted previously, it suffices to establish (2.5). The right side of this equality involves derivatives of $A(x)$. We can express these in terms of $A(x)$ using (2.4). Specifically, differentiating (2.4) yields

$$A'(x) = \frac{A(x)^3}{1 - 3xA(x)^2}. \quad (2.22)$$

Differentiating again gives

$$A''(x) = \frac{3A(x)^2(A'(x) + A(x)^3 - xA(x)^2A'(x))}{(1 - 3xA(x)^2)^2} \quad (2.23)$$

and repeating yields

$$A'''(x) = 3A(x)(3x^2A(x)^5A''(x) - 4xA(x)^3A''(x) + A(x)A''(x) - 6xA(x)^5A'(x) + 10A(x)^3A'(x) + 2xA(x)^2A'(x)^2 + 2A'(x)^2 + 6A(x)^6)(1 - 3xA(x)^2)^{-3}. \quad (2.24)$$

Now, in order to establish (2.5), apply (2.21), (2.22), (2.8), and (2.14) to express the left side as a rational function in x and $A(x)$. Applying (2.23), (2.24), and (2.22), we also express the right side as a rational function in x and $A(x)$. Simplifying, we obtain that the two sides are equal; we omit this computation here (but the proof of a more general identity may be found at the end of Section 3). \square

3 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 through a method similar to the one used when $m = 2$. We will suppose that $m > 1$, since the case $m = 1$ has been established by Stanley and Zanello [14]. Let us begin by defining several posets. For each nonnegative integer n , let $P_n = P_n^{(0)} = P_{n, mn+1}$. For each integer $j \in [1, m-1]$, let $P_n^{(j)}$ be the poset obtained from removing the elements $p \in P_{n+1}^{(0)}$ with $\lfloor p/(n+1) \rfloor < j$; equivalently, $P_n^{(j)} = P_{n+1} \setminus \bigcup_{h=0}^{j-1} \{h(n+1) + 1, h(n+1) + 2, \dots, h(n+1) + n\}$. If $m = 2$, then observe that $P_n^{(1)} = Q_n$ from the previous section. For each nonnegative integer n , let A_n denote the number of order ideals in P_n ; for each $j \in [0, m-1]$, let $A_n^{(j)}$ be the number of order ideals in $P_n^{(j)}$. Applying the theorem of Bizley (see [6]), we see that

$$A_n^{(0)} = \binom{mn + n + 1}{n} / (mn + n + 1). \quad (3.1)$$

Define the generating function $A^{(j)}(x) = \sum_{k=0}^{\infty} A_k^{(j)} x^k$, where x is a formal variable; let $A(x) = A^{(0)}(x)$. In order to obtain analogues of (2.3) and (2.4), we will apply a recursive method similar to the one used in the previous section.

For each integer $i \in [1, n]$, let $J_i(P_n) \subset J(P_n)$ be the set of order ideals of P_n that contain $\{1, 2, \dots, i-1\}$ but not i . For each integer $i \in [1, n+1]$ and $h \in [1, m-1]$, let $J_i(P_n^{(h)}) \subset J(P_n^{(h)})$ denote the set of order ideals of $P_n^{(h)}$ that contain $\{h(n+1) + 1, h(n+1) + 2, \dots, h(n+1) + i-1\}$ but not $h(n+1) + i$. When $m = 2$ and $h = 1$, we recover $J_i(Q_n)$ from the previous section. As in Section 2, we may partition $J(P_n) = \bigcup_{i=1}^n J_i(P_n)$ and $J(P_n^{(h)}) = \bigcup_{i=1}^{n+1} J_i(P_n^{(h)})$. We will use these decompositions to obtain the following result.

