COUNTING WITH IRRATIONAL TILES

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Abstract. We introduce and study the number of tilings of unit height rectangles with irrational tiles. We prove that the class of sequences of these numbers coincides with the class of *diagonals of* $\mathbb{N}$*-rational generating functions* and a class of certain *binomial multisums*. We then give asymptotic applications and establish connections to hypergeometric functions and Catalan numbers.

1. Introduction

The study of combinatorial objects enumerated by rational generating functions (GF) is classical and goes back to the foundation of Combinatorial Theory. Rather remarkably, this class includes a large variety of combinatorial objects, from integer points in polytopes and horizontally convex polyominoes, to magic squares and discordant permutations (see e.g. [Sta1, FS]). Counting the number of tilings of a *strip* (rectangle $[k \times n]$ with a fixed height k) is another popular example in this class, going back to Golomb, see [Gol, §7] (see also [BL, CCH, KM, MSV]). The nature of GFs of such tilings is by now completely understood (see Theorem 1.1 below).

In this paper we present an unusual generalization to tile counting functions with irrational tiles, of rectangles $[1 \times (n+\varepsilon)]$, where $\varepsilon \in \mathbb{R}$ is fixed. This class of functions turns out to be very rich and interesting; our main result (theorems 1.2 and 1.3 below) is a complete characterization of these functions. We then use this result to construct a number of tile counting functions useful for applications.

Let us first illustrate the notion of tile counting functions with several examples. Start with Fibonacci number F_n which count the number of tilings of $[1 \times n]$ with the set T of two rectangles $[1 \times 1]$ and $[1 \times 2]$, see Figure 1.

$[1 \times 10]$

Figure 1. Fibonacci tiles T and a tiling of $[1 \times 10]$.

Consider now a more generic set of tiles as in Figure 2, where each tile has height 1 and rational side lengths. Note that the dark shaded tiles here are *bookends*, i.e. every tiling of $[1 \times n]$ must begin and end with one, and they are not allowed to be in the middle. Also, no reflections or rotations are allowed, only parallel translations of the tiles. We then have exactly $f_T(n) = \binom{n-2}{2}$ tilings of $[1 \times n]$, since the two light tiles must be in this order and can be anywhere in the sequence of $(n-2)$ tiles.

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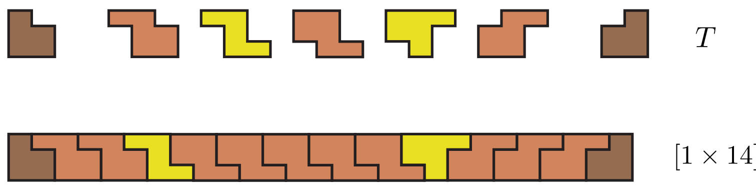

FIGURE 2. Set T of 5 rational tiles and two bookends; a tiling of $[1 \times 14]$ with T.

More generally, let $f_T(n)$ be the number of tilings on $[1 \times n]$ with a fixed set of rational tiles of height 1 and two bookends as above.[1] Denote by $\mathcal{F}_1$ the set of all such functions. It is easy to see via the transfer-matrix method (see e.g. [Sta1, §4.7]), that the GF $F_T(x) = f(0) + f(1)x + f(2)x^2 + \ldots$ is rational:

$$F_T(x) = \frac{P(x)}{Q(x)} \quad \text{for some} \quad P, Q \in \mathbb{Z}[x]\,.$$

In the two examples above, we have GFs $1/(1-x-x^2)$ and $x^4/(1-x)^3$, respectively.

Note, however, that the combinatorial nature of $f(n)$ adds further constraints on possible GFs $F_T(x)$. The following result gives a complete characterization of such GFs. Although never stated in this form, it is well known in a sense that it follows easily from several existing results (see §11.2 for references and details).

Theorem 1.1. *Function $f(n)$ is in $\mathcal{F}_1$, i.e. equal to $f_T(n)$ for all $n \geq 1$ and some rational set of tiles T as above, if and only if its* GF $F(x) = f(0)+f(1)x+f(2)x^2+\ldots$ *is $\mathbb{N}$-rational.*

Here the class $\mathcal{R}_1$ of $\mathbb{N}$-*rational functions* is defined to be the smallest class of GFs $G(x) = g(0) + g(1)x + g(2)x^2 + \ldots$, such that:

(1) $0, x \in \mathcal{R}_1$,
(2) $G_1, G_2 \in \mathcal{R}_1 \implies G_1 + G_2, G_1 \cdot G_2 \in \mathcal{R}_1$,
(3) $G \in \mathcal{R}_1$, $g(0) = 0 \implies 1/(1-G) \in \mathcal{R}_1$.

This class of rational GFs is classical and closely related to *deterministic finite automata* and *regular languages*, fundamental objets in the Theory of Computation (see e.g. [MM, Sip]), and Formal Language Theory (see e.g. [BR1, SS]).[2]

We are now ready to state the main result. Let T be a finite set of tiles as above (no bookends), which all have height 1 but now allowed to have irrational length intervals in the boundaries. Denote by $f(n) = f_{T,\varepsilon}(n)$ the number of tilings with T of rectangles $[1\times(n+\varepsilon)]$, where $\varepsilon \in \mathbb{R}$ is fixed. Denote by $\mathcal{F}$ the set of all such functions.

Observe that $\mathcal{F}$ is much larger than $\mathcal{F}_1$. For example, take 2 irrational tiles $\left[1 \times (\frac{1}{2} \pm \alpha)\right]$, for some $\alpha \notin \mathbb{Q}$, $0 < \alpha < 1/2$, and let $\varepsilon = 0$ (see Figure 3). Then $f(n) = \binom{2n}{n}$, and the GF equal to $F(x) = 1/\sqrt{1-4x}$.

Let $\mathcal{R}_k$ denote the multivariate $\mathbb{N}$-rational functions defined as a as the smallest class of GFs $F \in \mathbb{N}[[x_1, \ldots, x_k]]$, which satisfies condition

(1′) $0, x_1, \ldots, x_k \in \mathcal{R}_1$.

[1] For simplicity, we allow bookends in T to be *empty tiles*. In general, bookends play the role of boundary coloring for Wang tilings [Wang] (cf. [GaP, PY]). Note that irrational tilings are agile enough not to require them at all. This follows from our results, but the reader might enjoy finding a direct argument.

[2] Although we never state the connection explicitly, both theories give a motivation for this work, and are helpful in understanding the proofs (cf. §11.2).

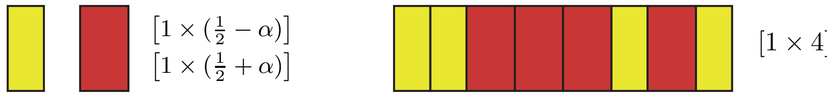

FIGURE 3. Set of 2 irrational tiles; a tiling of $[1 \times 4]$ with 8 tiles.

and conditions (2), (3) as above.

Main Theorem 1.2. *Function $f = f(n)$ is in $\mathcal{F}$ if and only if*

$$f(n) \, = \, \left[x_1^n \ldots x_k^n\right] F(x_1, \ldots, x_k) \quad \textit{for some} \quad F \in \mathcal{R}_k \, .$$

The theorem can be viewed as a multivariate version of Theorem 1.1, but strictly speaking it is not a generalization; here the number k of variables is not specified, and can in principle be very large even for small $|T|$ (cf. §11.7). Again, proving that $\mathcal{F}$ is a subset of diagonals of rational functions $F \in \mathbb{Z}[x_1, \ldots, x_k]$ is relatively straightforward by an appropriate modification of the transfer-matrix method, while our result is substantially stronger.

Main Theorem 1.3. *Function $f = f(n)$ is in $\mathcal{F}$ if and only if it can be written as*

$$f(n) \, = \sum_{(v_1,\ldots,v_d)\in\mathbb{Z}^d} \, \prod_{i=1}^{r} \binom{a_{i1}v_1 + \ldots + a_{id}v_d + a_i' n + a_i''}{b_{i1}v_1 + \ldots + b_{id}v_d + b_i' n + b_i''},$$

for some $r, d \in \mathbb{N}$, and $a_{ij}, b_{ij}, a_i', b_i', a_i'', b_i'' \in \mathbb{Z}$, for all $1 \le i \le r$, $1 \le j \le d$.[3]

The *binomial multisums* (multidimensional sums) as in the theorem is a special case of a very broad class of *holonomic functions* [PWZ], and a smaller class of *balanced multisums* defined in [Gar] (see §11.6). For examples of binomial multisums, take the *Delannoy numbers* D_n (sequence `A001850` in [OEIS]), and the *Apéry numbers* A_n (sequence `A005259` in [OEIS]) :

$$(\lozenge) \qquad D_n \, = \sum_{k=0}^{n} \binom{n+k}{n-k}\binom{2k}{k}, \qquad A_n \, = \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3.$$

In summary, Main Theorems 1.2 and 1.3 give two different characterizations of tile counting functions $f_{T,\varepsilon}(n)$, for some fixed $\varepsilon \in \mathbb{R}$ and an irrational set of tiles T. Theorem 1.2 is perhaps more structural, while Theorem 1.3 is easier to use to give explicit constructions (see Section 3). Curiously, neither direction of either main theorem is particularly easy.

The proof of the main theorems occupies much of the paper. We also present a number of applications of the main theorems, most notably to construction of tile counting function with given asymptotics (Section 4). This requires the full power of both theorems and their proofs. Specifically, we use the fact that this class of functions are closed under addition and multiplication – this is easy to see for the tile counting functions and the diagonals, but not for the binomial multisums.

The rest of the paper is structured as follows. We begin with definitions and notation (Section 2). In the next key Section 3, we expand on the definitions of classes $\mathcal{F}$, $\mathcal{B}$ and $\mathcal{R}_k$, illustrate them with examples and restate the main theorems. Then, in Section 4, we give applications to asymptotics of tile counting functions and to the Catalan numbers conjecture

[3] The binomial coefficients here are defined to be zero for negative parameters (see §2.1 for the precise definition); this allows binomial multisums in the r.h.s. to be finite.

(Conjecture 4.6). In the next four sections 5–8 we present the proof of the main theorems, followed by the proofs of applications (sections 9 and 10). We conclude with final remarks in Section 11.

2. Definitions and notation

2.1. **Basic notation.** Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{P} = \{1, 2, \ldots\}$, and let $\mathbb{A} = \overline{\mathbb{Q}}$ be the field of algebraic numbers. For a GF $G \in \mathbb{Z}[[x_1, \ldots, x_k]]$, denote by $\left[x_1^{c_1} \ldots x_k^{c_k}\right] G$ the coefficient of $x_1^{c_1} \ldots x_k^{c_k}$ in G, and by $[1]G$ the constant term in G.

For sequences $f, g : \mathbb{N} \to \mathbb{R}$, we use notation $f \sim g$ to denote that $f(n)/g(n) \to 1$ as $n \to \infty$. Here and elsewhere we only use the $n \to \infty$ asymptotics.

We assume that $0! = 1$, and $n! = 0$ for all $n < 0$. We also extend binomial coefficients to all $a, b \in \mathbb{Z}$ as follows:

$$\binom{a}{b} = \begin{cases} \frac{a!}{(a-b)!b!} & \text{if} \quad 0 \le b \le a\,, \\ 1 & \text{if} \quad a = -1,\ b = 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

CAVEAT: This is not the way binomial coefficients are normally extended to negative inputs; this notation allows us to use $\binom{a+b-1}{b}$ to denote the number of ways to distribute b identical objects into a distinct groups, for all $a, b \ge 0$.

2.2. **Tilings.** For the purposes of this paper, a *tile* is an axis-parallel simply connected (closed) polygon in $\mathbb{R}^2$. A *region* is a union of finitely many axis-parallel polygons. We use $|\tau|$ to denote the area of tile τ.

We consider only finite sets of tiles $T = \{\tau_1, \ldots, \tau_r\}$. A *tiling* of a region Γ with the set of tiles T, is a collection of non-overlapping translations of tiles in T (ignoring boundary intersections), which covers Γ. We use $\Phi_T(\Gamma)$ to denote the number of tilings of Γ with T.

A set of tiles T is called *tall* if every tile in T has height 1. We study only tilings with tall tiles of rectangular regions $\mathrm{R}_a = [1 \times a]$, where $a > 0$.

2.3. **Graphs.** Throughout the paper, we consider finite *directed weighted multi-graphs* $\mathrm{G} = (V, E)$. This means that between every two vertices $v, v' \in V$ there is a finite number of (directed) edges $v \to v'$, each with its own weight. A *path* γ in G is a sequence of oriented edges (v_1, v_2), (v_2, v_3), …, $(v_{\ell-1}, v_\ell)$; vertices v_1 and v_ℓ are called start and end of the path. A *cycle* is a path with $v_1 = v_\ell$. The *weight* of a path or a cycle, denoted $w(\gamma)$, is defined to be the sum of the weights of its edges.

