

EQUIVALENCE CLASSES OF NODES IN TREES AND RATIONAL GENERATING FUNCTIONS

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ABSTRACT. Let c_n denote the number of nodes at a distance n from the root of a rooted tree. A criterion for proving the rationality and computing the rational generating function of the sequence $\{c_n\}$ is described. This criterion is applied to counting the number of conjugacy classes of commuting tuples in finite groups and the number of isomorphism classes of representations of polynomial algebras over finite fields. The method for computing the rational generating functions, when applied to the study of point configurations in finite sets, gives rise to some classical combinatorial results on Bell numbers and Stirling numbers of the second kind. When applied to the study of vector configurations in a finite vector space, it reveals a connection between counting such configurations and Gaussian binomial coefficients.

1. INTRODUCTION

This paper begins by describing a technique for proving the rationality of, and often explicitly computing, ordinary generating functions of certain combinatorial sequences (Theorem 2). It applies to sequences whose n th term can be expressed as the number of nodes in a rooted tree at a distance n from the root. The rationality rests upon the finiteness of the number of what are called lineal isomorphism classes of nodes.

The counting of simultaneous conjugacy classes of commuting n -tuples in a finite group G is, in general, a difficult combinatorial problem. However, the rationality of the generating function associated to this count turns out to be an easy consequence of Theorem 2 (see Theorem 3). These generating functions are computed explicitly for the first five symmetric groups (Table 2).

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A slight variant of this result also shows that if $c_{q,m}(n)$ denotes the number of isomorphism classes of m -dimensional representations of the polynomial algebra $\mathbf{F}_q[x_1, \dots, x_n]$ (here \mathbf{F}_q is a finite field of order q), then $c_{q,m}(n)$ (as a sequence in n) has a rational generating function (Theorem 6).

The method from Theorem 2 for computing generating functions can sometimes be applied advantageously even to situations where rationality is easy to see by other methods. For instance, when applied to counting point configurations in finite sets, it leads to beautiful classical results concerning Bell numbers and Stirling numbers of the second kind. When applied to counting vector configurations in finite vector spaces, it leads to the discovery of a new interpretation of Gaussian binomial coefficients.

2. LINEAL EQUIVALENCE AND RATIONAL GENERATING FUNCTIONS

Let T denote the vertex set of a rooted tree with root vertex x_0 . Let T_n denote the set of vertices of T which are a distance n from x_0 . Then T is a disjoint union:

$$T = \coprod_{n=0}^{\infty} T_n.$$

We will give a sufficient condition for the formal generating function

$$(1) \quad f_T(t) = \sum_{n=0}^{\infty} |T_n| t^n$$

to be a rational function in t and a technique for its computation.

If $X \in T_n$ and $Y \in T_{n+1}$ are connected by an edge, then we say that Y is a *child* of X , and write $X \rightarrow Y$. More generally, if $X \in T_n$ and $Y \in T_{n+k}$ for some $k \geq 0$ are such that there exists a sequence

$$X = X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_k = Y,$$

then we say that Y is a *descendant* of X (under our definition X is a descendant of X).

For each $X \in T$, let $T(X)$ denote the full subtree consisting of the descendants of X . This is again a rooted tree, with root X .

Definition (Lineal Isomorphism). *Two vertices X and Y of T are said to be lineally isomorphic if the rooted trees $T(X)$ and $T(Y)$ are isomorphic (in other words, there is a graph isomorphism $T(X) \rightarrow T(Y)$ taking X to Y).*

Clearly, lineal isomorphism is an equivalence relation on T . The equivalence classes of this relation are called *lineal isomorphism classes*.

Theorem 1. *If X and Y are lineally isomorphic nodes in a rooted tree T , then for any lineal isomorphism class C , the number of children of X in C is equal to the number of children of Y in C .*

Proof. Since X and Y are lineally isomorphic, there exists an isomorphism $T(X) \rightarrow T(Y)$ of rooted trees. This isomorphism defines a bijection from the children of X in C to the children of Y in C . \square

Theorem 2. *Let T be a rooted tree with finitely many lineal isomorphism classes C_1, \dots, C_N , the root of T lying in C_1 . Let $\mathbf{B} = (b_{ij})$ be the $N \times N$ matrix where b_{ij} is the number of children that a node in the class C_j has in the class C_i . Then, for each $i \in \{1, \dots, N\}$,*

$$(2) \quad \sum_{n=0}^{\infty} |T_n \cap C_i| t^n = e'_i (I - \mathbf{B}t)^{-1} e_1.$$

Here, for each $i \in \{1, \dots, N\}$, e_i denotes the i th coordinate vector, viewed as an $N \times 1$ matrix, and e'_i its transpose. In particular, the sequence $\{|T_n \cap C_i|\}_{n=0}^{\infty}$ has a rational generating function for each i . Consequently,

$$(3) \quad \sum_{n=0}^{\infty} |T_n| t^n = \mathbf{1}' (I - \mathbf{B}t)^{-1} e_1.$$

Here $\mathbf{1}'$ is the $1 \times N$ all-ones row vector.

