A PROBLEM OF RANKIN ON SETS WITHOUT GEOMETRIC PROGRESSIONS

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Abstract. A *geometric progression of length k and integer ratio* is a set of numbers of the form $\{a, ar, \ldots, ar^{k-1}\}$ for some positive real number a and integer $r \geq 2$. For each integer $k \geq 3$, a greedy algorithm is used to construct a strictly decreasing sequence $(a_i)_{i=1}^{\infty}$ of positive real numbers with $a_1 = 1$ such that the set

$$G^{(k)} = \bigcup_{i=1}^{\infty} (a_{2i}, a_{2i-1}]$$

contains no geometric progression of length k and integer ratio. Moreover, $G^{(k)}$ is a maximal subset of $(0, 1]$ that contains no geometric progression of length k and integer ratio. It is also proved that there is a strictly increasing sequence $(A_i)_{i=1}^{\infty}$ of positive integers with $A_1 = 1$ such that $a_i = 1/A_i$ for all $i = 1, 2, 3, \ldots$.

The set $G^{(k)}$ gives a new lower bound for the maximum cardinality of a subset of the set of integers $\{1, 2, \ldots, n\}$ that contains no geometric progression of length k and integer ratio.

1. Real and integral geometric progressions

Let $\mathbf{R}$ denote the real numbers. For $t \in \mathbf{R}$, let $\mathbf{R}_{>t}$ denote the set of all real numbers $x > t$. Let $[x]$ denote the integer part of the real number x. For real numbers $u < v$, we define the intervals

$$(u, v] = \{x \in \mathbf{R} : u < x \leq v\} \quad \text{and} \quad [u, v) = \{x \in \mathbf{R} : u \leq x < v\}.$$

Let X be a set of positive real numbers, and let $u, v \in \mathbf{R}_{>0}$ with $u < v$. The *dilation of the set X by $q \in \mathbf{R}_{>0}$* is the set

$$q * X = \{qx : x \in X\}.$$

The *reciprocal* of the set X is the set

$$X^{-1} = \left\{x^{-1} : x \in X\right\}.$$

For example, $q * (u, v] = (qu, qv]$ and $(1/v, 1/u]^{-1} = [u, v)$.

If $A = (a_0, a_1, \ldots, a_{k-1})$ is a finite sequence of positive real numbers, then the *dilation of the sequence A by q* is the sequence $q * A = (qa_0, qa_1, \ldots, qa_{k-1})$ and the reciprocal of A is the sequence $A^{-1} = (1/a_0, 1/a_1, \ldots, 1/a_{k-1})$.

Let $\mathbf{N}$ denote the set of positive integers, and let $\mathbf{N}^{\sharp} = \mathbf{N} \setminus \{1\}$ denote the set of all integers $r > 1$. Let $k \in \mathbf{N}$ and let $r, a \in \mathbf{R}_{>0}$. A *geometric progression of length k and ratio r with first term a* is a sequence of the form

$$(a, ar, ar^2, \ldots, ar^{k-1}) = a * (1, r, r^2, \ldots, r^{k-1}).$$

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This is an *integer geometric progression of length* k if $ar^j \in \mathbf{N}$ for all $j \in \{0, 1, \ldots, k-1\}$. If $(a, ar, ar^2, \ldots, ar^{k-1})$ is an integer geometric progression, then the ratio r must be a rational number. For example, $(8, 12, 18, 27)$ is an integer geometric progression of length 4 with ratio $3/2$.

Note that the dilation by a positive real number q of the geometric progression $(a, ar, ar^2, \ldots, ar^{k-1})$ of length k, ratio r, and first term a is the geometric progression $(qa, qar, qar^2, \ldots, qar^{k-1})$ of length k, ratio r, and first term qa. The reciprocal of the geometric progression $(a, ar, ar^2, \ldots, ar^{k-1})$ is the geometric progression

$$\left(\frac{1}{a}, \frac{1}{a}\left(\frac{1}{r}\right), \frac{1}{a}\left(\frac{1}{r}\right)^2, \ldots, \frac{1}{a}\left(\frac{1}{r}\right)^{k-1}\right)$$

of length k, ratio $1/r$, and first term $1/a$.

The *reverse* of the sequence $(a_1, a_2, \ldots, a_{k-1}, a_k)$ is the sequence $(a_k, a_{k-1}, \ldots, a_2, a_1)$. The reverse of the reciprocal of the geometric progression $(a, ar, ar^2, \ldots, ar^{k-1})$ is the geometric progression $(b, br, br^2, \ldots, br^{k-1})$, where $b = 1/\left(ar^{k-1}\right)$.

Thus, a set G of real numbers contains no geometric progression of length k if and only if the dilation $q*G$ contains no geometric progression of length k for every positive real number q. Moreover, if a set G contains no geometric progression of length k, then no subset of G contains a geometric progression of length k. It follows that if a set G contains no geometric progression of length k, then, for every positive real number q, the set $(q * G) \cap \mathbf{N}$ is a set of positive integers that contains no geometric progression of length k. Similarly, if G contains no geometric progression of length k, then the set of $G^{-1} \cap \mathbf{N}$ is a set of positive integers that contains no geometric progression of length k.

A *geometric progression of length k with integer ratio* is a geometric progression of length k with ratio $r \in \mathbf{N}^\sharp$. An *integer geometric progression of length k with integer ratio* is a geometric progression of the form $(a, ar, ar^2, \ldots, ar^{k-1})$ with $a \in \mathbf{N}$ and $r \in \mathbf{N}^\sharp$.

For positive integers k and n, let $g_k(n)$ denote the cardinality of the largest subset of the set $\{1, 2, 3, \ldots, n\}$ that contains no integer geometric progression of length k with integer ratio, and let $\hat{g}_k(n)$ denote the cardinality of the largest subset of the set $\{1, 2, 3, \ldots, n\}$ that contains no integer geometric progression of length k with rational ratio.

