

Enumeration of Standard Young Tableaux

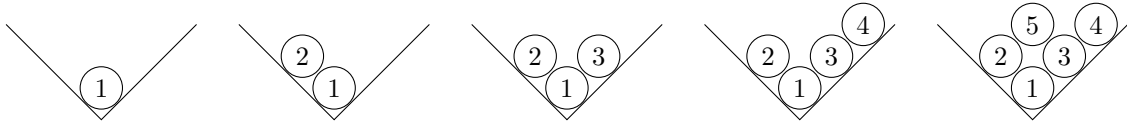
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1 Introduction

1.1 Appetizer

Consider throwing balls labeled $1, 2, \dots, n$ into a V-shaped bin with perpendicular sides.



Question 1.1. *What is the total number of resulting configurations? How many configurations are there of any particular shape?*

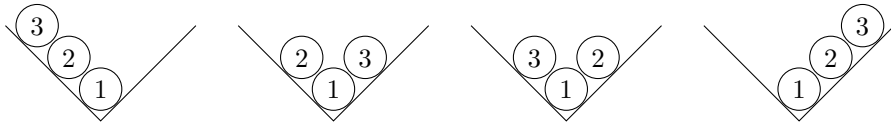
In order to answer these questions, at least partially, recall the symmetric group \mathcal{S}_n of all permutations of the numbers $1, \dots, n$. An **involution** is a permutation $\pi \in \mathcal{S}_n$ such that π^2 is the identity permutation.

Theorem 1.2. *The total number of configurations of n balls is equal to the number of involutions in the symmetric group \mathcal{S}_n .*

Theorem 1.2 may be traced back to Frobenius and Schur. A combinatorial proof will be outlined in Section 4 (see Corollary 4.12).

Example 1.3. *There are four configurations on three balls. Indeed,*

$$\{\pi \in \mathcal{S}_3 : \pi^2 = 1\} = \{123, 132, 213, 321\}.$$



The **inversion number** of a permutation π is defined by

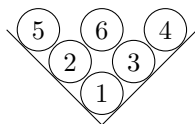
$$\text{inv}(\pi) := \#\{i < j : \pi(i) > \pi(j)\}.$$

The **left weak order** on \mathcal{S}_n is defined by

$$\pi \leq \sigma \iff \text{inv}(\pi) + \text{inv}(\sigma\pi^{-1}) = \text{inv}(\sigma).$$

The following surprising result was first proved by Stanley [112].

Theorem 1.4. *The number of configurations of $\binom{n}{2}$ balls which completely fill $n-1$ levels in the bin is equal to the number of maximal chains in the weak order on \mathcal{S}_n .*



The configurations of balls in a bin are called **standard Young tableaux**. We shall survey in this chapter results related to Question 1.1 and its refinements. Variants and extensions of Theorem 1.2 will be described in Section 4. Variants and extensions of Theorem 1.4 will be described in Section 11.

1.2 General

This chapter is devoted to the enumeration of standard Young tableaux of various shapes, both classical and modern, and to closely related topics. Of course, there is a limit as to how far afield one can go. We chose to include here, for instance, r -tableaux and q -enumeration, but many interesting related topics were left out. Here are some of them, with a minimal list of relevant references for the interested reader: Semi-standard Young tableaux [69][114], (reverse) plane partitions [114], solid (3-dimensional) standard Young tableaux [25], symplectic and orthogonal tableaux [55][20][127][12][128], oscillating tableaux [70][103][96][22][79], cylindrical (and toric) tableaux [86].

1.3 Acknowledgments

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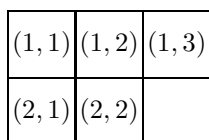
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2 Preliminaries

2.1 Diagrams and tableaux

Definition 2.1. A **diagram** is a finite subset D of the two-dimensional integer lattice \mathbb{Z}^2 . A point $c = (i, j) \in D$ is also called the **cell** in row i and column j of D ; write $\text{row}(c) = i$ and $\text{col}(c) = j$. Cells are usually drawn as squares with axis-parallel sides of length 1, centered at the corresponding lattice points.

Diagrams will be drawn here according to the “English notation”, by which i enumerates rows and increases downwards, while j enumerates columns and increases from left to right:



For alternative conventions see Subsection 2.4.

Definition 2.2. Each diagram D has a natural component-wise partial order, inherited from \mathbb{Z}^2 :

$$(i, j) \leq_D (i', j') \iff i \leq i' \text{ and } j \leq j'.$$

As usual, $c <_D c'$ means $c \leq_D c'$ but $c \neq c'$.

Definition 2.3. Let $n := |D|$, and consider the set $[n] := \{1, \dots, n\}$ with its usual linear order. A **standard Young tableau (SYT) of shape D** is a map $T : D \rightarrow [n]$ which is an order-preserving bijection, namely satisfies

$$c \neq c' \implies T(c) \neq T(c')$$

as well as

$$c \leq_D c' \implies T(c) \leq T(c').$$

Geometrically, a standard Young tableau T is a filling of the n cells of D by the numbers $1, \dots, n$ such that each number appears once, and numbers increase in each row (as the column index increases) and in each column (as the row index increases). Write $\text{sh}(T) = D$. Examples will be given below.

Let $\text{SYT}(D)$ be the set of all standard Young tableaux of shape D , and denote its size by

$$f^D := |\text{SYT}(D)|.$$

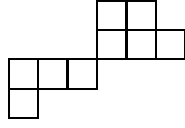
The evaluation of f^D (and some of its refinements) for various diagrams D is the main concern of the current chapter.

2.2 Connectedness and convexity

We now introduce two distinct notions of connectedness for diagrams, and one notion of convexity; for another notion of convexity see Observation 2.13.

Definition 2.4. Two distinct cells in \mathbb{Z}^2 are **adjacent** if they share a horizontal or vertical side; the cells adjacent to $c = (i, j)$ are $(i \pm 1, j)$ and $(i, j \pm 1)$. A diagram D is **path-connected** if any two cells in it can be connected by a **path**, which is a finite sequence of cells in D such that any two consecutive cells are adjacent. The maximal path-connected subsets of a nonempty diagram D are its **path-connected components**.

For example, the following diagram has two path-connected components:

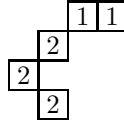


Definition 2.5. The **graph** of a diagram D has all the cells of D as vertices, with two distinct cells $c, c' \in D$ connected by an (undirected) edge if either $c <_D c'$ or $c' <_D c$. The diagram D is **order-connected** if its graph is connected. In any case, the **order-connected components** of D are the subsets of D forming connected components of its graph.

For example, the following diagram (in English notation) is order-connected:



while the following diagram has two order-connected components, with cells marked 1 and 2, respectively:



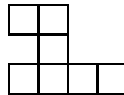
Of course, every path-connected diagram is also order-connected, so that every order-connected component is a disjoint union of path-connected components.

Observation 2.6. If D_1, \dots, D_k are the order-connected components of a diagram D , then

$$f^D = \binom{|D|}{|D_1|, \dots, |D_k|} \prod_{i=1}^k f^{D_i} = |D|! \cdot \prod_{i=1}^k \frac{f^{D_i}}{|D_i|!}.$$

Definition 2.7. A diagram D is **line-convex** if its intersection with every axis-parallel line is either empty or convex, namely if each of its rows $\{j \in \mathbb{Z} \mid (i, j) \in D\}$ (for $i \in \mathbb{Z}$) and columns $\{i \in \mathbb{Z} \mid (i, j) \in D\}$ (for $j \in \mathbb{Z}$) is either empty or an interval $[p, q] = \{p, p+1, \dots, q\} \subseteq \mathbb{Z}$.

For example, the following diagram is path-connected but not line-convex:



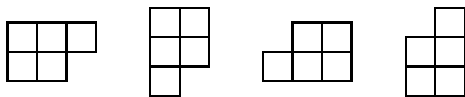
2.3 Invariance under symmetry

The number of SYT of shape D is invariant under some of the geometric operations (isometries of \mathbb{Z}^2) which transform D . It is clearly invariant under arbitrary translations $(i, j) \mapsto (i+a, j+b)$. The group of isometries of \mathbb{Z}^2 that fix a point, say $(0, 0)$, is the dihedral group of order 8. f^D is invariant under a subgroup of order 4.

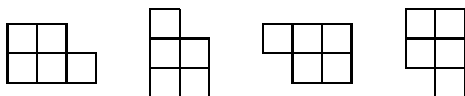
Observation 2.8. f^D is invariant under arbitrary translations of \mathbb{Z}^2 , as well as under

- reflection in a diagonal line: $(i, j) \mapsto (j, i)$ or $(i, j) \mapsto (-j, -i)$; and
- reflection in the origin (rotation by 180°): $(i, j) \mapsto (-i, -j)$.

Note that f^D is not invariant, in general, under reflections in a vertical or horizontal line $((i, j) \mapsto (i, -j)$ or $(i, j) \mapsto (-i, j)$) or rotations by 90° $((i, j) \mapsto (-j, i)$ or $(i, j) \mapsto (j, -i)$). Thus, for example, each of the following diagrams, interpreted according to the English convention (see Subsection 2.4),



has $f^D = 5$, whereas each of the following diagrams



has $f^D = 2$.

2.4 Ordinary, skew and shifted shapes

The best known and most useful diagrams are, by far, the ordinary ones. They correspond to partitions.

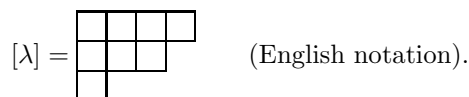
Definition 2.9. A **partition** is a weakly decreasing sequence of positive integers: $\lambda = (\lambda_1, \dots, \lambda_t)$, where $t \geq 0$ and $\lambda_1 \geq \dots \geq \lambda_t > 0$. We say that λ is a partition of **size** $n = |\lambda| := \sum_{i=1}^t \lambda_i$ and **length** $\ell(\lambda) := t$, and write $\lambda \vdash n$. The empty partition $\lambda = ()$ has size and length both equal to zero.

Definition 2.10. Let $\lambda = (\lambda_1, \dots, \lambda_t)$ be a partition. The **ordinary** (or **straight**, or **left-justified**, or **Young**, or **Ferrers**) **diagram of shape** λ is the set

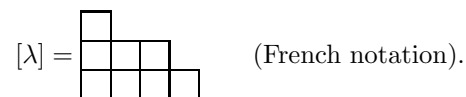
$$D = [\lambda] := \{(i, j) \mid 1 \leq i \leq t, 1 \leq j \leq \lambda_i\}.$$

We say that $[\lambda]$ is a diagram of **height** $\ell(\lambda) = t$.

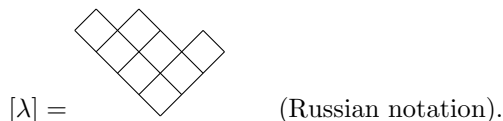
We shall adopt here the “English” convention for drawing diagrams, by which row indices increase from top to bottom and column indices increase from left to right. For example, in this notation the diagram of shape $\lambda = (4, 3, 1)$ is



An alternative convention is the “French” one, by which row indices increase from bottom to top (and column indices increase from left to right):



Note that the term “Young tableau” itself mixes English and French influences. There is also a “Russian” convention, rotated 45° :



This notation leads naturally to the “gravitational” setting used to introduce SYT at the beginning of Section 1.

A partition λ may also be described as an infinite sequence, by adding trailing zeros: $\lambda_i := 0$ for $i > t$. The partition λ' **conjugate** to λ is then defined by

$$\lambda'_j := |\{i \mid \lambda_i \geq j\}| \quad (\forall j \geq 1).$$

The diagram $[\lambda']$ is obtained from the diagram $[\lambda]$ by interchanging rows and columns. For the example above, $\lambda' = (3, 2, 2, 1)$ and

$$[\lambda'] = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$$

An ordinary diagram is clearly path-connected and line-convex. If $D = [\lambda]$ is an ordinary diagram of shape λ we shall sometimes write $\text{SYT}(\lambda)$ instead of $\text{SYT}(D)$ and f^λ instead of f^D .

Example 2.11.

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 8 \\ \hline 3 & 4 & 6 & \\ \hline 7 & & & \\ \hline \end{array} \in \text{SYT}(4, 3, 1).$$

Note that, by Observation 2.8, $f^\lambda = f^{\lambda'}$.

Definition 2.12. If λ and μ are partitions such that $[\mu] \subseteq [\lambda]$, namely $\mu_i \leq \lambda_i$ ($\forall i$), then the **skew diagram of shape λ/μ** is the set difference

$$D = [\lambda/\mu] := [\lambda] \setminus [\mu] = \{(i, j) \in [\lambda] : \mu_i + 1 \leq j \leq \lambda_i\}$$

of two ordinary shapes.

For example,

$$[(6, 4, 3, 1)/(4, 2, 1)] = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

A skew diagram is line-convex, but not necessarily path-connected. In fact, its path-connected components coincide with its order-connected components. If $D = [\lambda/\mu]$ is a skew diagram of shape λ/μ we shall sometimes write $\text{SYT}(\lambda/\mu)$ instead of $\text{SYT}(D)$ and $f^{\lambda/\mu}$ instead of f^D . For example,

$$T = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} \in \text{SYT}((6, 4, 3, 1)/(4, 2, 1)).$$

Skew diagrams have an intrinsic characterization.

Observation 2.13. A diagram D is skew if and only if it is **order-convex**, namely:

$$c, c'' \in D, c' \in \mathbb{Z}^2, c \leq c' \leq c'' \implies c' \in D,$$

where \leq is the natural partial order in \mathbb{Z}^2 , as in Definition 2.2.

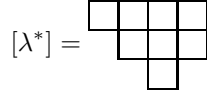
Another important class is that of shifted shapes, corresponding to strict partitions.

Definition 2.14. A partition $\lambda = (\lambda_1, \dots, \lambda_t)$ ($t \geq 0$) is **strict** if the part sizes λ_i are strictly decreasing: $\lambda_1 > \dots > \lambda_t > 0$. The **shifted diagram of shape λ** is the set

$$D = [\lambda^*] := \{(i, j) \mid 1 \leq i \leq t, i \leq j \leq \lambda_i + i - 1\}.$$

Note that $(\lambda_i + i - 1)_{i=1}^t$ is a weakly decreasing sequence of positive integers.

For example, the shifted diagram of shape $\lambda = (4, 3, 1)$ is



A shifted diagram is always path-connected and line-convex. If $D = [\lambda^*]$ is a shifted diagram of shape λ we shall sometimes write $\text{SYT}(\lambda^*)$ instead of $\text{SYT}(D)$ and g^λ instead of f^D . For example,

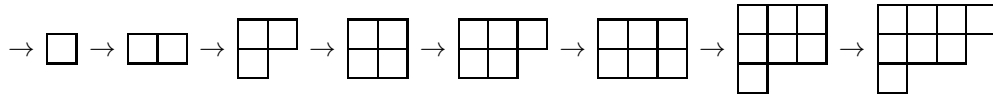
$$T = \begin{array}{cccc} \boxed{1} & \boxed{2} & \boxed{4} & \boxed{6} \\ & \boxed{3} & \boxed{5} & \boxed{8} \\ & & \boxed{7} & \end{array} \in \text{SYT}((4, 3, 1)^*).$$

2.5 Interpretations

There are various interpretations of a standard Young tableau, in addition to the interpretation (in Definition 2.3) as a linear extension of a partial order. Some of these interpretations play a key role in enumeration.

2.5.1 The Young lattice

A standard Young tableau of ordinary shape describes a **growth process** of diagrams of ordinary shapes, starting from the empty shape. For example, the tableau T in Example 2.11 corresponds to the process



Consider the **Young lattice** whose elements are all partitions, ordered by inclusion (of the corresponding diagrams). By the above, a SYT of ordinary shape λ is a maximal chain, in the Young lattice, from the empty partition to λ . The number of such maximal chains is therefore f^λ . More generally, a SYT of skew shape λ/μ is a maximal chain from μ to λ in the Young lattice.

A SYT of shifted shape can be similarly interpreted as a maximal chain in the **shifted Young lattice**, whose elements are strict partitions ordered by inclusion.

2.5.2 Ballot sequences and lattice paths

Definition 2.15. A sequence (a_1, \dots, a_n) of positive integers is a **ballot sequence**, or **lattice permutation**, if for any integers $1 \leq k \leq n$ and $r \geq 1$,

$$\#\{1 \leq i \leq k \mid a_i = r\} \geq \#\{1 \leq i \leq k \mid a_i = r + 1\},$$

namely: in any initial subsequence (a_1, \dots, a_k) , the number of entries equal to r is not less than the number of entries equal to $r + 1$.

A ballot sequence describes the sequence of votes in an election process with several candidates (and one ballot), assuming that at any given time candidate 1 has at least as many votes as candidate 2, who has at least as many votes as candidate 3, etc. For example, $(1, 1, 2, 3, 2, 1, 4, 2, 3)$ is a ballot sequence for an election process with 9 voters and 4 candidates.

For a partition λ of n , denote by $\text{BS}(\lambda)$ the set of ballot sequences (a_1, \dots, a_n) with $\#\{i \mid a_i = r\} = \lambda_r$ ($\forall r$).

Observation 2.16. The map $\phi : \text{SYT}(\lambda) \rightarrow \text{BS}(\lambda)$ defined by

$$\phi(T)_i := \text{row}(T^{-1}(i)) \quad (1 \leq i \leq n)$$

is a bijection.

For example, if T is the SYT in Example 2.11 then $\phi(T) = (1, 1, 2, 2, 1, 2, 3, 1)$.

Clearly, a ballot sequence $a = (a_1, \dots, a_n) \in \text{BS}(\lambda)$ with maximal entry t corresponds to a **lattice path** in \mathbb{R}^t , from the origin 0 to the point λ , where in step i of the path coordinate a_i of the point increases by 1. In fact, $\text{BS}(\lambda)$ is in bijection with the set of all lattice paths from 0 to λ which lie entirely in the cone

$$\{(x_1, \dots, x_t) \in \mathbb{R}^t \mid x_1 \geq \dots \geq x_t \geq 0\}.$$

A SYT of skew shape λ/μ corresponds to a lattice path in this cone from μ to λ .

A SYT of shifted shape corresponds to a **strict** ballot sequence, describing a lattice path within the cone

$$\{(x_1, \dots, x_t) \in \mathbb{R}^t \mid x_1 > \dots > x_s > x_{s+1} = \dots = x_t = 0 \text{ for some } 0 \leq s \leq t\}.$$

2.5.3 The order polytope

Using the partial order on a diagram D , as in Definition 2.2, one can define the corresponding **order polytope**

$$P(D) := \{f : D \rightarrow [0, 1] \mid c \leq_D c' \implies f(c) \leq f(c') \ (\forall c, c' \in D)\}.$$

The order polytope is a closed convex subset of the unit cube $[0, 1]^D$. Denoting $n := |D|$, each linear extension $c_1 < \dots < c_n$ of the partial order \leq_D corresponds to a simplex

$$\{f : D \rightarrow [0, 1] \mid f(c_1) \leq \dots \leq f(c_n)\}$$

of volume $1/n!$ inside $P(D)$. The union of these simplices is $P(D)$, and this union is almost disjoint: Any intersection of two or more simplices is contained in a hyperplane, and thus has volume 0. Each simplex, or linear extension, corresponds to a SYT of shape D . Therefore

Observation 2.17. *If $P(D)$ is the order polytope of a diagram D then*

$$\text{vol } P(D) = \frac{f^D}{|D|!}.$$

2.5.4 Other interpretations

The number of SYT of ordinary shape can be interpreted as a coefficient in a power series, or as the constant term in a Laurent series; see Remark 6.10.

SYT of certain ordinary, skew and shifted shapes may be interpreted as **reduced words** for suitable elements in the symmetric group. This interpretation will be developed and explained in Section 11.

SYT may be interpreted as **permutations**; f^λ then measures the size of certain subsets of the symmetric group, such as descent classes, sets of involutions, Knuth classes and pattern avoidance classes. Examples will be given in Sections 4 and 10.

The Young lattice has a vast generalization to the concept of **differential poset** [113][42].