Proof. Let $v_j^{(n)} = |C_j \cap T_n|$ for $n \geq 0$ and $1 \leq j \leq N$. Let $v^{(n)}$ denote the column vector with coordinates $(v_1^{(n)}, \dots, v_N^{(n)})$. The hypothesis that the root of T lies in C_1 implies that $v^{(0)} = e_1$.

Since each node in $C_j \cap T_{n-1}$ contributes b_{ij} elements to $C_i \cap T_n$,

$$(4) \quad v_i^{(n)} = |C_i \cap T_n| = \sum_{j=1}^N |C_j \cap T_{n-1}| b_{ij} = \sum_{j=1}^N b_{ij} v_j^{(n-1)}.$$

The recurrence relation (4) can be written in matrix form as:

$$v^{(n)} = \mathbf{B} v^{(n-1)},$$

upon iterating which (and using the fact that $v^{(0)} = e_1$), we get

$$v^{(n)} = \mathbf{B}^n e_1.$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} v^{(n)} t^n &= \sum_{n=0}^{\infty} \mathbf{B}^n t^n e_1 \\ &= (I - \mathbf{B}t)^{-1} e_1. \end{aligned}$$

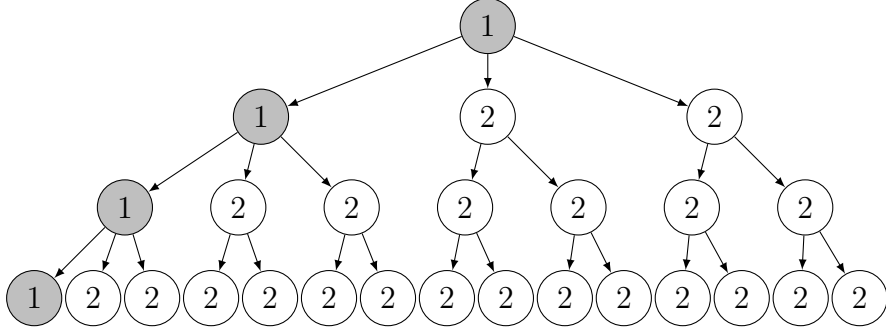


FIGURE 1. Part of a tree with two lineal isomorphism classes

Comparing the entries of the column vectors on the two sides of the above identity gives the identities (2). The identity (3) is their sum. \square

Example 1. Figure 1 shows the vertices within distance 3 from the root of a tree with two lineal equivalence classes, labelled ‘1’ and ‘2’. The vertices of type 1 have three children, one of type 1 and two of type 2, while the vertices of type 2 have no children of type 1 and two children of type 2. The “branching matrix” is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}.$$

Therefore, if $v_1^{(n)}$ and $v_2^{(n)}$ are the numbers of nodes of type 1 and 2 respectively which lie at distance n from the root, then

$$v^{(n)} := \begin{pmatrix} v_1^{(n)} \\ v_2^{(n)} \end{pmatrix} = \mathbf{B}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \sum_{n=0}^{\infty} v^{(n)} t^n &= (I - \mathbf{B}t)^{-1} e_1 \\ &= \begin{pmatrix} \frac{1}{1-2t} & 0 \\ \frac{1}{(1-t)(1-2t)} & \frac{1}{1-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{1-2t} \\ \frac{1}{(1-t)(1-2t)} \end{pmatrix}. \end{aligned}$$

If T_n is the set of nodes at a distance n from the root, we have

$$\sum_{n=0}^{\infty} |T_n| t^n = \frac{1}{(1-t)(1-2t)}$$

3. CONJUGACY CLASSES OF COMMUTING TUPLES IN GROUPS

Let G be a finite group. Then G acts on G^n for each non-negative integer n by simultaneous conjugacy:

$$(5) \quad g \cdot (x_1, \dots, x_n) = (gx_1g^{-1}, \dots, gx_ng^{-1}).$$

If a_n is the number of orbits for the action of G in G^n , it is not difficult to see that $f_G(t) = \sum_{n=0}^{\infty} a_n t^n$ is a rational function in t . Indeed, by Burnside's lemma

$$a_n = \frac{1}{|G|} \sum_{g \in G} |Z_G(g)|^n,$$

where $Z_G(g)$ denotes the centralizer of g in G . Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} a_n t^n &= \sum_{n=0}^{\infty} \frac{1}{|G|} \sum_{g \in G} |Z_G(g)|^n t^n \\ &= \frac{1}{|G|} \sum_{g \in G} \frac{1}{1 - |Z_G(g)|t}, \end{aligned}$$

which is a rational function in t .