We have $g_1(n) = \hat{g}_1(n) = 0$ for all $n \in \mathbf{N}$, and $g_k(n) = \hat{g}_k(n) = n$ if $n < k$. Moreover, $\hat{g}_2(n) = 1$ for $n \geq 2$. We compute $g_2(n)$ in the next section. In this paper we obtain new lower bounds for the function $g_k(n)$ for $k \geq 3$.

For every integer $k \geq 3$, there are four basic unsolved problems:

1. Determine the cardinality and the structure of the maximal subsets of $\{1, 2, \ldots, n\}$ that contain no geometric progression of length k with integer ratio. In particular, what is the maximum cardinality $g_k(n)$?
2. Determine the cardinality and the structure of the maximal subsets of $\{1, 2, \ldots, n\}$ that contain no geometric progression of length k with rational ratio. What is the maximum cardinality $\hat{g}_k(n)$?
3. Determine the density and structure of maximal infinite sets of positive integers that contain no geometric progression of length k with integer ratio. What is the least upper bound of the densities of such sets? Is this least upper bound achieved?

(4) Determine the density and structure of maximal infinite sets of positive integers that contain no geometric progression of length k with rational ratio. What is the least upper bound of the densities of such sets? Is this least upper bound achieved?

Very little is known about these problems. The literature consists mostly of lower bounds for the maximum cardinalities in Problems 1 and 2, and for the densities in Problems 3 and 4. In this paper we improve the lower bounds in Problem 1. Our method is to use a greedy algorithm to construct, for every integer $k \geq 3$, a unique maximal subset of the unit interval $(0, 1]$ that contains no geometric progression of length k with integer ratio, and to use the measure of this set to obtain new lower bounds for the finite sets considered in Problem 1.

The earliest discussion of sets with no k-term geometric progression is in a paper of Rankin [6] in 1960 that was concerned with sets of integers containing no k-term arithmetic progression.

2. Integral geometric progressions of length 2

We can quickly solve the problem of integer geometric progressions of length 2 with integer ratio. Every set $\{a, b\}$ of positive real numbers with $a < b$ is a geometric progression of length 2 with ratio $r = b/a$. In particular, every set $\{a, b\}$ of positive integers with $a < b$ is a geometric progression of length 2 with rational ratio $r = b/a$. The set $\{a, b\}$ is an integer geometric progression of length 2 with integer ratio if and only if $a, b \in \mathbf{N}$ and a divides b. Thus, a set S of positive integers contains no 2-term geometric progression if and only if S is *primitive* in the sense that no element of S divides another element of S.

The following is a classical result in combinatorial number theory.

Theorem 1. *Let $g_2(n)$ denote the cardinality of the largest primitive subset of $\{1, 2, \ldots, n\}$, that is, the largest subset of $\{1, 2, \ldots, n\}$ that contains no integer geometric progression of length 2 with integer ratio. Then $g_2(n) = \left[\frac{n+1}{2}\right]$.*

Proof. For every positive integer n, the interval

$$S = \left(\left[\frac{n}{2}\right], n\right] = \left\{\left[\frac{n}{2}\right] + 1, \left[\frac{n}{2}\right] + 2, \ldots, n-1, n\right\} \tag{1}$$

is primitive because $2\left(\left[\frac{n}{2}\right] + 1\right) \geq n + 1 > n$. The cardinality of this set is $\left[\frac{n+1}{2}\right]$, and so $g_2(n) \geq \left[\frac{n+1}{2}\right]$.

Let S be any primitive subset of $\{1, 2, \ldots, n\}$. Each element $s \in S$ can be written uniquely in the form $s = 2^{k(s)}a(s)$, where $k(s)$ is a nonnegative integer and $a(s)$ is an odd integer in $\{1, 2, \ldots, n\}$. If $a(s_1) = a(s_2)$ for integers $s_1, s_2 \in S$ with $s_1 < s_2$, then s_1 divides s_2. It follows that the cardinality of the primitive set S is at most the number of odd integers in $\{1, 2, \ldots, n\}$, and so $g_2(n) \leq \left[\frac{n+1}{2}\right]$. This completes the proof. □

3. Good set, bad set

Let k be an integer, $k \geq 3$. A *k-good set* is a set of positive real numbers that contains no geometric progression of length k with integer ratio. For example, the set

$$G_1^{(k)} = \left(\frac{1}{2^{k-1}}, 1\right]$$

is k-good because, if $x \in G_1^{(k)}$ and $r \in \mathbf{N}^\sharp$, then $xr^{k-1} \geq x2^{k-1} > 1$ and so $\{x, xr, xr^2, \ldots, xr^{k-1}\}$ is not a subset of $G_1^{(k)}$.

Let G be a k-good subset of $(0,1]$, and let $x \in (0,1] \setminus G$. The real number x is *k-bad with respect to G* if there exists an integer $r \in \mathbf{N}^\sharp$ such that $G \cup \{x\}$ contains the k-term geometric progression $(x, xr, xr^2, \ldots, xr^{k-1})$. Thus, if x is k-bad with respect to G, then the set $G \cup \{x\}$ is not k-good.

For example, the number $1/2^k$ is k-bad with respect to the k-good set $G_1^{(k)}$ because $(1/2^k, 1/2^{k-1}, 1/2^{k-2}, \ldots, 1/2, 1)$ is a k-term geometric progression with ratio $r = 2$ contained in $G_1^{(k)} \cup \{1/2^k\}$.