3. Three classes of functions

3.1. **Tile counting functions.** Fix $\varepsilon \ge 0$ and let T be a set of tall tiles. In the notation above, $f(n) = \Phi_T(\mathrm{R}_{n+\varepsilon})$ is the number of tiling of of rectangles $[1 \times (n + \varepsilon)]$ with T. We refer to $f(n)$ as the *tile counting function*. In notation of the introduction, $\mathcal{F}$ is the set of all such functions.

Example 3.1. We define functions $g_1, \ldots, g_6 : \mathbb{N} \to \mathbb{N}$ as follows:

$$g_1(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \qquad g_2(n) = 2\,, \qquad g_3(n) = n^2\,,$$

$$g_4(n) = 2^n \qquad g_5(n) = F_n \qquad g_6(n) = \binom{2n}{n},$$

where F_n is the n-th Fibonacci number. Let us show that these functions are all in $\mathcal{F}$.

First, function g_1 counts tilings of a length n rectangle by a single rectangle R_2. Second, consider a set of six tiles T_2 as in Figure 4, with dark shaded tiles of area $\alpha > 0$, $\alpha \notin \mathbb{Q}$, the light shaded tiles of area 1, and set $\varepsilon = 2\alpha$. Now observe that $\mathrm{R}_{n+\varepsilon}$ rectangle can be tiled with T_2 in exactly two ways: one way using either the first or the second triple of tiles.

Third, take any two rationally independent irrational numbers $\alpha > \beta > 0$, and set $\varepsilon = \alpha + \beta$. Consider the set of three rectangles $T_3 = \{\mathrm{R}_1, \mathrm{R}_{1+\alpha}, \mathrm{R}_{1+\beta}, \mathrm{R}_{1+\alpha+\beta}\}$. Now observe that there are exactly n^2 tilings of $\mathrm{R}_{n+\alpha+\beta}$. Fourth, take a set T_4 with one unit square and two tiles which can only form a unit square, and observe that R_n has exactly 2^n tilings. The remaining two examples are given in the introduction.

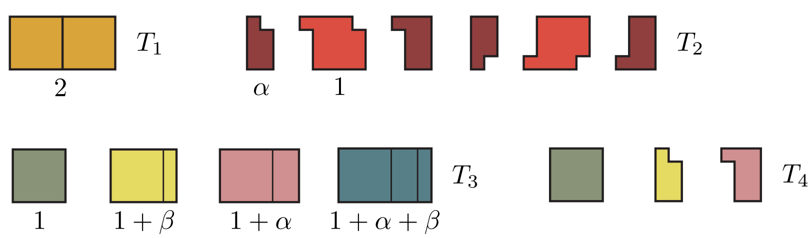

FIGURE 4. Tile sets $T_1, \ldots, T_4$ in the example.

3.2. **Diagonals of $\mathbb{N}$-rational generating functions.** As in the introduction, let $\mathcal{R}_k$ be the smallest class of GFs in k variables $x_1, \ldots, x_k$, satisfying

(1) $0,\ x_1, \ldots, x_k \in \mathcal{R}_k$,
(2) If $F, G \in \mathcal{R}_k$, then $F + F$ and $F \cdot G \in \mathcal{R}_k$.
(3) If $F \in \mathcal{R}_k$, and $[1]F = 0$, then $\frac{1}{1-F} \in \mathcal{R}_k$.

A GF in $\mathcal{R}_k$ is called an $\mathbb{N}$-*rational generating function in* k *variables.* Note that if $G(x_1, \ldots, x_k) \in \mathcal{R}_k$, then so is $G(x_1^m, \ldots, x_k^m)$,for all integer $m \geq 2$.

A *diagonal* of $G \in \mathbb{N}[[x_1, \ldots, x_k]]$ is a function $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(n) \ = \ \left[x_1^n \ldots x_k^n\right] G(x_1, \ldots, x_k)\,.$$

Denote by $\mathcal{D}$ the set of diagonals of all $\mathbb{N}$-rational generating functions, over all $k \in \mathbb{P}$.

Example 3.2. In notation of Example 3.1, let us show that $g_1, \ldots, g_6 \in \mathcal{D}$:

$$g_1(n) = \left[x^n\right]\frac{1}{1-x^2}\,,\ g_2(n) = \left[x^n\right]\,\frac{1}{1-x} + \frac{1}{1-x}\,,\ g_3(n) = \left[x^n y^n\right] x \left(\frac{1}{1-x}\right)^2 y \left(\frac{1}{1-y}\right)^2,$$

$$g_4(n) = \left[x^n\right]\,\frac{1}{1-2x}\,,\ g_5(n) = \left[x^n\right]\,\frac{1}{1-x-x^2}\,,\ g_6(n) = \left[x^n y^n\right]\,\frac{1}{1-x-y}\,.$$

3.3. **Binomial multisums.** Following the statement of Main Theorem 1.3, denote by $\mathcal{B}$ the set of all functions $f : \mathbb{N} \to \mathbb{N}$ that can be expressed as

$$f(n) \, = \, \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

for some $\alpha_i = \mathbf{a}_i v + a_i' n + a_i''$, $\beta_i = \mathbf{b}_i v + b_i' n + b_i''$, where $r, d \in \mathbb{N}$, $\mathbf{a}_i, \mathbf{b}_i : \mathbb{Z}^d \to \mathbb{Z}$ are integer linear functions, and $a_i', b_i', a_i'', b_i'' \in \mathbb{Z}$, for all i.

Note that the summation over all $v \in \mathbb{Z}^d$ is infinite, so it is unclear from the definition whether the multisums $f(n)$ are finite. However, the binomial coefficients are zero for the negative values of β_i and $\alpha_i - \beta_i$, so the summation is in fact over integer points in a convex polyhedron defined by these inequalities.

Example 3.3. In notation of Example 3.1, it follows from the definition that $g_2, g_6 \in \mathcal{B}$. To see $g_1, g_3, g_4, g_5 \in \mathcal{B}$, note that

$$g_1(n) \, = \, \sum_{v \in \mathbb{Z}} \binom{n}{2v}\binom{2v}{n}, \quad g_3(n) \, = \, \binom{n}{1}\binom{n}{1}, \quad g_4(n) \, = \, \sum_{v \in \mathbb{Z}} \binom{n}{v}, \quad g_5(n) \, = \, \sum_{v \in \mathbb{Z}} \binom{n-v}{v}.$$

For the last formula for the Fibonacci numbers $g_5(n) = F_n$ is classical, see e.g. [Rio, p. 14] or [Sta1, Exc. 1.37].

3.4. **Main theorems restated.** Surprisingly, the class $\mathcal{B}$ of binomial multisums as above coincides with both tile counting functions and diagonals of $\mathbb{N}$-rational functions, and plays an intermediate role connecting them.

Main Theorem 3.4. $\mathcal{F} = \mathcal{D} = \mathcal{B}$.

The proof of Main Theorem 3.4 is split into three parts. Lemmas 5.1, 6.1 and 7.1 state $\mathcal{F} \subseteq \mathcal{B}$, $\mathcal{D} \subseteq \mathcal{F}$ and $\mathcal{B} \subseteq \mathcal{D}$, respectively. Each is proved in a separate section, and together they imply the Main Theorem.

Corollary 3.5. *The classes of functions $\mathcal{F} = \mathcal{D} = \mathcal{B}$ are closed under addition and (point-wise) multiplication.*

This follows from the Main Theorem 3.4 and Lemma 7.2, which proves the claim for diagonals $f \in \mathcal{D}$.

Before we proceed to further applications, let us obtain the following elementary corollary of the Main Theorem 3.4. Note that each of these tile counting five functions can be constructed directly via ad hoc argument in the style of Example 3.1. We include it as an illustration of the versatility of the theorem.

Corollary 3.6. *The following functions $f_1, \ldots, f_5 : \mathbb{N} \to \mathbb{N}$ are tile counting functions:*

(i) f_1 *has finite support,*
(ii) f_2 *is periodic,*
(iii) $f_3(n) = a_p n^p + \ldots + a_1 n + a_0$, *where* $a_i \in \mathbb{N}$,
(iv) $f_4(n) = m^n$, *where* $m \in \mathbb{N}$,
(v) $f_5(n) = m^n - 1$, *where* $m \in \mathbb{N}$, $m \geq 1$.

Proof. By the main theorem, it suffices to show that each function f_i is in $\mathcal{D}$. Clearly, function f_1 is the diagonal of a polynomial, so $f_1 \in \mathcal{D}$.

The functions

$$f_{k,p}(m) = \begin{cases} 1 & \text{if } m = k \bmod p \\ 0 & \text{otherwise} \end{cases}$$

are the diagonals of the generating functions $\frac{x^k}{1-x^p}$, for all $0 \le k < p$, so are clearly in $\mathcal{D}$, and f_2 can be expressed as a sum of these $f_{k,p}$ functions. Since $\mathcal{D}$ is closed under addition, this implies $f_2 \in \mathcal{D}$. Similarly, the polynomial $f(n) = 1$ and $f(n) = n$ are the diagonals of $1/(1-x)$ and $x/(1-x)^2$ respectively, and thus in $\mathcal{D}$. Since $\mathcal{D}$ is closed under addition and multiplication, we have $f_3 \in \mathcal{D}$.

The function f_4 is the diagonal of $\frac{1}{1-mx}$, and therefore in $\mathcal{D}$. Similarly, the function f_5 satisfies the recurrence $f_5(n+1) = mf_5(n)+(m-1)$, and thus the diagonal of the generating function $G(x)$ satisfying $G = mxG + (m-1)/(1-x)$. Note that

$$G(x) \ = \ (m-1) \cdot \frac{1}{(1-x)} \cdot \frac{1}{(1-mx)}\,.$$

Therefore, $G \in \mathcal{R}_1$, which implies $f_5 \in \mathcal{D}$. $\square$

3.5. **Two more examples.** Recall that our definition of binomial coefficients is modified to have $\binom{-1}{0} = 1$, see §2.1. This normally does not affect any (usual) binomial sums, e.g. the Delannoy and Apéry numbers defined in the introduction remain unchanged when the summations in ($\Diamond$) are extended to all integers. Simply put, whenever $\binom{-1}{0}$ appears there, some other binomial coefficient in the product is equal to zero.

The proof of Main Theorem 3.4 is constructed by creating a large number of auxiliary variables for the $\mathbb{N}$-rational functions, and auxiliary indices for the binomial multisums. These auxiliary indices are often constrained to a small range, and $\binom{-1}{0}$ does appear in several cases.

Example 3.7. Denote by L_n the *Lucas numbers* $L_n = L_{n-1} + L_{n-2}$, where $L_1 = 1$ and $L_2 = 3$, see e.g. [Rio, §4.3] (sequence `A000204` in [OEIS]). They have a combinatorial interpretation as the number of matchings in an n-cycle, and are closely related to Fibonacci numbers F_n :

$$(\circledcirc) \qquad L_n \ = \ F_n + F_{n-2} \quad \text{for } n \ge 2\,.$$

From Corollary 3.5, the function $f(n) := L_n$ is in $\mathcal{F}$. In fact, it is immediate that $L_n \in \mathcal{R}_1$:

$$L_n \ = \ [x^n]\,\frac{1+x^2}{1-x-x^2}\,.$$

To see directly that Lucas numbers are in $\mathcal{F}$, take five tiles as in Figure 5, with two right bookends, emulating ($\circledcirc$). On the other hand, finding a binomial sum is less intuitive, as $\mathcal{B}$ is not obviously closed under addition. In fact, we have:

$$L_n \ = \sum_{(k,i)\in\mathbb{Z}^2} \binom{n-k-2i}{k}\binom{1}{i},$$

where we use ($\circledcirc$), the formula for $g_5(n)$ in Example 3.3, and make i constrained to $\{0,1\}$. Note that we avoid using $\binom{-1}{0}$.

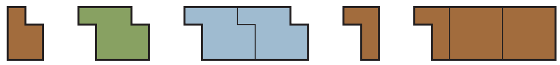

FIGURE 5. Five tiles giving Lucas numbers L_n.

Example 3.8. Let $f(n) = 2^n + 3^n$. Checking that $f \in \mathcal{F}$ and $f \in \mathcal{D}$ is straightforward and similar to $g_4(n)$ in the examples above. However, finding a binomial multisum is more difficult:

$$f(n) = \sum_{(i,j,k,\ell,m)\in\mathbb{Z}^5} \binom{n}{i}\binom{m}{j}\binom{1}{k}\binom{m-k}{m}\binom{\ell+k-1}{\ell}\binom{i}{m+\ell}\binom{m+\ell}{i}.$$

Note here that the term $\binom{1}{k}$ gives $k \in \{0,1\}$. Also, $\binom{i}{m+\ell}\binom{m+\ell}{i}$ terms give $m+\ell = i$. Similarly, $\binom{m-k}{m}\binom{\ell+k-1}{\ell}$ give that $m = 0$ if $k = 1$, and $\ell = 0$ if $k = 0$. Therefore,

$$f(n) = \sum_{(j,m)\in\mathbb{Z}^2} \binom{n}{m}\binom{m}{j} + \sum_{\ell\in\mathbb{Z}} \binom{n}{\ell} = 2^n + 3^n,$$

where two sums correspond to the cases $k = 0$ and $k = 1$, respectively. Note that $\binom{-1}{0} = 1$ is essential in this calculation. It would be interesting to see if Theorem 1.3 holds without modification.