Example 2. Taking $G = S_m$ (the symmetric group on m symbols),

$$\begin{aligned} f_{S_m}(t) &= \frac{1}{m!} \sum_{w \in S_m} \frac{1}{1 - Z_{S_m}(w)t} \\ &= \frac{1}{m!} \sum_{\lambda \vdash m} \frac{m!}{z_\lambda} \frac{1}{1 - z_\lambda t} \\ &= \sum_{\lambda \vdash m} \frac{1}{z_\lambda (1 - z_\lambda t)}. \end{aligned}$$

Here $\lambda \vdash m$ signifies that λ is a partition of m , and for each such partition, z_λ denotes the cardinality of the centralizer in S_m of a permutation with cycle type λ . If, for each positive integer i , m_i is the number of occurrences of i in λ , then

$$z_\lambda = \prod_{i=1}^{\infty} m_i! i^{m_i}.$$

The values of f_{S_m} for small values of m are given in Table 1.

A more subtle problem is that of counting simultaneous conjugacy classes of n -tuples of commuting elements in G . For each $n \geq 0$, let

$$G^{(n)} = \{(g_1, \dots, g_n) \in G^n \mid g_i g_j = g_j g_i \text{ for all } 1 \leq i, j \leq n\}.$$

In particular, $G^{(0)}$ is the trivial group.

m	$f_{S_m}(t)$
1	$\frac{1}{1-t}$
2	$\frac{1}{1-2t}$
3	$\frac{1-8t+14t^2}{(1-2t)(1-3t)(1-6t)}$
4	$\frac{1-34t+276t^2-584t^3}{(1-3t)(1-4t)(1-8t)(1-24t)}$
5	$\frac{1-148t+3746t^2-36984t^3+159200t^4-249792t^5}{(1-4t)(1-5t)(1-6t)(1-8t)(1-12t)(1-120t)}$

TABLE 1. Generating functions for simultaneous conjugacy classes in S_n .

Let c_n denote the number of orbits for the action of G on $G^{(n)}$ by simultaneous conjugation, as given in (5). Consider the generating function

$$h_G(t) = \sum_{n=0}^{\infty} c_n t^n.$$

Because the elements g_1, g_2, \dots, g_n are no longer independent, Burnside's lemma can no longer be used to prove the rationality of $h_G(t)$. However, Theorem 2 allows us to show that $h_G(t)$ is rational in t , and gives an algorithm to compute it for any finite group.

Theorem 3. *For every finite group G , the formal power series $h_G(t)$ defined above is a rational function of t .*

Proof. Let T_n^G denote the set of G -orbits in $G^{(n)}$. Say that $Y \in T_{n+1}^G$ is connected to $X \in T_n$ by an edge if there exists $(g_1, \dots, g_{n+1}) \in Y$ such that $(g_1, \dots, g_n) \in X$. This gives $T^G = \coprod_{n=0}^{\infty} T_n^G$ the structure of a rooted tree with root X_0 being the unique element of T_0^G . For $(g_1, \dots, g_n) \in G^{(n)}$, let

$$Z_G(g_1, \dots, g_n) = Z_G(g_1) \cap Z_G(g_2) \cap \dots \cap Z_G(g_n).$$

We will see in Theorem 4 below that the G -orbit of (g_1, \dots, g_n) is lineally isomorphic to the G -orbit of (s_1, \dots, s_l) in T^G if the group $Z_G(g_1, \dots, g_n)$ is isomorphic to the group $Z_G(s_1, \dots, s_l)$. Since each of these centralizers is a subgroup of the finite group G , there are only

finitely many possible isomorphism classes for them, and so only finitely many lineal isomorphism classes in T^G . Thus Theorem 2 applies, and $h_G(t)$ is a rational function of t . \square

We now come to Theorem 4 and its proof (which will complete the proof of Theorem 3).