The number $3/16$ is 3-bad with respect to the 3-good set $G_1^{(3)} = (1/4, 1]$ because, with $r = 2$,

$$\left\{ \frac{3}{16} r, \frac{3}{16} r^2 \right\} = \left\{ \frac{3}{8}, \frac{3}{4} \right\} \subseteq \left(\frac{1}{4}, 1 \right] = G_1^{(3)}$$

and so the set $G_1^{(3)} \cup \{3/16\}$ contains the 3-term geometric progression $(3/16, 3/8, 3/4)$. Similarly, $1/10$ is 3-bad with respect to $G_1^{(3)}$ because, with $r = 3$,

$$\left\{ \frac{1}{10}, \frac{1}{10} r, \frac{1}{10} r^2 \right\} = \left\{ \frac{1}{10}, \frac{3}{10}, \frac{9}{10} \right\} \subseteq G_1^{(3)} \cup \left\{ \frac{1}{10} \right\}.$$

Note that if G is a k-good subset of $(0,1]$ and if $x \in (0,1] \setminus G$ is k-bad with respect to G, then x is also k-bad with respect to the good set $G \cap (x, 1]$, because $x < r^j x$ for all $r \in \mathbf{N}^\sharp$ and $j \in \{1, \ldots, k-1\}$.

The real number $x \in (0,1] \setminus G$ is *k-good with respect to G* if x is not k-bad with respect to G. Thus, x is k-good with respect to G if and only if, for every $r \in \mathbf{N}^\sharp$, there exists $j \in \{1, 2, \ldots, k-1\}$ such that $xr^j \notin G$. Because $G \subseteq (0,1]$ and $xr^{k-1} \notin G$ if $r > (1/x)^{1/(k-1)}$, it follows that $x \in [0,1) \setminus G$ is k-good with respect to G if and only if, for every integer r with $2 \leq r \leq (1/x)^{1/(k-1)}$, there exists $j \in \{1, 2, \ldots, k-1\}$ with $xr^j \notin G$.

For every k-good set $G \subseteq (0,1]$, we define

$$\mathrm{Bad}(G) = \{x \in (0,1] \setminus G : \ x \text{ is } k\text{-bad with respect to } G\}.$$

Thus, $G \cup \{x\}$ is k-good for all $x \in (0,1] \setminus (G \cup \mathrm{Bad}(G))$. If G and G' are k-good sets with $G \subseteq G'$, then $\mathrm{Bad}(G) \subseteq \mathrm{Bad}(G')$.

For fixed k, we usually write "good" instead of "k-good" and "bad" instead of "k-bad."

4. Construction of a good set of real numbers

Fix the integer $k \geq 3$. We shall use a greedy algorithm to construct a large good set contained in the interval $(0,1]$. We begin with some simple observations about good and bad sets.

Lemma 1. *Let $k \geq 3$, let $0 < a < 1$, and let $\delta_k(a) = a^{(k-1)/(k-2)}$.*

(i) *For every $\delta > 0$, the interval $(0, \delta]$ is not good.*
(ii) *Every number in the interval $(0, a^2]$ is good with respect to the interval $(a, 1]$.*
(iii) *Let $x \in (0,1]$. If $xr^j \in (a, 1]$ for some $r \in \mathbf{N}^\sharp$ and all $j \in \{1, \ldots, k-1\}$, then $x > \delta_k(a)$.*
(iv) *If G is a good set with $G \subseteq (a, 1]$, then $(0, \delta_k(a)] \cap \mathit{Bad}(G) = \emptyset$.*

Note that $0 < \delta_k(a) < a$.

Proof. (i) We have $0 < 2^{1-k}\delta < \delta$. For every $x \in (0, 2^{1-k}\delta]$, we have

$$0 < x < 2x < \cdots < 2^{k-1}x \leq 2^{k-1}2^{1-k}\delta = \delta$$

and so $(0, \delta]$ contains the k-term geometric progression $\{x, 2x, \ldots, 2^{k-1}x\}$. Thus, the interval $(0, \delta]$ is not good.

(ii) If $x \in (0, a^2]$, $r \in \mathbf{N}^\sharp$ and $xr \in (a, 1]$, then $xr > a$ and $r > a/x$. It follows that

$$xr^{k-1} \geq xr^2 > x\left(\frac{a}{x}\right)^2 = \frac{a^2}{x} > 1$$

and so x is good with respect to $(a, 1]$.

x is bad with respect to $(a, 1]$, then there exists $r \in \mathbf{N}^\sharp$ such that $xr^i \in (a, 1]$

(iii) If $r \in \mathbf{N}^\sharp$ and

$$a < rx < \cdots < r^{k-1}x \leq 1$$

then

$$\frac{a}{r} < x \leq \frac{1}{r^{k-1}}$$

and so $1/r > a^{1/(k-2)}$. Therefore,

$$x > \frac{a}{r} > aa^{1/(k-2)} = a^{(k-1)/(k-2)} = \delta_k(a).$$

(iv) This follows immediately from (iii). □

Lemma 2. *Let $(a_i)_{i=1}^{2n}$ be a strictly decreasing sequence of positive real numbers with $a_1 \leq 1$ such that*

$$G_n = \bigcup_{i=1}^{n} (a_{2i}, a_{2i-1}]$$

is a good set. If $x \in Bad(G_n)$, then there exists $\delta > 0$ such that $(x-\delta, x] \subseteq Bad(G_n)$.