There are other deep algebraic and geometric interpretations. The interested reader is encouraged to start with the beautiful survey [9] and the excellent textbooks [53, 114, 99, 37, 69]. In the current survey we will focus on combinatorial aspects and mostly ignore the algebraic and geometric approaches.

2.6 Miscellanea

The concepts to be defined here are not directly related to standard Young tableaux, but will be used later in this survey.

A **composition** of a nonnegative integer n is a sequence (μ_1, \dots, μ_t) of positive integers such that $\mu_1 + \dots + \mu_t = n$. The components μ_i are *not* required to be weakly decreasing; in fact, every composition may be re-ordered to form a (unique) partition. $n = 0$ has a unique (empty) composition.

A permutation $\sigma \in \mathcal{S}_n$ **avoids a pattern** $\pi \in \mathcal{S}_k$ if the sequence $(\sigma(1), \dots, \sigma(n))$ does not contain a subsequence $(\sigma(t_1), \dots, \sigma(t_k))$ (with $t_1 < \dots < t_k$) which is order-isomorphic to π , namely: $\sigma(t_i) < \sigma(t_j) \iff \pi(i) < \pi(j)$. For example, $21354 \in \mathcal{S}_5$ is 312-avoiding, but 52134 is not (since 523 is order-isomorphic to 312).

3 Formulas for thin shapes

3.1 Hook shapes

A **hook shape** is an ordinary shape which is the union of one row and one column. For example,

$$[(6, 1^3)] = \begin{array}{|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & \\ \hline \square & & & & & \\ \hline \square & & & & & \\ \hline \end{array}$$

One of the simplest enumerative formulas is the following.

Observation 3.1. For every $n \geq 1$ and $0 \leq k \leq n - 1$,

$$f^{(n-k, 1^k)} = \binom{n-1}{k}.$$

Proof. The letter 1 must be in the corner cell. The SYT is uniquely determined by the choice of the other k letters in the first column. \square

Note that, in a hook shape $(n+1-k, 1^k)$, the letter $n+1$ must be in the last cell of either the first row or the first column. Thus

$$f^{(n+1-k, 1^k)} = f^{(n-k, 1^k)} + f^{(n+1-k, 1^{k-1})}.$$

By Observation 3.1, this is equivalent to Pascal's identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad (1 \leq k \leq n-1).$$

Observation 3.2. The total number of hook shaped SYT of size n is 2^{n-1} .

Proof. There is a bijection between hook shaped SYT of size n and subsets of $\{1, \dots, n\}$ containing 1: Assign to each SYT the set of entries in its first row.

Alternatively, a hook shaped SYT of size $n \geq 2$ is uniquely constructed by adding a cell containing n at the end of either the first row or the first column of a hook shaped SYT of size $n-1$, thus recursively multiplying the number of SYT by 2.

Of course, the claim also follows from Observation 3.1. \square

3.2 Two-rowed shapes

Consider now ordinary shapes with at most two rows.

Proposition 3.3. For every $n \geq 0$ and $0 \leq k \leq n/2$

$$f^{(n-k, k)} = \binom{n}{k} - \binom{n}{k-1},$$

where $\binom{n}{-1} = 0$ by convention. In particular,

$$f^{(m, m)} = f^{(m, m-1)} = C_m = \frac{1}{m+1} \binom{2m}{m},$$

the m -th Catalan number.

Proof. We shall outline two proofs, one by induction and one combinatorial.

For a proof by induction on n note first that $f^{(0,0)} = f^{(1,0)} = 1$.

If $0 < k < n/2$ then there are two options for the location of the letter n – at the end of the first row or at the end of the second. Hence

$$f^{(n-k,k)} = f^{(n-k-1,k)} + f^{(n-k,k-1)} \quad (0 < k < n/2).$$

Thus, by the induction hypothesis and Pascal's identity,

$$f^{(n-k,k)} = \binom{n-1}{k} - \binom{n-1}{k-1} + \binom{n-1}{k-1} - \binom{n-1}{k-2} = \binom{n-1}{k} - \binom{n-1}{k-2} = \binom{n}{k} - \binom{n}{k-1}.$$

The cases $k = 0$ and $k = n/2$ are left to the reader.

For a combinatorial proof, recall (from Subsection 2.5.2) the lattice path interpretation of a SYT and use André's reflection trick: A SYT of shape $(n-k, k)$ corresponds to a lattice path from $(0, 0)$ to $(n-k, k)$ which stays within the cone $\{(x, y) \in \mathbb{R}^2 \mid x \geq y \geq 0\}$, namely does not touch the line $y = x + 1$. The number of *all* lattice paths from $(0, 0)$ to $(n-k, k)$ is $\binom{n}{k}$. If such a path touches the line $y = x + 1$, reflect its “tail” (starting from the first touch point) in this line to get a path from $(0, 0)$ to the reflected endpoint $(k-1, n-k+1)$. The reflection defines a bijection between all the “touching” paths to $(n-k, k)$ and all the (necessarily “touching”) paths to $(k-1, n-k+1)$, whose number is clearly $\binom{n}{k-1}$. \square

Corollary 3.4. *The total number of SYT of size n and at most 2 rows is $\binom{n}{\lfloor n/2 \rfloor}$.*

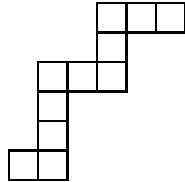
Proof. By Proposition 3.3,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} f^{(n-k,k)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\binom{n}{k} - \binom{n}{k-1} \right) = \binom{n}{\lfloor n/2 \rfloor}.$$

\square

3.3 Zigzag shapes

A **zigzag shape** is a path-connected skew shape which does not contain a 2×2 square. For example, every hook shape is zigzag. Here is an example of a zigzag shape of size 11:



The number of SYT of a specific zigzag shape has an interesting formula, to be presented in Subsection 7.1. The total number of SYT of given size and various zigzag shapes is given by the following folklore statement, to be refined later (Proposition 10.12).

Proposition 3.5. *The total number of zigzag shaped SYT of size n is $n!$.*

Proof. Define a map from zigzag shaped SYT of size n to permutations in \mathcal{S}_n by simply listing the entries of the SYT, starting from the SW corner and moving along the shape. This map is a bijection, since an obvious inverse map builds a SYT from a permutation $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathcal{S}_n$ by attaching a cell containing σ_{i+1} to the right of the cell containing σ_i if $\sigma_{i+1} > \sigma_i$, and above this cell otherwise. \square

4 Jeu de taquin and the RS correspondence

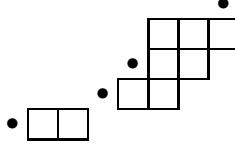
4.1 Jeu de taquin

Jeu de taquin is a very powerful combinatorial algorithm, introduced by Schützenberger [106]. It provides a unified approach to many enumerative results. In general, it transforms a SYT of skew shape into some other SYT of skew shape, using a sequence of **slides**. We shall describe here a version of it, using only **forward slides**, which transforms a SYT of skew shape into a (unique) SYT of ordinary shape. Our description follows [99].

Definition 4.1. Let D be a nonempty diagram of skew shape. An **inner corner** for D is a cell $c \notin D$ such that

1. $D \cup \{c\}$ is a skew shape, and
2. there exists a cell $c' \in D$ such that $c \leq c'$ (in the natural partial order of \mathbb{Z}^2 , as in Definition 2.2).

Example 4.2. Here is a skew shape with marked inner corners:



Here is the main jeu de taquin procedure:

Input: T , a SYT of arbitrary skew shape.

Output: T' , a SYT of ordinary shape.

```

1: procedure JdT( $T$ )
2:    $D \leftarrow \text{sh}(T)$ 
3:   Choose a cell  $c_0 = (i_0, j_0)$  such that  $D \subseteq (c_0)_+ := \{(i, j) \in \mathbb{Z}^2 : i \geq i_0, j \geq j_0\}$ 
4:   while  $c_0 \notin D$  do
5:     Choose  $c = (i, j) \in (c_0)_+ \setminus D$  which is an inner corner for  $D$ 
6:      $T \leftarrow \text{FORWARDSLIDE}(T, c)$ 
7:      $D \leftarrow \text{sh}(T)$ 
8:   end while
9:   return  $T$ 
10: end procedure

```

▷ Now $c_0 \in D \subseteq (c_0)_+$, so D has ordinary shape

and here is the procedure FORWARDSLIDE:

Input: (T_{in}, c_{in}) , where T_{in} is a SYT of skew shape D_{in} and c_{in} is an inner corner for D_{in} .

Output: T_{out} , a SYT of skew shape $D_{out} = D_{in} \cup \{c_{in}\} \setminus \{c'\}$ for some $c' \in D_{in}$.

```

1: procedure FORWARDSLIDE( $T_{in}, c_{in}$ )
2:    $T \leftarrow T_{in}, c \leftarrow c_{in}$ 
3:    $D \leftarrow \text{sh}(T)$ 
4:   if  $c = (i, j)$  then
5:      $c_1 \leftarrow (i + 1, j)$ 
6:      $c_2 \leftarrow (i, j + 1)$ 
7:   end if
8:   while at least one of  $c_1$  and  $c_2$  is in  $D$  do
9:      $c' \leftarrow \begin{cases} c_1, & \text{if } c_1 \in D \text{ but } c_2 \notin D, \text{ or } c_1, c_2 \in D \text{ and } T(c_1) < T(c_2) \\ c_2, & \text{if } c_2 \in D \text{ but } c_1 \notin D, \text{ or } c_1, c_2 \in D \text{ and } T(c_2) < T(c_1) \end{cases}$ 

```

```

10:    $D' \leftarrow D \cup \{c\} \setminus \{c'\}$   $\triangleright c \notin D, c' \in D$ 
11:   Define  $T' \in \text{SYT}(D')$  by:  $T' = T$  on  $D \setminus \{c'\}$  and  $T'(c) := T(c')$ 
12:    $D \leftarrow D', T \leftarrow T', c \leftarrow c'$ 
13:   if  $c = (i, j)$  then
14:      $c_1 \leftarrow (i + 1, j)$ 
15:      $c_2 \leftarrow (i, j + 1)$ 
16:   end if
17: end while
18: return  $T$ 
19: end procedure

```

The JDT algorithm employs certain random choices, but actually

Proposition 4.3. [106, 130, 131] *For any SYT T of skew shape, the resulting SYT $\text{JDT}(T)$ of ordinary shape is independent of the choices made during the computation.*

Example 4.4. *Here is an example of a forward slide, with the initial c_{in} and the intermediate cells c marked:*

$$T_{in} = \begin{array}{ccc} & \bullet & 3 & 6 \\ & 1 & 4 & 7 \\ 2 & 5 & 8 & \end{array} \rightarrow \begin{array}{ccc} & 1 & 3 & 6 \\ & \bullet & 4 & 7 \\ 2 & 5 & 8 & \end{array} \rightarrow \begin{array}{ccc} & 1 & 3 & 6 \\ & 4 & \bullet & 7 \\ 2 & 5 & 8 & \end{array} \rightarrow \begin{array}{ccc} & 1 & 3 & 6 \\ & 4 & 7 & \bullet \\ 2 & 5 & 8 & \end{array} = T_{out},$$

and here is an example of a full jeu de taquin (where each step is a forward slide):

$$T = \begin{array}{ccc} & \bullet & 3 & 6 \\ & 1 & 4 & 7 \\ 2 & 5 & 8 & \end{array} \rightarrow \begin{array}{ccc} & 1 & 3 & 6 \\ & \bullet & 4 & 7 \\ 2 & 5 & 8 & \end{array} \rightarrow \begin{array}{ccc} & \bullet & 1 & 3 & 6 \\ & 2 & 4 & 7 \\ 5 & 8 & \end{array} \rightarrow \begin{array}{ccc} & 1 & 3 & 6 \\ & 2 & 4 & 7 \\ 5 & 8 & \end{array} = \text{JDT}(T).$$

4.2 The Robinson-Schensted correspondence

The Robinson-Schensted (RS) correspondence is a bijection from permutations in \mathcal{S}_n to pairs of SYT of size n and same ordinary shape. Its original motivation was the study of the distribution of longest increasing subsequences in a permutation. For a detailed description see, e.g., the textbooks [114], [99] and [37]. We shall use the jeu de taquin algorithm to give an alternative description.

Definition 4.5. Denote $\delta_n := [(n, n - 1, n - 2, \dots, 1)]$. For a permutation $\pi \in \mathcal{S}_n$ let T_π the skew SYT of antidiagonal shape δ_n/δ_{n-1} in which the entry in the i -th column from the left is $\pi(i)$.

Example 4.6.

$$\pi = 53412 \implies T_\pi = \begin{array}{cccc} & & & 2 \\ & & & 1 \\ & & 4 & \\ & 3 & & \\ 5 & & & \end{array}.$$

Definition 4.7. (The Robinson-Schensted (RS) correspondence) For a permutation $\pi \in \mathcal{S}_n$ let

$$P_\pi := \text{JDT}(T_\pi) \quad \text{and} \quad Q_\pi := \text{JDT}(T_{\pi^{-1}}).$$

Example 4.8.

$$\pi = 2413 \implies T_\pi = \begin{array}{ccc} & & 3 \\ & & 1 \\ & 4 & \\ 2 & & \end{array}, \quad T_{\pi^{-1}} = \begin{array}{ccc} & & 2 \\ & & 4 \\ & 1 & \\ 3 & & \end{array}.$$

Then

$$T_\pi = \begin{array}{c} \bullet \\ \square 1 \\ \square 4 \\ \square 2 \end{array} \begin{array}{c} \square 3 \\ \square 4 \\ \square 2 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 1 \\ \square 4 \\ \square 2 \end{array} \begin{array}{c} \bullet \\ \square 3 \\ \square 4 \\ \square 2 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 1 \\ \square 4 \\ \square 2 \end{array} \begin{array}{c} \square 3 \\ \square 4 \\ \square 2 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 1 \\ \square 4 \\ \square 2 \end{array} \begin{array}{c} \square 3 \\ \square 4 \\ \square 2 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 1 \\ \square 4 \\ \square 2 \end{array} \begin{array}{c} \bullet \\ \square 3 \\ \square 4 \\ \square 2 \end{array} \rightarrow \begin{array}{c} \square 1 \\ \square 2 \\ \square 3 \\ \square 4 \end{array} = P_\pi$$

and

$$T_{\pi^{-1}} = \begin{array}{c} \bullet \\ \square 2 \\ \square 4 \\ \square 3 \end{array} \begin{array}{c} \square 1 \\ \square 4 \\ \square 3 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 2 \\ \square 4 \\ \square 3 \end{array} \begin{array}{c} \bullet \\ \square 1 \\ \square 4 \\ \square 3 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 2 \\ \square 4 \\ \square 3 \end{array} \begin{array}{c} \square 1 \\ \square 4 \\ \square 3 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 2 \\ \square 4 \\ \square 3 \end{array} \begin{array}{c} \square 1 \\ \square 4 \\ \square 3 \end{array} \rightarrow \begin{array}{c} \bullet \\ \square 2 \\ \square 4 \\ \square 3 \end{array} \begin{array}{c} \square 1 \\ \square 4 \\ \square 3 \end{array} \rightarrow \begin{array}{c} \square 1 \\ \square 3 \\ \square 2 \\ \square 4 \end{array} = Q_\pi.$$

Theorem 4.9. *The RS correspondence is a bijection from all permutations in \mathcal{S}_n to all pairs of SYT of size n and the same shape.*

Thus

Claim 4.10. *For every permutation $\pi \in \mathcal{S}_n$,*

- (i) $\text{sh}(P_\pi) = \text{sh}(Q_\pi)$.
- (ii) $\pi \leftrightarrow (P, Q) \implies \pi^{-1} \leftrightarrow (Q, P)$.

A very fundamental property of the RS correspondence is the following.

Proposition 4.11. [104] *The height of $\text{sh}(\pi)$ is equal to the size of the longest decreasing subsequence in π . The width of $\text{sh}(\pi)$ is equal to the size of the longest increasing subsequence in π .*

A version of the RS correspondence for shifted shapes was given, initially, by Sagan [100]. An improved algorithm was found, independently, by Worley [135] and Sagan [102]. See also [48].

4.3 Enumerative applications

In this section we list just a few applications of the above combinatorial algorithms.

Corollary 4.12.

- (1) *The total number of pairs of SYT of the same shape is $n!$. Thus*

$$\sum_{\lambda} (f^\lambda)^2 = n!$$

- (2) *The total number of SYT of size n is equal to the number of involutions in \mathcal{S}_n [110, A000085]. Thus*

$$\sum_{\lambda \vdash n} f^\lambda = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!!,$$

where $(2k-1)!! := 1 \cdot 3 \cdot \dots \cdot (2k-1)$.

- (3) *Furthermore, for every positive integer k , the total number of SYT of height $< k$ is equal to the number of $[k, k-1, \dots, 1]$ -avoiding involutions in \mathcal{S}_n .*

Proof. (1) follows from Claim 4.10(i), (2) from Claim 4.10(ii), and (3) from Proposition 4.11. □

A careful examination of the RS correspondence implies the following refinement of Corollary 4.12(2).

Theorem 4.13. *The total number of SYT of size n with $n - 2k$ odd rows is equal to $\binom{n}{2k}(2k - 1)!$, the number of involutions in \mathcal{S}_n with $n - 2k$ fixed points.*

Corollary 4.14. *The total number of SYT of size $2n$ and all rows even is equal to $(2n - 1)!$, the number of fixed point free involutions in \mathcal{S}_{2n} .*

For further refinements see, e.g., [120, Ex. 45–46, 85].

Recalling the simple formula for the number of two-rowed SYT (Corollary 3.4), it is tempting to look for the total number of SYT of shapes with more rows.

Theorem 4.15. [87] *The total number of SYT of size n and at most 3 rows is*

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k,$$

the n -th Motzkin number [110, A001006].

Proof. By Observation 2.6 together with Proposition 3.3, the number of SYT of skew shape $(n - k, k, k)/(k)$ is equal to

$$\binom{n}{2k} C_k,$$

where C_k is k -th Catalan number. On the other hand, by careful examination of the jeu de taquin algorithm, one can verify that it induces a bijection from the set of all SYT of skew shape $(n - k, k, k)/(k)$ to the set of all SYT of shapes $(n - k - j, k, j)$ for $0 \leq j \leq \min(k, n - 2k)$. Thus

$$\sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq 3}} f^\lambda = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_j f^{(n-k-j, k, j)} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k,$$

completing the proof. □

See [28] for a bijective proof of Theorem 4.15 via a map from SYT of height at most 3 to Motzkin paths. The n -th Motzkin number also counts non-crossing involutions in \mathcal{S}_n . It follows that

Corollary 4.16. *The total number of SYT of height at most 3 is equal to the number of non-crossing involutions in \mathcal{S}_n .*

Somewhat more complicated formulas have been found for shapes with more rows.

Theorem 4.17. [45]

1. *The total number of SYT of size n and at most 4 rows is equal to $C_{\lfloor (n+1)/2 \rfloor} C_{\lceil (n+1)/2 \rceil}$.*
2. *The total number of SYT of size n and at most 5 rows is equal to $6 \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k \frac{(2k+2)!}{(k+2)!(k+3)!}$.*

The following shifted analogue of Corollary 4.12(1) was proved by Schur [105], more than a hundred years ago, in a representation theoretical setting. A combinatorial proof, using the shifted RS correspondence, was given by Sagan [100]. An improved shifted RS algorithm was found, independently, by Worley [135] and Sagan [102]. See the end of Subsection 2.4 for the notation g^λ .

Theorem 4.18.

$$\sum_{\text{strict } \lambda} 2^{n-\ell(\lambda)} (g^\lambda)^2 = n!$$

5 Formulas for classical shapes

There is an explicit formula for the number of SYT of each classical shape – ordinary, skew or shifted. In fact, there are several equivalent formulas, all unusually elegant. These formulas, with proofs, will be given in this section. Additional proof approaches (mostly for ordinary shapes) will be described in Section 6.

5.1 Ordinary shapes

In this subsection we consider ordinary shapes $D = [\lambda]$, corresponding to partitions λ . Recall the notation $f^\lambda := |\text{SYT}(\lambda)|$ for the number of standard Young tableaux of shape λ . Several explicit formulas are known for this number – a product formula, a hook length formula and a determinantal formula.