Theorem 4. *Suppose that $(g_1, \dots, g_n) \in G^{(n)}$ lies in the G -orbit $X \in T_n^G$, then the full subtree of $T^G(X)$ (rooted at X) consisting of descendants of X in T^G is isomorphic to the rooted tree $T^{Z_G(g_1, \dots, g_n)}$ associated to the group $Z_G(g_1, \dots, g_n)$.*

Proof. Let $S = Z_G(g_1, \dots, g_n)$. Define a map $S^{(l)} \rightarrow G^{(n+l)}$ by

$$(s_1, \dots, s_l) \mapsto (g_1, \dots, g_n, s_1, \dots, s_l).$$

It is easy to check that this map induces an isomorphism of rooted trees $T^S \rightarrow T^G(X)$. \square

We now consider the examples of symmetric groups: since S_2 is abelian, $h_{S_2}(t) = f_{S_2}(t)$, which was computed earlier in this section. Now S_3 has three conjugacy classes, which lie in different lineal isomorphism classes in T^{S_3} since they have non-isomorphic centralizers. The centralizers are given in the following table:

λ	$Z_{S_3}(x)$
(1, 1, 1)	S_3
(2, 1)	C_2
(3)	C_3

With the exception of the class of the identity element (with cycle type (1, 1, 1)) each of these centralizers is abelian. If the orbit of a tuple has abelian centralizer, then all its descendants are lineally isomorphic to it. For the singleton orbit of the identity element, we once again have three children, one corresponding to each partition of 3. Thus every pair of commuting elements on S_3 has centralizer isomorphic to that of an element of S_3 . The branching matrix of Theorem 2 is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

A routine calculation shows that

$$h_{S_3}(t) = \frac{1 - 3t + t^2}{(1 - t)(1 - 2t)(1 - 3t)}.$$

The group S_4 five conjugacy classes with centralizers given by:

λ	$Z_{S_4}(x)$
(1, 1, 1, 1)	S_4
(2, 1, 1)	$C_2 \times C_2$
(2, 2)	$C_2 \wr S_2$
(3, 1)	C_3
(4)	C_4

The only troublesome case here is $\lambda = (2, 2)$. The centralizer group in this case is a non-abelian group of order 8, which we now proceed to analyse: for concreteness, consider the centralizer of the permutation $(12)(34) \in S_4$. The centralizer subgroup consists of the permutations:

$$H = \{1, (12)(34), (12), (34), (13)(24), (14)(23), (1423), (1324)\}.$$

This group has five conjugacy classes, with centralizers given by:

class	centralizer
1	$C_2 \wr S_2$
(12)(34)	$C_2 \wr S_2$
(12), (34)	$C_2 \times C_2$
(14)(23), (13)(24)	$C_2 \times C_2$
(1324), (1423)	C_4

Thus, as in the case of S_3 , the centralizers of pairs are all centralizers of elements in S_4 . The branching matrix is

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 1 & 0 & 1 & 0 & 4 \end{pmatrix},$$

which gives:

$$h_{S_4}(t) = \frac{1 - 5t + 6t^2 - t^3}{(1-t)(1-2t)(1-3t)(1-4t)}.$$

Similarly for S_5 we have

λ	$Z_{S_5}(x)$
(1 ⁵)	S_5
(2, 1 ³)	$C_2 \times S_3$
(2, 2, 1)	$C_2 \wr S_2$
(3, 1, 1)	$C_3 \times S_2$
(3, 2)	$C_3 \times C_2$
(4, 1)	C_4
(5)	C_5

The classes corresponding to the partitions $(3, 1, 1)$ and $(3, 2)$ can be clubbed, as they have isomorphic centralizers (and therefore are lineally isomorphic). In $C_2 \times S_3$, we have the following count of classes and their centralizers:

centralizer	no. of classes
$C_2 \times S_3$	2
$C_2 \times C_2$	2
$C_2 \times C_3$	2

The only centralizer here which is not the centralizer of an element of S_5 is $C_2 \times C_2$, which is abelian. The branching matrix is given by:

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 6 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 4 \end{pmatrix}$$

whence one may compute:

$$h_{S_5}(t) = \frac{1 - 11t + 34t^2 - 21t^3 + 2t^4}{(1-t)(1-2t)(1-4t)(1-5t)(1-6t)}.$$

Our calculations of h_{S_m} for small values of m are summarized in Table 2. The techniques at hand are not strong enough to derive an analog of

m	$h_{S_m}(t)$
1	$\frac{1}{1-t}$
2	$\frac{1}{1-2t}$
3	$\frac{1-3t+t^2}{(1-t)(1-2t)(1-3t)}$
4	$\frac{1-5t+6t^2-t^3}{(1-t)(1-2t)(1-3t)(1-4t)}$
5	$\frac{1-11t+34t^2-21t^3+2t^4}{(1-t)(1-2t)(1-4t)(1-5t)(1-6t)}$

TABLE 2. Generating functions for simultaneous conjugacy classes of commuting elements in S_m .

the formula in Example 2 for h_{S_m} for general m .