4. APPLICATIONS

4.1. **Balanced multisums.** Define a *positive multisum* to be a function $g : \mathbb{N} \to \mathbb{N}$ that can be expressed as

$$g(n) = \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r} \frac{\alpha_i(v,n)!}{\beta_i(v,n)!\,\gamma_i(v,n)!},$$

for some $\alpha_i = \mathbf{a}_i v + a_i' n + a_i''$, $\beta_i = \mathbf{b}_i v + b_i' n + b_i''$, $\gamma_i = \mathbf{c}_i v + c_i' n + c_i''$, where $r, d \in \mathbb{N}$, $\mathbf{a}_i, \mathbf{b}_i, \mathbf{c}_i : \mathbb{Z}^d \to \mathbb{Z}$ are integer linear functions, and $a_i', \ldots, c_i'' \in \mathbb{Z}$, for all i. Here the sum is over all $v \in \mathbb{Z}^d$ for which $\alpha_i(v,n), \beta_i(v,n), \gamma_i(v,n) \geq 0$, for all i.

Positive multisum is called *balanced* if $\alpha_i = \beta_i + \gamma_i$ for all i. Denote by $\mathcal{B}'$ the set of finite sums of balanced positive multisums:

$$f(n) = g_1(n) + \ldots + g_k(n).$$

Theorem 4.1. $\mathcal{B} = \mathcal{B}'$.

The Delannoy and Apéry numbers defined in equation ($\lozenge$) in the introduction are examples of balanced multisums, as are Lucas numbers, see Example 3.7. These formulas use only one balanced positive multisum, i.e. have $k = 1$. However, as Example 3.8 suggests, the sums $f(n) = 2^n + 3^n$ can we written with $k = 2$, as the lengthy binomial multisum for $f(n)$ involves using the $\binom{-1}{0} = 1$ notation. Therefore, one can think of Theorem 4.1 as a *tradeoff*: we prohibit using the $\binom{-1}{0}$ notation, but now allow taking finite sums of balanced multisums (cf. §11.6).

We give direct proof of the theorem in Section 10. Note that $\mathcal{B}'$ is trivially closed under addition and multiplication, so Theorem 4.1 together with the main theorem immediately implies Corollary 3.5.

4.2. **Growth of tile counting functions.** We say that a function f is *eventually polynomial* if there exist an $N \in \mathbb{N}$ and a polynomial q such that for all $n \geq N$, we have $f(n) = q(n)$. We say that a function f *grows exponentially*, if there exist $c_1, c_2 > 0$ and $N \in \mathbb{N}$, such that for all $n \geq N$, we have $e^{c_1 n} \leq f(n) \leq e^{c_2 n}$.

Theorem 4.2. *Let $f \in \mathcal{F}$ be a tile counting function. There exists an integer $m \geq 1$, such that every function $f_i(n) := f(nm + i)$ either grows exponentially or is eventually polynomial, where $0 \leq i \leq m - 1$.*

In particular, Theorem 4.2 implies that the growth of f is at most exponential. Further, if the growth of f is subexponential, then f must have polynomial growth. This rules out many natural combinatorial and number theoretic sequences, e.g. the number of partitions $p(n)$, or the n-th prime p_n, cf. [FGS].

The proof of Theorem 4.2 uses the geometry of integer points in convex polyhedra; it is given in Section 9. The theorem should be contrasted with the following asymptotic characterization of diagonals of rational functions, which follows from several known results:

Theorem 4.3 (See §11.3)**.** *Let $f(n)$ be a diagonal of P/Q, where $P, Q \in \mathbb{Z}[x_1, \ldots, x_k]$. Suppose further that $f(n) = \exp O(n)$ as $n \to \infty$. Then there exists an integer $m \geq 1$, s.t.*

$$f(n) \sim A\lambda^n n^\alpha (\log n)^\beta, \quad \text{for all} \quad n = i \mod m, \quad 0 \leq i \leq m-1,$$

where $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$, and $\lambda \in \mathbb{A}$.

In our case, the subexponential growth implies $\lambda = 1$, which gives asymptotics $An^\alpha(\log n)^\beta$. Theorem 4.2 implies further that $\alpha \in \mathbb{N}$, $\beta = 0$, and $A \in \mathbb{Q}$ in that case.

Example 4.4. The following binomial sums show that nontrivial exponents $\alpha \notin \mathbb{Z}$ and $\beta > 0$ can indeed appear for $f \in \mathcal{F}$ and $\lambda > 1$:

$$\binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 4^n n^{-1/2}, \qquad \sum_{k=1}^{n} \binom{2k}{k}^2 16^{n-k} \sim \frac{1}{\pi} 16^n \log n.$$

Following these examples, we conjecture that α is always half-integer:

Conjecture 4.5. *Let $f \in \mathcal{F}$ be a tile counting function. Then there exists an integer $m \geq 1$, s.t.*

$$f(n) \sim A\lambda^n n^\alpha (\log n)^\beta, \quad \text{for all} \quad n = i \mod m, \quad 0 \leq i \leq m-1,$$

where $\alpha \in \mathbb{Z}/2$, $\beta \in \mathbb{N}$, and $\lambda \in \mathbb{A}$.

See §11.3 for a brief overview of related asymptotic results.

4.3. **Catalan numbers.** Recall the Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

We make the following mesmerizing conjecture.

Conjecture 4.6. *The Catalan numbers C_n is* not *a tile counting function.*

Several natural approaches to the conjecture can be proved not to work. First, we show that the naive asymptotic approach cannot be used to prove Conjecture 4.6.

Proposition 4.7. *For every $\epsilon > 0$, there exists a tile counting function $f \in \mathcal{F}$, s.t.*

$$f(n) \sim A \cdot C_n \qquad \text{for some} \qquad A \in (1-\epsilon, 1+\epsilon).$$

In a different direction, we show that Conjecture 4.6 does not follow from elementary number theory considerations.

Proposition 4.8. *For every $m \in \mathbb{N}$, there exists a tile counting function $f \in \mathcal{F}$, s.t. $f(n) = C_n$ mod m.*

Proposition 4.9. *For every prime p, there exists a tile counting function $f \in \mathcal{F}$, s.t. $\mathrm{ord}_p\big(f(n)\big) = \mathrm{ord}_p(C_n)$, where $\mathrm{ord}_p(m) = \max\{d : p^d | m\}$.*

The results in this subsection are proved in Section 10. See §11.10 for more on the last proposition.

4.4. **Hypergeometric functions.** We use the following special case of the *generalized hypergeometric function*:

$$ {}_{p+1}F_p(a_1, \ldots, a_p, 1; b_1, \ldots, b_p; r) \;=\; \sum_{m=0}^{\infty} \prod_{k=0}^{m-1} \frac{(k+a_1)(k+a_2)\ldots(k+a_p)r}{(k+b_1)(k+b_2)\ldots(k+b_p)}. $$

Let p be a positive integer and $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash p$ be a partition of p. Denote by Υ_λ the following multiset of p rational numbers:

$$ \Upsilon_\lambda \;=\; \bigcup_{i=1}^{\ell} \left\{ \frac{1}{\lambda_i}, \frac{2}{\lambda_i}, \ldots, \frac{\lambda_i - 1}{\lambda_i}, 1 \right\}. $$

For example, if $\lambda = (5, 4, 2, 1) \vdash 12$, then

$$ \Upsilon_\lambda \;=\; \left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{1}{2}, 1, 1 \right\}. $$

Theorem 4.10. *Let $\mu = (\mu_1, \ldots, \mu_k) \vdash p$, and let $\nu = (\nu_1, \ldots, \nu_\ell) \vdash p$ be a* refinement *of μ. Write*

$$ \Upsilon_\mu = \{a_1, \ldots, a_p\}, \quad \Upsilon_\nu = \{b_1, \ldots, b_p\}, $$

and fix $r = r_1/r_2 \in \mathbb{Q}$. Denote $A = {}_{p+1}F_p(a_1, \ldots, a_p, 1; b_1, \ldots, b_p; r)$, and suppose that $A < \infty$ is well defined. Finally, let $c \in \mathbb{N}$ be a multiple of all prime factors of $\mu_1 \cdot \mu_2 \cdot \ldots \cdot \mu_k \cdot r_2$. Then, there exists a tile counting function $f \in \mathcal{F}$, s.t. $f(n) \sim A c^n$.

The proof of Theorem 4.10 is given in Section 10.

Corollary 4.11. *There exists a tile counting function $f \in \mathcal{F}$, such that*

$$ f(n) \sim \frac{\sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)}\, 128^n. $$

Proof. Let $p = 4$, let $\mu = (4)$, $\nu = (2, 1, 1)$, and set $r = 1/2$. Then $\Upsilon_\mu = \{1/4, 1/2, 3/4, 1\}$ and $\Upsilon_\nu = \{1/2, 1, 1, 1\}$. Since any even c is allowed, we can take $c = 128$. Then, by Theorem 4.10, there exists $f \in \mathcal{F}$, s.t.

$$ \frac{f(n)}{128^n} \sim {}_5F_4\Big(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, 1; \frac{1}{2}, 1, 1, 1; \frac{1}{2}\Big) \;=\; {}_2F_1\Big(\frac{1}{4}, \frac{3}{4}; 1; \frac{1}{2}\Big) \;=\; \frac{\sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)}, $$

as desired. □

Since the proof of Theorem 4.10 is constructive, we can obtain an explicit tile counting function $f(n)$ as in the corollary:

$$f(n) \, = \, \sum_{k=0}^{n} \binom{4k}{k}\binom{3k}{k} \, 128^{n-k} \, \sim \, \frac{\sqrt{\pi}}{\Gamma(5/8)\Gamma(7/8)} \, 128^n \, .$$

The corollary and the theorem suggest that there is no easy characterization of constants A in Conjecture 4.5, at least not enough to obtain Conjecture 4.6 this way. Here is yet another quick variation on the theme.

Corollary 4.12. *There exists a tile counting function $f \in \mathcal{F}$, such that*

$$f(n) \, \sim \, \frac{\Gamma(3/4)^3}{\sqrt[3]{2}\,\pi} \, 6^n.$$

Proof. Take $p = 3$, $\mu = (3)$, $\nu = (1,1,1)$, and proceed as above. □

5. Tile counting functions are binomial multisums

In this section, we prove the following result towards the proof of Main Theorem 3.4.

Lemma 5.1. $\mathcal{F} \subseteq \mathcal{B}$.

The proof first restates the lemma in the language of counting cycles in multi-graphs G (see §2.3), and then uses graph theoretic tools to give a binomial-multisum formula for the latter.

5.1. **Cycles in graphs.** We first show how to compute tile counting functions in the language of cycles in weighted graph.

Lemma 5.2. *For every tile counting function $f(n)$ there exists a finite weighted directed multi-graph G_T with vertices $v_0, \ldots, v_N$, such that $f(n)$ is the number of paths of weight $n + \varepsilon$, which start and end at v_0.*

Proof. Fix an $f(n) = \Phi_T(\mathrm{R}_{n+\varepsilon})$. Recall that each tile $\tau \in T$ has height 1. Denote by $\partial_L(\tau)$ and $\partial_R(\tau)$ the left boundary and right boundary curves of τ of height 1, respectively. A sequence of tiles $(\tau_0, \ldots, \tau_\ell)$ is a tiling of $\mathrm{R}_{n+\varepsilon}$ if and only if

1. $\partial_R(\tau_i) = \partial_L(\tau_{i+1})$ for $0 \le i \le \ell - 1$,
2. $\partial_L(\tau_0)$ is a vertical line,
3. $\partial_R(\tau_\ell)$ is a vertical line,
4. $|\tau_0| + \ldots + |\tau_\ell| = n + \varepsilon$.

Here the first condition implies that all the tiles fit together with no gaps, the second and third conditions imply that the union of the tiles is actually a rectangle, and the fourth condition implies that the rectangle has length $n + \varepsilon$.

We now construct a weighted directed multi-graph G_T corresponding to T as follows. The vertices of G_T are exactly the set of left or right boundaries of tiles (up to translation). Denote them $v_0, \ldots, v_N$, where v_0 is the vertical line. Let the edge $e_{ij} = (v_i, v_j)$ in G_T correspond to tile $\tau \in T$, such that $\partial_L(\tau) = v_i$, $\partial_R(\tau) = v_j$, and let $\mathrm{weight}(e_{ij}) = |\tau|$. Note that edges e_{ij} and e'_{ij}, corresponding to tiles τ and τ', can have different weight. By construction, the paths in G_T of weight $n + \varepsilon$, which start and end at v_0, are in bijection with tilings of $\mathrm{R}_{n+\varepsilon}$. □

5.2. **Irreducible cycles.** To count the number of cycles in a directed multi-graph starting at v_0 of weight $n+\varepsilon$, we factor the cycles into irreducible cycles.