Historically, ordinary tableaux were introduced by Young in 1900 [136]. The first explicit formula for the number of SYT of ordinary shape was the product formula. It was obtained in 1900 by Frobenius [36, eqn. 6] in an algebraic context, as the degree of an irreducible character χ^λ of \mathcal{S}_n . Independently, MacMahon [71, p. 175] in 1909 (see also [72, §103]) obtained the same formula for the number of ballot sequences (see Definition 2.15 above), which are equinumerous with SYT. In 1927 Young [137, pp. 260–261] showed that $\deg(\chi^\lambda)$ is actually equal to the number of SYT of shape λ , and also provided his own proof [137, Theorem II] of MacMahon’s result.

Theorem 5.1. (Ordinary product formula) *For a partition $\lambda = (\lambda_1, \dots, \lambda_t)$, let $\ell_i := \lambda_i + t - i$ ($1 \leq i \leq t$). Then*

$$f^\lambda = \frac{|\lambda|!}{\prod_{i=1}^t \ell_i!} \cdot \prod_{(i,j): i < j} (\ell_i - \ell_j).$$

The best known and most influential of the explicit formulas is doubtlessly the Frame-Robinson-Thrall hook length formula, published in 1954 [34]. The story of its discovery is quite amazing [99]: Frame was led to conjecture the formula while discussing the work of Staal, one of Robinson’s students, during Robinson’s visit to him in May 1953. Robinson could not believe, at first, that such a simple formula exists, but became convinced after trying some examples, and together they proved it. A few days later, Robinson gave a lecture followed by a presentation of the new result by Frame. Thrall, who was in the audience, was very surprised because he had just proved the same result on the same day!

Definition 5.2. *For a cell $c = (i, j) \in [\lambda]$ let*

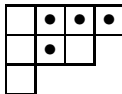
$$H_c := [\lambda] \cap (\{(i, j)\} \cup \{(i, j') \mid j' > j\} \cup \{(i', j) \mid i' > i\})$$

be the corresponding hook, and let

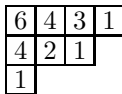
$$h_c := |H_c| = \lambda_i + \lambda'_j - i - j + 1.$$

be the corresponding hook length.

For example, in the following diagram the cells of the hook $H_{(1,2)}$ are marked:



and in the following diagram each cell is labeled by the corresponding hook length:



Theorem 5.3. (Ordinary hook length formula) *For any partition $\lambda = (\lambda_1, \dots, \lambda_t)$,*

$$f^\lambda = \frac{|\lambda|!}{\prod_{c \in [\lambda]} h_c}.$$

Last, but not least, is the determinantal formula. Remarkably, it also has a generalization to the skew case; see the next subsection.

Theorem 5.4. (Ordinary determinantal formula) *For any partition $\lambda = (\lambda_1, \dots, \lambda_t)$,*

$$f^\lambda = |\lambda|! \cdot \det \left[\frac{1}{(\lambda_i - i + j)!} \right]_{i,j=1}^t,$$

using the convention $1/k! := 0$ for negative integers k .

We shall now show that all these formulas are equivalent. Their validity will then follow from a forthcoming proof of Theorem 5.6, which is a generalization of Theorem 5.4. Other proof approaches will be described in Section 6.

Claim 5.5. *The formulas in Theorems 5.1, 5.3 and 5.4 are equivalent.*

Proof. To prove the equivalence of the product formula (Theorem 5.1) and the hook length formula (Theorem 5.3), it suffices to show that

$$\prod_{c \in [\lambda]} h_c = \frac{\prod_{i=1}^t (\lambda_i + t - i)!}{\prod_{(i,j): i < j} (\lambda_i - \lambda_j - i + j)}.$$

This follows by induction on the number of columns, once we show that the product of hook lengths for all the cells in the first column of $[\lambda]$ satisfies

$$\prod_{i=1}^t h_{(i,1)} = \prod_{i=1}^t (\lambda_i + t - i);$$

and this readily follows from the obvious

$$h_{(i,1)} = \lambda_i + t - i \quad (\forall i).$$

Actually, one also needs to show that the ordinary product formula is valid even when the partition λ has trailing zeros (so that t in the formula may be larger than the number of nonzero parts in λ). This is not difficult, since adding one zero part $\lambda_{t+1} = 0$ (and replacing t by $t+1$) amounts, in the product formula, to replacing each $\ell_i = \lambda_i + t - i$ by $\ell_i + 1$ ($1 \leq i \leq t$) and adding $\ell_{t+1} = 0$, which multiplies the RHS of the formula by

$$\frac{1}{\prod_{i=1}^t (\ell_i + 1) \cdot \ell_{t+1}!} \cdot \prod_{i=1}^t (\ell_i + 1 - \ell_{t+1}) = 1.$$

To prove equivalence of the product formula (Theorem 5.1) and the determinantal formula (Theorem 5.4), it suffices to show that

$$\det \left[\frac{1}{(\ell_i - t + j)!} \right]_{i,j=1}^t = \frac{1}{\prod_{i=1}^t \ell_i!} \cdot \prod_{(i,j): i < j} (\ell_i - \ell_j) \quad ,$$

where

$$\ell_i := \lambda_i + t - i \quad (1 \leq i \leq t)$$

as in Theorem 5.1. Using the **falling factorial** notation

$$(a)_n := \prod_{i=1}^n (a + 1 - i) \quad (n \geq 0),$$

this claim is equivalent to

$$\det [(\ell_i)_{t-j}]_{i,j=1}^t = \prod_{(i,j): i < j} (\ell_i - \ell_j)$$

which, in turn, is equivalent (under suitable column operations) to the well known evaluation of the Vandermonde determinant

$$\det [\ell_i^{t-j}]_{i,j=1}^t = \prod_{(i,j): i < j} (\ell_i - \ell_j).$$

See [99, pp. 132–133] for an inductive proof avoiding explicit use of the Vandermonde. □

5.2 Skew shapes

The determinantal formula for the number of SYT of an ordinary shape can be extended to apply to a general skew shape. The formula is due to Aitken [5, p. 310], and was rediscovered by Feit [29]. No product or hook length formula is known in this generality (but a product formula for a staircase minus a rectangle has been found by DeWitt [21]; see also [61]). Specific classes of skew shapes, such as zigzags and strips of constant width, have interesting special formulas; see Section 7.

Theorem 5.6. (Skew determinantal formula) [5][29][114, Corollary 7.16.3] *The number of SYT of skew shape λ/μ , for partitions $\lambda = (\lambda_1, \dots, \lambda_t)$ and $\mu = (\mu_1, \dots, \mu_s)$ with $\mu_i \leq \lambda_i$ ($\forall i$), is*

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^t,$$

with the conventions $\mu_j := 0$ for $j > s$ and $1/k! := 0$ for negative integers k .

The following proof is inductive. There is another approach that uses the Jacobi-Trudi identity.

Proof. (Adapted from [29])

By induction on the size $n := |\lambda/\mu|$. Denote

$$a_{ij} := \frac{1}{(\lambda_i - \mu_j - i + j)!}.$$

For $n = 0$, $\lambda_i = \mu_i$ ($\forall i$). Thus

$$i = j \implies \lambda_i - \mu_i - i + i = 0 \implies a_{ii} = 1$$

and

$$i > j \implies \lambda_i - \mu_j - i + j < \lambda_i - \mu_j = \lambda_i - \lambda_j \leq 0 \implies a_{ij} = 0.$$

Hence the matrix (a_{ij}) is upper triangular with diagonal entries 1, and $f^{\lambda/\mu} = 1 = 0! \det(a_{ij})$.

For the induction step assume that the claim holds for all skew shapes of size $n - 1$, and consider a shape λ/μ of size n with t rows. The cell containing n must be the last cell in its row and column. Therefore

$$f^{\lambda/\mu} = \sum_{i'} f^{(\lambda/\mu)_{i'}}$$

where $(\lambda/\mu)_{i'}$ is the shape λ/μ minus the last cell in row i' , and summation is over all the rows i' which are nonempty and whose last cell is also last in its column. Explicitly, summation is over all i' such that $\lambda_{i'} > \mu_{i'}$ as well as $\lambda_{i'} > \lambda_{i'+1}$. By the induction hypothesis,

$$f^{\lambda/\mu} = (n-1)! \sum_{i'} \det(a_{ij}^{(i')})$$

where $a_{ij}^{(i')}$ is the analogue of a_{ij} for the shape $(\lambda/\mu)_{i'}$ and summation is over the above values of i' . In fact,

$$a_{ij}^{(i')} = \begin{cases} a_{ij}, & \text{if } i \neq i'; \\ (\lambda_i - \mu_j - i + j) \cdot a_{ij}, & \text{if } i = i'. \end{cases}$$

This holds for all values (positive, zero or negative) of $\lambda_i - \mu_j - i + j$. The rest of the proof consists of two steps.

Step 1: The above formula for $f^{\lambda/\mu}$ holds with summation extending over all $1 \leq i' \leq t$. Indeed, it suffices to show that

$$\lambda_{i'} = \mu_{i'} \text{ or } \lambda_{i'} = \lambda_{i'+1} \implies \det(a_{ij}^{(i')}) = 0.$$

If $\lambda_{i'} = \lambda_{i'+1}$ then

$$\lambda_{i'+1} - \mu_j - (i' + 1) + j = (\lambda_{i'} - 1) - \mu_j - i' + j \quad (\forall j),$$

so that the matrix $(a_{ij}^{(i')})$ has two equal rows and hence its determinant is 0. If $\lambda_{i'} = \mu_{i'}$ then

$$j \leq i' < i \implies \lambda_i - \mu_j - i + j < \lambda_i - \mu_j \leq \lambda_i - \mu_{i'} \leq \lambda_{i'} - \mu_{i'} = 0$$

and

$$j \leq i' = i \implies (\lambda_{i'} - 1) - \mu_j - i' + j < \lambda_{i'} - \mu_j \leq \lambda_{i'} - \mu_{i'} = 0.$$

Thus the matrix $(a_{ij}^{(i')})$ has a zero submatrix corresponding to $j \leq i' \leq i$, which again implies that its determinant is zero – e.g., by considering the determinant as a sum over permutations $\sigma \in \mathcal{S}_t$ and noting that, by the pigeon hole principle, there is no permutation satisfying $j = \sigma(i) > i'$ for all $i \geq i'$.

Step 2: Let A_{ij} be the (i, j) -cofactor of the matrix $A = (a_{ij})$, so that

$$\det A = \sum_j a_{ij} A_{ij} \quad (\forall i)$$

and also

$$\det A = \sum_i a_{ij} A_{ij} \quad (\forall j).$$

Then, expanding along row i' ,

$$\det(a_{ij}^{(i')}) = \sum_j a_{i'j}^{(i')} A_{i'j} = \sum_j (c_{i'} - d_j) a_{i'j} A_{i'j}$$

where $c_i := \lambda_i - i$ and $d_j := \mu_j - j$. Thus

$$\begin{aligned}
\frac{f^{\lambda/\mu}}{(n-1)!} &= \sum_{i'=1}^t \det(a_{ij}^{(i')}) = \sum_{i'} \sum_j (c_{i'} - d_j) a_{i'j} A_{i'j} \\
&= \sum_{i'} \sum_j c_{i'} a_{i'j} A_{i'j} - \sum_{i'} \sum_j d_j a_{i'j} A_{i'j} \\
&= \sum_{i'} c_{i'} \det A - \sum_j d_j \det A = \left(\sum_{i'} c_{i'} - \sum_j d_j \right) \det A \\
&= \left(\sum_{i'} \lambda_{i'} - \sum_j \mu_j \right) \det A = |\lambda/\mu| \det A = n \det A
\end{aligned}$$

which completes the proof. □

5.3 Shifted shapes

For a strict partition λ , let $g^\lambda := |\text{SYT}(\lambda^*)|$ be the number of standard Young tableaux of shifted shape λ . Like ordinary shapes, shifted shapes have three types of formulas – product, hook length and determinantal. The product formula was proved by Schur [105], using representation theory, and then by Thrall [132], using recursion and combinatorial arguments.

Theorem 5.7. (Schur's shifted product formula) [105][132][69, p. 267, eq. (2)] *For any strict partition $\lambda = (\lambda_1, \dots, \lambda_t)$,*

$$g^\lambda = \frac{|\lambda|!}{\prod_{i=1}^t \lambda_i!} \cdot \prod_{(i,j): i < j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$

Definition 5.8. *For a cell $c = (i, j) \in [\lambda^*]$ let*

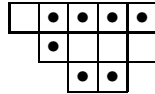
$$H_c^* := [\lambda^*] \cap (\{(i, j)\} \cup \{(i, j') \mid j' > j\} \cup \{(i', j) \mid i' > i\} \cup \{(j+1, j') \mid j' \geq j+1\})$$

be the corresponding shifted hook; note that the last set is relevant only for $j < t$. Let

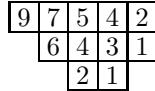
$$h_c^* := |H_c^*| = \begin{cases} \lambda_i + \lambda_{j+1}, & \text{if } j < t; \\ \lambda_i - j + |\{i' \mid i' \geq i, \lambda_{i'} + i' \geq j+1\}|, & \text{if } j \geq t. \end{cases}$$

be the corresponding shifted hook length.

For example, in the following diagram the cells in the shifted hook $H_{(1,2)}^*$ are marked



and in the following diagram each cell is labeled by the corresponding shifted hook length.



Theorem 5.9. (Shifted hook length formula) [69, p. 267, eq. (1)] *For any strict partition $\lambda = (\lambda_1, \dots, \lambda_t)$,*

$$g^\lambda = \frac{|\lambda|!}{\prod_{c \in [\lambda^*]} h_c^*}.$$

Theorem 5.10. (Shifted determinantal formula) *For any strict partition $\lambda = (\lambda_1, \dots, \lambda_t)$,*

$$g^\lambda = \frac{|\lambda|!}{\prod_{(i,j): i < j} (\lambda_i + \lambda_j)} \cdot \det \left[\frac{1}{(\lambda_i - t + j)!} \right]_{i,j=1}^t,$$

using the convention $1/k! := 0$ for negative integers k .

The formulas in Theorems 5.7, 5.9 and 5.10 can be shown to be equivalent in much the same way as was done for ordinary shapes in Subsection 5. Note that the factors of the first denominator in the determinantal formula (Theorem 5.10) are precisely the shifted hook lengths h_c^* for cells $c = (i, j)$ in the region $j < t$.

6 More proofs of the hook length formula

6.1 A probabilistic proof

Probabilistic proofs rely on procedures for a random choice of an object from a set. The key observation is that a uniform distribution implies an exact evaluation and “almost uniform” distributions yield good bounds.

A seminal example is the Greene-Nijenhuis-Wilf probabilistic proof of the ordinary hook length formula, to be described here. Our outline follows Sagan’s description, in the first edition of [99], of the original proof of Greene, Nijenhuis and Wilf [46].

We start with a procedure that generates a random SYT of a given ordinary shape D . Recall from Definition 5.2 the notions of **hook** H_c and **hook length** h_c corresponding to a cell $c \in D$. A **corner** of D is a cell which is last in its row and in its column (equivalently, has hook length 1).

Input: D , a diagram of ordinary shape.

Output: A random $T \in \text{SYT}(D)$.

```

1: procedure RANDOMSYT( $D$ )
2:   while  $D$  is not empty do
3:      $n \leftarrow |D|$ 
4:     Choose randomly a cell  $c \in D$  ▷ with uniform probability  $1/n$ 
5:     while  $c$  is not a corner of  $D$  do
6:       Choose randomly a cell  $c' \in H_c \setminus \{c\}$  ▷ with uniform probability  $1/(h_c - 1)$ 
7:        $c \leftarrow c'$ 
8:     end while
9:      $T(c) \leftarrow n$ 
10:     $D \leftarrow D \setminus \{c\}$ 
11:  end while
12:  return  $T$ 
13: end procedure

```

We claim that this procedure produces each SYT of shape D with the same probability. More precisely,

Lemma 6.1. *The procedure RANDOMSYT produces each SYT of shape D with probability*

$$p = \frac{1}{|D|!} \prod_{c \in D} h_c.$$

Proof. By induction on $n := |D|$. The claim clearly holds for $n = 0, 1$.

Suppose that the claim holds for all shapes of size $n - 1$, and let D be an ordinary shape of size n . Let $T \in \text{SYT}(D)$, and assume that $T(v) = n$ for some corner $v = (\alpha, \beta)$. Denote $D' := D \setminus \{v\}$, and let $T' \in \text{SYT}(D')$ be the restriction of T to D' .

In order to produce T , the algorithm needs to first produce v (in rows 4–8, given D), and then move on to produce T' from D' . By the induction hypothesis, it suffices to show that the probability that rows 4–8 produce the corner $v = (\alpha, \beta)$ is

$$\frac{\prod_{c \in D} h_c/n!}{\prod_{c \in D'} h'_c/(n-1)!} = \frac{1}{n} \prod_{c \in D'} \frac{h_c}{h'_c} = \frac{1}{n} \prod_{i=1}^{\alpha-1} \frac{h_{i,\beta}}{h_{i,\beta}-1} \prod_{j=1}^{\beta-1} \frac{h_{\alpha,j}}{h_{\alpha,j}-1},$$

where h'_c denotes hook length in D' . This is equal to

$$\frac{1}{n} \prod_{i=1}^{\alpha-1} \left(1 + \frac{1}{h_{i,\beta}-1}\right) \prod_{j=1}^{\beta-1} \left(1 + \frac{1}{h_{\alpha,j}-1}\right) = \frac{1}{n} \sum_{\substack{A \subseteq [\alpha-1] \\ B \subseteq [\beta-1]}} \prod_{i \in A} \frac{1}{h_{i,\beta}-1} \prod_{j \in B} \frac{1}{h_{\alpha,j}-1}.$$

Following Sagan [99] we call any possible sequence of cells of D obtained by lines 4–8 of the procedure (starting at a random c and ending at the given corner v) a **trial**. For each trial τ , let

$$A(\tau) := \{i < \alpha : \exists j \text{ s.t. } (i, j) \text{ is a cell in the trial } \tau\} \subseteq [1, \alpha - 1]$$

be its **horizontal projection** and let

$$B(\tau) := \{j < \beta : \exists i \text{ s.t. } (i, j) \text{ is a cell in the trial } \tau\} \subseteq [1, \beta - 1].$$

be its **vertical projection**.

It then suffices to show that for any given $A \subseteq [1, \alpha - 1]$ and $B \subseteq [1, \beta - 1]$, the sum of probabilities of all trials τ ending at $v = (\alpha, \beta)$ such that $A(\tau) = A$ and $B(\tau) = B$ is

$$\frac{1}{n} \prod_{i \in A} \frac{1}{h_{i,\beta}-1} \prod_{j \in B} \frac{1}{h_{\alpha,j}-1}.$$

This may be proved by induction on $|A \cup B|$. □

Lemma 6.1 says that the algorithm produces each $T \in \text{SYT}(D)$ with the same probability p . The number of SYT of shape D is therefore $1/p$, proving the hook length formula (Theorem 5.3).

For a fully detailed proof see [46] or the first edition of [99].

A similar method was applied in [101] to prove the hook length formula for shifted shapes (Theorem 5.9 above).

6.2 Bijective proofs

There are several bijective proofs of the (ordinary) hook length formula. Franzblau and Zeilberger [35] gave a bijection which is rather simple to describe, but breaks the row-column symmetry of hooks. Remmel [95] used the Garsia-Milne involution principle [40] to produce a composition of maps, “bijectivizing” recurrence relations. Zeilberger [140] then gave a bijective version of the probabilistic proof of Greene, Nijenhuis and Wilf [46] (described in the previous subsection). Krattenthaler [58] combined the Hillman-Grassl algorithm [51] and Stanley’s (P, ω) -partition theorem with the involution principle. Novelli, Pak and Stoyanovskii [76] gave a complete proof of a bijective algorithm previously outlined by Pak and Stoyanovskii [80]. A generalization of their method was given by Krattenthaler [59].

Bijective proofs for the shifted hook length formula were given by Krattenthaler [58] and Fischer [30].

A bijective proof of the ordinary determinantal formula was given by Zeilberger [139]; see also [66] and [57].