Theorem 3 can also be stated for finite algebras:

Theorem 5. *Let A be a finite ring, A^* its multiplicative group of units, and let*

$$A^{(n)} = \{(a_1, \dots, a_n) \in A^n \mid a_i a_j = a_j a_i \text{ for } 1 \leq i, j \leq n\}.$$

Then A^ acts on $A^{(n)}$ by simultaneous similarity:*

$$u \cdot (a_1, \dots, a_n) = (ua_1u^{-1}, \dots, ua_nu^{-1}).$$

Let $c_A(n)$ denote the number of orbits for the action of A^ on $A^{(n)}$. Then the generating function*

$$h_A(t) = \sum_{n=0}^{\infty} c_A(n)t^n$$

is a rational function of t .

Proof. The proof is similar to Theorem 3: Let

$$Z_A(a_1, \dots, a_n) = Z_A(a_1) \cap \dots \cap Z_A(a_n),$$

where $Z_A(a)$ denotes the subring of elements of A that commute with a . The A^* -orbits of (a_1, \dots, a_n) and (b_1, \dots, b_l) are linearly isomorphic if the rings $Z(a_1, \dots, a_n)$ and $Z(b_1, \dots, b_l)$ are isomorphic. \square

Let \mathbf{F}_q be a finite field of order q . Taking A to be the algebra $M_m(\mathbf{F}_q)$ of $m \times m$ matrices with entries in \mathbf{F}_q in Theorem 5 gives simultaneous similarity classes of commuting n -tuples in $M_m(\mathbf{F}_q)$. An n -tuple of commuting matrices is nothing but an $\mathbf{F}_q[x_1, \dots, x_n]$ -module. Two modules are isomorphic if and only if the corresponding n -tuples are simultaneously similar. So we have:

Theorem 6. *Let \mathbf{F}_q denote the finite field with q elements, and for each positive integer m let $c_{q,m}(n)$ denote the number of isomorphism classes of m -dimensional modules for the polynomial algebra $\mathbf{F}_q[x_1, \dots, x_n]$. Then the generating function:*

$$h_{q,m}(t) = \sum_{n=0}^{\infty} c_{q,m}(n)t^n$$

is a rational function of t .

The polynomials $h_{q,m}(t)$ are quite difficult to compute for $m \geq 4$, but seem to have very interesting combinatorial properties, an investigation

of which is the subject of [11]. For example,

$$\begin{aligned} h_{q,1}(t) &= \frac{1}{1-qt} \\ h_{q,2}(t) &= \frac{1}{(1-qt)(1-q^2t)} \\ h_{q,3}(t) &= \frac{1+q^2t^2}{(1-qt)(q-q^2t)(1-q^3t)}. \end{aligned}$$

The details of these (and further) calculations can be found in [11].

As with groups, counting of A^* -orbits in A^n (instead of $A^{(n)}$) is much easier, because of the applicability of Burnside's lemma. When $A = M_m(\mathbf{F}_q)$ this becomes the problem of counting isomorphism classes of m -dimensional representations of the free algebra $\mathbf{F}_q\langle x_1, \dots, x_n \rangle$, which in turn is a special case of the problem of counting representations of a quiver with the fixed dimension vector, a well-developed program which was started by Kac [7] in 1983 and culminated in the recent work of Hausel, Letellier and Rodriguez-Villegas [5] and Mozgovoy [9]. In contrast, we do not even know that $c_{q,m}(n)$ is a polynomial in q for $m > 4$.

The counting of isomorphism classes of $\mathbf{F}_q[x_1, \dots, x_n]$ -modules appears to be related to the counting of similarity classes of matrices in finite quotients of discrete valuation rings. Let R be a discrete valuation ring with residue field \mathbf{F}_q . Let P denote the maximal ideal of R . Matrices $A, B \in M_m(R/P^n)$ are said to be similar if $B = gAg^{-1}$ for some $g \in GL_m(R/P^n)$. From the work of Singla [12], Jambor and Plesken [6] and Prasad, Singla and Spallone [10], we know that the number of isomorphism classes of m -dimensional $\mathbf{F}_q[x_1, x_2]$ -modules (what we have called $c_{q,m}(2)$ above) is equal to the number of similarity classes in $M_m(R/P^2)$. Further, by comparing the values for $h_{q,m}(t)$ quoted above with the results obtained by Avni, Onn, Prasad and Vaserstein [2] we find that for $m \leq 3$, $c_{q,m}(n)$ matches the number of similarity classes in $M_m(R/P^n)$ for $m \leq 3$ and all n .