Let $\mathrm{G} = (V, E)$ be a finite directed multi-graph, and let $V = \{v_0, \ldots, v_N\}$. A cycle γ in G is called *positive* if it starts and ends at v_i, and only passes through vertices v_j with $j \geq i$. Cycle γ is called *irreducible* if it is a positive and contains no positive shorter cycle γ'; we refer to γ' as *subcycle* of γ.

Lemma 5.3. *There are finitely many irreducible cycles in* G.

Proof. We proceed by induction on the number $N+1$ of vertices in G. The claim is trivial for $N = 0$. Suppose $N \geq 1$ and let γ be an irreducible cycle in G. If γ does not contain every vertex in G, we can delete an unvisited vertex v_i and apply inductive assumption to a smaller graph $\mathrm{G}' = \mathrm{G} - v_i$. Thus we can assume that γ contains all vertices.

Since γ is positive and contains all vertices, it must start at v_0. Since γ is irreducible, it never come back to v_0 until the end. Note that γ passes through v_1 exactly one, since otherwise it is not irreducible. Identify vertices v_0 and v_1, and denote by H the resulting smaller graph. The cycle γ is then mapped into a concatenation of two irreducible cycles in H. Applying inductive assumption to H gives the result. □

5.3. **Multiplicities of irreducible cycles.** Let ρ be an irreducible subcycle of a positive cycle γ. Define $\gamma - \rho$ to be the positive cycle given by traversing γ, but skipping over ρ. The *multiplicity* of ρ in γ, denoted $m(\rho, \gamma)$, is defined to be:

$$\rho(\gamma) = \begin{cases} 1 & \text{if } \gamma = \rho, \\ 0 & \text{if } \gamma \text{ is irreducible and not equal to } \rho, \\ m(\rho, \gamma') + m(\rho, \gamma - \gamma') & \text{if } \gamma' \text{ is an irreducible positive subcycle of } \gamma. \end{cases}$$

Lemma 5.4. *The multiplicity* $m(\rho, \gamma)$ *is well defined.*

In other words, the multiplicity $m(\rho, \gamma)$ represents the number of times ρ appears in the decomposition of γ. This allows us to count cycles in G_T, which start and end at v_0, that decompose into a given list of irreducible cycles.

Proof of Lemma 5.4. By contradiction, assume γ is the smallest positive cycle with irreducible decompositions $\rho_1, \ldots, \rho_k$ and $\rho'_1, \ldots, \rho'_\ell$, giving different multiplicities.

We claim that ρ'_1 must appear on the first list as ρ_i and does not intersect (edge-wise) any of the previous cycles ρ_j, $j < i$. Indeed, neither ρ_j can contain ρ'_1 or vice versa since both are irreducible. However, if they have non-empty overlap, one of them must contain the end of another which contradicts positivity. Since the edges of ρ'_1 have to be eventually removed, we have the claim.

By construction, we now have a new positive cycle $\gamma' = \gamma - \rho'_1$ with irreducible decompositions $\rho_1, \ldots, \rho_{i-1}, \rho_{i+1}, \ldots, \rho_k$ and $\rho'_2, \ldots, \rho'_\ell$, giving different multiplicities. This contradicts the assumption that γ is minimal. □

5.4. **Counting cycles.** Let T be a tall set of tiles and $f(n) = \Phi_T(\mathrm{R}_{n+\varepsilon})$. Consider graph G_T constructed in Lemma 5.2, and let $\rho_1, \ldots, \rho_r$ be the list of irreducible cycles in G_T, ordered lexicographically. Denote by $B_T(z_1, \ldots, z_r)$ the number of cycles γ in G_T, which start at v_0 and have multiplicity $m(\rho_i, \gamma) = z_i$.

For each $0 < j < i$, let $a_{i,j}$ be the number of times the first vertex in ρ_i is visited in ρ_j, where the first and last vertex in ρ_j is considered to be visited exactly once. Let $a_{i,0} = 1$ if the first vertex of ρ_i is v_0 and let $a_{i,0} = 0$ otherwise.

Lemma 5.5. *We have*

$$B_T(z_1,\ldots,z_r) \,=\, \prod_{i=1}^{r} \binom{a_{i,0}+a_{i,1}z_1+\ldots+a_{i,i-1}z_{i-1}+z_i-1}{z_i}.$$

Proof. Given a cycle γ in G_T starting at v_0 with $m(\rho_j,\gamma) = z_j$ for all j, we can remove irreducible subcycles of γ one at a time, until we are left with the empty cycle at v_0. Reversing the process, we can also speak of "adding" irreducible cycles to build γ.

Let i, $0 \le i \le r$, be maximal index, such that $z_i > 0$. By definition, every vertex in ρ_i has index greater than every vertex at the start of a irreducible cycle with positive multiplicity in γ. Thus, irrespectively of order in which we add the irreducible cycles, no cycles are inserted in the middle of a copy of ρ_i. Therefore, we may assume that the copies of ρ_i are added last. Further, one can take the cycle γ and determine the cycle with all of the copies of ρ_i removed, and the locations where the $\rho_i(\gamma)$ copies of ρ_i were inserted.

Let γ' be the cycle γ with all copies of ρ_i removed, and let v_k be the start of ρ_i. Note that the number of v_k in γ' is exactly

$$a_{i,0}+a_{i,1}z_1+\ldots+a_{i,i-1}z_{i-1}\,,$$

since adding the cycle ρ_j adds $a_{i,j}$ more vertices v_k. Therefore, the number of ways to add z_i copies of ρ_i to γ' is equal to

$$(\star) \qquad \binom{a_{i,0}+a_{i,1}z_1+\ldots+a_{i,i-1}z_{i-1}+z_i-1}{z_i}.$$

We conclude that $B_T(z_1,\ldots,z_r)$ is $(\star)$ times the number of possible strings you can get after removing all z_i copies of ρ_i. This gives the recursive formula:

$$B_T(z_1,\ldots,z_i,0,\ldots,0) \,=\, \binom{a_{i,0}+a_{i,1}z_1+\ldots+a_{i,i-1}z_{i-1}+z_i-1}{z_i} B_T(z_1,\ldots,z_{i-1},0,\ldots,0).$$

Since $B_T(0,\ldots,0)=1$, iterating the above formula gives the result. □

We can now count all cycles γ which start at v_0 by summing over all lists of irreducible cycles as above giving decompositions of γ.

Lemma 5.6. *Every tile counting function $f \in \mathcal{F}$ can be written as*

$$f(n) \,=\, \sum \prod_{i=1}^{r} \binom{a_{i,0}+a_{i,1}z_1+\ldots+a_{i,i-1}z_{i-1}+z_i-1}{z_i},$$

where the sum is over all $(z_1,\ldots,z_r) \in \mathbb{Z}^r$ satisfying $c_1z_1+\ldots+c_rz_r = n+\varepsilon$, where all $c_i \in \mathbb{R}$ and $a_{i,j} \in \mathbb{N}$.

Proof. By Lemma 5.2, the function $f(n)$ counts the number of cycles γ in G_T which start at v_0 of weight $n+\varepsilon$. In notation above, we have for such γ :

$$w(\gamma) \,=\, w(\rho_1)m(\rho_1,\gamma)+\ldots+w(\rho_r)m(\rho_r,\gamma) \,=\, n+\varepsilon.,$$

Therefore,

$$f(n) \,=\, \sum B_T(z_1,\ldots,z_r)\,,$$

where the summation is over all $(z_1,\ldots,z_r)$ such that $w(\rho_1)z_1+\ldots+w(\rho_r)z_r \,=\, n+\varepsilon$. Now Lemma 5.5 implies the result. □

5.5. **Proof of Lemma 5.1.** In notation above, denote by Z_n the set of of all vectors $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{Z}^r$ satisfying $c_1 z_1 + \ldots + c_r z_r = n + \varepsilon$, where all $c_i \in \mathbb{R}$ and $a_{i,j} \in \mathbb{N}$. By Lemma 5.6, every $f \in \mathcal{F}$ can be written as

$$f(n) \, = \, \sum_{\mathbf{z} \in \mathrm{Z}_n} \prod_{i=1}^{r} \binom{a_{i,0} + a_{i,1} z_1 + \ldots + a_{i,i-1} z_{i-1} + z_i - 1}{z_i}.$$

Without loss of generality, we may assume that $c_r = \varepsilon$ and $c_{r-1} = 1$, since if this were not the case, we could add two tiles to T of area ε and 1, each with a new boundary that only fits together with itself. This adds two disjoint loops to G, and we can label the vertices so that these two disjoint loops are the last two irreducible cycles. Note that for any n, the set Z_n is nonempty. In particular, it contains the vector $(0, 0, \ldots, n, 1)$.

Consider the set $W \subset \mathbb{Z}^r$ of all integer vectors $(w_1, \ldots w_r)$ with $c_1 w_1 + \ldots + c_r w_r = 0$. This set forms a lattice, and therefore has a basis, $\mathbf{b}_1, \ldots, \mathbf{b}_d$. Note that the set Z_n is exactly the set of all vectors $\mathbf{z} = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + \ldots + v_d \mathbf{b}_d + (0, \ldots, n, 1)$, with each $v_i \in \mathbb{Z}$, and each vector is expressible uniquely in this way. Thus, each coordinate $z_i = \beta_i(v_1, \ldots, v_d, n)$ is an integer coefficient affine function of $(v_1, \ldots, v_d, n)$. This implies

$$a_{i,0} + a_{i,1} z_1 + \ldots + a_{i,i-1} z_{i-1} + z_i - 1 \, = \, \alpha_i(v_1, \ldots, v_d, n)\,,$$

where α_i is also integer coefficient affine functions of $(v_1, \ldots, v_d, n)$. Therefore,

$$f(n) = \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

where α_i, β_i and $r, d \in \mathbb{P}$ are as desired. $\square$

6. Diagonals of $\mathbb{N}$-rational functions are tile counting functions

In this section, we make the next step towards the proof of Main Theorem 3.4.

Lemma 6.1. $\mathcal{D} \subseteq \mathcal{F}$.

In other words, we prove that every diagonal f of an $\mathbb{N}$-rational generating function, is also a tile counting function.

6.1. **Paths in networks.** Let $\mathrm{W} = (V, E)$ be a directed weighted multi-graph with a unique *source* v_1 and *sink* v_2. Further, assume that the edges of W are colored with k colors. We call such graph a *k-network*. We need the following technical lemma.

Lemma 6.2. *Let $G(x_1, \ldots, x_k) \in \mathcal{R}_k$. Then there exists a k-network W, such that for all $(n_1, \ldots, n_k) \in \mathbb{N}^k$, $n_1 + \ldots + n_k \geq 1$, the number of paths from v_1 to v_2 with exactly n_i edges of color i is equal to $\left[x_1^{n_1} \ldots x_k^{n_k}\right] G$.*

Proof. Let $\mathcal{Q}_k$ be the set of GFs, for which there is a k-network as in the lemma. We show that $\mathcal{Q}_k$ satisfies the three conditions in the definition of $\mathbb{N}$-rational generating function, which proves the result.

Condition (1) is trivial. To get $0 \in \mathcal{R}'_k$, take the graph with vertices v_1 and v_2 and no edges. Similarly, to get $x_i \in \mathcal{R}'_k$, take the graph with vertices v_1 and v_2 and a unique edge (v_1, v_2) of color i.

For (2), let $F, G \in \mathcal{Q}_k$, and let U, W be the corresponding k-networks. Attaching sinks and sources, and the rest of U and W in parallel, gives $F + G \in \mathcal{Q}_k$ (see Figure 6). Similarly, if $[1]F = [1]G = 0$, attaching U and W sequentially gives GF $F \cdot G$. More generally, for $a = [1]F$ and $b = [1]G$ not necessarily zero, write $aG = G + \ldots + G$ (a times), and use

$$F \cdot G = (F - a) \cdot (G - b) + aG + bF.$$

to obtain the desired k-network.

For (3), let $F \in \mathcal{Q}_k$ and $[1]F = 0$. To obtain $1/(1 - F)$, write:

$$\frac{1}{1-F} = 1 + F + F \cdot F + \frac{F^3}{1-F^2} + F \cdot \frac{F^3}{1-F^2}.$$

For $\frac{F^3}{1-F^2}$, arrange four copies of k-network U as shown in Figure 6. The details are straightforward. □

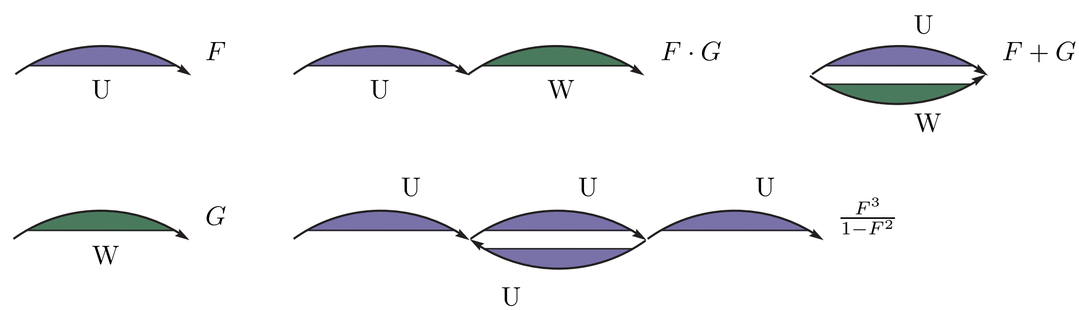

FIGURE 6. Networks giving $F \cdot G$, $F + G$ and $F^3/(1 - F^2)$.