We shall briefly describe here the bijections of Franzblau-Zeilberger and of Novelli-Pak-Stoyanovskii. Only the algorithms (for the map in one direction) will be specified; the interested reader is referred to the original papers (or to [99]) for more complete descriptions and proofs.

The basic setting for both bijections is the following.

Definition 6.2. Let λ be a partition of n and A a set of positive integers such that $|A| = |\lambda|$. A **Young tableau** of (ordinary) shape λ and image A is a bijection $R : [\lambda] \rightarrow A$, not required to be order-preserving. A **pointer tableau** (or **hook function**) of shape λ is a function $P : [\lambda] \rightarrow \mathbb{Z}$ which assigns to each cell $c \in [\lambda]$ a pointer $p(c')$ which encodes some cell c' in the hook H_c of c (see Definition 5.2). The pointer corresponding to $c' \in H_c$ is defined as follows:

$$p(c') := \begin{cases} j, & \text{if } c' \text{ is } j \text{ columns to the right of } c, \text{ in the same row;} \\ 0, & \text{if } c' = c; \\ -i, & \text{if } c' \text{ is } i \text{ rows below } c, \text{ in the same column.} \end{cases}$$

Let $\text{YT}(\lambda, A)$ denote the set of all Young tableaux of shape λ and image A , $\text{PT}(\lambda)$ the set of all pointer tableaux of shape λ , and $\text{SPT}(\lambda, A)$ the set of all pairs (T, P) (“standard and pointer tableaux”) where $T \in \text{SYT}(\lambda, A)$ and $P \in \text{PT}(\lambda)$. $\text{YT}(\lambda)$ is a shorthand for $\text{YT}(\lambda, [n])$ where $n = |\lambda|$, and $\text{SPT}(\lambda)$ a shorthand for $\text{SPT}(\lambda, [n])$.

Example 6.3. A typical hook, with each cell marked by its pointer:

0	1	2	3	4
-1				
-2				

The hook length formula that we want to prove may be written as

$$n! = f^\lambda \cdot \prod_{c \in [\lambda]} h_c.$$

The LHS of this formula is the size of $\text{YT}(\lambda)$, while the RHS is the size of $\text{SPT}(\lambda)$. Any explicit bijection $f : \text{YT}(\lambda) \rightarrow \text{SPT}(\lambda)$ will prove the hook length formula. As promised, we shall present algorithms for two such bijections.

The Franzblau-Zeilberger algorithm [35]: The main procedure, `FZ-SORTTABLEAU`, “sorts” a YT R of ordinary shape, column by column from right to left, to produce a SPT (T, P) of the same shape. The pointer tableau P records each step of the sorting, keeping just enough information to enable reversal of the procedure. \emptyset denotes the empty tableau.

Input: $R \in \text{YT}(\lambda)$.

Output: $(T, P) \in \text{SPT}(\lambda)$.

```

1: procedure FZ-SORTTABLEAU( $R$ )
2:    $(T, P) \leftarrow (\emptyset, \emptyset)$  ▷ Initialize
3:    $m \leftarrow$  number of columns of  $R$ 
4:   for  $j \leftarrow m$  downto 1 do ▷ Add columns from right to left
5:      $c \leftarrow$  column  $j$  of  $R$ 
6:      $(T, P) \leftarrow \text{INSERTCOLUMN}(T, P, c)$ 
7:   end for
8:   return  $(T, P)$ 
9: end procedure

```

The algorithm makes repeated use of the following procedure `INSERTCOLUMN`:

Input: (T, P, c) , where $(T, P) \in \text{SPT}(\mu, A)$ for some ordinary shape μ and some set A of positive numbers of size $|A| = |\mu|$ such that all the rows of T are increasing, and $c = (c_1, \dots, c_m)$ is a vector of distinct positive integers $c_i \notin A$ whose length $m \geq \ell(\mu)$.

Output: $(T', P') \in \text{SPT}(\mu', A')$, where $A' = A \cup \{c_1, \dots, c_m\}$ and μ' is obtained from μ by attaching a new first column of length m .

```

1: procedure INSERTCOLUMN( $T, P, c$ )
2:   for  $i \leftarrow 1$  to  $m$  do
3:      $T \leftarrow \text{INSERT}(T, i, c_i)$  ▷ Insert  $c_i$  into row  $i$  of  $T$ , keeping the row entries increasing
4:      $d_i \leftarrow (\text{new column index of } c_i) - 1$  ▷ Initialize the pointer  $d_i$ 
5:   end for
6:   while  $T$  is not a Standard Young Tableau do
7:      $(k, x) \leftarrow T^{-1}(\min\{T(i, j) \mid T(i-1, j) > T(i, j)\})$  ▷ The smallest entry out of order
8:      $y \leftarrow d_{k-1} + 1$  ▷ Claim:  $y > 0$ 
9:      $T \leftarrow \text{EXCHANGE}(T, (k, x), (k-1, y))$ 
10:     $y' \leftarrow \text{new column index of the old } T(k-1, y)$  ▷ The new row index is  $k$ 
11:    ▷ Update the pointers  $d_{k-1}$  and  $d_k$ 
12:     $d_{k-1} \leftarrow \begin{cases} v, & \text{if } d_k = v \geq 0, v \neq x-1; \\ -1, & \text{if } d_k = x-1; \\ -(u+1), & \text{if } d_k = -u < 0. \end{cases}$ 
13:     $d_k \leftarrow y' - 1$ 
14:   end while
15:    $P \leftarrow \text{ATTACH}(d, P)$  ▷ Attach  $d$  to  $P$  as a first column
16:   return  $(T, P)$  ▷  $T$  is now a SYT
17: end procedure

```

This procedure makes use of some elementary operations, which may be described as follows:

- $\text{INSERT}(T, i, c_i)$ inserts the entry c_i into row i of T , reordering this row to keep it increasing.
- $\text{EXCHANGE}(T, (k, x), (\ell, y))$ exchanges the entries in cells (k, x) and (ℓ, y) of T and then reorders rows k and ℓ to keep them increasing.
- $\text{ATTACH}(d, P)$ attaches the vector d to the pointer tableau P as a new first column.

Example 6.4. An instance of $\text{INSERTCOLUMN}(T, P, c)$ with

$$T = \begin{array}{|c|c|} \hline 1 & 8 \\ \hline 4 & \\ \hline 7 & \\ \hline \end{array}, \quad P = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & \\ \hline 0 & \\ \hline \end{array}, \quad c = \begin{array}{|c|} \hline 12 \\ \hline 5 \\ \hline 3 \\ \hline 6 \\ \hline \end{array}$$

proceeds as follows (with the smallest entry out of order set in boldface):

$$(T, d) = \begin{array}{|c|c|c|} \hline 1 & 8 & 12 \\ \hline 4 & 5 & \\ \hline \mathbf{3} & 7 & \\ \hline 6 & & \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 8 & 12 \\ \hline 3 & 4 & \\ \hline 5 & 7 & \\ \hline 6 & & \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline -1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 4 & 8 \\ \hline 3 & 12 & \\ \hline 5 & \mathbf{7} & \\ \hline 6 & & \\ \hline \end{array} \begin{array}{|c|} \hline -2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 4 & 8 \\ \hline 3 & 7 & \\ \hline 5 & 12 & \\ \hline 6 & & \\ \hline \end{array} \begin{array}{|c|} \hline -2 \\ \hline 0 \\ \hline 1 \\ \hline 0 \\ \hline \end{array}$$

and yields

$$T = \begin{array}{|c|c|c|} \hline 1 & 4 & 8 \\ \hline 3 & 7 & \\ \hline 5 & 12 & \\ \hline 6 & & \\ \hline \end{array}, \quad P = \begin{array}{|c|c|c|} \hline -2 & 1 & 0 \\ \hline 0 & 0 & \\ \hline 1 & 0 & \\ \hline 0 & & \\ \hline \end{array}.$$

An instance of $\text{FZ-SORTTABLEAU}(R)$ with

$$R = \begin{array}{|c|c|c|c|} \hline 9 & 12 & 8 & 1 \\ \hline 2 & 5 & 4 & \\ \hline 11 & 3 & 7 & \\ \hline 10 & 6 & & \\ \hline \end{array}$$

proceeds as follows (the second step being the instance above):

$$\begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline 1 & 8 \\ \hline 4 & \\ \hline 7 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & \\ \hline 0 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 4 & 8 \\ \hline 3 & 7 & \\ \hline 5 & 12 & \\ \hline 6 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline -2 & 1 & 0 \\ \hline 0 & 0 & \\ \hline 1 & 0 & \\ \hline 0 & & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 8 \\ \hline 2 & 7 & 9 & \\ \hline 5 & 10 & 12 & \\ \hline 6 & 11 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 0 & -2 & 1 & 0 \\ \hline 2 & 0 & 0 & \\ \hline -1 & 1 & 0 & \\ \hline 1 & 0 & & \\ \hline \end{array} = (T, P) .$$

The Novelli-Pak-Stoyanovskii algorithm [76]: Again, we prove the hook length formula

$$n! = f^\lambda \cdot \prod_{c \in [\lambda]} h_c$$

by building an explicit bijection $f : \text{YT}(\lambda) \rightarrow \text{SPT}(\lambda)$. However, instead of building the tableaux column by column, we shall use a modified jeu de taquin to unscramble the entries of $R \in \text{YT}(\lambda)$ so that rows and columns increase. Again, a pointer tableau will keep track of the process so as to make it invertible. Our description will essentially follow [99].

First, define a linear (total) order on the cells of a diagram D of ordinary shape by defining

$$(i, j) \preceq (i', j') \iff \text{either } j > j' \text{ or } j = j' \text{ and } i \geq i'.$$

For example, the cells of the following diagram are labelled 1 to 7 according to this linear order:

7	4	2
6	3	1
5		

If $R \in \text{YT}(\lambda)$ and $c \in D := [\lambda]$, let $R^{\preceq c}$ (respectively $R^{\prec c}$) be the tableau consisting of all cells $b \in D$ with $b \preceq c$ (respectively, $\prec c$).

Define a procedure MFORWARDSLIDE which is the procedure FORWARDSLIDE from the description of jeu de taquin, with the following two modifications:

1. Its input is (T, c) with $T \in \text{YT}$ rather than $T \in \text{SYT}$.
2. Its output is (T, c) (see there), rather than just T .

Input: $R \in \text{YT}(\lambda)$.

Output: $(T, P) \in \text{SPT}(\lambda)$.

- 1: **procedure** NPS(R)
- 2: $T \leftarrow R$
- 3: $D \leftarrow \text{sh}(T)$
- 4: $P \leftarrow 0 \in \text{PT}(D)$ ▷ A pointer tableau of shape D filled with zeros
- 5: **while** T is not standard **do**
- 6: $c \leftarrow$ the \prec -maximal cell such that $T^{\prec c}$ is standard
- 7: $(T', c') \leftarrow \text{MFORWARDSLIDE}(T^{\prec c}, c)$
- 8: **for** $b \in D$ **do** ▷ Replace $T^{\preceq c}$ by T' , except that $T(c') \leftarrow$ the old $T(c)$
- 9: $T(b) \leftarrow \begin{cases} T(b), & \text{if } b \triangleright c; \\ T'(b), & \text{if } b \preceq c \text{ and } b \neq c'; \\ T(c), & \text{if } b = c'. \end{cases}$
- 10: **end for**
- 11: Let $c = (i_0, j_0)$ and $c' = (i_1, j_1)$ ▷ Necessarily $i_0 \leq i_1$ and $j_0 \leq j_1$
- 12: **for** i **from** i_0 **to** $i_1 - 1$ **do**

```

13:          $P(i, j_0) \leftarrow P(i + 1, j_0) - 1$ 
14:     end for
15:      $P(i_1, j_0) \leftarrow j_1 - j_0$ 
16: end while
17: return  $T$ 
18: end procedure

```

▷ Now $c_0 \in D \subseteq (c_0)_+$, so D has ordinary shape

Example 6.5. For

$$R = \begin{array}{|c|c|} \hline 6 & 2 \\ \hline 4 & 3 \\ \hline 5 & 1 \\ \hline \end{array},$$

here is the sequence of pairs (T, P) produced during the computation of $\text{NPS}(R)$ (with c in boldface):

$$\begin{array}{|c|c|} \hline 6 & 2 \\ \hline 4 & \mathbf{3} \\ \hline 5 & 1 \\ \hline \end{array}
\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}
\rightarrow
\begin{array}{|c|c|} \hline 6 & \mathbf{2} \\ \hline 4 & 1 \\ \hline 5 & 3 \\ \hline \end{array}
\begin{array}{|c|c|} \hline 0 & 0 \\ \hline 0 & -1 \\ \hline 0 & 0 \\ \hline \end{array}
\rightarrow
\begin{array}{|c|c|} \hline 6 & 1 \\ \hline 4 & 2 \\ \hline \mathbf{5} & 3 \\ \hline \end{array}
\begin{array}{|c|c|} \hline 0 & -2 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array}
\rightarrow$$

$$\rightarrow
\begin{array}{|c|c|} \hline 6 & 1 \\ \hline \mathbf{4} & 2 \\ \hline 3 & 5 \\ \hline \end{array}
\begin{array}{|c|c|} \hline 0 & -2 \\ \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}
\rightarrow
\begin{array}{|c|c|} \hline \mathbf{6} & 1 \\ \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}
\begin{array}{|c|c|} \hline 0 & -2 \\ \hline 1 & 0 \\ \hline 1 & 0 \\ \hline \end{array}
\rightarrow
\begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}
\begin{array}{|c|c|} \hline 0 & -2 \\ \hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}.$$

6.3 Partial difference operators

MacMahon [72] has originally used partial difference equations, also known as recurrence relations, to solve various enumeration problems – among them the enumeration of ballot sequences, or equivalently SYT of an ordinary shape (see Subsection 2.5.2). Zeilberger [138] improved on MacMahon’s proof by extending the domain of definition of the enumerating functions, thus simplifying the boundary conditions: In PDE terminology, a Neumann boundary condition (zero normal derivatives) was replaced by a Dirichlet boundary condition (zero function values). He also made explicit use of the algebra of partial difference operators; we shall present here a variant of his approach.

Consider, for example, the two dimensional ballot problem – finding the number $F(m_1, m_2)$ of lattice paths from $(0, 0)$ to (m_1, m_2) which stay in the region $m_1 \geq m_2 \geq 0$. MacMahon [72, p. 127] has set the partial difference equation

$$F(m_1, m_2) = F(m_1 - 1, m_2) + F(m_1, m_2 - 1) \quad (m_1 > m_2 > 0)$$

with the boundary conditions

$$F(m_1, m_2) = F(m_1, m_2 - 1) \quad (m_1 = m_2 > 0)$$

and

$$F(m_1, 0) = 1 \quad (m_1 \geq 0).$$

By extending F to the region $m_1 \geq m_2 - 1$, the recursion can be required to hold for almost all $m_1 \geq m_2 \geq 0$:

$$F(m_1, m_2) = F(m_1 - 1, m_2) + F(m_1, m_2 - 1) \quad (m_1 \geq m_2 \geq 0, (m_1, m_2) \neq (0, 0))$$

with

$$F(m_1, m_2) = 0 \quad (m_1 = m_2 - 1 \text{ or } m_2 = -1)$$

and

$$F(0, 0) = 1.$$

In general, consider functions $f : \mathbb{Z}^n \rightarrow \mathbb{C}$ and define the fundamental **shift operators** X_1, \dots, X_n by

$$(X_i f)(m_1, \dots, m_n) := f(m_1, \dots, m_i + 1, \dots, m_n) \quad (1 \leq i \leq n).$$

For $\alpha \in \mathbb{Z}^n$ write $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, so that $(X^\alpha f)(m) = f(m + \alpha)$. A typical linear partial difference operator with constant coefficients has the form

$$P = p(X_1^{-1}, \dots, X_n^{-1}) = \sum_{\alpha \geq 0} a_\alpha X^{-\alpha},$$

for some polynomial $p(z)$ with complex coefficients, so that $a_\alpha \in \mathbb{C}$ for each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $a_\alpha \neq 0$ for only finitely many values of α . We also assume that $p(0) = a_0 = 1$.

Definition 6.6. Define the **discrete delta function** $\delta : \mathbb{Z}^n \rightarrow \mathbb{C}$ by

$$\delta(m) = \begin{cases} 1, & \text{if } m = 0; \\ 0, & \text{otherwise.} \end{cases}$$

A function $f : \mathbb{Z}^n \rightarrow \mathbb{C}$ satisfying $Pf = \delta$ is called a **fundamental solution** corresponding to the operator P . If f is supported in \mathbb{N}^n , it is called a **canonical fundamental solution**.

It is clear that each operator P as above has a unique canonical fundamental solution.

In the following theorem we consider a slightly more general type of operators, which can be written as $X^\alpha p(X_1^{-1}, \dots, X_n^{-1})$ for a polynomial p and some $\alpha \geq 0$.

Theorem 6.7. (A variation on [138, Theorem 2]) Let $F_n = F_n(m_1, \dots, m_n)$ be the canonical fundamental solution corresponding to an operator $P = p(X_1^{-1}, \dots, X_n^{-1})$, where $p(z)$ is a symmetric polynomial with $p(0) = 1$. Denote

$$\Delta_n := \prod_{(i,j): i < j} (I - X_i X_j^{-1}).$$

Then $G_n = \Delta_n F_n$ is the unique solution of the equation $Pg = 0$ in the region

$$\{(m_1, \dots, m_n) \in \mathbb{Z}^n \mid m_1 \geq \dots \geq m_n \geq 0\} \setminus \{(0, \dots, 0)\}$$

subject to the boundary conditions

$$(\exists i) m_i = m_{i+1} - 1 \implies g(m_1, \dots, m_n) = 0,$$

$$m_n = -1 \implies g(m_1, \dots, m_n) = 0$$

and

$$g(0, \dots, 0) = 1.$$

Proof. Since each X_i commutes with the operator P , so does Δ_n . Since $m_1 + \dots + m_n$ is invariant under $X_i X_j^{-1}$ and F_n is a solution of $Pf = 0$ in the complement of the hyperplane $m_1 + \dots + m_n = 0$, so is G_n . It remains to verify that G_n satisfies the prescribed boundary conditions. Now, by definition,

$$G_n(m) = (I - X_1 X_2^{-1}) A_{1,2} F_n(m)$$

where the operator $A_{1,2}$ is symmetric with respect to X_1, X_2 . Since $F_n(m)$ is a symmetric function, we can write

$$G_n(m) = (I - X_1 X_2^{-1}) H(m)$$

where H is symmetric with respect to m_1 and m_2 . Suppressing the dependence on m_3, \dots, m_n ,

$$G_n(m_1, m_2) = (I - X_1 X_2^{-1}) H(m_1, m_2) = H(m_1, m_2) - H(m_1 + 1, m_2 - 1) = 0$$

whenever $m_1 = m_2 - 1$, by the symmetry of H . Similarly, for $i = 1, \dots, n-1$, $G_n(m) = 0$ on $m_i = m_{i+1} - 1$. On $m_n = -1$,

$$G_n(m_1, \dots, m_{n-1}, -1) = \prod_{(i,j): 1 \leq i < j \leq n-1} (I - X_i X_j^{-1}) \prod_{1 \leq i \leq n-1} (I - X_i X_n^{-1}) F_n(m_1, \dots, m_{n-1}, -1).$$

Since $F_n(m) = 0$ for $m_n < 0$,

$$G_n(m_1, \dots, m_{n-1}, -1) = 0.$$

Finally, $F_n(m_1, \dots, m_n) = 0$ on all of the hyperplane $m_1 + \dots + m_n = 0$ except the origin $0 = (0, \dots, 0)$. Therefore

$$G_n(0) = \Delta_n F_n(0) = F_n(0) = 1.$$

□

For every function $f : \mathbb{Z}^n \rightarrow \mathbb{C}$ whose support is contained in a translate of \mathbb{N}^n (i.e., such that there exists $N \in \mathbb{Z}$ such that $f(m_1, \dots, m_n) = 0$ whenever $m_i < N$ for some i) there is a corresponding generating function (formal Laurent series)

$$\text{gf}(f) := \sum_m f(m) z^m \in \mathbb{C}((z_1, \dots, z_n)).$$

Let $p(z_1, \dots, z_n)$ be a polynomial with complex coefficients and $p(0) = 1$, and let $P = p(X_1^{-1}, \dots, X_n^{-1})$ be the corresponding operator. Since $\text{gf}(X^{-\alpha} f) = z^\alpha \text{gf}(f)$ we have

$$\text{gf}(Pf) = p(z_1, \dots, z_n) \cdot \text{gf}(f),$$

and therefore $Pf = \delta$ implies $\text{gf}(f) = 1/p(z_1, \dots, z_n)$.