One is led to the following conjecture:

Conjecture. *The number of similarity classes in $M_m(R/P^n)$ is equal to the number of isomorphism classes of m -dimensional $\mathbf{F}_q[x_1, \dots, x_n]$ -modules for all positive integers m and n .*

If $C_{q,m}(n)$ denotes the number of similarity classes in $M_m(R/P^n)$, then du Sautoy [3] has shown (using model theory) that when R has characteristic zero, then $C_{q,m}(n)$, as a sequence in n , has a rational generating function. His result is the analog of Theorem 6 for similarity classes in $M_m(R/P^n)$.

4. POINT AND VECTOR CONFIGURATIONS

The symmetric group S_m acts on the set $\mathbf{m} = \{1, \dots, m\}$. An n -point configuration in \mathbf{m} is, by definition, an orbit of S_m for its action on \mathbf{m}^n by

$$w \cdot (x_1, \dots, x_n) = (w \cdot x_1, \dots, w \cdot x_n).$$

For example, there are two 2-point configurations in \mathbf{m} for $m \geq 2$: either the points x_1 and x_2 coincide, or they are distinct. Likewise, there are five 3-point configurations in \mathbf{m} for $m \geq 3$, represented by

$$(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1) \text{ and } (1, 2, 3).$$

Let $c_m(n)$ denote the number of n -point configurations in \mathbf{m} .

We may compute $c_m(n)$ using Burnside's lemma. With the notation of Example 2,

$$\begin{aligned} c_m(n) &= \frac{1}{m!} \sum_{w \in S_m} (\text{no. of fixed points of } w)^n \\ &= \sum_{\lambda \vdash m} \frac{m_1(\lambda)^n}{z_\lambda}, \end{aligned}$$

so that

$$\sum_{n=0}^{\infty} c_m(n)t^n = \sum_{\lambda \vdash m} \frac{1}{z_\lambda} \frac{1}{1 - m_1(\lambda)t}.$$

However, we shall see below that using Theorem 2 for the same computation leads to the standard ordinary generating functions and recurrence relations for Bell numbers [13, A000110] and Stirling numbers of the second kind [13, A008277].

Let $T_n^{(m)}$ denote the set of S_m orbits in \mathbf{m}^n , and $T^{(m)} = \coprod_{n=0}^{\infty} T_n^{(m)}$. Say that $Y \in T_{n+1}^{(m)}$ is a child of $X \in T_n^{(m)}$ if there exists $(x_1, \dots, x_{n+1}) \in Y$ such that $(x_1, \dots, x_n) \in X$. We say that $X \in T_n^{(m)}$ has type i if, for any $(x_1, \dots, x_n) \in X$, the number of distinct elements in the set $\{x_1, \dots, x_n\}$ is i . Clearly, if X has type i , then each of its children has type either i or $i+1$. Also, $(x_1, \dots, x_n, x_{n+1})$ and $(x_1, \dots, x_n, x'_{n+1})$ lie in the same S_m -orbit if and only if there exists a permutation which fixes x_1, \dots, x_n and maps x_{n+1} to x'_{n+1} .

Now suppose that X is of type i and $(x_1, \dots, x_n) \in X$. If, for some $x_{n+1} \in \mathbf{m}$, the orbit of (x_1, \dots, x_{n+1}) is also of type i , then x_{n+1} coincides with one of x_1, \dots, x_n and is therefore fixed by any $w \in S_m$ which fixes them. Thus, a node of type i has i children of type i . On the other hand, if the orbit of (x_1, \dots, x_{n+1}) is of type $i+1$, then x_{n+1} is different from each of x_1, \dots, x_n and can therefore be permuted to any

other element of \mathbf{m} that is distinct from x_1, \dots, x_n while fixing them. It follows that a node of type i has 1 child of type $i + 1$.

The branching matrix is given by

$$(6) \quad \mathbf{B} = \begin{pmatrix} 0 & & & & & & & & \\ 1 & 1 & & & & & & & \\ & & 1 & 2 & & & & & \\ & & & & 1 & 3 & & & \\ & & & & & & \ddots & & \\ & & & & & & & & m-1 \\ & & & & & & & & & 1 & m \end{pmatrix},$$

a matrix whose diagonal entries are $0, \dots, m$, with 1's just below the diagonal and with all other entries zero. One easily computes

$$(7) \quad e_i(1 - \mathbf{B}t)^{-1}e_1 = \prod_{r=1}^i \frac{t}{1 - rt}.$$

We obtain:

Theorem 7. *The sequence $c_m(n)$ has generating function*

$$\sum_{n=0}^{\infty} c_m(n)t^n = \sum_{i=0}^m \prod_{r=0}^i \frac{t}{1 - rt}$$

Each n -tuple (x_1, \dots, x_n) gives rise to an equivalence relation on \mathbf{n} ; indices $i, j \in \mathbf{n}$ are equivalent if $x_i = x_j$. Two tuples are in the same S_m -orbit if and only if they give rise to the same equivalence relation on \mathbf{n} . Thus the number of n -point configurations in \mathbf{m} is nothing but the number of equivalence relations on \mathbf{n} with at most m equivalence classes. For $m \geq n$, this number is the well-known Bell number B_n . Under the correspondence between n -point configurations in \mathbf{m} and equivalence relations on \mathbf{n} with at most m equivalence classes, point configurations of type i map to equivalence relations with exactly i equivalence classes. The number of equivalence relations with exactly i equivalence classes is the well-known *Stirling number of the second kind*, usually denoted $S(n, i)$ or $\left\{ \begin{matrix} n \\ i \end{matrix} \right\}$ [14, Section 1.9]. The identity (7) becomes a well-known generating function for Stirling numbers of the second kind [14, Eq. (1.94c)]:

$$\sum_{n=0}^{\infty} S(n, i)t^n = t^i \prod_{r=0}^i \frac{1}{1 - rt}.$$

This identity reflects the obvious fact that $S(n, i) = 0$ for $i > n$. The branching rule (6) gives the standard recurrence relation for Stirling numbers of the second kind [14, Eq. (1.93)]:

$$(8) \quad S(n, k) = S(n-1, k-1) + kS(n-1, k).$$

By definition, the Bell number B_n is the number of equivalence relations on a set of order n . Clearly, $B_n = \sum_{i=0}^{\infty} S(n, i)$ which equals $c_m(n)$ provided that $m \geq n$. Thus the ordinary generating function for Bell numbers (see, e.g., [8, Lemma 8]) is obtained:

$$\sum_{n=0}^{\infty} B_n t^n = \sum_{i=0}^{\infty} \prod_{r=0}^i \frac{t}{1-rt}.$$

Let \mathbf{F}_q denote a finite field with q elements. The general linear group $GL_m(\mathbf{F}_q)$ acts on the vector space \mathbf{F}_q^m , and therefore also on n -tuples of vectors in it:

$$g \cdot (x_1, \dots, x_n) = (g(x_1), \dots, g(x_n)),$$

for $g \in GL_m(\mathbf{F}_q)$ and for $x_1, \dots, x_n \in \mathbf{F}_q^m$. A configuration of n vectors in \mathbf{F}_q^m is an orbit of $GL_m(\mathbf{F}_q)$ on $(\mathbf{F}_q^m)^n$.

Let $T_n^{q,m}$ denote the set of $GL_m(\mathbf{F}_q)$ -orbits in $(\mathbf{F}_q^m)^n$. Say that $Y \in T_{n+1}^{q,m}$ is a child of $X \in T_n^{q,m}$ if there exists $(x_1, \dots, x_{n+1}) \in Y$ such that $(x_1, \dots, x_n) \in X$. Let $T^{q,m} = \coprod_{n=0}^{\infty} T_n^{q,m}$. We say that $X \in T_{q,m}$ has type i if, for any $(x_1, \dots, x_n) \in X$, the dimension of the subspace spanned by the set $\{x_1, \dots, x_n\}$ is i . If X is of type i , then a child of X must be of type i or $i+1$. If $(x_1, \dots, x_n, x_{n+1})$ is of type i , then x_{n+1} lies in the span of x_1, \dots, x_n . Therefore any element of $GL_m(\mathbf{F}_q)$ that fixes x_1, \dots, x_n fixes x_{n+1} as well. Therefore, a tuple of type i has q^i children of type i . If, on the other hand, (x_1, \dots, x_{n+1}) and (x_1, \dots, x'_{n+1}) both have type $i+1$, then x_{n+1} and x'_{n+1} are linearly independent of x_1, \dots, x_n , so there exists $g \in GL_m(\mathbf{F}_q)$ mapping x_{n+1} to x'_{n+1} while fixing x_1, \dots, x_n . Therefore, a tuple of type i has only one child of type $i+1$.