Proof. We have $G_n \subseteq (0, 1]$. If $x \in \mathrm{Bad}(G_n)$, then there exists $r \in \mathbf{N}^\sharp$ such that $xr^j \in G_n$ for all $j \in \{1, \ldots, k-1\}$. It follows that, for each $j \in \{1, \ldots, k-1\}$, there exists $i_j \in \{1, \ldots, n\}$ such that

$$xr^j \in (a_{2i_j}, a_{2i_j-1}]$$

or, equivalently,

$$\frac{a_{2i_j}}{r^j} < x \leq \frac{a_{2i_j-1}}{r^j}.$$

Choose $\delta > 0$ such that

$$\frac{a_{2i_j}}{r^j} < x - \delta < x \leq \frac{a_{2i_j-1}}{r^j}$$

for all $j \in \{1, \ldots, k-1\}$. If $y \in (x-\delta, x]$, then

$$a_{2i_j} < (x-\delta)r^j < yr^j \leq xr^j \leq a_{2i_j-1}$$

and so $yr^j \in (a_{2i_j}, a_{2i_j-1}] \subseteq G_n$ for all $j \in \{1, \ldots, k-1\}$. Thus, $(x-\delta, x] \subseteq \mathrm{Bad}(G_n)$. □

Lemma 3. *Let $(a_i)_{i=1}^{2n+1}$ be a strictly decreasing sequence of positive real numbers with $a_1 \leq 1$ such that*

$$G_n = \bigcup_{i=1}^{n} (a_{2i}, a_{2i-1}]$$

is a good set, and

$$\bigcup_{i=1}^{n}(a_{2i+1}, a_{2i}] \subseteq Bad(G_n).$$

If $x \in (a_{2n+1}/2, a_{2n+1}]$ *is good with respect to* G_n*, then there exists* $\delta > 0$ *such that* $(x-\delta, x] \cup G_n$ *is good.*

Proof. Let $x \in (a_{2n+1}/2, a_{2n+1}]$ be good with respect to G_n. For each $r \in \mathbf{N}^\sharp$ there exists $j_r \in \{1, \ldots, k-1\}$ such that $xr^{j_r} \notin G_n$. Let r_0 be the smallest integer such that $r_0 \geq 2$ and $xr_0^{k-1} > a_1$. Then $x > a_1/r_0^{k-1}$, and there exists $\delta_0 > 0$ such that $x - \delta_0 > a_1/r_0^{k-1}$. If $y \in (x-\delta_0, x]$ and $r \geq r_0$, then

$$yr^{k-1} > (x-\delta_0)r_0^{k-1} > a_1$$

and so $yr^{k-1} \notin G_n$.

For each integer r such that $2 \leq r < r_0$, we have

$$a_{2n+1} < 2x \leq xr^{j_r} \leq xr^{k-1} \leq a_1$$

and so there exists $i_r \in \{1, \ldots, n\}$ such that

$$a_{2i_r+1} < xr^{j_r} \leq a_{2i_r}.$$

Equivalently,

$$\frac{a_{2i_r+1}}{r^{j_r}} < x \leq \frac{a_{2i_r}}{r^{j_r}}.$$

Choose $0 < \delta_1 < x/2$ such that

$$\frac{a_{2i_r+1}}{r^{j_r}} < x - \delta_1 < x \leq \frac{a_{2i_r}}{r^{j_r}}$$

for all $r \in \mathbf{N}^\sharp$ with $r < r_0$. If $y \in (x-\delta_1, x]$ and $r < r_0$, then

$$a_{2i_r+1} < (x-\delta_1)r^{j_r} < yr^{j_r} \leq xr^{j_r} \leq a_{2i_r}$$

and so $yr^{j_r} \notin G_n$. Let $\delta = \min(\delta_0, \delta_1)$. It follows that if $y \in (x-\delta, x]$, then y is good with respect to G_n. This completes the proof. □

Theorem 2. *Let* $k \geq 3$*. There exists a unique strictly decreasing sequence* $(a_i)_{i=1}^{\infty}$ *of positive real numbers with* $a_1 = 1$ *such that*

$$G = \bigcup_{i=1}^{\infty}(a_{2i}, a_{2i-1}]$$

is a good set, and

$$Bad(G) = \bigcup_{i=1}^{\infty}(a_{2i+1}, a_{2i}].$$

Proof. We construct the sequence $(a_i)_{i=1}^{\infty}$ by induction.

Let $a_1 = 1$. If $x > 2^{1-k}$, then for all $r \in \mathbf{N}^\sharp$ we have

$$r^{k-1}x > 2^{k-1}2^{1-k} = 1$$

and so $(2^{1-k}, 1]$ is a good set. Therefore,

$$a_2 = \inf\{x \in (0,1] : (x, a_1] \text{ is good}\} \leq 2^{1-k}.$$

We observe that $[2^{1-k}, 1]$ is not a good set because, with $y = 2^{1-k}$, we have $\{y, y2, \ldots, y2^{k-1}\} \subseteq [2^{1-k}, 1]$. Therefore,

$$a_2 = \frac{1}{2^{k-1}} \in \mathrm{Bad}(G_1)$$

where

$$G_1 = (a_2, a_1] = \left(\frac{1}{2^{k-1}}, 1 \right].$$

We define

$$a_3 = \inf\{x \in (0,1] : (x, a_2] \subseteq \text{Bad}(G_1)\}.$$

It follows from Lemmas 1 and 2 that $0 < \delta_k(a_2) \leq a_3 < a_2$ and $a_3 \notin \text{Bad}(G_1)$.