Definition 6.8. (MacMahon [72]) *Let $A \subseteq \mathbb{Z}^n$ and $f : A \rightarrow \mathbb{C}$. A formal Laurent series $\sum_m a(m) z^m$ is a **redundant generating function for f on A** if $f(m) = a(m)$ for all $m \in A$.*

Theorem 6.9. (MacMahon [72, p. 133]) *Let $g(m)$ be the number of lattice paths from 0 to m , where travel is restricted to the region*

$$A = \{(m_1, \dots, m_n) \in \mathbb{Z}^n \mid m_1 \geq m_2 \geq \dots \geq m_n \geq 0\}.$$

Then

$$\prod_{(i,j): i < j} \left(1 - \frac{z_j}{z_i}\right) \cdot \frac{1}{1 - z_1 - \dots - z_n}$$

is a redundant generating function for g on A and therefore

$$g(m) = \frac{(m_1 + \dots + m_n)!}{(m_1 + n - 1)! \dots m_n!} \prod_{(i,j): i < j} (m_i - m_j + j - i).$$

This gives, of course, the ordinary product formula (Theorem 5.1).

Proof. Apply Theorem 6.7 with $P = I - X_1^{-1} - \dots - X_n^{-1}$. The canonical fundamental solution of $Pf = \delta$ is easily seen to be the multinomial coefficient

$$F_n(m_1, \dots, m_n) = \begin{cases} \frac{(m_1 + \dots + m_n)!}{m_1! \dots m_n!}, & \text{if } m_i \geq 0 \ (\forall i); \\ 0, & \text{otherwise,} \end{cases}$$

with generating function $(1 - z_1 - \dots - z_n)^{-1}$.

The number of lattice paths in the statement of Theorem 6.9 clearly satisfies the conditions on g in Theorem 6.7, and therefore

$$g = G_n = \Delta_n F_n \quad (\text{on } A).$$

This implies the claimed redundant generating function for g on A .

To get an explicit expression for $g(m)$ note that $(m_1 + \dots + m_n)!$ is invariant under $X_i X_j^{-1}$, so that

$$\begin{aligned} g(m_1, \dots, m_n) &= \prod_{(i,j): i < j} (I - X_i X_j^{-1}) \left[\frac{(m_1 + \dots + m_n)!}{m_1! \dots m_n!} \right] \\ &= (m_1 + \dots + m_n)! \cdot \prod_{(i,j): i < j} (I - X_i X_j^{-1}) \left[\frac{1}{m_1! \dots m_n!} \right]. \end{aligned}$$

Consider

$$\begin{aligned} H(m_1, \dots, m_n) &:= \prod_{(i,j): i < j} (I - X_i X_j^{-1}) \left[\frac{1}{m_1! \dots m_n!} \right] \\ &= \prod_{(i,j): i < j} (X_i^{-1} - X_j^{-1}) \cdot \prod_i X_i^{n-i} \left[\frac{1}{m_1! \dots m_n!} \right] \\ &= \prod_{(i,j): i < j} (X_i^{-1} - X_j^{-1}) \left[\frac{1}{\ell_1! \ell_2! \dots \ell_n!} \right], \end{aligned}$$

where $\ell_i := m_i + n - i$ ($1 \leq i \leq n$). Clearly H is an alternating (anti-symmetric) function of ℓ_1, \dots, ℓ_n , which means that

$$H(m_1, \dots, m_n) = \frac{Q(\ell_1, \ell_2, \dots, \ell_n)}{\ell_1! \ell_2! \dots \ell_n!},$$

where Q is an alternating polynomial of degree $n - 1$ in each of its variables. g , and therefore also H and Q , vanish on each of the hyperplanes $m_i = m_{i+1} - 1$, namely $\ell_i = \ell_{i+1}$ ($1 \leq i \leq n - 1$). Hence $\ell_i - \ell_{i+1}$, and by symmetry also $\ell_i - \ell_j$ for each $i \neq j$, divide Q . Hence

$$Q(\ell) = c \prod_{(i,j): i < j} (\ell_i - \ell_j)$$

and

$$g(m) = c \frac{(m_1 + \dots + m_n)!}{(m_1 + n - 1)! \dots m_n!} \prod_{(i,j): i < j} (m_i - m_j - i + j)$$

for a suitable constant c , which is easily found to be 1 by evaluating $g(0)$. □

Remark 6.10. *Theorem 6.9 gives an expression of the number of SYT of ordinary shape $\lambda = (\lambda_1, \dots, \lambda_t)$ as the coefficient of z^ℓ (where $\ell_i = \lambda_i + t - i$) in the power series*

$$\prod_{(i,j): i < j} (z_i - z_j) \cdot \frac{1}{1 - z_1 - \dots - z_t},$$

or as the constant term in the Laurent series

$$\prod_i z_i^{-\ell_i} \prod_{(i,j): i < j} (z_i - z_j) \cdot \frac{1}{1 - z_1 - \dots - z_t}.$$

7 Formulas for skew strips

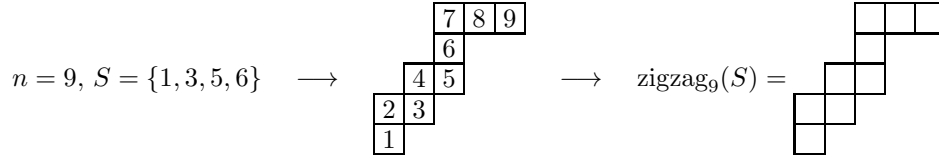
We focus our attention now on two important families of skew shapes, which are of special interest: Zigzag shapes and skew strips of constant width.

7.1 Zigzag shapes

Recall (from Subsection 3.3) that a **zigzag shape** is a path-connected skew shape which does not contain a 2×2 square.

Definition 7.1. For any subset $S \subseteq [n-1] := \{1, \dots, n-1\}$ define a zigzag shape $D = \text{zigzag}_n(S)$, with cells labeled $1, \dots, n$, as follows: Start with an initial cell labeled 1. For each $1 \leq i \leq n-1$, given the cell labeled i , add an adjacent cell labeled $i+1$ above cell i if $i \in S$, and to the right of cell i otherwise.

Example 7.2.



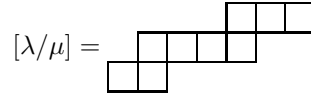
This defines a bijection between the set of all subsets of $[n-1]$ and the set of all zigzag shapes of size n (up to translation). The set S consists of the labels of all the cells in the shape $\text{zigzag}_n(S)$ such that the cell directly above is also in the shape. These are exactly the last (rightmost) cells in all the rows except the top row.

Recording the lengths of all the rows in the zigzag shape, from bottom up, it follows that zigzag shapes of size n are also in bijection with all the **compositions** of n . In fact, given a subset $S = \{s_1, \dots, s_k\} \subseteq [n-1]$ (with $s_1 < \dots < s_k$), the composition corresponding to $\text{zigzag}_n(S)$ is simply $(s_1, s_2 - s_1, \dots, s_k - s_{k-1}, n - s_k)$.

Theorem 7.3. [72, Vol. 1, p. 190][119, Example 2.2.4] Let $S = \{s_1, \dots, s_k\} \subseteq [n-1]$ ($s_1 < \dots < s_k$) and set $s_0 := 0$ and $s_{k+1} := n$. Then

$$f^{\text{zigzag}_n(S)} = n! \cdot \det \left[\frac{1}{(s_{j+1} - s_i)!} \right]_{i,j=0}^k = \det \left[\binom{n - s_i}{s_{j+1} - s_i} \right]_{i,j=0}^k = \det \left[\binom{s_{j+1}}{s_{j+1} - s_i} \right]_{i,j=0}^k.$$

For example, the zigzag shape

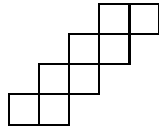


corresponds to $n = 9$ and $S = \{2, 6\}$, and therefore

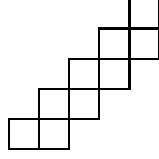
$$f^{\lambda/\mu} = 9! \cdot \det \begin{bmatrix} \frac{1}{2!} & \frac{1}{6!} & \frac{1}{9!} \\ 1 & \frac{1}{4!} & \frac{1}{7!} \\ 0 & 1 & \frac{1}{3!} \end{bmatrix} = \det \begin{bmatrix} \binom{9}{2} & \binom{9}{6} & \binom{9}{9} \\ \binom{7}{0} & \binom{7}{4} & \binom{7}{7} \\ 0 & \binom{3}{0} & \binom{3}{3} \end{bmatrix} = \det \begin{bmatrix} \binom{2}{2} & \binom{6}{6} & \binom{9}{9} \\ \binom{2}{0} & \binom{6}{4} & \binom{9}{7} \\ 0 & \binom{6}{6} & \binom{9}{3} \end{bmatrix}$$

Theorem 7.3 is a special case of the determinantal formula for skew shapes (Theorem 5.6). We shall now consider a specific family of examples.

Example 7.4. Consider, for each nonnegative integer n , one special zigzag shape D_n . For n even it has all rows of length 2:



and for n odd it has a top row of length 1 and all others of length 2:



Clearly, by Definition 7.1, $D_n = \text{zigzag}_n(S)$ for $S = \{2, 4, 6, \dots\} \subseteq [n-1]$.

Definition 7.5. A permutation $\sigma \in S_n$ is **up-down** (or **alternating**, or **zigzag**) if

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \dots \quad .$$

Observation 7.6. If we label the cells of D_n as in Definition 7.1 (see Example 7.2), then clearly each standard Young tableau $T : D_n \rightarrow [n]$ becomes an up-down permutation, and vice versa. (For an extension of this phenomenon see Proposition 10.12.)

Up-down permutations were already studied by André [6, 7] in the nineteenth century. He showed that their number A_n [110, A000111] satisfies

Proposition 7.7.

$$\sum_{n=0}^{\infty} \frac{A_n x^n}{n!} = \sec x + \tan x.$$

They are therefore directly related to the **secant** (or **zig**, or **Euler**) numbers E_n [110, A000364], the **tangent** (or **zag**) numbers T_n [110, A000182] and the **Bernoulli numbers** B_n [110, A000367 and A002445] by

$$A_{2n} = (-1)^n E_{2n} \quad (n \geq 0)$$

and

$$A_{2n-1} = T_n = \frac{(-1)^{n-1} 2^{2n} (2^{2n} - 1)}{2n} B_{2n} \quad (n \geq 1).$$

Note that there is an alternative convention for denoting Euler numbers, by which $E_n = A_n$ for all n .

Proposition 7.8.

$$2A_{n+1} = \sum_{k=0}^n \binom{n}{k} A_k A_{n-k} \quad (n \geq 1)$$

with $A_0 = A_1 = 1$.

Proof. In a SYT of the required shape and size $n+1$, the cell containing $n+1$ must be the last in its row and column. Removing this cell leaves at most two path-connected components, with the western/southern one necessarily of odd size (for $n \geq 1$). It follows that

$$A_{n+1} = \sum_{\substack{k=0 \\ k \text{ odd}}}^n \binom{n}{k} A_k A_{n-k} \quad (n \geq 1).$$

Applying a similar argument to the cell containing 1 gives

$$A_{n+1} = \sum_{\substack{k=0 \\ k \text{ even}}}^n \binom{n}{k} A_k A_{n-k} \quad (n \geq 0),$$

and adding the two formulas gives the required recursion. □

Indeed, the recursion for A_n (Proposition 7.8) can be seen to be equivalent to the generating function (Proposition 7.7) since $f(x) = \sec x + \tan x$ satisfies the differential equation

$$2f'(x) = 1 + f(x)^2$$

with $f(0) = 1$.

Proposition 7.3 thus gives the determinantal formulas

$$(-1)^n E_{2n} = (2n)! \cdot \det \begin{pmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \cdot & \cdot & \frac{1}{(2n-4)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n)!} \\ 1 & \frac{1}{2!} & \frac{1}{4!} & \cdot & \cdot & \cdot & \frac{1}{(2n-4)!} & \frac{1}{(2n-2)!} \\ 0 & 1 & \frac{1}{2!} & \cdot & \cdot & \cdot & \cdot & \frac{1}{(2n-4)!} \\ \vdots & \vdots & \vdots & \cdot & \cdot & \cdot & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & \frac{1}{2!} \end{pmatrix}$$

and

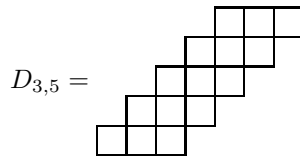
$$T_n = (2n-1)! \cdot \det \begin{pmatrix} \frac{1}{2!} & \frac{1}{4!} & \frac{1}{6!} & \cdot & \cdot & \frac{1}{(2n-4)!} & \frac{1}{(2n-2)!} & \frac{1}{(2n-1)!} \\ 1 & \frac{1}{2!} & \frac{1}{4!} & \cdot & \cdot & \cdot & \frac{1}{(2n-4)!} & \frac{1}{(2n-3)!} \\ 0 & 1 & \frac{1}{2!} & \cdot & \cdot & \cdot & \cdot & \frac{1}{(2n-5)!} \\ \vdots & \vdots & \vdots & \cdot & \cdot & \cdot & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & \frac{1}{1!} \end{pmatrix}$$

7.2 Skew strips of constant width

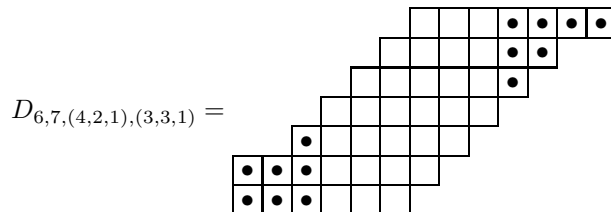
The **basic skew strip of width m and height n** is the skew diagram

$$D_{m,n} = [(n+m-1, n+m-2, \dots, m+1, m)/(n-1, n-2, \dots, 1, 0)].$$

It has n rows, of length m each, with each row shifted one cell to the left with respect to the row just above it. For example, $D_{2,n}$ is D_{2n} from Example 7.4 above while



The **general skew strip of width m and height n** (m -strip, for short), $D_{m,n,\lambda,\mu}$, has arbitrary partitions λ and μ , each of height at most $k := \lfloor m/2 \rfloor$, as “head” (northeastern tip) and “tail” (southwestern tip), respectively, instead of the basic partitions $\lambda = \mu = (k, k-1, \dots, 1)$. For example,



where $m = 6$, $n = 7$, $k = 3$ and the head and tail have marked cells.

The determinantal formula for skew shapes (Theorem 5.6) expresses f^D as an explicit determinant of order n , the number of rows. Baryshnikov and Romik [11], developing an idea of Elkies [27], gave an alternative determinant of order k , half the length of a typical row. This is a considerable improvement if m , k , λ and μ are fixed while n is allowed to grow.

The general statement needs a bit of notation. Denote, for a non-negative integer n ,

$$A'_n := \frac{A_n}{n!}, \quad A''_n := \frac{A'_n}{2^{n+1} - 1}, \quad A'''_n := \frac{(2^n - 1)A''_n}{2^n},$$

where A_n are André's alternating numbers (as in the previous subsection); and let

$$\epsilon(n) := \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Define, for nonnegative integers N , p and q ,

$$\begin{aligned} X_N^{(0)}(p, q) &:= \sum_{i=0}^{\lfloor p/2 \rfloor} \sum_{j=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{i+j} A'_{N+2i+2j+1}}{(p-2i)!(q-2j)!} + \epsilon(p+1) \sum_{j=0}^{\lfloor q/2 \rfloor} \frac{(-1)^j A'_{N+p+2j+1}}{(q-2j)!} \\ &+ \epsilon(q+1) \sum_{i=0}^{\lfloor p/2 \rfloor} \frac{(-1)^i A'_{N+2i+q+1}}{(p-2i)!} + \epsilon(p+1)\epsilon(q+1)A'_{N+p+q+1} \end{aligned}$$

and

$$\begin{aligned} X_N^{(1)}(p, q) &:= \sum_{i=0}^{\lfloor p/2 \rfloor} \sum_{j=0}^{\lfloor q/2 \rfloor} \frac{(-1)^{i+j} A'''_{N+2i+2j+1}}{(p-2i)!(q-2j)!} + \epsilon(p) \sum_{j=0}^{\lfloor q/2 \rfloor} \frac{(-1)^j A'''_{N+p+2j+1}}{(q-2j)!} \\ &+ \epsilon(q) \sum_{i=0}^{\lfloor p/2 \rfloor} \frac{(-1)^i A'''_{N+2i+q+1}}{(p-2i)!} + \epsilon(p)\epsilon(q)A'''_{N+p+q+1} \quad . \end{aligned}$$

Theorem 7.9. [11, Theorem 4] *Let $D = D_{m,n,\lambda,\mu}$ be an m -strip with head and tail partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_k)$, where $k := \lfloor m/2 \rfloor$. For $(1 \leq i \leq k)$ define $L_i := \lambda_i + k - i$ and $M_i := \mu_i + k - i$, and denote*

$$m\%2 := \begin{cases} 0, & \text{if } m \text{ is even;} \\ 1, & \text{if } m \text{ is odd.} \end{cases}$$

Then

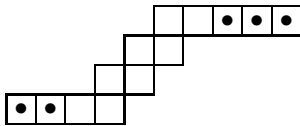
$$f^D = (-1)^{\binom{k}{2}} |D|! \cdot \det \left[X_{2n-m+1}^{(m\%2)}(L_i, M_j) \right]_{i,j=1}^k .$$

Note that $X_N^{(\epsilon)}(p, q)$ are linear combinations of $A_{N+1}, \dots, A_{N+p+q+1}$, so that f^D is expressed as a polynomial in the A_i whose complexity depends on the row length m but not on the number of rows n .

The impressive formal definitions of $X_N^{(0)}$ have simple geometric interpretations:

$$X_{2n-1}^{(0)}(p, q) = \frac{f^D}{|D|!}$$

where

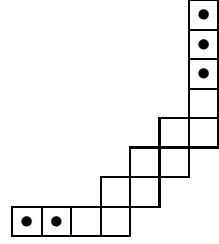
$$D = \text{zigzag}_{2n+p+q}(\{p+2, p+4, \dots, p+2n-2\}) =$$


(p marked southwestern cells in a row, $2n$ unmarked cells, and q marked northeastern cells in a row), and

$$X_{2n}^{(0)}(p, q) = \frac{f^D}{|D|!}$$

where

$$D = \text{zigzag}_{2n+p+q+1}(\{p+2, p+4, \dots, p+2n, p+2n+1, \dots, p+2n+q\}) =$$



(p marked southwestern cells in a row, $2n+1$ unmarked cells, and q marked northeastern cells in a column). These are 2-strips, i.e. zigzag shapes. It is possible to define $X_N^{(1)}(p, q)$ similarly in terms of 3-strips, a task left as an exercise to the reader [11].

Here are some interesting special cases.