Proposition 3.1. *We have that $A^{(j)}(x) = A(x)^{m-j+1}$ for each integer $j \in [1, m-1]$. Moreover, $xA(x)^{m+1} - A(x) + 1 = 0$.*

Proof. To verify the first equality, it suffices to check that $A^{(h)}(x) = A(x)A^{(h+1)}(x)$ for each integer $h \in [1, m-1]$, where the index h is taken modulo m . Let $i \in [1, n+1]$ and $h \in [1, m-1]$ be integers and let $I \in J_i(P_n^{(h)})$ be an order ideal. As in the previous section, I can be partitioned as the disjoint union $\{h(n+1) + 1, h(n+1) + 2, \dots, h(n+1) + i-1\} \cup I_1 \cup I_2$, where I_1 consists of the elements of I greater than some $j \in \{h(n+1) + 1, h(n+1) + 2, \dots, h(n+1) + i-1\}$ and incomparable to each $j \in \{h(n+1) + i, h(n+1) + i + 2, \dots, h(n+1) + n\}$ (where the ordering is with respect to the poset $P_n^{(h)}$) and I_2 consists of the elements of I that are incomparable to each

$j \in \{h(n+1) + 1, h(n+1) + 2, \dots, h(n+1) + i\}$. Observe that I_1 is an order ideal in a poset isomorphic to $P_{i-1}^{(h+1)}$ and that I_2 is an order ideal in a poset isomorphic to P_{n-i+1} . Hence,

$$A_n^{(h)} = \sum_{i=1}^{n+1} \sum_{I \in J_i(P_n^{(h)})} 1 = \sum_{i=1}^{n+1} \sum_{I_1 \in J(P_{i-1}^{(h+1)})} \sum_{I_2 \in J(P_{n-i+1})} 1 = \sum_{i=0}^n A_i^{(h+1)} A_{n-i}.$$

This recursion yields the relation $A^{(h)}(x) = A(x)A^{(h+1)}(x)$ for all integers $h \in [1, m-1]$, thereby establishing the first statement of the proposition. The second statement of the proposition follows from the equality $A(x) = xA^{(1)}(x)A(x) + 1$, which can be verified through a similar recursive method. \square

For each integer $n \geq 0$ and each $p \in P_n$, let $\rho_{n,0}(p) = \lfloor p/n \rfloor$. For each integer $j \in [1, m-1]$ and element $q \in P_n^{(j)}$, let $\rho_{n,j}(q) = \rho_{n,0}(q)$. For each integer $j \in [0, m-1]$, define the sums

$$T_n^{(j)} = \sum_{I \in J(P_n^{(j)})} |I|; \quad R_n^{(j)} = \sum_{I \in J(P_n^{(j)})} \sum_{i \in I} \rho_{n,j}(i); \quad G_n^{(j)} = \sum_{I \in J(P_n^{(j)})} \left(\sigma(I) - \binom{|I|}{2} \right).$$

Also define the generating functions

$$T_j(x) = \sum_{k=0}^{\infty} T_k^{(j)} x^k; \quad R_j(x) = \sum_{k=0}^{\infty} R_k^{(j)} x^k; \quad G_j(x) = \sum_{k=0}^{\infty} G_k^{(j)} x^k.$$

Analogous to Theorem 2.1 (which is equivalent to (2.5)), Theorem 1.3 is equivalent to an algebraic identity involving $G_0(x)$ and derivatives of $A(x)$. Specifically, due to (3.1), it suffices to establish the equality

$$m(m+1)x^3 A'''(x) + m(2m+4)x^2 A''(x) - 24G_0(x) = 0 \quad (3.2)$$

in order to prove Theorem 1.3. As in Section 2, we will deduce (3.2) by expressing $T_j(x)$, $R_j(x)$, and $G_0(x)$ as rational functions in x and $A(x)$. Let us begin with $T_j(x)$.