Let $n \geq 1$, and assume that there is a unique strictly decreasing sequence $(a_i)_{i=1}^{2n+1}$ of positive real numbers with $a_1 = 1$ such that

$$G_n = \bigcup_{i=1}^{n} (a_{2i}, a_{2i-1}]$$

is a good set,

$$\bigcup_{i=1}^{n} (a_{2i+1}, a_{2i}] \subseteq \text{Bad}(G_n).$$

and

$$a_{2n+1} = \inf\{x \in (0,1] : (x, a_{2n}] \subseteq \text{Bad}(G_n)\}.$$

By Lemma 2, the number a_{2n+1} is good with respect to G_n. Let

$$a_{2n+2} = \inf\{x \in (0, a_{2n+1}] : (x, a_{2n+1}] \text{ is good with respect to } G_n\}.$$

Let

$$G_{n+1} = G_n \cup (a_{2n+2}, a_{2n+1}].$$

Lemmas 1 and 3 imply that $0 < a_{2n+2} < a_{2n+1}$, and that $a_{2n+2} \in \text{Bad}(G_{n+1})$. We define

$$a_{2n+3} = \inf\{x \in (0,1] : (x, a_{2n+2}] \subseteq \text{Bad}(G_n)\}.$$

This completes the induction. □

Theorem 3. *Let $(a_i)_{i=1}^{2n}$ be a strictly decreasing sequence of positive real numbers such that*

$$G_n = \bigcup_{i=1}^{n} (a_{2i}, a_{2i-1}]$$

is a good set, and

$$\bigcup_{i=1}^{n-1} (a_{2i+1}, a_{2i}] \subseteq Bad(G_n).$$

If A_1 and A_2 are positive integers such that $a_1 = 1/A_1$ and $a_2 = 1/A_2$, then there is a strictly increasing sequence $(A_i)_{i=1}^{2n}$ of positive integers such that

$$a_i = \frac{1}{A_i}$$

for $i = 1, \ldots, 2n$.

Proof. The proof is by induction on i. Let $2 \leq i \leq n$ and assume that there are positive integers $A_1 < \cdots < A_{2i-2}$ such that $a_j = 1/A_j$ for $j = 1, \ldots, 2i-2$. We shall prove that there are positive integers A_{2i-1} and A_{2i} such that $a_{2i-1} = 1/A_{2i-1}$ and $a_{2i} = 1/A_{2i}$.

Consider the good number a_{2i-1}. If $h \in \mathbf{N}$ and $h \geq (a_{2i-2} - a_{2i-1})^{-1}$, then

$$a_{2i-1} + \frac{1}{h} \in (a_{2i-1}, a_{2i-2}] \subseteq \text{Bad}(G_n)$$

and so there exists $r_h \in \mathbf{N}^\sharp$ such that, for all $j \in \{1, 2, \ldots, k-1\}$,

$$\left(a_{2i-1} + \frac{1}{h}\right) r_h^j \in G_n$$

and

$$a_{2i-1} < a_{2i-1} r_h \leq a_{2i-1} r_h^j < \left(a_{2i-1} + \frac{1}{h}\right) r_h^{k-1} \leq a_1.$$

Therefore,

$$2 \leq r_h < \frac{a_1}{a_{2i-1}}.$$

Because $a_{2i-1} \in G_n$, there exists $j_h \in \{1, 2, \ldots, k-1\}$ such that

$$a_{2i-1} r_h^{j_h} \notin G_n.$$

There are only finitely many choices for r_h and j_h. By the pigeonhole principle, there are integers $r \in \mathbf{N}^\sharp$ and $j \in \{1, 2, \ldots, k-1\}$ and there is a strictly increasing infinite sequence $(h_\ell)_{\ell \in \mathbf{N}}$ of positive integers such that

$$r_{h_\ell} = r \qquad \text{and} \qquad j_{h_\ell} = j$$

for all $\ell \in \mathbf{N}$. Because $a_{2i-1} r^j \notin G_n$ and $a_{2i-1} < a_{2i-1} r^j < a_1$, there is a unique positive integer $t \leq i$ such that $a_{2i-1} r^j \in (a_{2t-1}, a_{2t-2}]$. Because $(a_{2i-1} + 1/h_\ell)\, r^j \in G_n$, it follows that

$$a_{2i-1} r^j \leq a_{2t-2} < \left(a_{2i-1} + \frac{1}{h_\ell}\right) r^j$$

or, equivalently,

$$\frac{a_{2t-2}}{r^j} - \frac{1}{h_\ell} < a_{2i-1} \leq \frac{a_{2t-2}}{r^j}.$$

By the induction hypothesis, there is a positive integer A_{2h-2} such that $a_{2t-2} = 1/A_{2t-2}$. Letting $\ell \to \infty$, we obtain

$$a_{2i-1} = \frac{a_{2t-2}}{r^j} = \frac{1}{r^j A_{2t-2}} = \frac{1}{A_{2i-1}}$$

with $A_{2i-1} = r^j A_{2t-2}$.

Next we consider the bad number a_{2i}. There exists $r \in \mathbf{N}^\sharp$ such that $a_{2i} r^j \in G_n$ for all $j \in \{1, 2, \ldots, k-1\}$. If $h \geq (a_{2i-1} - a_{2i})^{-1}$, then

$$a_{2i} + \frac{1}{h} \in (a_{2i}, a_{2i-1}] \subseteq G_n$$

and so there exists $j_h \in \{1, 2, \ldots, k-1\}$ such that

$$\left(a_{2i} + \frac{1}{h}\right) r^{j_h} \notin G_n.$$

By the pigeonhole principle, there is an integer $j \in \{1, 2, \ldots, k-1\}$ and there is a strictly increasing infinite sequence $(h_\ell)_{\ell \in \mathbf{N}}$ of positive integers such that $j_{h_\ell} = j$ for all $\ell \in \mathbf{N}$. Because $a_{2i} r^j \in G_n$, there is a unique positive integer $t \leq i$ such that $a_{2i} r^j \in (a_{2t}, a_{2t-1}]$. Because $(a_{2i} + 1/h_\ell)\, r^j \notin G_n$, it follows that, for all $\ell \in \mathbf{N}$, we have

$$a_{2i} r^j \leq a_{2t-1} < \left(a_{2i} + \frac{1}{h_\ell}\right) r^j$$

or, equivalently,

$$\frac{a_{2t-1}}{r^j} - \frac{1}{h_\ell} < a_{2i} \leq \frac{a_{2t-1}}{r^j}.$$

By the induction hypothesis, there is a positive integer A_{2t-1} such that $a_{2t-1} = 1/A_{2t-1}$. Letting $\ell \to \infty$, we obtain