Corollary 7.10. [11, Theorem 1] *3-strips:*

$$f^{D_{3,n,(0),(0)}} = \frac{(3n-2)! T_n}{(2n-1)! 2^{2n-2}},$$

$$f^{D_{3,n,(1),(0)}} = \frac{(3n-1)! T_n}{(2n-1)! 2^{2n-1}},$$

$$f^{D_{3,n}} = f^{D_{3,n,(1),(1)}} = \frac{(3n)! (2^{2n-1} - 1) T_n}{(2n-1)! 2^{2n-1} (2^{2n} - 1)}.$$

Corollary 7.11. [11, Theorem 2] *4-strips:*

$$f^{D_{4,n,(0),(0)}} = (4n-2)! \left(\frac{T_n^2}{(2n-1)!^2} + \frac{E_{2n-2} E_{2n}}{(2n-2)! (2n)!} \right),$$

$$f^{D_{4,n}} = f^{D_{4,n,(1),(1)}} = (4n)! \left(\frac{E_{2n}^2}{(2n)!^2} - \frac{E_{2n-2} E_{2n+2}}{(2n-2)! (2n+2)!} \right).$$

Corollary 7.12. [11, Theorem 3] *5-strip:*

$$f^{D_{5,n,(0),(0)}} = \frac{(5n-6)! T_{n-1}^2}{(2n-3)!^2 2^{4n-6} (2^{2n-2} - 1)}.$$

Proof of Theorem 7.9 (sketch). The proof uses transfer operators, following Elkies [27]. Elkies considered, essentially, the zigzag shapes (2-strips) D_n from Example 7.4, whose SYT correspond to alternating (up-down) permutations. Recall from Subsection 2.5.3 the definition of the order polytope $P(D_n)$, whose volume is, by Observation 2.17,

$$\text{vol } P(D_n) = \frac{f^{D_n}}{n!}.$$

This polytope can be written as

$$P(D_n) = \{(x_1, \dots, x_n) \in [0, 1]^n : x_1 \leq x_2 \geq x_3 \leq x_4 \geq \dots\},$$

and therefore its volume can also be computed by an iterated integral:

$$\text{vol } P(D_n) = \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \cdots.$$

Some manipulations now lead to the expression

$$\text{vol } P(D_n) = \langle T^{n-1}(\mathbf{1}), \mathbf{1} \rangle$$

where $\mathbf{1} \in L^2[0, 1]$ is the function with constant value 1, $\langle \cdot, \cdot \rangle$ is the standard inner product on $L^2[0, 1]$, and $T : L^2[0, 1] \rightarrow L^2[0, 1]$ is the compact self-adjoint operator defined by

$$(Tf)(x) := \int_0^{1-x} f(y) dy \quad (\forall f \in L^2[0, 1]).$$

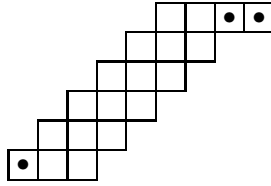
The eigenvalues λ_k and corresponding orthonormal eigenfunctions ϕ_k of T can be computed explicitly, leading to the explicit formula

$$\text{vol } P(D_n) = \sum_k \lambda_k^{n-1} \langle \mathbf{1}, \phi_k \rangle^2 = \frac{2^{n+2}}{\pi^{n+1}} \sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^{n+1}} \quad (n \geq 1)$$

which gives a corresponding expression for

$$A_n = f^{D_n} = n! \text{vol } P(D_n).$$

Baryshnikov and Romik extended this treatment of a 2-strip to general m -strips, using additional ingredients. For instance, the iterated integral for a 3-strip



gives

$$\text{vol } P(D_{3,n,\lambda,\mu}) = \langle (BA)^{n-1} T_\mu(\mathbf{1}), T_\lambda(\mathbf{1}) \rangle$$

where $\Omega := \{(u, v) \in [0, 1]^2 : u \leq v\}$, the transfer operators $A : L^2[0, 1] \rightarrow L^2(\Omega)$ and $B : L^2(\Omega) \rightarrow L^2[0, 1]$ are defined by

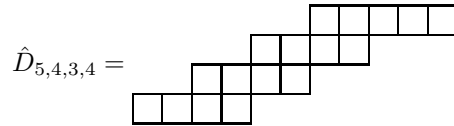
$$(Af)(u, v) := \int_u^v f(x) dx$$

and

$$(Bg)(x) := \int_0^x \int_x^1 g(u, v) dv du,$$

and T_λ, T_μ are operators corresponding to the “head” and “tail” partitions λ and μ . □

In a slightly different direction, Stanley [118] defines $\hat{D}_{a,b,c,n}$ to be the skew shape whose diagram has n rows, with a cells in the top row and b cells in each other row, and such that each row is shifted $c - 1$ columns to the left with respect to the preceding (higher) row. For example,



Theorem 7.13. [118, Corollary 2.5] For a, b and c with $c \leq b < 2c$,

$$\sum_{n \geq 0} f^{\hat{D}_{a,b,c,n+1}} \frac{x^{n+1}}{(a+nb)!} = \frac{x \sum_{n \geq 0} \frac{(-x)^n}{(b+nc)!}}{(b-c)! - x \sum_{n \geq 0} \frac{(-x)^n}{(a+nc)!}}.$$

Two special cases deserve special attention: $a = b$ and $b = c$.
 For $a = b$ all the rows are of the same length.

Corollary 7.14. *For a and c with $c \leq a < 2c$,*

$$1 + \sum_{n \geq 1} f^{\hat{D}_{a,a,c,n}} \frac{x^n}{(na)!} = \left(1 - \frac{x}{(a-c)!} \sum_{n \geq 0} \frac{(-x)^n}{(a+nc)!} \right)^{-1}.$$

In particular, for $a = b = 3$ and $c = 2$, $\hat{D}_{3,3,2,n} = D_{3,n}$ as in Theorem 7.10:

$$\sum_{n \geq 0} f^{D_{3,n}} \frac{x^{2n}}{(3n)!} = \left(\sum_{n \geq 0} \frac{(-x^2)^n}{(2n+1)!} \right)^{-1} = \frac{x}{\sin x}.$$

This result was already known to Gessel and Viennot [43].

For $b = c$ one obtains a zigzag shape: $\hat{D}_{a,c,c,n+1} = \text{zigzag}_{a+nc}(S)$ for $S = \{c, 2c, \dots, nc\}$.

Corollary 7.15. *For any positive a and c ,*

$$\sum_{n \geq 0} f^{\text{zigzag}_{a+nc}(\{c, 2c, \dots, nc\})} \frac{x^{n+1}}{(a+nc)!} = \frac{x \sum_{n \geq 0} \frac{(-x)^n}{(c+nc)!}}{1 - x \sum_{n \geq 0} \frac{(-x)^n}{(a+nc)!}}.$$

8 Truncated and other non-classical shapes

Definition 8.1. *A diagram of **truncated shape** is a line-convex diagram obtained from a diagram of ordinary or shifted shape by deleting cells from the NE corner (in the English notation, where row lengths decrease from top to bottom).*

For example, here are diagrams of a truncated ordinary shape

$$[(4, 4, 2, 1) \setminus (1)] = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array}$$

and a truncated shifted shape:

$$[(4, 3, 2, 1)^* \setminus (1, 1)] = \begin{array}{cccc} & \square & \square & \square \\ & & \square & \square \\ & & & \square \\ & & & & \square \\ & & & & & \square \end{array}$$

Modules associated to truncated shapes were introduced and studied in [54, 94]. Interest in the enumeration of SYT of truncated shapes was recently enhanced by a new interpretation [4]: The number of geodesics between distinguished pairs of antipodes in the flip graph of inner-triangle-free triangulations is twice the number of SYT of a corresponding truncated shifted staircase shape. Motivated by this result, extensive computations were carried out for the number of SYT of these and other truncated shapes. It was found that, in some cases, these numbers are unusually “smooth”, i.e., all their prime factors are relatively very small. This makes it reasonable to expect a product formula. Subsequently, such formulas were conjectured and proved for rectangular and shifted staircase shapes truncated by a square, or nearly a square, and for rectangular shapes truncated by a staircase; see [1, 81, 124, 123].

8.1 Truncated shifted staircase shape

In this subsection, $\lambda = (\lambda_1, \dots, \lambda_t)$ (with $\lambda_1 > \dots > \lambda_t > 0$) will be a strict partition, with g^λ denoting the number of SYT of shifted shape λ .

For any nonnegative integer n , let $\delta_n := (n, n-1, \dots, 1)$ be the corresponding shifted staircase shape. By Schur's formula for shifted shapes (Theorem 5.7),

Corollary 8.2. *The number of SYT of shifted staircase shape δ_n is*

$$g^{\delta_n} = N! \cdot \prod_{i=0}^{n-1} \frac{i!}{(2i+1)!},$$

where $N := |\delta_n| = \binom{n+1}{2}$.

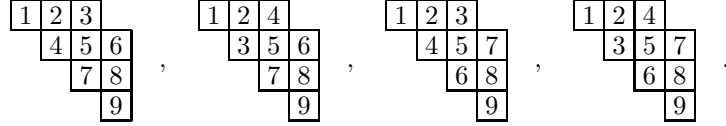
The following enumeration problem was actually the original motivation for the study of truncated shapes, because of its combinatorial interpretation, as explained in [4].

Theorem 8.3. [1, Corollary 4.8][81, Theorem 1] *The number of SYT of truncated shifted staircase shape $\delta_n \setminus (1)$ is equal to*

$$g^{\delta_n \setminus (1)} = \frac{C_n C_{n-2}}{2 C_{2n-3}},$$

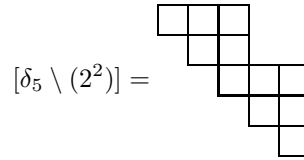
where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number.

Example 8.4. *There are $g^{\delta_4} = 12$ SYT of shape δ_4 , but only 4 SYT of truncated shape $\delta_4 \setminus (1)$:*



Theorem 8.3 may be generalized to a truncation of a $(k-1) \times (k-1)$ square from the NE corner of a shifted staircase shape δ_{m+2k} .

Example 8.5. *For $m = 1$ and $k = 3$, the truncated shape is*



Theorem 8.6. [1, Corollary 4.8] *The number of SYT of truncated shifted staircase shape $\delta_{m+2k} \setminus ((k-1)^{k-1})$ is*

$$g^{(m+k+1, \dots, m+3, m+1, \dots, 1)} g^{(m+k+1, \dots, m+3, m+1)} \cdot \frac{N!M!}{(N-M-1)!(2M+1)!},$$

where $N = \binom{m+2k+1}{2} - (k-1)^2$ is the size of the shape and $M = k(2m+k+3)/2 - 1$.

Similar results were obtained in [1] for truncation by “almost squares”, namely by $\kappa = (k^{k-1}, k-1)$.

8.2 Truncated rectangular shapes

In this section, $\lambda = (\lambda_1, \dots, \lambda_m)$ (with $\lambda_1 \geq \dots \geq \lambda_m \geq 0$) will be a partition with (at most) m parts. Two partitions which differ only in trailing zeros will be considered equal.

For any nonnegative integers m and n , let $(n^m) := (n, \dots, n)$ (m times) be the corresponding rectangular shape. The Frobenius-Young formula (Theorem 5.1) implies the following.

Observation 8.7. *The number of SYT of rectangular shape (n^m) is*

$$f^{(n^m)} = (mn)! \cdot \frac{F_m F_n}{F_{m+n}},$$

where

$$F_m := \prod_{i=0}^{m-1} i!.$$

Consider truncating a $(k-1) \times (k-1)$ square from the NE corner of a rectangular shape $((n+k-1)^{m+k-1})$.

Example 8.8. *Let $n = 3$, $m = 2$ and $k = 3$. Then*

$$[(5^4) \setminus (2^2)] = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

Theorem 8.9. [1, Corollary 5.7] *The number of SYT of truncated rectangular shape $((n+k-1)^{m+k-1}) \setminus ((k-1)^{k-1})$ is*

$$\frac{N!(mk-1)!(nk-1)!(m+n-1)!k}{(mk+nk-1)!} \cdot \frac{F_{m-1}F_{n-1}F_{k-1}}{F_{m+n+k-1}},$$

where N is the size of the shape and F_n is as in Observation 8.7.

In particular,

Corollary 8.10. *The number of SYT of truncated rectangular shape $((n+1)^{m+1}) \setminus (1)$ is*

$$\frac{N!(2m-1)!(2n-1)! \cdot 2}{(2m+2n-1)!(m+n+2)!} \cdot \frac{F_{m-1}F_{n-1}}{F_{m+n+1}},$$

where $N = (m+1)(n+1) - 1$ is the size of the shape and F_n is as in Observation 8.7.

Similar results were obtained in [1, 81] for truncation by almost squares $\kappa = (k^{k-1}, k-1)$.

Not much is known for truncation by rectangles. The following formula was conjectured in [1] and proved by Sun [125] using complex integrals.

Proposition 8.11. [125] *For $n \geq 2$*

$$f^{(n^n) \setminus (2)} = \frac{(n^2-2)!(3n-4)!^2 \cdot 6}{(6n-8)!(2n-2)!(n-2)!^2} \cdot \frac{F_{n-2}^2}{F_{2n-4}},$$

where F_n is as in Observation 8.7.

The following result was proved by Snow [111].

Proposition 8.12. [111] *For $n \geq 2$ and $k \geq 0$*

$$f^{(n^{k+1}) \setminus (n-2)} = \frac{(kn-k)!(kn+n)!}{(kn+n-k)!} \cdot \frac{F_k F_n}{F_{n+k}},$$

where F_n is as in Observation 8.7.

A different method to derive product formulas, for other families of truncated shapes, has been developed by Panova [81]. Consider a rectangular shape truncated by a staircase shape.

Example 8.13.

$$[(4^5) \setminus \delta_2] = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

Theorem 8.14. [81, Theorem 2] *Let $m \geq n \geq k$ be positive integers. The number of SYT of truncated shape $(n^m) \setminus \delta_k$ is*

$$\binom{N}{m(n-k-1)} f^{(n-k-1)^m} g^{(m, m-1, \dots, m-k)} \frac{E(k+1, m, n-k-1)}{E(k+1, m, 0)},$$

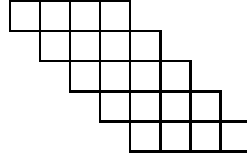
where $N = mn - \binom{k+1}{2}$ is the size of the shape and

$$E(r, p, s) = \begin{cases} \prod_{r < l < 2p-r+2} \frac{1}{(l+2s)^{r/2}} \prod_{2 \leq l \leq r} \frac{1}{((l+2s)(2p-l+2s+2))^{\lfloor l/2 \rfloor}}, & \text{if } r \text{ is even;} \\ \frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!} E(r-1, p, s), & \text{if } r \text{ is odd.} \end{cases}$$

8.3 Other truncated shapes

The following elegant result regarding **shifted strips** was recently proved by Sun.

Theorem 8.15. [126, §4.2] *The number of SYT of truncated shifted shape with n rows and 4 cells in each row*



is the $(2n-1)$ -st Pell number [110, A000129]

$$\frac{1}{2\sqrt{2}} \left((1 + \sqrt{2})^{2n-1} - (1 - \sqrt{2})^{2n-1} \right).$$

Sun applied a probabilistic version of computations of volumes of order polytopes to enumerate SYT of truncated and other exotic shapes. In [123] he obtained product formulas for the number of SYT of certain truncated skew shapes. This includes the shape $((n+k)^{r+1}, n^{m-1}) / (n-1)^r$ truncated by a rectangle or an “almost rectangle”, the truncated shape $((n+1)^3, n^{m-2}) / (n-2) \setminus (2^2)$, and the truncated shape $(n+1)^2 / n^{m-2} \setminus (2)$.

Modules associated with non-line-convex shapes were considered in [54]. The enumeration of SYT of such shapes is a very recent subject of study. Special non-line-convex shapes with one box removed at the end or middle of a row were considered in [124]. For example,

Proposition 8.16. [124, Theorem 5.2] *For $m \geq 0$, the number of SYT of shape $(m+3, 3, 3)$ with middle box in the second row removed, is*

$$\frac{m+5}{10} \binom{m+2}{2} \binom{m+9}{2}.$$

There are very few known results in this direction; problems in this area are wide open.

8.4 Proof approaches for truncated shapes

Different approaches were applied to prove the above product formulas. We will sketch one method and remark on another.

The **pivoting approach** of [1] is based on a combination of two different bijections from SYT to pairs of smaller SYT:

- (i) Choose a pivot cell P in the NE boundary of a truncated shape ζ and subdivide the entries of a given SYT T into those that are less than the entry of P and those that are greater.
- (ii) Choose a letter t and subdivide the entries in a SYT T into those that are less than or equal to t and those that are greater than t .

Proofs are obtained by combining applications of the first bijection to truncated shapes and the second to corresponding non-truncated ones. Here is a typical example.

Proof of Theorem 8.3 (sketch). First, apply the second bijection to a SYT of a shifted staircase shape.

Let n and t be nonnegative integers, with $t \leq \binom{n+1}{2}$. Let T be a SYT of shifted staircase shape δ_n , let T_1 be the set of all cells in T with values at most t , and let T_2 be obtained from $T \setminus T_1$ by transposing the shape (reflecting in an anti-diagonal) and replacing each entry i by $N - i + 1$, where $N = |\delta_n| = \binom{n+1}{2}$. Clearly T_1 and T_2 are shifted SYT.

Here is an example with $n = 4$ and $t = 5$.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 6 \\ \hline & 3 & 5 & 8 \\ \hline & & 7 & 9 \\ \hline & & & 10 \\ \hline \end{array} \rightarrow \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 6 \\ \hline 8 \\ \hline 7 \\ \hline 9 \\ \hline 10 \\ \hline \end{array} \right) \rightarrow \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 10 & 9 & 8 & 6 \\ \hline & 7 & & \\ \hline \end{array} \right) \rightarrow \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline & 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline & 4 & & \\ \hline \end{array} \right).$$

Notice that, treating strict partitions as sets, $\delta_4 = (4, 3, 2, 1)$ is the disjoint union of $\text{sh}(T_1) = (3, 2)$ and $\text{sh}(T_2) = (4, 1)$. This is not a coincidence.

Claim. *Treating strict partitions as sets, δ_n is the disjoint union of the shape of T_1 and the shape of T_2 .*

In order to prove the claim notice that the borderline between T_1 and $T \setminus T_1$ is a lattice path of length exactly n , starting at the NE corner of the staircase shape δ_n , and using only S and W steps, and ending at the SW boundary. If the first step is S then the first part of $\text{sh}(T_1)$ is n , and the rest corresponds to a lattice path in δ_{n-1} . Similarly, if the first step is W then the first part of $\text{sh}(T_2)$ is n , and the rest corresponds to a lattice path in δ_{n-1} . Thus exactly one of the shapes of T_1 and T_2 has a part equal to n . The claim follows by induction on n .

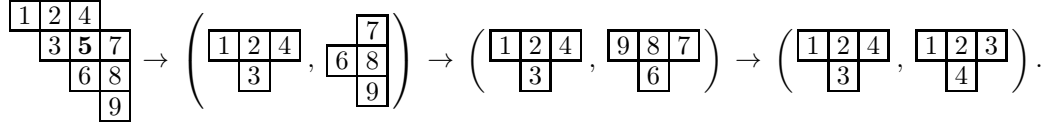
We deduce that, for any nonnegative integers n and t with $t \leq \binom{n+1}{2}$,

$$\sum_{\substack{\lambda \subseteq \delta_n \\ |\lambda| = t}} g^\lambda g^{\lambda^c} = g^{\delta_n}. \quad (1)$$

Here summation is over all strict partitions λ with the prescribed restrictions, and λ^c is the complement of λ in $\delta_n = \{1, \dots, n\}$ (where strict partitions are treated as sets). In particular, the LHS is independent of t .

Next apply the first bijection on SYT of truncated staircase shape $\delta_n \setminus (1)$. Choose as a pivot the cell $c = (2, n - 1)$, just SW of the missing NE corner. The entry $t = T(c)$ satisfies $2n - 3 \leq t \leq \binom{n}{2} - 2n + 2$. Let T be a SYT of shape $\delta_n \setminus (1)$ with entry t in P . One subdivides the other entries of T into those that are (strictly) less than t and those that are greater than t . The entries less than t constitute T_1 . To obtain T_2 , replace each entry $i > t$ of T by $N - i + 1$, where N is the total number of entries in T , and suitably transpose the resulting array. It is easy to see that both T_1 and T_2 are shifted SYT.

Example 8.17.



Next notice that the shape of T_1 is $(m-1, m-3) \cup \lambda$ while the shape of T_2 is $(m-1, m-3) \cup \lambda^c$, where λ is a strict partition contained in δ_{n-2} and λ^c is its complement in δ_{n-2} .