Proposition 3.2. *For each integer $j \in [1, m-2]$,*

$$T_j(x) = A(x)T_{j+1}(x) + (m-j)x A'(x)A(x)^{m-j} + A(x)^{m-j} T_0(x). \quad (3.3)$$

Moreover,

$$T_0(x) = xA(x)T_1(x) + mx^2 A'(x)A(x)^m + xA(x)^m T_0(x) \quad (3.4)$$

and

$$T_{m-1}(x) = 2A(x)T_0(x) + xA'(x)A(x). \quad (3.5)$$

Proof. Following the proof of Proposition 2.2, one obtains that

$$T_n^{(j)} = \sum_{i=0}^n (T_i^{(j+1)} A_{n-i} + iA_i^{(j+1)} A_{n-i} + A_i^{(j+1)} T_{n-i}^{(0)}).$$

for each integer $j \in [1, m - 2]$;

$$T_n^{(0)} = \sum_{i=0}^{n-1} (T_i^{(1)} A_{n-i-1} + i A_i A_{n-i-1} + A_i^{(1)} T_{n-i-1}^{(0)});$$

and

$$T_n^{(m-1)} = \sum_{i=0}^n (T_i^{(0)} A_{n-i} + i A_i A_{n-i} + A_i T_{n-i}^{(0)}).$$

These recursive relations imply the proposition. \square

Corollary 3.3. For each integer $j \in [1, m - 1]$,

$$T_j(x) = (m + 1 - j)A(x)^{m-j}T_0(x) + \binom{m+1-j}{2}xA'(x)A(x)^{m-j} \quad (3.6)$$

and

$$T_0(x) = \frac{\binom{m+1}{2}x^2A'(x)^2}{A(x)}. \quad (3.7)$$

Moreover,

$$\sum_{j=1}^{m-1} A(x)^{j-1}T_j(x) = A(x)^{m-1} \left(\frac{(m^2 + m - 2)T_0(x)}{2} + \binom{m+1}{3}xA'(x) \right). \quad (3.8)$$

Proof. Using (3.3) and induction on $m - j$ (the base case $m - j = 1$ is given by (3.5)), we obtain (3.6). Multiplying (3.6) by $A(x)^{j-1}$ and summing over j yields (3.8). Inserting (3.6), with $j = 1$, into (3.4) gives (3.7). \square

Corollary 3.4. We have that

$$\begin{aligned} \sum_{j=1}^{m-1} A(x)^j T_j'(x) = A(x)^{m-1} & \left(\frac{(m^2 + m - 2)A(x)T_0'(x)}{2} + \frac{(m-1)m(m+1)A'(x)T_0(x)}{3} \right. \\ & + \binom{m+1}{3}A'(x)A(x) + \binom{m+1}{3}xA''(x)A(x) \\ & \left. + \frac{(m-1)m(m+1)(3m-2)xA'(x)^2}{24} \right). \end{aligned} \quad (3.9)$$

Proof. Differentiating (3.6) gives

$$\begin{aligned} T_j'(x) = (m + 1 - j)A(x)^{m-j}T_0'(x) + (m + 1 - j)(m - j)A'(x)A(x)^{m-j-1}T_0(x) \\ + \binom{m+1-j}{2}(A'(x)A(x)^{m-j} + xA''(x)A(x)^{m-j} + (m - j)xA'(x)^2A(x)^{m-j-1}) \end{aligned}$$

for each integer $j \in [1, m - 1]$. Multiplying this equality by $A(x)^j$ and summing over j yields (3.9). \square

Next, we will find $R_j(x)$.

Proposition 3.5. For each integer $j \in [1, m-2]$,

$$R_j(x) = A(x)R_{j+1}(x) + jA(x)^{m-j}T_0(x) + j(m-j)x A'(x)A(x)^{m-j} + A(x)^{m-j}R_0(x). \quad (3.10)$$