$$a_{2i} = \frac{a_{2t-1}}{r^j} = \frac{1}{r^j A_{2t-1}} = \frac{1}{A_{2i}}$$

with $A_{2i} = r^j A_{2t-1}$. This completes the proof. $\square$

Theorem 4. *Let $(a_i)_{i \in \mathbf{N}}$ be a strictly decreasing infinite sequence of positive real numbers such that*

$$G = \bigcup_{i=1}^{\infty} (a_{2i}, a_{2i-1}]$$

is a good set, and

$$Bad(G) = \bigcup_{i=1}^{\infty} (a_{2i+1}, a_{2i}].$$

If A_1 and A_2 are positive integers such that $a_1 = 1/A_1$ and $a_2 = 1/A_2$, then there is a strictly increasing infinite sequence $(A_i)_{i \in \mathbf{N}}$ of positive integers such that

$$a_i = \frac{1}{A_i}$$

for all $i \in \mathbf{N}$. Moreover,

$$\lim_{i \to \infty} a_i = 0.$$

Proof. Apply Theorem 3 to the good set $G_n = \bigcup_{i=1}^{n} (a_{2i}, a_{2i-1}]$.

Because there is a strict increasing sequence $(A_i)_{i=1}^{\infty}$ of positive integers such that $a_i = 1/A_i$, it follows that

$$\lim_{i \to \infty} a_i = \lim_{i \to \infty} \frac{1}{A_i} = 0.$$

This completes the proof. $\square$

Theorem 5. *Let $k \geq 3$. There exists a unique strictly increasing sequence $\left(A_i^{(k)}\right)_{i=1}^{\infty}$ of positive integers with $A_1^{(k)} = 1$ such that*

$$G^{(k)} = \bigcup_{i=1}^{\infty} \left(\frac{1}{A_{2i}^{(k)}}, \frac{1}{A_{2i-1}^{(k)}} \right] \tag{2}$$

is a k-good set and

$$Bad\left(G^{(k)}\right) = \bigcup_{i=1}^{\infty} \left(\frac{1}{A_{2i+1}^{(k)}}, \frac{1}{A_{2i}^{(k)}} \right].$$

Proof. The existence and uniqueness of the sequence $\left(A_i^{(k)}\right)_{i=1}^{\infty}$ follows immediately from Theorems 2 and 4. $\square$

Note that

$$\inf G^{(k)} = \inf \mathrm{Bad}\left(G^{(k)}\right) = 0$$

because $\lim_{i \to \infty} A_i^{(k)} = \infty$.

We have already proved that $A_2^{(k)} = 2^{k-1}$. We can also determine the integers $A_3^{(k)}$ and $A_4^{(k)}$. The proofs use a simple arithmetic inequality: If $k \geq 3$, then

$$\frac{3^{k-1}}{2^k} = \frac{1}{2}\left(\frac{3}{2}\right)^{k-1} \geq \frac{1}{2}\left(\frac{3}{2}\right)^2 = \frac{9}{8} > 1$$

and so $2^k < 3^{k-1}$. Note that between two consecutive integral powers of 2 there is at most one integral power of 3.

Theorem 6. *If $k \geq 3$, then*

$$A_3^{(k)} = 2^{k-1}.$$

Proof. Let $G_1^{(k)} = (1/2^{k-1}, 1]$. If

$$\frac{1}{2^k} < x \leq \frac{1}{2^{k-1}}$$

then

$$\frac{1}{2^{k-1}} = \frac{2}{2^k} < 2x < 2^2x < \cdots < 2^{k-1}x \leq \frac{2^{k-1}}{2^{k-1}} = 1$$

and so $\{2^i x : i = 1, 2, \ldots, k-1\} \subseteq G_1^{(k)}$, that is, x is k-bad with respect to $G_1^{(k)}$.

If $x = 1/2^k$, then $2x = 1/2^{k-1} \notin G_1^{(k)}$. If $r \geq 3$, then

$$r^{k-1}x \geq \frac{3^{k-1}}{2^k} > 1$$

and so $r^{k-1}x \notin G_1^{(k)}$. Therefore, $1/2^k$ is k-good with respect to $G_1^{(k)}$, and $A_3^{(k)} = 2^k$. □

Theorem 7. *Let $k \geq 3$. If there is no integral power of 3 between 2^{k-1} and 2^k, then*

$$A_4^{(k)} = 3^{k-1}.$$

If there is an integral power of 3 between 2^{k-1} and 2^k, and if ℓ is the positive integer such that

$$2^{k-1} < 3^\ell < 2^k \tag{3}$$

then $2 \leq \ell \leq k-2$ and

$$A_4^{(k)} = 2^k 3^{k-1-\ell} = 3^{k-1}\left(\frac{2^k}{3^\ell}\right).$$

Inequality (3) is equivalent to $1 < 2^k/3^\ell < 2$.

For positive integers k, the following are equivalent:

(i) There is an integral power of 3 between 2^{k-1} and 2^k.
(ii) The fractional part of $k \log_3 2$ is less than $\log_3 2$.
(iii) k is in the set $\{[\ell \log_2 3] + 1 : \ell = 1, 2, \ldots\}$. Thus, the formula for $A_4^{(k)}$ depends on diophantine properties of logarithms.

Proof. We have $A_3^{(k)} = 1/2^k$ by Theorem 6. Let

$$\frac{1}{4^{k-1}} < x \leq \frac{1}{2^k}.$$

For $r \geq 4$ we have

$$xr^{k-1} \geq x4^{k-1} > \frac{4^{k-1}}{4^{k-1}} = 1$$

and so $\{xr^i : i = 1, 2, \ldots, k-1\} \not\subseteq G_1^{(k)}$.