We deduce that

$$g^{\delta_n \setminus (1)} = \sum_t \sum_{\substack{\lambda \subseteq \delta_{n-2} \\ |\lambda|=t}} g^{(n-1, n-3) \cup \lambda} g^{(n-1, n-3) \cup \lambda^c}. \quad (2)$$

Here summation is over all strict partitions λ with the prescribed restrictions.

Finally, by Schur's formula (Theorem 5.7), for any strict partitions λ and $\mu = (\mu_1, \dots, \mu_k)$ with $\mu_1 > \dots > \mu_k > m$ and $\lambda \cup \lambda^c = \delta_m$, the following holds.

$$g^{\mu \cup \lambda} g^{\mu \cup \lambda^c} = c(\mu, |\lambda|, |\lambda^c|) \cdot g^\lambda g^{\lambda^c}, \quad (3)$$

where

$$c(\mu, |\lambda|, |\lambda^c|) = \frac{g^{\mu \cup \delta_m} g^\mu}{g^{\delta_m}} \cdot \frac{|\delta_m|! (|\mu| + |\lambda|)! (|\mu| + |\lambda^c|)!}{(|\mu| + |\delta_m|)! |\mu|! |\lambda|! |\lambda^c|!}$$

depends only on the sizes $|\lambda|$ and $|\lambda^c|$ and not on the actual partitions λ and λ^c .

Combining Equations (2), (1) and (3) together with some binomial identities completes the proof. \square

For a detailed proof and applications of the method to other truncated shapes see [1].

A different proof was presented by Panova [81]. Panova's approach is sophisticated and involved and will just be outlined. The proof relies on a bijection from SYT of the truncated shape to semi-standard Young tableaux of skew shapes. This bijection translates the enumeration problem to evaluations of sums of Schur functions at certain specializations. These evaluations are then reduced to computations of complex integrals, which are carried out by a comparison to another translation of the original enumerative problem to a volume of the associated order polytope.

9 Rim hook and domino tableaux

9.1 Definitions

The following concept generalizes the notion of SYT. Recall from Subsection 3.3 the definition of a zigzag shape.

Definition 9.1. *Let r and n be positive integers and let $\lambda \vdash rn$. An r -rim hook tableau of shape λ is a filling of the cells of the diagram $[\lambda]$ by the letters $1, \dots, n$ such that*

1. *each letter i fills exactly r cells, which form a zigzag shape called the i -th rim hook (or border strip); and*
2. *for each $1 \leq k \leq n$, the union of the i -th rim hooks for $1 \leq i \leq k$ is a diagram of ordinary shape.*

Denote by f_r^λ the number of r -rim hook tableaux of shape $\lambda \vdash rn$.

The n -th rim hook forms a path-connected subset of the **rim** (SE boundary) of the diagram $[\lambda]$, and removing it leads inductively to a similar description for the other rim hooks.

1-rim hook tableaux are ordinary SYT; 2-rim hook tableaux are also called **domino tableaux**.

Example 9.2. Here is a domino tableau of shape $(5, 5, 4)$:

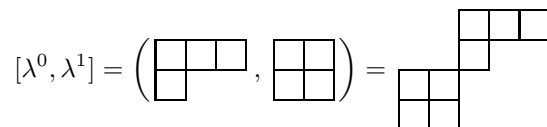
1	1	3	3	6
2	4	5	5	6
2	4	7	7	

and here is a 3-rim hook tableau of shape $(5, 4, 3)$:

1	1	3	3	3
1	2	4	4	
2	2	4		

Definition 9.3. An **r -partition** of n is a sequence $\lambda = (\lambda^0, \dots, \lambda^{r-1})$ of partitions of total size $|\lambda^0| + \dots + |\lambda^{r-1}| = n$. The corresponding **r -diagram** $[\lambda^0, \dots, \lambda^{r-1}]$ is the sequence $([\lambda^0], \dots, [\lambda^{r-1}])$ of ordinary diagrams. It is sometimes drawn as a skew diagram, with $[\lambda^i]$ lying directly southwest of $[\lambda^{i-1}]$ for every $1 \leq i \leq r-1$.

Example 9.4. The 2-diagram of shape $(\lambda^0, \lambda^1) = ((3, 1), (2, 2))$ is



Definition 9.5. A **standard Young r -tableau** (**r -SYT**) $T = (T^0, \dots, T^{r-1})$ of shape $\lambda = (\lambda^0, \dots, \lambda^{r-1})$ and total size n is obtained by inserting the integers $1, 2, \dots, n$ as entries into the cells of the diagram $[\lambda]$ such that the entries increase along rows and columns.

9.2 The r -quotient and r -core

Definition 9.6. Let λ be a partition, and $D = [\lambda]$ the corresponding ordinary diagram. The **boundary sequence** of λ is the 0/1 sequence $\partial(\lambda)$ constructed as follows: Start at the SW corner of the diagram and proceed along the edges of the SE boundary up to the NE corner. Each horizontal (east-bound) step is encoded by 1, and each vertical (north-bound) step by 0.

Example 9.7.

$$\lambda = (3, 1) \quad \rightarrow \quad D = [\lambda] = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & & \\ \hline \end{array} \quad \rightarrow \quad \partial(\lambda) = (1, 0, 1, 1, 0)$$

The boundary sequence starts with 1 and ends with 0 – unless λ is the empty partition, for which $\partial(\lambda)$ is the empty sequence.

Definition 9.8. The **extended boundary sequence** $\partial_*(\lambda)$ of λ is the doubly-infinite sequence obtained from $\partial(\lambda)$ by prepending to it the sequence $(\dots, 0, 0)$ and appending to it the sequence $(1, 1, \dots)$.

Geometrically, these additions represent a vertical ray and a horizontal ray, respectively, so that the tour of the boundary of $[\lambda]$ actually “starts” at the far south and “ends” at the far east.

Example 9.9. If $\lambda = (3, 1)$ then $\partial_*(\lambda) = (\dots, 0, 0, 1, 0, 1, 1, 0, 1, 1, \dots)$, and if λ is empty then $\partial_*(\lambda) = (\dots, 0, 0, 1, 1, \dots)$.

∂_* is clearly a bijection from the set of all partitions to the set of all doubly-infinite 0/1 sequences with initially only 0-s and eventually only 1-s.

Definition 9.10. *There is a natural indexing of any (extended) boundary sequence, as follows: The index k of an element is equal to the number of 1-s weakly to its left minus the number of 0-s strictly to its right.*

Example 9.11.

$$\begin{array}{rcccccccccccc} \text{0/1 sequence:} & \dots & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & \dots \\ & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ \text{Indexing:} & \dots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \dots \end{array}$$

Definition 9.12. *Let λ be a partition, r a positive integer, and $s := \partial_*(\lambda)$.*

1. *The r -quotient $q_r(\lambda)$ is a sequence of r partitions obtained as follows: For each $0 \leq i \leq r-1$ let s^i be the subsequence of s corresponding to the indices which are congruent to $i \pmod{r}$, and let $\lambda^i := \partial_*^{-1}(s^i)$. Then $q_r(\lambda) := (\lambda^0, \dots, \lambda^{r-1})$.*
2. *The r -core (or r -residue) $c_r(\lambda)$ is the partition $\lambda' = \partial_*^{-1}(s')$, where s' is obtained from s by a sequence of moves which interchange a 1 in position i with a 0 in position $i+r$ (for some i), as long as such a move is still possible.*

Denote $|q_r(\lambda)| := |\lambda^0| + \dots + |\lambda^{r-1}|$.

Theorem 9.13. $|\lambda| = r \cdot |q_r(\lambda)| + |c_r(\lambda)|$.

Example 9.14. *For $\lambda = (6, 4, 2, 2, 2, 1)$ and $r = 2$,*

$$s = \partial_*(\lambda) = (\dots, 0, 0, 0, 1, 0, 1, 0, 0, \hat{0}, 1, 1, 0, 1, 1, 0, 1, 1, 1, \dots)$$

with a hat over the entry indexed 0. It follows that

$$s^0 = (\dots, 0, 0, 0, 0, 0, 1, 1, 0, 1, \dots) \quad \text{and} \quad s^1 = (\dots, 0, 1, 1, 0, 1, 0, 1, 1, 1, \dots).$$

The 2-quotient is therefore $q_2(\lambda) = ((2), (3, 2))$. The 2-core is

$$c_2(\lambda) = \partial_*^{-1}(s') = \partial_*^{-1}(\dots, 0, 0, 0, 0, 0, 0, 0, 1, \hat{0}, 1, 0, 1, 1, 1, 1, 1, 1, \dots) = (2, 1).$$

Indeed, $|\lambda| = 17 = 2 \cdot 7 + 3 = 2 \cdot |q_2(\lambda)| + |c_2(\lambda)|$.

It is easy to see that, in this example, there are no 2-rim hook tableaux of shape λ .

Theorem 9.15. $f_r^\lambda \neq 0 \iff$ the r -core $c_r(\lambda)$ is empty.

Example 9.16. *Let $\lambda = (4, 2)$, $n = 3$ and $r = 2$. Then*

$$s = \partial_*(\lambda) = (\dots, 0, 0, 0, 1, \hat{1}, 0, 1, 1, 0, 1, 1, 1, \dots),$$

so that the 2-core

$$s' = \partial_*^{-1}(\dots, 0, 0, 0, 0, \hat{0}, 1, 1, 1, 1, 1, 1, \dots)$$

is empty and the 2-quotient is

$$q_2(\lambda) = (\partial_*^{-1}(\dots, 0, 0, 1, 1, 0, 1, \dots), \partial_*^{-1}(\dots, 0, 1, 0, 1, 1, 1, \dots)) = ((2), (1)).$$

Of course, here $|\lambda| = 6 = 2 \cdot 3 = r \cdot |q_2(\lambda)|$.

In this example there are three domino tableaux of shape $(4, 2)$, and also three 2-SYT of shape $((2), (1))$:

$$\begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline 1 & 2 & & \\ \hline \end{array} \longleftrightarrow \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array}$$

This is not a coincidence, as the following theorem shows.

Theorem 9.17. *Let λ be a partition with empty r -core, and let $q_r(\lambda)$ be its r -quotient. Then*

$$f_r^\lambda = f^{q_r(\lambda)}.$$

Theorem 9.17 may be combined with the hook length formula for ordinary shapes (Theorem 5.3) to obtain the following.

Theorem 9.18. [53, p. 84] *If $f_r^\lambda \neq 0$ then*

$$f_r^\lambda = \frac{(|\lambda|/r)!}{\prod_{c \in [\lambda]: r|h_c} h_c/r}.$$

Proof. λ has an empty r -core; let $q_r(\lambda) = (\lambda^0, \dots, \lambda^{r-1})$ be its r -quotient. By the hook length formula for ordinary shapes (Theorem 5.3) together with Observation 2.6,

$$f^{(\lambda^0, \dots, \lambda^{r-1})} = \left(\begin{array}{c} |\lambda|/r \\ |\lambda^0|, \dots, |\lambda^{r-1}| \end{array} \right) \prod_{i=0}^{r-1} f^{\lambda^i} = \frac{(|\lambda|/r)!}{\prod_{c \in [\lambda^0, \dots, \lambda^{r-1}]} h_c}.$$

A careful examination of the r -quotient implies that it induces a bijection from the cells in λ with hook length divisible by r to all the cells in $(\lambda^0, \dots, \lambda^{r-1})$, such that every cell $c \in [\lambda^0, \dots, \lambda^{r-1}]$ with hook length h_c corresponds to a cell $c' \in [\lambda]$ with hook length $h_{c'} = rh_c$. This completes the proof. \square

Stanton and White [121] generalized the RS correspondence to a bijection from r -colored permutations (i.e., elements of the wreath product $\mathbb{Z}_r \wr \mathcal{S}_n$) to pairs of r -rim hook tableaux of the same shape. The Stanton-White bijection, together with Theorem 9.17, implies the following generalization of Corollary 4.12.

Theorem 9.19.

$$(1) \quad \sum_{\lambda \vdash rn} (f_r^\lambda)^2 = r^n n!$$

and

$$(2) \quad \sum_{\lambda \vdash rn} f_r^\lambda = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! r^{n-k}.$$

In particular, the total number of domino tableaux of size $2n$ is equal to number of involutions in the hyperoctahedral group B_n . By similar arguments, the number of SYT of size n and unordered 2-partition shape is equal to the number of involutions in the Weyl group D_n .

An important inequality for the number of rim hook tableaux has been found by Fomin and Lulov.

Theorem 9.20. [33] *For any $\lambda \vdash rn$,*

$$f_r^\lambda \leq r^n n! \left(\frac{f^\lambda}{(rn)!} \right)^{1/r}.$$

See also [68][97][65].

10 q -Enumeration

This section deals primarily with three classical combinatorial parameters – inversion number, descent number and major index. These parameters were originally studied in the context of permutations (and, more generally, words). The major index, for example, was introduced by MacMahon [72]. These permutation statistics were studied extensively by Foata and Schützenberger [31, 32], Garsia and Gessel [39], and others. Only later were these concepts defined and studied for standard Young tableaux.

10.1 Permutation statistics

We start with definitions of the main permutation statistics.

Definition 10.1. *The descent set of a permutation $\pi \in \mathcal{S}_n$ is*

$$\text{Des}(\pi) := \{i : \pi(i) > \pi(i+1)\},$$

the descent number of π is

$$\text{des}(\pi) := |\text{Des}(\pi)|,$$

and the major index of π is

$$\text{maj}(\pi) := \sum_{i \in \text{Des}(\pi)} i.$$

The inversion set of π is

$$\text{Inv}(\pi) := \{(i, j) : 1 \leq i < j \leq n, \pi(i) > \pi(j)\},$$

and the inversion number of π is

$$\text{inv}(\pi) := |\text{Inv}(\pi)|.$$

We also use standard q -notation: For a nonnegative integer n and nonnegative integers k_1, \dots, k_t summing up to n denote

$$[n]_q := \frac{q^n - 1}{q - 1}, \quad [n]_q! := \prod_{i=1}^n [i]_q \quad (\text{including } [0]_q! := 1), \quad \left[\begin{matrix} n \\ k_1, \dots, k_t \end{matrix} \right]_q := \frac{[n]_q!}{\prod_{i=1}^t [k_i]_q!}.$$

Theorem 10.2. [72] (MacMahon's fundamental equidistribution theorem) *For every positive integer n*

$$\sum_{\pi \in \mathcal{S}_n} q^{\text{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\text{inv}(\pi)} = [n]_q!$$

A bijective proof was given in the classical paper of Foata [31]. Refinements and generalizations were given by many. In particular, Foata's bijection was applied to show that the major index and inversion number are equidistributed over inverse descent classes [32]. A different approach was suggested by Garsia and Gessel, who proved the following.

Theorem 10.3. [39] *For every subset $S = \{k_1, \dots, k_t\} \subseteq [n-1]$*

$$\sum_{\substack{\pi \in \mathcal{S}_n \\ \text{Des}(\pi^{-1}) \subseteq S}} q^{\text{maj}(\pi)} = \sum_{\substack{\pi \in \mathcal{S}_n \\ \text{Des}(\pi^{-1}) \subseteq S}} q^{\text{inv}(\pi)} = \left[\begin{matrix} n \\ k_1, k_2 - k_1, k_3 - k_2, \dots, n - k_t \end{matrix} \right]_q.$$

The following determinantal formula [119, Example 2.2.5] follows by the inclusion-exclusion principle.

$$\begin{aligned} \sum_{\substack{\pi \in \mathcal{S}_n \\ \text{Des}(\pi^{-1}) = S}} q^{\text{maj}(\pi)} &= \sum_{\substack{\pi \in \mathcal{S}_n \\ \text{Des}(\pi^{-1}) = S}} q^{\text{inv}(\pi)} = [n]_q! \det \left(\frac{1}{[s_{j+1} - s_i]_q!} \right)_{i,j=0}^k \\ &= \det \left(\left[\begin{matrix} n - s_i \\ s_{j+1} - s_i \end{matrix} \right]_q \right)_{i,j=0}^k. \end{aligned}$$

10.2 Statistics on tableaux

We start with definitions of descent statistics for SYT. Let T be a standard Young tableau of shape D and size $n = |D|$. For each entry $1 \leq t \leq n$ let $\text{row}(T^{-1}(t))$ denote the index of the row containing the cell $T^{-1}(t)$.

Definition 10.4. *The descent set of T is*

$$\text{Des}(T) := \{1 \leq i \leq n-1 \mid \text{row}(T^{-1}(i)) < \text{row}(T^{-1}(i+1))\},$$

the descent number of T is

$$\text{des}(T) := |\text{Des}(T)|,$$

and the major index of T is

$$\text{maj}(T) := \sum_{i \in \text{Des}(T)} i.$$

Example 10.5. *Let*

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 5 & 8 \\ \hline 3 & 4 & 6 & \\ \hline 7 & & & \\ \hline \end{array}.$$

Then $\text{Des}(T) = \{2, 5, 6\}$, $\text{des}(T) = 3$ and $\text{maj}(T) = 2 + 5 + 6 = 13$.

For a permutation $\pi \in \mathcal{S}_n$, recall from Section 4 the notation T_π for the skew (anti-diagonal) SYT which corresponds to π and the notation (P_π, Q_π) for the pair of ordinary SYT which corresponds to π under the Robinson-Schensted correspondence. By definition,

$$\text{Des}(T_\pi) = \text{Des}(\pi^{-1}).$$

The jeu de taquin algorithm preserves the descent set of a SYT, and therefore

Proposition 10.6. *For every permutation $\pi \in \mathcal{S}_n$,*

$$\text{Des}(P_\pi) = \text{Des}(\pi^{-1}) \quad \text{and} \quad \text{Des}(Q_\pi) = \text{Des}(\pi).$$

When it comes to inversion number, there is more than one possible definition for SYT.

Definition 10.7. *An inversion in T is a pair (i, j) such that $1 \leq i < j \leq n$ and the entry for j appears strictly south and strictly west of the entry for i :*

$$\text{row}(T^{-1}(i)) < \text{row}(T^{-1}(j)) \quad \text{and} \quad \text{col}(T^{-1}(i)) > \text{col}(T^{-1}(j)).$$

The inversion set of T , $\text{Inv}(T)$, consists of all the inversions in T , and the inversion number of T is

$$\text{inv}(T) := |\text{Inv}(T)|.$$

The sign of T is

$$\text{sign}(T) := (-1)^{\text{inv}(T)}.$$

Definition 10.8. *A weak inversion in T is a pair (i, j) such that $1 \leq i < j \leq n$ and the entry for j appears strictly south and weakly west of the entry for i :*

$$\text{row}(T^{-1}(i)) < \text{row}(T^{-1}(j)) \quad \text{and} \quad \text{col}(T^{-1}(i)) \geq \text{col}(T^{-1}(j)).$$

The weak inversion set of T , $\text{Winv}(T)$, consists of all the weak inversions in T , and the weak inversion number of T is

$$\text{winv}(T) := |\text{Winv}(T)|.$$

Observation 10.9. For every standard Young tableaux T of ordinary shape λ ,

$$\text{winv}(T) = \text{inv}(T) + \sum_j \binom{\lambda'_j}{2}.$$

Here λ'_j is the length of the j -th column of the diagram $[\lambda]$.

Example 10.10. For T as in Example 10.5, $\text{Inv}(T) = \{(2, 3), (2, 7), (4, 7), (5, 7), (6, 7)\}$, $\text{inv}(T) = 5$ and $\text{sign}(T) = -1$. The weak inversion set consist of the inversion set plus all pairs of entries in the same column. Thus $\text{winv}(T) = \text{inv}(T) + \binom{3}{2} + \binom{2}{2} + \binom{2}{2} + \binom{1}{2} = 4 + 3 + 1 + 1 = 9$.

For another (more complicated) inversion number on SYT see [47].

10.3 Thin shapes

We begin with refinements and q -analogues of results from Section 3.

10.3.1 Hook shapes

It is easy to verify that

Observation 10.11. For any $1 \leq k \leq n - 1$

$$\sum_{\text{sh}(T)=(n-k, 1^k)} \mathbf{x}^{\text{Des}(T)} = e_k,$$

where

$$e_k := \sum_{1 \leq i_1 < i_2 < \dots < i_k < n} x_{i_1} \cdots x_{i_k}$$

are the elementary symmetric functions.