6.2. **Proof of Lemma 6.1.** Let $G \in \mathcal{R}_k$ be such that $f(n) = \left[x_1^n \ldots x_k^n\right] G$. By Lemma 6.2, there is a k-network W with source v_1 and sink v_2, such that there are exactly $f(n)$ paths from v_1 to v_2, which pass through n edges of color i, for all i.

Let $\varepsilon, \alpha_1, \ldots, \alpha_k > 0$ be irrational numbers, such that the only rational linear dependence between them is $\alpha_1 + \ldots + \alpha_k = 1$. Assign weight α_i to each edge in W with color i. Add vertex v_0 and edges (v_0, v_1) (v_2, v_0), both of weight $\varepsilon/2$. Denote the resulting graph by W_0.

Note that cycles in W_0 which start at v_0 of weight $n + \varepsilon$ are in bijection with paths from v_1 to v_2 in W with exactly n edges of each color. Therefore, there are exactly $f(n)$ of them.

In notation of the proof of Lemma 5.2, we associate a different tile boundary ∂_i with vertices v_i, $1 \le i \le N$, and the vertical line segment with the vertex v_0. We can always ensure $\mathrm{width}(\partial_i) < \frac{1}{3} \min\{\varepsilon/2, \alpha_1, \ldots, \alpha_k\}$.

For every edge $e = (v_i, v_j)$ in W_0 with weight w_e, denote by τ_e the unique tile with height 1, $\partial_L(\tau_e) = \partial_i$, $\partial_R(\tau_e) = \partial_j$ and area $|\tau_e| = w_e$. Note that such tile exists by the width condition above. Let T be the set of tiles τ_e. From above, for all $n \ge 1$, the number of tilings of $\mathrm{R}_{n+\varepsilon}$ by T is equal to the number of cycles in W_0 starting at v_0 of weight $n + \varepsilon$, which is equal to $f(n)$ by assumption.

When $n = 0$, this tile set has zero tilings. Since $a := [1]F \in \mathbb{N}$, we can make the number of tilings of R_ε equal to a by adding a copies of a $1 \times \varepsilon$ rectangle to T. This does not change the number of tilings for any $n \ge 0$, since every tiling for $n \ge 0$ must already has two tiles of area $\varepsilon/2$, and thus cannot contain more tiles of area ε.

Finally, if T has multiple copies of the same tile, replace each copy with two tiles which only fit together with each other, to make a copy of that tile. We can always do this in such as way to make all new tiles distinct. This implies that $f \in \mathcal{F}$, as desired. □

7. Binomial multisums are diagonals of $\mathbb{N}$-rational functions

In this section, we prove the following result towards the proof of Main Theorem 3.4.

Lemma 7.1. $\mathcal{B} \subseteq \mathcal{D}$.

The proof of the lemma follows easily from five sub-lemmas, three on diagonals and two on binomial multisums. While the former are somewhat standard, the latter are rather technical; we prove them in the next section.

7.1. Diagonals.

We start with the following three simple results.

Lemma 7.2. *The set of diagonals of an* $\mathbb{N}$*-rational generating functions is closed under addition and multiplication.*

Proof. Let $f, g \in \mathcal{D}$. We have

$$f(n) = \big[x_1^n \dots x_k^n\big]\, F(x_1, \dots, x_k), \quad g(n) = \big[y_1^n \dots y_\ell^n\big]\, G(y_1, \dots, y_\ell)\,,$$

for some $F \in \mathcal{R}_k$ and $G \in \mathcal{R}_\ell$. Consider $A(x_1, \dots x_k, y_1, \dots, y_\ell)$ defined as

$$A(x_1, \dots x_k, y_1, \dots, y_\ell) = \left(\prod_{i=1}^{\ell} \frac{1}{1-y_i}\right) F\big(x_1, \dots, x_k\big) + \left(\prod_{i=1}^{k} \frac{1}{1-x_i}\right) G\big(y_1, \dots, y_\ell\big)\,.$$

It follows from the definition of $\mathbb{N}$-rational functions, that $A \in \mathcal{R}_{k+\ell}$. We have

$$\begin{aligned}\big[x_1^n \dots x_k^n y_1^n \dots y_\ell^n\big]\, A \,&=\, \big[x_1^n \dots x_k^n\big]\, F\big(x_1, \dots, x_k\big) \\ &+\, \big[y_1^n \dots y_\ell^n\big]\, G\big(y_1, \dots, y_\ell\big) \,=\, f(n) + g(n)\,.\end{aligned}$$

Similarly, define

$$B(x_1, \dots x_k, y_1, \dots, y_\ell) \,=\, F\big(x_1, \dots, x_k\big) \cdot G\big(y_1, \dots, y_\ell\big)\,,$$

and observe that

$$\big[x_1^n \dots x_k^n y_1^n \dots y_\ell^n\big]\, B \,=\, f(n) \cdot g(n)\,,$$

as desired. □

A function f is called a *quasi-diagonal* of an $\mathbb{N}$-rational generating function if

$$f(n) \,=\, \big[x_1^{cn} \dots x_k^{cn}\big]\, F(x_1, \dots, x_k),$$

for some fixed constant $c \in \mathbb{P}$.

Lemma 7.3. *For every function* f *which is the quasi-diagonal of* $F \in \mathcal{R}_k$*, there exists* $\ell \in \mathbb{P}$ *and* $G \in \mathcal{R}_\ell$*, such that* f *is the diagonal of* G*.*

Proof. First, we show that for every $F \in \mathcal{R}_k$, there exists a function $F_\circ \in \mathcal{R}_{k+1}$, such that for all $c_0, c_1, \dots, c_k \in \mathbb{N}$:

$$\big[x_0^{c_0} x_1^{c_1} x_2^{c_2} \dots x_k^{c_k}\big]\, F_\circ(x_0, x_1, x_2, \dots, x_k) = \begin{cases} \big[x_1^{c_0+c_1} x_2^{c_2} \dots x_k^{c_k}\big]\, F(x_1, \dots, x_k) & \text{if } c_1 \geq 1, \\ 0 & \text{otherwise.}\end{cases}$$

We prove this by structural induction in the definition of $\mathcal{R}_k$. For case (1), let $F_\circ = 0$ for $F = x_i$, $i \geq 2$, and $F_\circ = x_1$ for $F = x_1$. In case (2), let $F_\circ = G_\circ + H_\circ$ for $F = G + H$. For $F = G \cdot H$, let

$$F_\circ(x_0, \dots, x_k) \,=\, G_\circ(x_0, \dots, x_k)\, H(x_1, x_2, \dots, x_k) + G(x_0, x_2, \dots, x_k)\, H_\circ(x_0, \dots, x_k).$$

For case (3), for $F = 1/(1-G)$, let

$$F_\circ(x_0, \dots, x_k) \,=\, F(x_0, x_2, \dots, x_k)\, G_\circ(x_0, \dots, x_k)\, F(x_1, x_2, \dots, x_k).$$

It is clear that this works for case (1), for $F = G + H$, and when $c_1 = 0$, so we only need to worry about the cases where $c_1 > 0$ and where $F = G \cdot H$ or $F = 1/(1 - G)$.

For $F = G \cdot H$, recall that

$$\left[x_1^{c_0+c_1} x_2^{c_2} \ldots x_k^{c_k}\right] F(x_1, \ldots, x_k)$$

equals the sum over all $(d_1 + e_1, \ldots, d_k + e_k) = (c_0 + c_1, c_2, \ldots, c_k)$

$$\left[x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}\right] G(x_1, \ldots, x_k) \left[x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}\right] H(x_1, \ldots, x_k).$$

Note that in the formula for $F_\circ$, all instances of x_0 in $G_\circ(x_0, \ldots, x_k) H(x_1, x_2, \ldots, x_k)$, come from the $G_\circ(x_0, \ldots, x_k)$. Thus, if there are d_1 instances of x_0 or x_1 in the first summand, exactly $d_1 - c_0$ instances of x_1 come from $G_\circ$. Similarly, if there are e_1 instances of x_0 or x_1 in the second summand, exactly $e_1 - c_1$ instances of x_0 must come from $H_\circ$.

We break the contributions to $F_\circ$ into two cases: (a) with $d_1 > c_0$, and (b) with $d_1 \le c_0$. In the case (a), note that

$$\left[x_0^{c_0} x_1^{d_1-c_0} x_2^{d_2} \ldots x_k^{d_k}\right] G_\circ(x_0, \ldots, x_k) \;=\; \left[x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}\right] G(x_1, \ldots, x_k),$$

and

$$\left[x_0^{e_1-c_1} x_1^{c_1} x_2^{e_2} \ldots x_k^{e_k}\right] H_\circ(x_0, \ldots, x_k) \;=\; 0.$$

In the case (b), we similarly have:

$$\left[x_0^{c_0} x_1^{d_1-c_0} x_2^{d_2} \ldots x_k^{d_k}\right] G_\circ(x_0, \ldots, x_k) \;=\; 0,$$

and

$$\left[x_0^{e_1-c_1} x_1^{c_1} x_2^{e_2} \ldots x_k^{e_k}\right] H_\circ(x_0, \ldots, x_k) \;=\; \left[x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}\right] H(x_1, \ldots, x_k).$$

Therefore, in the case (a), we have

$$\begin{aligned}&\left[x_0^{c_0} x_1^{d_1-c_0} x_2^{d_2} \ldots x_k^{d_k}\right] G_\circ(x_0, \ldots, x_k) \left[x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}\right] H(x_1, \ldots, x_k) \\ &= \left[x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}\right] G(x_1, \ldots, x_k) \left[x_1^{e_1} x_2^{e_2} \ldots x_k^{e_k}\right] H(x_1, \ldots, x_k),\end{aligned}$$

and

$$\left[x_1^{d_1} x_2^{d_2} \ldots x_k^{d_k}\right] G(x_1, \ldots, x_k) \left[x_0^{e_1-c_1} x_1^{c_1} x_2^{e_2} \ldots x_k^{e_k}\right] H_\circ(x_0, \ldots, x_k) \;=\; 0.$$

In the case (b), we get a similar result with the r.h.s.'s interchanged. We conclude:

$$\left[x_0^{c_0} x_1^{c_1} x_2^{c_2} \ldots x_k^{c_k}\right] F_\circ(x_0, x_1, x_2, \ldots, x_k) \;=\; \left[x_1^{c_0+c_1} x_2^{c_2} \ldots x_k^{c_k}\right] F(x_1, \ldots, x_k).$$

For case (3), we think of any contribution to F as coming from G^r for some $r \in \mathbb{N}$, and break into cases based on how many copies of G we go through in this G^r before we have seen more than c_0 instances of x_1. We then proceed similarly.

Now, for every fixed $m \ge 2$, $F \in \mathcal{R}_k$ and a function $f(n)$ defined as

$$f(n) \;=\; \left[x_1^{mn} \ldots x_k^{mn}\right] F(x_1, \ldots, x_k),$$

we may recursively apply the above result to split each variable x_i into m variables x_{ij}, for $j = 1, \ldots, m$. We get a function $G \in \mathcal{R}_{mk}$ satisfying

$$\left[x_{11}^{c_{11}} x_{12}^{c_{12}} \cdots x_{km}^{c_{km}}\right] G(x_{11}, x_{12}, \ldots, x_{km}) \;=\; \left[x_1^{c_1} \ldots x_k^{c_k}\right] F(x_1, \ldots, x_k),$$

whenever $c_{i1} + \ldots + c_{im} = c_i$ and $c_{ij} \ge 1$, for all i and j. In particular, for all $c_{ij} = n \ge 1$, this gives

$$\left[x_{11}^{n} x_{12}^{n} \cdots x_{km}^{n}\right] G(x_{11}, x_{12}, \ldots, x_{km}) \;=\; f(n).$$

Further $[1]G = 0$, so we can simply add the constant term $f(0)$ to get the desired function with the diagonal $f(n)$. □

Lemma 7.4. *Let $f \in \mathcal{D}$ and $g : \mathbb{N} \to \mathbb{N}$ satisfy $f(n) = g(n)$ for all $n \ge 1$. Then $g \in \mathcal{D}$.*

Proof. The functions

$$j(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad h(n) = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{otherwise} \end{cases}$$

are trivially diagonals of functions 1 and $x/(1-x) \in \mathcal{R}_1$. Writing

$$g(n) \, = \, g(0) j(n) \cdot f(n) \, + \, h(n) \cdot f(n) \, ,$$

implies the result by Lemma 7.2.

7.2. **Finiteness of binomial multisums.** Next, we find a bound on which terms can contribute to a binomial multisum.