With $r = 2$ we have

$$x2^{k-1} > \frac{2^{k-1}}{4^{k-1}} = \frac{1}{2^{k-1}}.$$

Let j be the smallest integer such that $x2^j > 1/2^{k-1}$. Then $j \leq k-1$. Because $2x \leq 2/2^k = 1/2^{k-1}$, it follows that $j \geq 2$. If

$$x2^{j-1} \leq \frac{1}{2^k}$$

then

$$x2^j \leq \frac{1}{2^{k-1}} < x2^j$$

which is absurd. Therefore,

$$\frac{1}{2^k} < x2^{j-1} \leq \frac{1}{2^{k-1}}$$

and and so $\{x2^i : i = 1, 2, \ldots, k-1\} \not\subseteq G_1^{(k)}$.

The remaining case is the ratio $r = 3$ and the geometric progression $\{x3^i : i = 1, 2, \ldots, k-1\}$. If $x > 1/3^{k-1}$, then $x3^{k-1} > 1$ and x is good with respect to $G_1^{(k)}$. Therefore, $A_4^{(k)} \geq 3^{k-1}$.

Let $x = 1/3^{k-1}$. If there exists $j \in \{1, 2, \ldots, k-1\}$ such that

$$\frac{1}{2^k} < x3^j = \frac{3^j}{3^{k-1}} < \frac{1}{2^{k-1}}$$

then

$$2^{k-1} < 3^{k-1-j} < 2^k.$$

Thus, if there is no power of 3 between 2^{k-1} and 2^k, then for all $i \in \{1, 2, \ldots, k-1\}$, either $x < 3^i x < 1/2^k$ or $1/2^{k-1} < 3^i x \leq 1$. Thus, $1/3^{k-1}$ is k-bad, and $A_4^{(k)} = 3^{k-1}$.

Suppose that there is a power of 3 between 2^{k-1} and 2^k, and that ℓ is the unique positive integer that satisfies (3). We observe that $k \geq 4$ because there is no power of 3 between $2^2 = 4$ and $2^3 = 8$, and that $2 \leq \ell \leq k-2$ because $2^3 < 3^2 \leq 3^\ell$ and $2^{k-1} < 3^\ell \leq 3^{k-2}$. Let

$$j = k - 1 - \ell.$$

Then $1 \leq j \leq k-3$. For $k \geq 4$ we have

$$\left(\frac{4}{3}\right)^{k-1} \geq \left(\frac{4}{3}\right)^3 > 2$$

and so

$$\left(\frac{2}{3}\right)^{k-1} > \frac{1}{2^{k-2}}.$$

Let

$$x_0 = \frac{1}{2^k 3^j} = \frac{3^\ell}{2^k 3^{k-1}} > \frac{2^{k-1}}{2^k 3^{k-1}} = \frac{1}{2^k}\left(\frac{2}{3}\right)^{k-1} > \frac{1}{4^{k-1}}.$$

If

$$x_0 < x \leq \frac{1}{3^{k-1}}$$

then

$$\frac{1}{2^k} = x_0 3^j < x3^j \leq \frac{3^j}{3^{k-1}} = \frac{1}{3^\ell} < \frac{1}{2^{k-1}}$$

and so x is good with respect to $G_1^{(k)}$.

It remains only to prove that x_0 is bad. If $1 \leq i \leq j$, then

$$x_0 < x_0 3^i \leq x_0 3^j = \frac{1}{2^k}.$$

If $j+1 \leq i \leq k-1$, then

$$\frac{1}{2^{k-1}} < \frac{3}{2^k} = 3^{j+1}x_0 \leq 3^i x_0 \leq 3^{k-1}x_0 = \frac{3^\ell}{2^k} < 1.$$

Thus, $x_0 = 1/(2^k 3^j)$ is bad and $A_4^{(k)} = 2^k 3^j$. This completes the proof. $\square$

5. Integer sequences with no k-term geometric progression

If a and b are real numbers with $a \leq b$, then the number of integers in the interval $(a, b]$ is $b - a + \theta$ with $|\theta| < 1$.

Recall that, for positive integers k and n, the arithmetic function $g_k(n)$ denotes the cardinality of the largest subset of the set $\{1, 2, 3, \ldots, n\}$ that contains no integer geometric progression of length k with integer ratio.

Theorem 8. *Let $k \geq 3$, and let $\left(A_i^{(k)}\right)_{i=1}^{\infty}$ be the strictly increasing sequence of positive integers constructed in Theorem 5. Then*

$$\gamma_k = \liminf_{n\to\infty} \frac{g_k(n)}{n} \geq \sum_{i=1}^{\infty} \left(\frac{1}{A_{2i-1}^{(k)}} - \frac{1}{A_{2i}^{(k)}} \right).$$

In particular,

$$\gamma_k \geq 1 - \frac{1}{2^k} - \frac{1}{3^k}.$$

Proof. For every positive integer h, the set

$$G_h^{(k)} = \bigcup_{i=1}^{h} \left(\frac{1}{A_{2i}^{(k)}}, \frac{1}{A_{2i-1}^{(k)}} \right]$$

is a k-good subset of $(0, 1]$. For every positive integer n, the dilated set

$$n * G_h^{(k)} = n * \bigcup_{i=1}^{h} \left(\frac{1}{A_{2i}}, \frac{1}{A_{2i-1}} \right] = \bigcup_{i=1}^{h} \left(\frac{n}{A_{2i}}, \frac{n}{A_{2i-1}} \right]$$

is a disjoint union of intervals, and so

$$\begin{aligned} \left| (n * G_h^{(k)}) \cap \mathbf{N} \right| &= \sum_{i=1}^{h} \left| \left(\frac{n}{A_{2i}}, \frac{n}{A_{2i-1}} \right] \cap \mathbf{N} \right| \\ &= n \sum_{i=1}^{h} \left(\frac{1}{A_{2i-1}} - \frac{1}{A_{2i}} \right) + \theta_h \end{aligned}$$