Moreover,

$$R_0(x) = xA(x)R_1(x) + xA(x)^m R_0(x) \quad (3.11)$$

and

$$R_{m-1}(x) = 2A(x)R_0(x) + (2m-1)A(x)T_0(x) + (m-1)x A'(x)A(x). \quad (3.12)$$

Proof. Following the proof of Proposition 2.4, one obtains that

$$R_n^{(j)} = \sum_{i=0}^n (A_{n-i}R_i^{(j+1)} + ijA_i^{(j+1)}A_{n-i} + A_i^{(j+1)}(R_{n-i}^{(0)} + jT_{n-i}^{(0)}))$$

for each integer $j \in [1, m-2]$;

$$R_n^{(0)} = \sum_{i=0}^{n-1} (A_{n-i-1}R_i^{(1)} + A_i^{(1)}R_{n-i-1}^{(0)});$$

and

$$R_n^{(m-1)} = \sum_{i=0}^{n-1} (A_{n-i}(R_i^{(0)} + mT_i^{(0)}) + i(m-1)A_iA_{n-i} + A_i(R_{n-i}^{(0)} + (m-1)T_{n-i}^{(0)})).$$

These recursive relations imply the proposition. \square

Corollary 3.6. We have that

$$R_0(x) = \frac{\binom{m+1}{2}xA'(x)T_0(x) + \binom{m+1}{3}x^2A'(x)^2}{A(x)} \quad (3.13)$$

and

$$\sum_{j=1}^{m-1} A(x)^{j-1}R_j(x) = A(x)^{m-1} \left(\frac{(m^2 + m - 2)R_0(x)}{2} + \frac{m(2m^2 + 3m - 5)T_0(x)}{6} + \frac{(m-1)m^2(m+1)xA'(x)}{12} \right). \quad (3.14)$$

Proof. Using (3.10) and induction on $m-j$ (the base case $m-j=1$ is given by (3.12)), we obtain that

$$R_j(x) = (m-j+1)A(x)^{m-j}R_0(x) + \frac{(m-j+1)(m+j)A(x)^{m-j}T_0(x)}{2} + \frac{m+2j-1}{3} \binom{m-j+1}{2} xA'(x)A(x)^{m-j} \quad (3.15)$$

for each integer $j \in [1, m-1]$. Multiplying (3.15) by $A(x)^{j-1}$ and summing over j yields (3.14). Inserting (3.15), with $j=1$, into (3.11) gives (3.13). \square

We may now evaluate $G_0(x)$.

Proposition 3.7. *For each integer $j \in [1, m-2]$,*

$$\begin{aligned} G_i(x) &= A(x)G_{i+1}(x) + xA'(x)R_{i+1}(x) - xA(x)T'_{i+1}(x) \\ &\quad + A(x)^{m-i}G_0(x) + A(x)^{m-i}R_0(x) + (m-i)x A'(x)A(x)^{m-i-1}R_0(x) \\ &\quad + (i+1)A(x)^{m-i}T_0(x) + i(m-i)x A'(x)A(x)^{m-i-1}T_0(x) + ix A(x)^{m-i}T'_0(x) \\ &\quad + (2i+1)(m-i)x A'(x)A(x)^{m-i} + i(m-i)x^2 A''(x)A(x)^{m-i} \\ &\quad + i(m-i)^2 x^2 A'(x)^2 A(x)^{m-i-1} - T_0(x)T_{i+1}(x). \end{aligned}$$

Moreover,

$$\begin{aligned} G_0(x) &= xA(x)G_1(x) + x^2 A'(x)R_1(x) - x^2 A(x)T'_1(x) \\ &\quad + xA(x)^m G_0(x) + xA(x)^m R_0(x) + mx^2 A'(x)A(x)^{m-1}R_0(x) \\ &\quad + xA(x)^m T_0(x) + mx^2 A'(x)A(x)^m - xT_0(x)T_1(x), \end{aligned}$$

and

$$\begin{aligned} G_{m-1}(x) &= 2A(x)G_0(x) + 2xA'(x)R_0(x) + 2A(x)R_0(x) \\ &\quad + (2m+1)A(x)T_0(x) + (2m-1)x A'(x)T_0(x) + (2m-2)x A(x)T'_0(x) \\ &\quad + (2m-1)x A'(x)A(x) + (m-1)x^2 A''(x)A(x) \\ &\quad + (m-1)x^2 A'(x)^2 - T_0(x)^2. \end{aligned}$$