Lemma 7.5. *In notation of* §3.3*, let*

$$f(n) \, = \, \sum_{v \in \mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)}$$

be finite for all $n \in \mathbb{N}$*. Then there exists a constant* $c \in \mathbb{N}$*, such that for all* $n \in \mathbb{P}$*, and* $|v_i| > cn$ *for all* i*, the product on the right hand side is zero.*

Finally, we show that binomial sums bounded as in Lemma 7.5 are in fact quasi-diagonals of $\mathbb{N}$-rational generating functions.

Lemma 7.6. *In notation of* §3.3*, let* $f(n) : \mathbb{N} \to \mathbb{N}$ *be defined as*

$$f(n) \, = \, \sum_{v \in \mathbb{Z}^d, |v_i| \le cn} \, \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

for all $n \in \mathbb{P}$*. Then* $f(n)$ *agrees with a quasi-diagonal of an* $\mathbb{N}$*-rational generating function at all* $n \geq 1$*.*

Both lemmas are proved in the next section.

7.3. **Proof of Lemma 7.1.** By Lemmas 7.5 and 7.6, every function $f(n)$ as in Lemma 7.5 agrees with a quasi-diagonal of an $\mathbb{N}$-rational generating function at all $n \geq 1$. By Lemma 7.3, any function of this form agrees with a diagonal of an $\mathbb{N}$-rational generating function at all $n \geq 1$. By Lemma 7.4, any such function is in $\mathcal{D}$. □

8. Proofs of lemmas 7.5 and 7.6

8.1. **A geometric lemma.** We first need the following simple result; we include a short proof for completeness.

Lemma 8.1. *Let* $\alpha_1, \ldots, \alpha_r : \mathbb{R}^d \to \mathbb{R}$ *be integer coefficient affine functions. Let* $P \subset \mathbb{R}^d$ *be the (possibly unbounded) polyhedron of points satisfying* $\alpha_i \geq 0$ *for all* i*. If* P *contains a positive finite number of integer lattice points, then* P *is bounded.*

Proof. Suppose P is not bounded. Without loss of generality, assume that at the origin $O \in P$. Consider the *base cone* C_P of all infinite rays in P starting at O (see e.g. [Pak, §25.5]). Since P is a rational polyhedron, the cone C_P is also rational and contains at least one ray of rational slope. This ray contains an integer point and has a rational slope, and therefore contains infinitely many integer points, a contradiction. □

8.2. **Proof of Lemma 7.5.** Let S be a subset of $\{1, \ldots, r\}$, and let P_S be the set of all points $(v, n) \in \mathbb{R}^{d+1}$ satisfying $\alpha_i(v, n), \beta_i(v, n) \geq 0$ for $i \notin S$, satisfying $\alpha_i(v, n) = -1$ and $\beta_i(v, n) = 0$ for $i \in S$, and satisfying $n \geq 0$. Let $|v|$ denote $\max_i |v_i|$.

Note that we have 2^r polytopes P_S, and the integer lattice points in P_S form a cover for the set of all $(v, n) \in \mathbb{Z}^{d+1}$ which contribute a positive amount to $f(n)$. For each P_S, we prove that there exists a constant c such that any integer lattice point in P_S, with $n \geq 1$ satisfies $|v_i| \leq cn$. Since f is finite, we know that P_S contains finitely many integer lattice points for any fixed value of n.

We can assume that there are two distinct values $n_1 < n_2$, such that there exist integer lattice points in P_S with $n = n_1$ and with $n = n_2$, since otherwise P_S only contains finitely many lattice points. Let (v_1, n_1) be an integer point in P_S. Consider the set of all points in P_S satisfying $n = n_2$. This is a not necessarily bounded polytope with a positive finite number of integer lattice points, so by Lemma 8.1, it is bounded. Thus, there exists a c_S, such that $|v_2 - v_1| < c_S$ for all (v_2, n_2) in P_S.

This implies that $|v - v_1| < c_S(n - n_1)$ for all (v, n) in P_S with $n > n_2$. Indeed, otherwise the line segment connecting (v_1, n_1) to (v, n) would intersect the hyperplane $n = n_2$ at a point (v_2, n_2) in P_S but not satisfying $|v_2 - v_1| < c_S$.

Take a c'_S such that $c'_S > c$, and all finitely many integer lattice points (v, n) in P_S with $1 \leq n < n_2$, including (v_1, n_1), satisfy $|v| \leq c'_S n$. Then, all integer lattice points (v, n) in P_S with $n \geq 1$, satisfy $|v| \leq c'_S n$. Taking $c = \max_S \{c'_S\}$, proves the result. $\square$

8.3. **Proof of Lemma 7.6.** Let $f(n) : \mathbb{N} \to \mathbb{N}$ be a function such that for all $n \geq 1$,

$$f(n) = \sum_{v \in \mathbb{Z}^d, |v_i| \leq cn} \prod_{i=1}^{r} \binom{\alpha_i(v, n)}{\beta_i(v, n)},$$

where α_i and β_i are integer coefficient affine functions of v and n. We construct a function g in $d + 2r + 1$ variables $x_1, \ldots, x_d, a_1, \ldots a_r, b_1, \ldots, b_r, y$. Let

$$G(x_1, \ldots, x_d, a_1, \ldots a_r, b_1, \ldots, b_r, y) \, = \, \Pi_1 \cdot \Pi_2 \cdot \Pi_3 \cdot \Pi_4 \,,$$

where

$$\Pi_1 = \prod_{j=1}^{d} \frac{1}{1 - x_j h_j}\,, \qquad \Pi_2 = \prod_{j=1}^{d} \frac{1}{1 - x_j h'_j}\,,$$

$$\Pi_3 = \prod_{i=1}^{r} \left(1 + a_i \, \frac{a_i + b_i a_i}{1 - (a_i + b_i a_i)} \right)\,, \quad \text{and} \quad \Pi_4 = yq \, \frac{1}{1 - yq'}\,.$$

The terms h_j, h'_j, q, and q' are monomials in variables a_i and b_i, to be determined later.

We consider the coefficients of terms in which the exponent on each x_j variable is $2cn$. The Π_1 part will contribute some number of these x_j factors, and the Π_2 will contribute the rest. This choice will represent the variable v_j. We will use $cn + v_j$ to denote the number of factors of x_j coming from Π_1, so then $cn - v_j$ will be the number of factors of x_j coming from Π_2. Note that v_j can be any integer between $-cn$ and cn.

Define all the monomials in such a way that the a_i monomial needs to be repeated $\alpha_i(v, n) + 1$ times in the Π_3 term, while that b_i monomial needs to be repeated $\beta_i(v, n)$ times in the Π_3 term. By the definition in §2.1, this is exactly $\binom{\alpha_i}{\beta_i}$.

We choose the monomials h_j, h'_j, q, and q' a follows. Let

$$\beta_i(v, n) = \beta_{i,0} + \beta_{i,1} v_1 + \ldots + \beta_{i,d} v_d + \beta_{i,d+1} n.$$

First, consider the case where $\beta_{i,j} \le 0$, for all $0 \le j \le d$. In this case, we put b_i in h_j with multiplicity $|\beta_{i,j}|$, put β_i in q with multiplicity $|\beta_{0,j}|$, and not put any b_i terms in h'_j or q'. This implies that outside of the Π_3 term, the number of times b_i appears is exactly

$$-\beta_{i,0} + \sum_{j=1}^{d} (cn + v_j)(-\beta_{i,j}) \, = \, n \left(-\beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j} \right) - \beta(v,n).$$

Therefore, for the coefficient of a term with total multiplicity $n\left(-\beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j}\right)$ of b_i, we have $\beta(v,n)$ of the b_i terms must come from the Π_3 term.

We only consider coefficients where the multiplicity of the y term is n. If $\beta_{i,0}$ is positive, then we can swap the multiplicity of b_i in q and q'. Since the q' term is necessarily repeated $n-1$ times in the Π_4 term, this implies that outside of the Π_3 term, b_i appears exactly

$$(n-1)\beta_{i,0} + \sum_{j=1}^{d} (cn + v_j)(-\beta_{i,j}) \, = \, n \left(\beta_{i,0} - \beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j} \right) - \beta(v,n)$$

times. Therefore, for the coefficient of a term with $n\left(\beta_{i,0} - \beta_{i,d+1} - \sum_{j=1}^{d} c\beta_{i,j}\right)$ total multiplicity of b_i, we again have $\beta(v,n)$ of the b_i terms must come from the Π_3 term.

If any of the $\beta_{i,j}$ terms, with $1 \le j \le d$ are actually positive, we swap the multiplicity of b_i in h_j and h'_j, which gives the same analysis with v_j negated. Using the same method, we can require that the number of a_i terms coming from the Π_3 term is $\alpha_i + 1$, for all i.

In summary,

$$f(n) \, = \, \left[x_1^{nc_1} \ldots x_d^{nc_d} a_1^{nc_{d+1}} \ldots a_r^{nc_{d+r}} b_1^{nc_{d+r+1}} \ldots b_r^{nc_{d+2r}} y^{nc_{d+2r+1}}\right] G.$$

Take c' to be a common multiple of all c_i and make a substitution $x_i \leftarrow x_i^{c'/c_i}$, $a_j \leftarrow a_j^{c'/c_{d+j}}$, etc. This gives a desired quasi-diagonal. $\square$

9. Proof of Theorem 4.2

9.1. **Preliminaries.** We start with the following simple result:

Lemma 9.1. *Let $f \in \mathcal{F}$ be a tile counting function. Then $f(n) \le C^n$, for all $n \in \mathbb{P}$ for some $C > 0$.*

Proof. Let $T = \{\tau_1, \ldots, \tau_s\}$, and let $\mu = \min_i |\tau_i|$ be the minimum area of a tile in T. Every tiling of $\mathrm{R}_{n+\varepsilon}$ with T corresponds to a unique sequence of tiles in the tilings, listed from left to right. The length of this sequence is at most $(n+\varepsilon)/m$. Therefore, $f_T(n) \le (s+1)^{(n+\varepsilon)/\mu} = e^{O(n)}$. $\square$

Theorem 4.2 shows that this upper bound is usually tight, and every function growing slower than this must be eventually quasi-polynomial. The following lemma is a special case of Theorem 1.1 in [CLS];

Lemma 9.2 ([CLS]). *Let $g(n)$ be the number of integer points $(x_1, \ldots, x_r, n) \in \mathbb{Z}^{r+1}$ satisfying m inequalities $a_i(x,n) > c_i$ where a_i is an integer coefficient linear function and c_i is an integer for all $i = 1, \ldots, m$. Then $g(n)$ is eventually quasi-polynomial.*

The proof of the lemma uses a generalization of *Ehrhart polynomials*. We refer to [Bar] for a review of the area and further references.

9.2. **Proof setup.** From Main Theorem 3.4, function f can be expressed as

$$(\circledast) \qquad f(n) = \sum_{v\in\mathbb{Z}^d}\prod_{i=1}^{r}\binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

where each α_i and β_i is an integer coefficient affine function of v and n.

From Lemma 9.1, function $f \leq e^{cn}$ for some c. Therefore, it suffice to show that f is either greater than e^{cn} for some $c > 0$, or is eventually polynomial. Furthermore, it suffices to show that f is either greater than e^{cn} for some c, or eventually quasi-polynomial. We can decompose f into even more functions, by multiplying p by the periods of all of the quasi-polynomials, which proves that each component function that does not grow exponentially is eventually polynomial.

Denote by k the number of indices i in $(\circledast)$, such that α_i, β_i, and $\gamma_i = \alpha_i - \beta_i$ are three non-constant functions. We use induction on k.

9.3. **Step of Induction.** Let M be a constant integer satisfying $M > |\beta_i(0,0)|, |\gamma_i(0,0)|$, for all i. We decompose f into a sum of $(M+2)^{2r}$ functions depending on the values of β_i and γ_i, for all i. For each β_i and γ_i, we either require that $\beta_i \geq M$ or that the value of β_i be some constant $< M$. There are $M+2$ possibilities for each function, since only values ≥ -1 give non-zero binomial coefficients, giving $(M+2)^{2r}$ bound as above.

To ensure that $\beta_i = z \geq -1$ for some constant z, we replace $\beta_i(v,n)$ by z, and multiply the binomial coefficients $\binom{\beta_i(v,n)+1}{z+1}$ and $\binom{z+1}{\beta_i(v,n)+1}$ to the existing product. This works because $\binom{\beta_i(v,n)+1}{z+1}\binom{z+1}{\beta_i(v,n)+1}$ is 1 if $\gamma_i = z$, and 0 otherwise.

Similarly, to ensure that $\gamma_i = z \geq -1$, we replace $\alpha_i(v,n)$ with $\beta_i + z$, and multiply the binomial coefficients $\binom{\gamma_i(v,n)+1}{z+1}$ and $\binom{z+1}{\gamma_i(v,n)+1}$ to the existing product.

Finally, to ensure that $\beta_i \geq M$, we multiply the binomial coefficient $\binom{\beta_i - M - 1}{0}$ to the existing product. This binomial coefficient is 1 if $\beta_i - M - 1 \geq -1$, and 0 otherwise. Similarly for enforcing that $\gamma_i \geq M$.