with $|\theta_h| < h$. Because the dilation of a k-good set is k-good, and a subset of a k-good set is k-good, it follows that $(n * G_h^{(k)}) \cap \mathbf{N}$ is a k-good set of positive integers. Moreover, $A_1 = 1$ implies that $(n * G_h^{(k)}) \cap \mathbf{N}$ is a subset of $\{1, 2, \ldots, n\}$. Therefore,

$$\left| (n * G_h^{(k)}) \cap \mathbf{N} \right| \leq g_k(n)$$

and so

$$\sum_{i=1}^{h}\left(\frac{1}{A_{2i-1}}-\frac{1}{A_{2i}}\right)=\lim_{n\to\infty}\frac{\left|(n*G_h^{(k)})\cap\mathbf{N}\right|}{n}\leq\liminf_{n\to\infty}\frac{g_k(n)}{n}. \tag{4}$$

This inequality holds for all $h\in\mathbf{N}$, and so

$$\sum_{i=1}^{\infty}\left(\frac{1}{A_{2i-1}}-\frac{1}{A_{2i}}\right)\leq\liminf_{n\to\infty}\frac{g_k(n)}{n}=\gamma_k.$$

Applying inequality (4) with $h=2$ and the values for $A_3^{(k)}$ and $A_4^{(k)}$ computed in Theorems 6 and 7, we obtain

$$\gamma_k\geq\left(1-\frac{1}{2^{k-1}}\right)+\left(\frac{1}{2^k}-\frac{1}{3^k}\right)=1-\frac{1}{2^k}-\frac{1}{3^k}.$$

This completes the proof. □

It is a finite calculation to determine explicit values of the integers $A_i^{(k)}$ for small values of i and k. Table 1 contains all values of $A_i^{(k)}$ for $3\leq k\leq 9$ with $A_i^{(k)}<10^6$. Applying inequality (4) in Theorem 8, we can use these values to get lower bounds for γ_k that improve results obtained previously by Rankin [6] and Riddell [7]. For $k=3$, McNew [3] has the current best lower bound. Related results have been obtained by Brown and Gordon [2], Beiglböck, Bergelson, Hindman, and Strauss [1], and Nathanson and O'Bryant [4, 5].

The following table records upper and lower bounds for γ_k.

	Lower bounds on γ_k					Upper bounds on γ_k		
k	Rankin	Riddell	This paper	McNew	k	McNew	From r_k	Riddell
3	0.719 745		0.815 870	0.818 410	3	0.819222	0.846 376	0.857 143
4	0.862 601	0.895 283	0.919 818		4		0.928 874	0.933 334
5	0.931 652	0.958 056	0.963 737		5		0.967 742	0.967 742
6	0.966 324	0.980 371	0.982 877		6		0.983 871	0.984 126
7	0.983 438	0.991 159	0.991 805		7			0.992 126
8	0.991 841	0.995 717	0.995 913		8			0.996 079
9	0.995 969	0.997 939	0.998 012		9			0.998 044

References

[1] M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss, *Multiplicative structures in additively large sets*, J. Combin. Theory Ser. A **113** (2006), no. 7, 1219–1242.

[2] B. E. Brown and D. M. Gordon, *On sequences without geometric progressions*, Math. Comp. **65** (1996), no. 216, 1749–1754.

[3] N. McNew, *On sets of integers which contain no three terms in geometric progression*, arXiv: 1310.2277, 2013.

[4] M. B. Nathanson and K. O'Bryant, *On sequences without geometric progressions*, Integers **13** (2013), #A73, 1–5.

[5] ———, *Irrational numbers associated to sequences without geometric progressions*, Integers **14** (2014).

[6] R. A. Rankin, *Sets of integers containing not more than a given number of terms in arithmetical progression*, Proc. Roy. Soc. Edinburgh Sect. A **65** (1960/1961), 332–344 (1960/61).

[7] J. Riddell, *Sets of integers containing no n terms in geometric progression*, Glasgow Math. J. **10** (1969), 137–146.

	k						
i	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256
3	8	16	32	64	128	256	512
4	9	48	96	243	1152	2304	6561
5	12	200	144	288	1728	3456	6912
6	24	216	576	576	8192	16384	13824
7	27	288	4032	729	28800	32768	19683
8	32	1200	4096	1152	172800	163840	131072
9	36	1296	4608	2048	248832	288000	221184
10	40	1400	32256	3645	307328	331776	492075
11	45	1512	32768	4000	395136	497664	655360
12	48	1600	36288	10240		884736	
13	2208	1728	36864	20736		995328	
14	2209	1800	40320	21952			
15	2256	1944	40960	92160			
16	8832	2000	41472	100000			
17	8836	62400	129600	102400			
18	9024	63936	131072	207360			
19	17664	73800	147456	219520			
20	17672	74088	157216	518400			
21	18048	75600	166464	548800			
22	19872	79704		921600			
23	19881	80688					
24	20304	81648					
25	26496	88000					
26	26508	499200					
27	27072	511488					
28	52992	590400					
29	53016	592704					
30	54144	604800					
31	59616	637632					
32	59643	645504					
33	60912	653184					
34	70656	704000					
35	70688	998400					
36	72192						
37	79488						
38	79524						
39	81216						
40	88320						
41	88360						
42	90240						
43	99360						
44	99405						
45	101520						
46	103776						
47	103823						
48	105984						
49	106032						
50	108192						
51	108241						
52	108288						

TABLE 1. For $3 \leq k \leq 9$, the table contains all integers $A_i^{(k)}$ satisfying Theorem 5 that are less than 10^6. These numbers are sequences A235054-60 in the OEIS.

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