Proof. Following the proof of Proposition 2.6, one obtains that

$$\begin{aligned} G_n^{(j)} &= \sum_{i=0}^n \left(A_{n-i} \left(G_i^{(j+1)} + (n-i)R_i^{(j+1)} - iT_i^{(j+1)} \right) \right. \\ &\quad \left. + A_i^{(j+1)} \left(G_{n-i}^{(0)} + (i+1)R_{n-i}^{(0)} + (j(n+1)+1)T_{n-i}^{(0)} \right) \right. \\ &\quad \left. + (j(n+1)+1)iA_i^{(j+1)} A_{n-i} - T_i^{(j+1)} T_{n-i}^{(0)} \right) \end{aligned}$$

for each integer $j \in [1, m-2]$;

$$\begin{aligned} G_n^{(0)} &= \sum_{i=0}^n \left(A_{n-i-1} \left(G_i^{(1)} + (n-i-1)R_i^{(1)} - iT_i^{(1)} \right) \right. \\ &\quad \left. + A_i^{(1)} \left(G_{n-i-1}^{(0)} + (i+1)R_{n-i-1}^{(0)} + T_{n-i-1}^{(0)} \right) \right. \\ &\quad \left. + iA_i^{(1)} A_{n-i-1} - T_i^{(1)} T_{n-i-1}^{(0)} \right); \end{aligned}$$

and

$$\begin{aligned} G_n^{(m-1)} &= \sum_{i=0}^n \left(A_{n-i} \left(G_i^{(0)} + (n-i+1)R_i^{(0)} + (m(n+1)+1-i)T_i^{(0)} \right) \right. \\ &\quad \left. + A_i \left(G_{n-i}^{(0)} + (i+1)R_{n-i}^{(0)} + ((m-1)(n+1)+1)T_{n-i}^{(0)} \right) \right. \\ &\quad \left. + ((m-1)(n+1)+1)iA_i A_{n-i} - T_i^{(0)} T_{n-i}^{(0)} \right). \end{aligned}$$

These recursive relations imply the proposition. \square

Corollary 3.8. *We have that*

$$\begin{aligned}
G_0(x) &= \left((m+1)xA(x)^m R_0(x) + (m^2+m)x^2 A'(x)A(x)^{m-1} R_0(x) \right. \\
&\quad + \binom{m+2}{2} xA(x)^m T_0(x) + \binom{m+1}{2} x^2 A'(x)A(x)^{m-1} T_0(x) \\
&\quad + \binom{m+2}{3} x^2 A'(x)A(x)^m + \binom{m+2}{4} x^3 A'(x)^2 A(x)^{m-1} \\
&\quad \left. - \binom{m+1}{2} xA(x)^{m-1} T_0(x)^2 \right) (1 - (m+1)xA(x)^m)^{-1}. \tag{3.16}
\end{aligned}$$

Proof. Using Proposition 3.7 and the equality

$$G_0(x) = (G_0(x) - xA(x)G_1(x)) + x \sum_{j=1}^{m-2} (A(x)^j G_j(x) - A(x)^{j+1} G_{j+1}(x)) + xA(x)^{m-1} G_{m-1}(x)$$