Note that when we require that β_i or γ_i equal to a constant, we reduce k by 1, and when we specify that $\beta_i \geq M$ or $\gamma_i \geq M$, we keep k the same. Therefore of these $(M+2)^{2r}$ functions which add to g, we get by induction that all but one of them is quasi-polynomial. The only one we have to worry about is the function g in which each β_i and γ_i is specified to be at least $\mathcal{M}$, and we show that this function is identically 0.

Assume by way of contradiction that g is not identically 0. Then, there exists some (v_1, n_1) such that

$$\prod_{i=1}^{r}\binom{\alpha_i(v_1,n_1)}{\beta_i(v_1,n_1)} > 0, \quad \beta_i(v_1,n_1) - \beta_i(0,0) > 0, \quad \text{and} \quad \gamma_i(v_1,n_1) - \gamma_i(0,0) > 0,$$

for all i. Adding (v_1, n_1) to any point (v, n) would increase every $\beta_i(v,n)$ and $\gamma_i(v,n)$ by at least 1. Consider the sequence of points $(v_t, n_t) = (tv_1, tn_1)$, where t is a positive integer. Note that every $\beta_i(v_t, n_t)$ and every $\gamma_i(v_t, n_t)$ is at least t. Therefore, $\binom{\alpha_1(v_t,n_t)}{\beta_1(v_t,n_t)} \geq \binom{2t}{t}$, so $f(n_1 t) \geq \binom{2t}{t} \geq 2^t$, contradicting the fact that $f(n) < e^{cn}$ for all positive c.

9.4. **Base of Induction:** Now consider the case $k = 0$. In notation of $(\circledast)$, this means

$$f(n) = \sum_{v\in\mathbb{Z}^d}\prod_{i=1}^{r}\binom{\alpha_i(v,n)}{\beta_i(v,n)},$$

where for each i, at least one of α_i, β_i, and γ_i is constant. Without loss of generality, we can assume that either α_i or β_i are constant.

When α_i is a nonzero constant, then we can write f as a sum of functions where we condition on the value of β_i to be z, by replacing β_i with z and multiplying the existing product by $\binom{0}{\beta_i(v,n)-z}$. Since $\binom{\alpha_i}{z}$ is a constant, we can again express our functions as a sum of $\binom{\alpha_i}{z}$ copies of that function with the $\binom{\alpha_i}{z}$ term removed. Therefore, we can assume that if α_i is constant, that constant is 0.

We can also replace every $\binom{0}{\beta_i(v,n)}$ with $\binom{\beta_i(v,n)-1}{0}\binom{-\beta_i(v,n)-1}{0}$, since we are replacing the indicator that $\beta_i = 0$ with the indicators $\beta_i - 1 \geq -1$ and $-\beta_i - 1 \geq -1$. Therefore, we may assume that every β_i is a constant.

In summary,

$$f(n) \, = \, \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r_1} \binom{\alpha_i(v,n)}{0} \prod_{i=r_1+1}^{r} \binom{\alpha_i(v,n)}{z_i},$$

where each α_i and β_i is an integer coefficient affine function of v and n, and each $z_i \in \mathbb{P}$.

Note that each $\binom{\alpha_i(v,n)}{0}$ term is just an indicator function that $\alpha_i(v,n) \geq -1$. Therefore,

$$f(n) \, = \, \sum_{v\in P_n} \prod_{i=r_1+1}^{r_2} \binom{\alpha_i(v,n)}{z_i},$$

where P_n is the polytope of all integer points such that $\alpha_i(v,n) \geq -1$ for all $1 \leq i \leq d_1$.

Also, note that $\binom{\alpha_i(v,n)}{z_i}$ is equal to the number of integer points $(x_1,\ldots,x_{z_i})$, such that

$$0 \leq x_1 < x_2 < \ldots < x_{z_i} < \alpha_i(v,n).$$

Therefore, $f(n)$ is equal to the number of points in the polytope

$$P_n \times \mathbb{Z}^{z_{r_1+1}} \times \ldots \times \mathbb{Z}^{z_r},$$

where v is the point in P_n, and the coordinates $(x_1,\ldots,x_{z_i})$ satisfying

$$0 \leq x_1 < x_2 < \ldots < x_{z_i} < \alpha_i(v,n).$$

By Lemma 9.2, $f(n)$ is eventually quasi-polynomial. This proves the base of induction, and completes the proof of Theorem 4.2.

10. Proofs of applications

10.1. **Proof of Theorem 4.1.** First, we show that $\mathcal{B}' \subseteq \mathcal{B}$. Since $\mathcal{B}$ is closed under addition, it suffices to show that every balanced multisum

$$g(n) \, = \, \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r} \frac{\alpha_i(v,n)!}{\beta_i(v,n)!\gamma_i(v,n)!}$$

is in $\mathcal{B}$. This follows since

$$\frac{\alpha_i(v,n)!}{\beta_i(v,n)!\gamma_i(v,n)!} \, = \, \binom{\alpha_i(v,n)}{\beta_i(v,n)}\binom{\alpha_i(v,n)-1}{0}.$$

The second factor ensures that $\alpha_i \geq 0$, so the first factor is never $\binom{-1}{0}$.

To show that $\mathcal{B} \subseteq \mathcal{B}'$, take

$$g(n) \, = \, \sum_{v\in\mathbb{Z}^d} \prod_{i=1}^{r} \binom{\alpha_i(v,n)}{\beta_i(v,n)}.$$

Denote by S the set of all subsets of $\{1,\ldots,r\}$, and $\gamma_i = \alpha_i - \beta_i$. For each $s \in S$, let

$$g_s(n) \,=\, \sum_{v\in\mathbb{Z}^d} e_s(v,n)\cdot \prod_{i\in s} \frac{\alpha_i(v,n)!}{\beta_i(v,n)!\gamma_i(v,n))!}\,, \qquad \text{where}$$

$$e_s(v,n) \,=\, \prod_{i\notin s}\left[\frac{0!}{(\alpha_i(v,n)+1)!(-\alpha_i(v,n)-1)!}\right]\left[\frac{0!}{\beta_i(v,n)!(-\beta_i(v,n))!}\right].$$

Observe that $e_s(v,n)=1$ if $\alpha_i(v,n)=-1$ and $\beta_i(v,n)=0$, and $e_s(v,n)=0$ otherwise.

For every v, let s_v be the set of indices $i\in\{1,\ldots,r\}$ for which $\alpha_i(v,n),\beta_i(v,n)\ge 0$. Then

$$e_s(v,n)\cdot\prod_{i\in s}\frac{\alpha_i(v,n)!}{\beta_i(v,n)!\gamma_i(v,n)!} \,=\, \binom{\alpha_i(v,n)}{\beta_i(v,n)} \quad \text{when} \quad s=s_v\,,$$

and 0 otherwise. Therefore,

$$g(n) \,=\, \sum_{s\in S} g_s(n)\,.$$

This implies that $g\in\mathcal{B}'$, and completes the proof. □

10.2. **Getting close to Catalan numbers.** Before we prove Proposition 4.7, we need the following weaker result.

Lemma 10.1. *There exists a tile counting function f such that*

$$f(n) \,\sim\, \frac{3\sqrt{3}}{\pi}\,C_n\,, \qquad \textit{as} \qquad n\to\infty\,.$$

Proof. Consider the following three binomial multisums $f_1, f_2, f_3 \in \mathcal{B}$:

$$f_1(n) \,=\, \sum_{v\in\mathbb{Z}}\binom{n}{3v}\binom{3v}{n}\binom{2v}{v}^3, \quad f_2(n) \,=\, 4\sum_{v\in\mathbb{Z}}\binom{n-1}{3v}\binom{3v}{n-1}\binom{2v}{v}^3,$$

$$\text{and} \qquad f_3(n) \,=\, 16\sum_{v\in\mathbb{Z}}\binom{n-2}{3v}\binom{3v}{n-2}\binom{2v}{v}^3.$$

Let $f=f_1+f_2+f_3$. By Main Theorem 3.4 and Corollary 3.5, we know that $f\in\mathcal{F}$.

Observe that $f_1(n)\ne 0$ only when n is a multiple of 3. We have:

$$f_1(n) \,=\, \binom{2n/3}{n/3}^3 \,\sim\, \left(\frac{4^{n/3}}{\sqrt{n/3}\sqrt{\pi}}\right)^3 \,\sim\, \frac{3\sqrt{3}}{\pi}\,C_n\,, \qquad \text{for} \quad 3|n.$$

Analyzing f_2 and f_3 gives the same result when $n=1,2 \bmod 3$, respectively. □

10.3. **Proof of Proposition 4.7.** Let $f\in\mathcal{F}$ be the tile counting function from Lemma 10.1. For each $i\in\mathbb{N}$, let $g_i(n)=f(n-i)$ if $n\ge i$, and let $g_i(n)=0$ otherwise. Each g_i is also a tile counting function, since we can take the exact same tile set, and replace ε with $\varepsilon+i$.

Denote $\xi=3\sqrt{3}/\pi$. Note that $g_i(n)\sim f(n)/4^i\sim C_n\xi/4^i$. Given any $\varepsilon>0$, we can take i large enough so that $\xi/4^i<\varepsilon$, and $m\in\mathbb{P}$ such that $1-\epsilon<m\xi/4^i<1+\epsilon$. This gives $mg_i(n)\sim C_n\xi m/4^i$ which is between $1-\epsilon$ and $1+\epsilon$, as desired. Finally, we have $mg_i\in\mathcal{F}$ since $\mathcal{B}=\mathcal{F}$ is closed under addition. □

10.4. **Proof of Proposition 4.8.** Given an $m \geq 1$, let

$$f(n) = \binom{2n}{n} + (m-1)\binom{2n}{n+1}.$$

Note that f is a tile counting function, since it is a finite sum of binomial coefficients of affine functions of n. Since $C_n = \binom{2n}{n} - \binom{2n}{n+1}$, we have that $f(n)$ and C_n differ by $m\binom{2n}{n}$, and are therefore congruent modulo m. □

10.5. **Proof of Proposition 4.9.** Given a prime $p \geq 2$, let

$$f(n) = \binom{2n}{n} + (p^{2n}-1)\binom{2n}{n+1}.$$

By Corollary 3.6, $p^{2n} - 1 \in \mathcal{B}$, and binomial coefficients are in $\mathcal{B}$ by definition. Since $\mathcal{B}$ is closed under addition and multiplication, we obtain $f \in \mathcal{B}$. Note that $p^{2n} > C_n$, so adding or subtracting an integer multiple of p^{2n} to C_n does not change the order of p. Therefore,

$$\operatorname{ord}_p(C_n) \, = \, \operatorname{ord}_p\left(C_n + p^{2n}\binom{2n}{n+1}\right) \, = \, \operatorname{ord}_p(f(n)),$$

as desired. □

10.6. **Proof of Theorem 4.10.** We start with the case where $k = 1$ and $\ell = 2$. Let $r = r_1/r_2$, $c = \mu_1^{\mu_1} r_2$, and let

$$f(n) = \sum_{\ell=0}^{n} \binom{\mu_1\ell}{\nu_1\ell} c^{n-\ell}(\nu_1^{\nu_1}\nu_2^{\nu_2}r_1)^\ell \, = \sum_{\ell\in\mathbb{Z}} \binom{\ell-1}{0}\binom{n-\ell-1}{0}\binom{\mu_1\ell}{\nu_1\ell} c^{n-\ell}(\nu_1^{\nu_1}\nu_2^{\nu_2}r_1)^\ell.$$

First, let us prove that f is a tile counting function. Replace $c^{n-\ell}$ with

$$\sum_{v_1,\ldots,v_c\in\mathbb{Z}} \binom{n-\ell}{v_1}\binom{n-\ell-v_1}{v_2}\binom{n-\ell-v_1-v_2}{v_3}\cdots\binom{n-\ell-v_1-v_2-\ldots-v_{c-1}}{v_c}.$$

We can ignore the fact that $\binom{-1}{0} = 1$, since if we take the least i such that $n-\ell-v_1-v_2-\ldots-v_i = -1$, we have $\binom{n-\ell-v_1-v_2-\ldots-v_{i-1}}{v_i} = 0$. We then make a similar replacement for $(\nu_1^{\nu_1}\nu_2^{\nu_2}r_1)^\ell$. Therefore, $f \in \mathcal{F}$ by the Main Theorem 3.4.