The branching matrix is given by

$$(9) \quad \mathbf{B} = \begin{pmatrix} 1 & & & & & \\ 1 & q & & & & \\ & 1 & q^2 & & & \\ & & 1 & q^3 & & \\ & & & \ddots & \ddots & \\ & & & & q^{m-1} & \\ & & & & & 1 & q^m \end{pmatrix}.$$

We obtain:

$$(10) \quad e_i(I - \mathbf{B}t)^{-1}e_1 = t^i \prod_{r=0}^i \frac{1}{1 - q^r t} \text{ for } 1 \leq i \leq m.$$

Theorem 8. *The generating function for $\{|T_n^{q,m}|\}_n$ is*

$$\sum_{n=0}^{\infty} |T_n^{q,m}| t^n = \sum_{i=0}^m t^i \prod_{r=0}^i \frac{1}{1 - q^r t}$$

The quantity $|T_n^{q,m}|$ does not depend on m so long as $m \geq n$. The stable value of this quantity $B_{q,n} := |T_n^{q,n}|$ may be regarded as an analog of the Bell number for which we get the ordinary generating function:

$$(11) \quad \sum_{n=0}^{\infty} B_{q,n} t^n = \sum_{i=0}^{\infty} t^i \prod_{r=0}^i \frac{1}{1 - q^r t}.$$

Likewise, if we only count those n -vector configurations in \mathbf{F}_q^m which span an i -dimensional subspace of \mathbf{F}_q^m when $m \geq i$ (given by (10)), these clearly do not depend on m so long as $m \geq i$) we obtain an analog of the Stirling number of the second kind:

$$\sum_{n=0}^{\infty} S_q(n, i) t^n = t^i \prod_{r=0}^i \frac{1}{1 - q^r t}.$$

The branching rule (9) gives the recurrence relation

$$(12) \quad S_q(n, i) = S_q(n-1, i-1) + q^i S_q(n-1, i) \text{ for } 0 < i < n.$$

which is not the same as the recurrence relation for the usual q -Stirling numbers [4, Eq. (3.8)]; it does not specialize to (8) at $q = 1$. Instead it is one of the Pascal identities for Gaussian binomial coefficients:

$$\binom{n}{i}_q = \binom{n-1}{i-1}_q + q^i \binom{n-1}{i}_q.$$

Also, just like Gaussian binomial coefficients, $S_q(n, 0) = S_q(n, n) = 1$, so we have:

Theorem 9. *The number of n -vector configurations in \mathbf{F}_q^m whose span has dimension i when $m \geq i$ is equal to the number of i -dimensional subspaces of an n -dimensional vector space:*

$$(13) \quad S_q(n, i) = \binom{n}{i}_q.$$

The identity (12) becomes

$$\sum_{n=0}^{\infty} \binom{n}{i}_q t^n = t^i \prod_{r=0}^{i-1} \frac{1}{1 - q^r t},$$

a well-known generating function for Gaussian binomial coefficients [14, p. 74].

For a bijective proof of Theorem 9, given an n -tuple (x_1, \dots, x_n) of vectors in \mathbf{F}_q^m , write a matrix \mathbf{X} whose columns are the coordinates of these vectors; if $x_j = (x_{1j}, x_{2j}, \dots, x_{mj})$, then

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}.$$

Similarly, if (x'_1, \dots, x'_n) is another n -tuple of vectors, and \mathbf{X}' is the corresponding matrix, then (x_1, \dots, x_n) and (x'_1, \dots, x'_n) lie in the same $GL_m(\mathbf{F}_q)$ -orbit if and only if there exists $g \in GL_m(\mathbf{F}_q)$ such that

$$(14) \quad \mathbf{X}' = g\mathbf{X}.$$

However, the condition (14) is also necessary and sufficient for two $m \times n$ matrices \mathbf{X} and \mathbf{X}' to have the same row space. Thus the function that takes the tuple (x_1, \dots, x_n) to the row space of \mathbf{X} descends to a bijective map from the set of n -vector configurations in \mathbf{F}_q^m to the set of subspaces of \mathbf{F}_q^n which are spanned by m vectors, in other words, subspaces of dimension m or less. If $m \geq n$, then this is the set of all subspaces of \mathbf{F}_q^n . Since the row rank of a matrix is equal to its column rank, if the n -tuple (x_1, \dots, x_n) spans an i -dimensional vector space, the row space of \mathbf{X} is an i -dimensional subspace of \mathbf{F}_q^m . This sets up a bijection from the set of configurations of n vectors in \mathbf{F}_q^m of type i to the set of all i -dimensional subspaces of \mathbf{F}_q^m when $m \geq i$.

Anilkumar and Prasad [1] studied the number of configurations of pairs in finite abelian p -groups. They conjectured that these numbers are represented by polynomials in p with non-negative integer coefficients. It would be interesting to try to generalize the ideas behind Theorems 8 and 9 to counting configurations of tuples in finite abelian p -groups.

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