gives

$$\begin{aligned}
G_0(x) &= \left((m+1)xA(x)^m R_0(x) + \frac{(m^2+m+2)x^2 A'(x)A(x)^{m-1} R_0(x)}{2} \right. \\
&\quad + x^2 A'(x) \sum_{j=1}^{m-1} A(x)^{j-1} R_j(x) - x^2 \sum_{j=1}^{m-1} A(x)^j T_j'(x) \\
&\quad + \binom{m+2}{2} xA(x)^m T_0(x) + \frac{m(m^2+5)x^2 A'(x)A(x)^{m-1} T_0(x)}{6} \\
&\quad + \frac{(m+2)(m-1)x^2 A(x)^m T_0'(x)}{2} + \frac{m(2m+1)(m+1)x^2 A'(x)A(x)^m}{6} \\
&\quad + \binom{m+1}{3} x^3 A''(x)A(x)^m + \frac{(m-1)m^2(m+1)x^3 A'(x)^2 A(x)^{m-1}}{12} \\
&\quad \left. - xA(x)^{m-1} T_0(x)^2 - xT_0(x) \sum_{j=1}^{m-1} A(x)^{j-1} T_j(x) \right) (1 - (m+1)xA(x)^m)^{-1}.
\end{aligned}$$

Applying (3.8), (3.9), and (3.14) to the above yields (3.16). \square

We may now prove Theorem 1.3.

Proof of Theorem 1.3. As stated previously, it suffices to establish (3.2). The left side of this equality involves derivatives of $A(x)$, which can be expressed in terms of $A(x)$ using Proposition 3.1. Specifically, differentiating the second equality stated in Proposition 3.1, we obtain that

$$A'(x) = \frac{A(x)^{m+1}}{1 - (m+1)xA(x)^m}. \tag{3.17}$$

Differentiating again yields

$$A''(x) = \frac{(m+1)A(x)^m (A'(x) + A(x)^{m+1} - xA(x)^m A'(x))}{(1 - (m+1)xA(x)^m)^2} \tag{3.18}$$

and repeating gives

$$\begin{aligned}
A'''(x) &= (m+1)A(x)^{m-1}(A(x)A''(x) + (m-1)xA(x)^m A'(x)^2 + (4m+2)A(x)^{m+1}A'(x) \\
&\quad + mA'(x)^2 - (m+2)xA(x)^{m+1}A''(x) + (m+1)x^2A(x)^{2m+1}A''(x) \\
&\quad - 2(m+1)xA'(x)A(x)^{2m+1} + 2(m+1)A(x)^{2m+2}(1 - (m+1)xA(x)^m)^{-3}. \quad (3.19)
\end{aligned}$$

Now, we may express the left side of (3.2) as a rational function in x and $A(x)$ using (3.16), (3.17), (3.18), (3.19), (3.7), and (3.13). After inserting these identities into the left side and simplifying, one obtains 0. The below Sage code verifies this claim since its output is 0.

```

A,m=var('A','m')
A1=A^(m+1)/(1-(m+1)*x*A^m)
A2=(m+1)*A^m*(A1+A^(m+1)-x*A^m*A1)/(1-(m+1)*x*A^m)^2
A3=(m+1)*A^(m-1)*(A*A2+(m-1)*m*x*A^m*A1^2+(4*m+2)*A^(m+1)*A1+m*A1^2
-(m+2)*x*A^(m+1)*A2+(m+1)*x^2*A^(2*m+1)*A2-2*(m+1)*x*A^(2*m+1)*A1
+2*(m+1)*A^(2*m+2))/(1-(m+1)*x*A^m)^3
T=binomial(m+1,2)*x^2*(A1)^2/A
R=(binomial(m+1,2)*x*A1*T+binomial(m+1,3)*x^2*(A1)^2)/A
G=((m+1)*x*R*A^m+(m^2+m)*x^2*A1*R*A^(m-1)+binomial(m+2,2)*x*A^m*T
+binomial(m+1,2)*x^2*A1*A^(m-1)*T+binomial(m+2,3)*x^2*A1*A^m
+binomial(m+2,4)*x^3*(A1)^2*A^(m-1)-binomial(m+1,2)*x*A^(m-1)*T^2)/(1-(m+1)*x*A^m)
d=m*(m+1)*x^3*A3+m*(2*m+4)*x^2*A2-24*G
d.full_simplify()

```

□

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