Letting

$$g(\ell) = \binom{\mu_1\ell}{\nu_1\ell} c^{-\ell}(\nu_1^{\nu_1}\nu_2^{\nu_2}r_1)^\ell,$$

we get

$$f(n) \, = \, \sum_{\ell=0}^{n} g(\ell)c^n.$$

Note that $g(0) = 1$, and

$$g(\ell+1)/g(\ell) \, = \, \frac{\nu_1^{\nu_1}\nu_2^{\nu_2}\, r \, \prod_{i=1}^{\mu_1}(\mu_1\ell+i)}{\mu_1^{\mu_1}\, \prod_{i=1}^{\nu_1}(\nu_1\ell+i)\, \prod_{i=1}^{\nu_2}(\nu_2\ell+i)} \, = \, \frac{(\ell+a_1)(\ell+a_2)\ldots(\ell+a_p)r}{(\ell+b_1)(\ell+b_2)\ldots(\ell+b_p)}.$$

Therefore,

$$f(n)/c^n \, = \, \sum_{\ell=0}^{n}\prod_{k=0}^{\ell-1} \frac{(k+a_1)(k+a_2)\ldots(k+a_p)r}{(k+b_1)(k+b_2)\ldots(k+b_p)} \, \to \, A \quad \text{as} \quad n\to\infty.$$

In general, let each part μ_i be subdivided into ℓ_i parts $\nu_{i,1}, \ldots, \nu_{i,\ell_i}$. Let $r = r_1/r_2$, and let $c = (\mu_1)^{\mu_1} \ldots (\mu_k)^{\mu_k} r_2$. Similarly to in the previous case, we define f as

$$f(n) \, = \, \sum_{\ell=0}^{n} c^n r_1^\ell r_2^{-\ell} \prod_{i=1}^{p} g_i(\ell),$$

where each

$$g_i(\ell) = \binom{\mu_i \ell}{\nu_{i,1}} \binom{\mu_i \ell - \nu_{i,1}\ell}{\nu_{i,2}\ell} \cdots \binom{\mu_i \ell - \nu_{i,1}\ell - \ldots - \nu_{i,\ell_i - 1}\ell}{\nu_{i,l_i}} (\mu_i^{\mu_i})^{-\ell} \left(\nu_{i,1}^{\nu_{i,1}} \ldots \nu_{i,\ell_i}^{\nu_{i,\ell_i}}\right)^\ell .$$

This is a tile counting function for reasons similar to in the previous case. Note that all the negative exponents are canceled out by the c^n term. We have:

$$g_i(\ell+1)/g(\ell) \, = \, \frac{\nu_{i,1}^{\nu_{i,1}} \ldots \nu_{i,\ell_i}^{\nu_{i,\ell_i}} \prod_{j=1}^{\mu_i} (\mu_i \ell + j)}{\mu_i^{\mu_i} \prod_{j=1}^{\nu_{i,1}} (\nu_{i,1}\ell + j) \cdots \prod_{j=1}^{\nu_{i,\ell_1}} (\nu_{i,\ell_i}\ell + j)} \, .$$

Therefore,

$$\frac{r_1^{\ell+1} r_2^{-(\ell+1)} \prod_{i=1}^p g_i(\ell)}{r_1^\ell r_2^{-\ell} \prod_{i=1}^p g_i(\ell)} = \frac{(\ell+a_1)(\ell+a_2) \, \cdots \, (\ell+a_p) r}{(\ell+b_1)(\ell+b_2) \ldots (\ell+b_p)} \, .$$

Since

$$r_1^0 r_2^0 \prod_{i=1}^{p} g_i(0) \, = \, 1,$$

we have

$$f(n)/c^n = \sum_{\ell=0}^{n} \prod_{k=0}^{\ell-1} \frac{(k+a_1)(k+a_2)\ldots(k+a_p) r}{(k+b_1)(k+b_2)\ldots(k+b_p)} \to A \quad \text{as} \quad n \to \infty \, .$$

The base of exponent we get from this construction is $c = (\mu_1)^{\mu_1} \ldots (\mu_k)^{\mu_k} r_2$. However, note that it is easy to multiply c by any positive integer N, simply by multiplying f by N^n. In particular, let L be the product of all primes which are factors of $\mu_1 \ldots \mu_k r_2$. Then there exists some positive integer d, such that L^d is a multiple of $(\mu_1)^{\mu_1} \ldots (\mu_k)^{\mu_k} r_2$. This implies that there exists a function $h \in \mathcal{F}$ with $h(n) \sim A L^{dn}$.

Note now, that we can scale all the tiles horizontally by d, and scale ε by d, to get a new function $f_0(n)$ such that $f_0(dn) = h(n)$. We may assume that $f_0(n) = 0$ when n is not a multiple of d, because we can multiply f_0 by the indicator that $d|n$. We can similarly get a function f_i for $i = 1, \ldots, d-1$, such that $f_i(n)$ is nonzero only when $n = i \bmod d$, and $f_i(nd+i) = L^i h(n)$. We have

$$f(n) = \sum_{i=0}^{d-1} f_i(n) \in \mathcal{F},$$

and by the way we constructed f we obtain $f(n) \sim A L^n$. Thus, we can take $c = L$ or any integer multiple of L, as desired. $\square$

11. Final Remarks

11.1. The idea of irrational tilings was first introduced by Korn, who found a bijection between *Baxter permutations* and tilings of large rectangles with three fixed irrational rectangles [Korn, §6].

11.2. For Theorem 1.1, much of the credit goes to Schützenberger [Schü] who proved the equivalence between *regular languages* and the (weakest in power) *deterministic finite automata* (DFA). He used the earlier work of Kleene (1956) and the language of semirings; the GF reformulation in the language of $\mathbb{N}$-rational functions came later, see [SS]. We refer to [BR1, SS] for a thorough treatment of the subject and connections to GFs, and to [Pin] for a more recent survey.

Now, the relationship between (polyomino) tilings of the strip, regular languages (as well as DFAs) were proved more recently in [BL, MSV]. Theorem 1.1 now follows as combination of these results in several different ways.

Let us mention here that in the usual polyomino tiling setting there is no height condition, so in fact Theorem 1.1 remains unchanged when rational tiles of smaller height are allowed. In the irrational tiling setting, the standard "finite number of cut paths" argument fails. Still, we conjecture that Theorem 1.2 also extends to tiles with smaller heights.

11.3. The history of Theorem 4.3 is somewhat confusing. In fact, it holds for integer *G-sequences* defined as integer *D-finite* (*holonomic*) *sequences* with at most at most exponential growth. It is stated in this form since diagonals of all rational GFs are D-finite [Ges1] and at most exponential. We refer to [FS, Sta1] for more on D-finite sequences, examples and applications, and to [DGS, Gar] for G-sequences.

The asymptotics of D-finite GFs go back to Birkhoff and Trjitzinsky (1932), and Turrittin (1960). See [FS, §VIII.7] and [Odl, §9.2] for various formulations of general asymptotic estimates, and an extensive discussion of priority and validity issues. However, for G-sequences, the result seems to be accepted and well understood, see [BRS, §2.2] and [Gar].

11.4. Note that D-finite sequences can be superexponential, e.g. $n!$. They can also have $\exp(n^\gamma)$ terms with $\gamma \in \mathbb{Q}$, e.g. the number a_n of involutions in S_n :

$$a_n \sim 2^{-1/2} e^{-1/4} \left(\frac{n}{e}\right)^{n/2} e^{\sqrt{n}},$$

(see [Sta1] and `A000085` in [OEIS]).

In notation of Theorem 4.3, the $\alpha \in \mathbb{Q}$ conclusion cannot be substantially strengthened even for $k = 2$ variables. To understand this, recall Furstenberg's theorem (see [Sta1, §6.3]) that every algebraic function is a diagonal of $P(x,y)/Q(x,y)$, and that by Theorem 2 in [BD] there exist algebraic functions with asymptotics $A\lambda^n n^\alpha$, for all $\alpha \in \mathbb{Q} \smallsetminus \{-1,-2,\ldots\}$. For example, the number $g(n)$ of *Gessel walks* (see `A135404` in [OEIS]) is famously algebraic [BK], and has asymptotics

$$g(n) \sim \frac{2^{2/3}\Gamma(\frac{1}{3})}{3\pi} 16^n n^{-7/3}.$$

11.5. There is more than one way a sequence can be a diagonal of a rational function. For example, the Catalan numbers C_n are the diagonals of

$$\frac{1-x/y}{1-x-y} \quad \text{and} \quad \frac{y(1-2xy-2xy^2)}{1-x-2xy-xy^2}.$$

The former follows from $C_n = \binom{2n}{n} - \binom{2n}{n-1}$, while the second is given in [RY].

11.6. There is a vast literature on binomials sums and multisums, both classical and modern, see e.g. [PWZ, Rio]. It was shown by Zeilberger [Zei] (see also [WZ]), that under certain restrictions, the resulting functions are D-finite, a crucial discovery which paved a way to *WZ algorithm*, see [PWZ, WZ]. A subclass of *balanced multisums*, related but larger than $\mathcal{B}'$, was defined and studied in [Gar]. Note that the positivity is the not the only constraint we add. For example, balanced multisums in [Gar] easily contain Catalan numbers:

$$\frac{(2n)!\,1!}{n!\,(n+1)!} = C_n .$$

We refer to [B+, §5.1] and [BLS] for the recent investigations of binomial multisums which are diagonals of rational functions, but without $\mathbb{N}$-rationality restriction.

11.7. The class $\mathcal{R}_1$ of $\mathbb{N}$-rational GFs does not contain *all* of $\mathbb{N}[[x]]$ (Berstel, 1971); see [BR2, Ges2] for some examples. These are rare, however; e.g. Koutschan investigated "about 60" nonnegative rational GFs from [OEIS], and found all of them to be in $\mathcal{R}_1$, see [Kou, §4.4]. In fact, there is a complete characterization of $\mathcal{R}_1$ by analytic means, via the Berstel (1971) and Soittola (1976) theorems. We refer to [BR1, SS] for these results and further references, and to [Ges2] for a friendly introduction.

Unfortunately, there is no such characterization of $\mathcal{D}$, nor we expect there to be one, as singularities in higher dimensions are most daunting [FS, PW]. Even the most natural questions remain open in that case (cf. Conjecture 4.5). Here is one such question.

Open Problem 11.1. *Let $f \in \mathcal{F}$ such that the corresponding* GF $F(x) \in \mathbb{N}[[x]]$. *Does it follow that $f \in \mathcal{F}_1$?*

Personally, we favor a negative answer. In [Ges2], Gessel asks whether there are (nonnegative) rational GF which have a *combinatorial interpretation*, but are not $\mathbb{N}$-rational. Thus, a negative answer to Problem 11.1 would give a positive answer to Gessel's question.[4] Of course, what's a combinatorial interpretation is in the eye of the beholder; here we are implicitly assuming that our irrational tilings or paths in graphs (see §5.1) are *nice* enough to pass this test (cf. [Ges2]). We plan to revisit this problem in the future.

11.8. There are over 200 different combinatorial interpretation of Catalan numbers [Sta2], some of them 1-dimensional such as the *ballot sequences*. A quick review suggests that none of them can be verified with a bounded memory read only TM. For example, for the ballot 0–1 sequences one must remember the running differences $(\#0 - \#1)$, which can be large. This gives some informal support in favor of our Conjecture 4.6. Let us make following, highly speculative and priceless claim.[5]

Conjecture 11.2. *There is no tile counting function $f \in \mathcal{F}$ which is* asymptotically Catalan:

$$f(n) \sim C_n \quad \textit{as} \quad n \to \infty .$$

We initially tried to disprove the conjecture. Recall that by Lemma 10.1 and the technology in Section 10, it suffices to obtain the constant $\frac{\pi}{3\sqrt{3}}$ as the product of values of the hypergeometric functions given in Theorem 4.10. While $\frac{1}{3\sqrt{3}}$ is easy to obtain, our hypergeometric sums seem too specialized to give value π. This is somewhat similar to the conjecture that $\frac{1}{\pi}$ is not a *period* [KZ].

We should mention here that it is rare when we can say anything at all about the constant A in Conjecture 4.5. The constant in Corollary 4.12 is an exception: is known to be transcendental by the celebrated 1996 result of Nesterenko on algebraic independence of π and $\Gamma(\frac{1}{4})$, see [NP].

11.9. The proof of Lemma 5.3 uses a generalization of a standard argument in combinatorial linear algebra, for computing the number of rational tilings:

$$F(x) \,=\, \sum_{n=0}^{\infty} f_T(n) x^n \,=\, \sum_{n=0}^{\infty} (M^n)_{00}\, x^n \,=\, \left(\frac{1}{1 - Mx}\right)_{00} \,=\, \frac{\det(1 - M^{00} x)}{\det(1 - Mx)},$$

where M is the weighted adjacency matrix of G_T. It is thus not surprising that we use a cycle decomposition argument somewhat similar but more general than that in [CF, KP].

In the same vein, the "well-defined multiplicities" argument in the proof of Lemma 5.4 is similar to the "cycle popping" argument in [Wil] (see also [GoP, Mar]). The details are quite different, however.

[4]Christophe Reutenauer writes to us that according the "general metamathematical principle that goes back to Schützenberger" (see [BR2, p. 149]), the logic must be reversed: a negative answer to Gessel's question implies that the answer to Problem 11.1 must be positive.

[5]Cf. `http://tinyurl.com/mc3h8tn`.

11.10. The values $\mathrm{ord}_p(C_n)$ in Proposition 4.9 were computed by Kummer (1852); see [DS] for a recent combinatorial proof.

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