Proof. Let T be a SYT of hook shape. Then for every $1 \leq i < n$, i is a descent in T if and only if the letter $i + 1$ lies in the first column. \square

Thus

$$\sum_{\text{sh}(T)=\text{hook of size } n} \mathbf{x}^{\text{Des}(T)} = \prod_{i=1}^{n-1} (1 + x_i). \quad (4)$$

It follows that

$$\sum_{\text{sh}(T)=\text{hook of size } n} t^{\text{des}(T)} q^{\text{maj}(T)} = \prod_{i=1}^{n-1} (1 + tq^i). \quad (5)$$

Notice that for a T of hook shape and size n , $\text{sh}(T) = (n - k, 1^k)$ if and only if $\text{des}(T) = k$. Combining this observation with Equation (5), the q -binomial theorem implies that

$$\sum_{\text{sh}(T)=(n-k, 1^k)} q^{\text{maj}(T)} = q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q.$$

Finally, the statistics winv and maj are equal on all SYT of hook shape. For most Young tableaux of non-hook zigzag shapes these statistics are not equal. However, the equidistribution phenomenon may be generalized to all zigzags. This will be shown below.

10.3.2 Zigzag shapes

Recall from Subsection 7.1 that each subset $S \subseteq [n-1]$ defines a zigzag shape $\text{zigzag}_n(S)$ of size n . The following statement refines Proposition 3.5.

Proposition 10.12. *For any $S \subseteq [n-1]$,*

$$f^{\text{zigzag}_n(S)} = \#\{\pi \in \mathcal{S}_n : \text{Des}(\pi) = S\}.$$

Proof. Standard Young tableaux of the zigzag shape encoded by S are in bijection with permutations in \mathcal{S}_n whose descent set is exactly S . The bijection converts such a tableau into a permutation by reading the cell entries starting from the southwestern corner. For example,

$$T = \begin{array}{ccccc} & & & 2 & 6 & 8 \\ & & & 5 & & \\ & & 3 & 7 & & \\ & 1 & 4 & & & \\ 9 & & & & & \end{array} \mapsto \pi = [914375268].$$

□

Notice that in this example, $\pi^{-1} = [274368591]$ and $\text{Des}(T) = \text{Des}(\pi^{-1}) = \{2, 3, 6, 8\}$. Also, $\text{Winv}(T) = \text{Inv}(\pi)$. This is a general phenomenon. Indeed,

Observation 10.13. *Let T be a SYT of zigzag shape and let π be its image under the bijection described in the proof of Proposition 10.12. Then*

$$\text{Des}(T) = \text{Des}(\pi^{-1}), \quad \text{Winv}(T) = \text{Inv}(\pi).$$

By Observation 10.13, there is a maj-winv preserving bijection from SYT of given zigzag shape to permutations in the corresponding descent class. Combining this with Theorem 10.3 one obtains

Proposition 10.14. *For every zigzag z encoded by $S = \{s_1, \dots, s_k\} \subseteq [n-1]$ set $s_0 := 0$ and $s_{k+1} := n$. Then*

$$\sum_{\text{sh}(T)=z} q^{\text{maj}(T)} = \sum_{\text{sh}(T)=z} q^{\text{winv}(T)} = [n]_q! \cdot \det \left(\frac{1}{[s_{j+1} - s_i]_q!} \right)_{i,j=0}^k.$$

10.3.3 Two-rowed shapes

The major index and (weak) inversion number are not equidistributed over SYT of two-rowed shapes. However, q -enumeration of both is nice. Two different q -Catalan numbers appear in the scene. First, consider, enumeration by major index.

Proposition 10.15. *For every $n \in \mathbb{N}$ and $0 \leq k \leq n/2$*

$$\sum_{\text{sh}(T)=(n-k,k)} q^{\text{maj}(T)} = \begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

In particular,

$$\sum_{\text{sh}(T)=(m,m)} q^{\text{maj}(T)} = q^m C_m(q)$$

where

$$C_m(q) = \frac{1}{[m+1]_q} \begin{bmatrix} 2m \\ m \end{bmatrix}_q$$

is the m -th F\"urlinger-Hofbauer q -Catalan number [38].

Hence

Corollary 10.16.

$$\sum_{\text{height}(T) \leq 2} q^{\text{maj}(T)} = \left[\begin{matrix} n \\ \lfloor n/2 \rfloor \end{matrix} \right]_q.$$

For a bijective proof and refinements see [10].

The descent set is invariant under jeu de taquin. Hence the proof of Theorem 4.15 may be lifted to a q -analogue. Here is a q -analogue of Theorem 4.15.

Theorem 10.17. *The major index generating function over SYT of size n and height ≤ 3 is equal to*

$$m_n(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} q^k \left[\begin{matrix} n \\ 2k \end{matrix} \right]_q C_k(q).$$

Furthermore, the following strengthened version of Corollary 4.12(3) holds.

Corollary 10.18. *For every positive integer k*

$$\sum_{\{T \in \text{SYT}_n : \text{height}(T) < k\}} \mathbf{x}^{\text{Des}(T)} = \sum_{\{\pi \in \text{Avoid}_n(\sigma_k) : \pi^2 = \text{id}\}} \mathbf{x}^{\text{Des}(T)},$$

where $\text{Avoid}_n(\sigma_k)$ is the subset of all permutations in \mathcal{S}_n which avoid $\sigma_k := [k, k-1, \dots, 1]$.

Proof. Combine Theorem 4.11 with Proposition 10.6. □

Counting by inversions is associated with another q -Catalan number.

Definition 10.19. [19] *Define the Carlitz-Riordan q -Catalan number $\tilde{C}_n(q)$ by the recursion*

$$\tilde{C}_{n+1}(q) := \sum_{k=0}^n q^k \tilde{C}_k(q) \tilde{C}_{n-k}(q)$$

with $\tilde{C}_0(q) := 1$.

These polynomials are, essentially, generating functions for the area under Dyck paths of order n .

Proposition 10.20. [108]

$$\sum_{\text{sh}(T)=(n,n)} q^{\text{inv}(T)} = \tilde{C}_n(q).$$

Proposition 10.21. [108] *For $0 \leq k \leq n/2$ denote $G_k(q) := \sum_{\text{sh}(T)=(n-k,k)} q^{\text{inv}(T)}$. Then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} q^{\binom{n-2k}{2}} G_k(q)^2 = \tilde{C}_n(q).$$

Enumeration of two-rowed SYT by descent number was studied by Barahovski.

Proposition 10.22. [8] *For $m \geq k \geq 1$,*

$$\sum_{\text{sh}(T)=(m,k)} t^{\text{des}(T)} = \sum_{d=1}^k \frac{m-k+1}{k} \binom{k}{d} \binom{m}{d-1} t^d$$

10.4 The general case

10.4.1 Counting by descents

There is a nice formula, due to Gessel, for the number of SYT of a given shape λ with a given descent set. Since it involves as a scalar product of symmetric functions, which are out of the scope of the current survey, we refer the reader to the original paper [41, Theorem 7].

There is also a rather complicated formula of Kreweras [62, 63] for the generating function of descent number on SYT of a given shape. However, the first moments of the distribution of this statistic may be calculated quite easily.

Proposition 10.23. *For every partition $\lambda \vdash n$ and $1 \leq k \leq n-1$*

$$\#\{T \in \text{SYT}(\lambda) : k \in \text{Des}(T)\} = \left(\frac{1}{2} - \frac{\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2}}{n(n-1)} \right) f^\lambda.$$

Here λ_i is the length of the i -th row in the diagram $[\lambda]$, and λ'_j is the length of the j -th column.

For proofs see [50, 3].

One deduces that $\#\{T \in \text{SYT}(\lambda) : k \in \text{Des}(T)\}$ is independent of k . This phenomenon may be generalized as follows: For any composition $\mu = (\mu_1, \dots, \mu_t)$ of n let

$$S_\mu := \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \dots + \mu_{t-1}\} \subseteq [1, n-1].$$

The underlying partition of μ is obtained by reordering the parts in a weakly decreasing order.

Theorem 10.24. *For every partition $\lambda \vdash n$ and any two compositions μ and ν of n with same underlying partition,*

$$\sum_{\text{sh}(T)=\lambda} \mathbf{x}^{\text{Des}(T) \setminus S_\mu} = \sum_{\text{sh}(T)=\lambda} \mathbf{x}^{\text{Des}(T) \setminus S_\nu}.$$

Proposition 10.23 implies that

Corollary 10.25. *The expected descent number of a random SYT of shape $\lambda \vdash n$ is equal to*

$$\frac{n-1}{2} - \frac{1}{n} \left(\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2} \right).$$

The variance of descent number was computed in [3, 50], implying a concentration around the mean phenomenon. The proofs in [3] involve character theory, while those in [50] follow from a careful examination of the hook length bijection of Novelli, Pak and Stoyanovskii [76], described in Subsection 6.2 above.

10.4.2 Counting by major index

Counting SYT of general ordinary shape by descents is difficult. Surprisingly, it was discovered by Stanley that counting by major index leads to a natural and beautiful q -analogue of the ordinary hook length formula (Proposition 5.3).

Theorem 10.26. (q -Hook Length Formula) [114, Corollary 7.21.5] *For every partition $\lambda \vdash n$*

$$\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = q^{\sum_i \binom{\lambda'_i}{2}} \frac{[n]_q!}{\prod_{c \in [\lambda]} [h_c]_q}.$$

This result follows from a more general identity, showing that the major index generating function for SYT of a skew shape is essentially the corresponding skew Schur function [114, Proposition 7.19.11]. If $|\lambda/\mu| = n$ then

$$s_{\lambda/\mu}(1, q, q^2, \dots) = \frac{\sum_{T \in \text{SYT}(\lambda/\mu)} q^{\text{maj}(T)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

An elegant q -analogue of Schur's shifted product formula (Proposition 5.7) was found by Stembridge.

Theorem 10.27. [122, Corollary 5.2] *For every strict partition $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$*

$$\sum_{T \in \text{SYT}(\lambda^*)} q^{n \cdot \text{des}(T) - \text{maj}(T)} = \frac{[n]_q!}{\prod_{i=1}^t [\lambda_i]_q!} \cdot \prod_{(i,j): i < j} \frac{q^{\lambda_j} - q^{\lambda_i}}{1 - q^{\lambda_i + \lambda_j}}.$$

Theorem 10.26 may be easily generalized to r -tableaux.

Corollary 10.28. *For every r -partition $\lambda = (\lambda^1, \dots, \lambda^r)$ of total size n*

$$\sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)} = q^{\sum_i \binom{\lambda_i}{2}} \cdot \frac{[n]_q!}{\prod_{c \in [\lambda]} [h_c]_q}.$$

The proof relies on a combination of Theorem 10.3 with the Stanton-White bijection for colored permutations.

10.4.3 Counting by inversions

Unlike descent statistics, not much is known about enumeration by inversion statistics in the general case. The following result was conjectured by Stanley [116] and proved, independently, by Lam [64], Reifegerste [92] and Sjöstrand [109].

Theorem 10.29.

$$\sum_{\lambda \vdash n} \sum_{T \in \text{SYT}(\lambda)} \text{sign}(T) = 2^{\lfloor n/2 \rfloor}.$$

A generalization of Foata's bijective proof of MacMahon's fundamental equidistribution theorem (Theorem 10.2) to SYT of any given shape was described in [47], using a more involved concept of inversion number for SYT.

11 Counting reduced words

An interpretation of SYT as reduced words is presented in this section. This interpretation is based on Stanley's seminal paper [112] and follow ups. For further reading see [15, §7.4-7.5] and [16, §7].

11.1 Coxeter generators and reduced words

Recall that the symmetric group \mathcal{S}_n is generated by the set $S := \{s_i : 1 \leq i < n\}$ subject to the defining Coxeter relations:

$$s_i^2 = 1 \quad (1 \leq i < n); \quad s_i s_j = s_j s_i \quad (|j - i| > 1); \quad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \leq i < n - 1).$$

The elements in S are called simple reflections and may be identified as adjacent transposition in \mathcal{S}_n , where $s_i = (i, i + 1)$.

The Coxeter length of a permutation $\pi \in \mathcal{S}_n$ is

$$\ell(\pi) := \min\{t : s_{i_1} \cdots s_{i_t} = \pi\}$$

the minimal length of an expression of π as a product of simple reflections.

Claim 11.1. For every $\pi \in \mathcal{S}_n$

$$\ell(\pi) = \text{inv}(\pi).$$

A series of Coxeter generators $(s_{i_1}, \dots, s_{i_t})$ is a reduced word of $\pi \in \mathcal{S}_n$ if the resulting product is a factorization of minimal length of π , that is $s_{i_1} \cdots s_{i_t} = \pi$ and $t = \ell(\pi)$. In this section, the enumeration of reduced words will be reduced to enumeration of SYT.

11.2 Ordinary and skew shapes

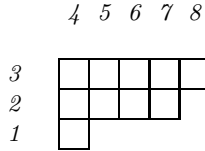
A **shuffle** of the two sequences $(1, 2, \dots, k)$ and $(k+1, k+2, \dots, \ell)$ is a permutation $\pi \in \mathcal{S}_\ell$ in which both sequences appear as subsequences. For example ($k=3, \ell=7$): 4516237.

Proposition 11.2. [26] *There exists a bijection $\lambda \mapsto \pi_\lambda$ from the set of all partitions to the set of all fixed point free shuffles such that:*

1. *If λ has height k , width (length of first row) $\ell - k$ and size n then π_λ is a fixed point free shuffle of $(1, 2, \dots, k)$ and $(k+1, k+2, \dots, \ell)$ with $\text{inv}(\pi_\lambda) = n$.*
2. *The number of SYT of shape λ is equal to the number of reduced words of π_λ .*

Proof sketch. For the first claim, read the permutation from the shape as follows: Encode the rows by $1, 2, \dots, k$ from bottom to top, and the columns by $k+1, k+2, \dots, \ell$ from left to right. Then walk along the SE boundary from bottom to top. If the i -th step is horizontal set $\pi_\lambda(i)$ to be its column encoding; otherwise set $\pi_\lambda(i)$ to be its row encoding.

Example 11.3. *The shape*

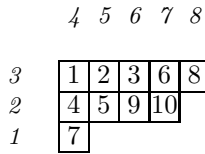


corresponds to the shuffle permutation

$$\pi = 41567283.$$

For the second claim, read the reduced word from the SYT as follows: If the letter $1 \leq j \leq n$ lies on the i -th diagonal (from the left), set the j -th letter in the word (from right to left) to be s_i .

Example 11.4. *The SYT*



corresponds to the reduced word (in adjacent transpositions)

$$s_5 s_4 s_7 s_1 s_6 s_3 s_2 s_5 s_4 s_3 = 41567283.$$

The proof that this map is a bijection from all SYT of shape λ to all reduced words of π_λ is obtained by induction on the size of λ . □

Corollary 11.5. *For every pair of positive integers $1 \leq k \leq \ell$, the number of reduced words of the permutation $[k+1, k+2, \dots, \ell, 1, 2, \dots, k]$ is equal to the number of SYT of rectangular shape $(k^{\ell-k})$.*

Proposition 11.6. *There exists an injection from the set of all 321-avoiding permutations to the set of all skew shapes, which satisfies the following property: For every 321-avoiding permutation π there exists a skew shape λ/μ such that the number of reduced words of π is equal to the number of SYT of shape λ/μ .*

The following theorem was conjectured and first proved by Stanley using symmetric functions [112]. A bijective proof was given later by Edelman and Greene [24].

Theorem 11.7. [112, Corollary 4.3] *The number of reduced words (in adjacent transpositions) of the longest permutation $w_0 := [n, n-1, \dots, 1]$ is equal to the number of SYT of staircase shape $\delta_{n-1} = (n-1, n-2, \dots, 1)$.*

Corollary 11.5 and Theorem 11.7 are special instances of the following remarkable result.

Theorem 11.8. [112, 24]

1. For every permutation $\pi \in \mathcal{S}_n$, the number of reduced words of π can be expressed as

$$\sum_{\lambda \vdash \text{inv}(\pi)} m_\lambda f^\lambda$$

where m_λ are nonnegative integers canonically determined by π .

2. The above sum is a unique f^{λ_0} (i.e., $m_\lambda = \delta_{\lambda, \lambda_0}$) if and only if π is 2143-avoiding.

Reiner [93] applied Theorem 11.7 to show that the expected number of subwords of type $s_i s_{i+1} s_i$ and $s_{i+1} s_i s_{i+1}$ in a random reduced word of the longest permutation is exactly one. He conjectured that the distribution of their number is Poisson. For some recent progress see [129].

A generalization of Theorem 11.7 to type B involves square shapes. The following theorem was conjectured by Stanley and proved by Haiman.

Theorem 11.9. [49] *The number of reduced words (in the alphabet of Coxeter generators) of the longest signed permutation $w_0 := [-1, -2, \dots, -n]$ in B_n is equal to the number of SYT of square shape.*

For a recent application see [82].

11.3 Shifted shapes

An interpretation of the number of SYT of a shifted shape was given by Edelman. Recall the left weak order from Section 1.1, and recall that σ covers π in this order if $\sigma = s_i \pi$ and $\ell(\sigma) = \ell(\pi) + 1$. Edelman considered a modification of this order in which we further require that the letter moved to the left be larger than all letters that precede it.

Theorem 11.10. [23, Theorem 3.2] *The number of maximal chains in the modified weak order is equal to the number of SYT of shifted staircase shape.*

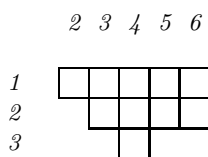
A related interpretation of SYT of shifted shapes was given in [26]. A permutation $\pi \in \mathcal{S}_n$ is **unimodal** if $\text{Des}(\pi^{-1}) = \{1, \dots, j\}$ for some $0 \leq j \leq n-1$. Consider U_n , the set of all unimodal permutations in \mathcal{S}_n , as a poset under the left weak order induced from \mathcal{S}_n .

Proposition 11.11. [26] *There exists a bijection $\lambda \mapsto \pi_\lambda$ from the set of all shifted shapes contained in the shifted staircase $\delta_{n-1} = (n-1, n-2, \dots, 1)$ to the set U_n of all unimodal permutations in \mathcal{S}_n such that:*

1. $|\lambda| = \text{inv}(\pi_\lambda)$.
2. The number of SYT of shifted shape λ is equal to the number of maximal chains in the interval $[id, \pi_\lambda]$ in U_n .

Proof sketch. Construct the permutation π_λ from the shape λ as follows: Encode the rows of λ by $1, 2, \dots$ from top to bottom, and the columns by $2, 3, \dots$ from left to right. Then walk along the SE boundary from bottom to top. If the i -th step is horizontal, set $\pi_\lambda(i)$ to be its column encoding; otherwise set $\pi_\lambda(i)$ to be its row encoding.

Example 11.12. *The shifted shape*

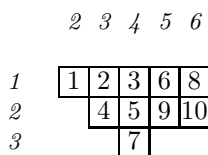


corresponds to the permutation

$$\pi = 4356217.$$

Now construct the reduced word from the SYT T as follows: If the letter j lies in the i -th diagonal (from left to right) of T then set the j -th letter in the word (from right to left) to be s_i .

Example 11.13. *The SYT*



corresponds to the reduced word (in adjacent transpositions)

$$s_4 s_3 s_5 s_1 s_4 s_2 s_1 s_3 s_2 s_1 = 4356217.$$

□

12 Appendix 1: Representation theoretic aspects

Representation theory may be considered as the birthplace of SYT; in fact, one cannot imagine group representations without the presence of SYT. Representation theory has been intimately related to combinatorics since its early days. The pioneering work of Frobenius, Schur and Young made essential use of integer partitions and tableaux. In particular, formulas for restriction, induction and decomposition of representations, as well as many character formulas, involve SYT. On the other hand, it is well known that many enumerative problems may be solved using representations. In this survey we restricted the discussion to combinatorial approaches. It should be noted that most results have representation theoretic proofs, and in many cases the discovery of the enumerative results was motivated by representation theoretic problems.

In this section we briefly point on several connections, assuming basic knowledge in non-commutative algebra, and give a very short sample of applications.

12.1 Degrees and enumeration

A SYT T of shape λ has an associated group algebra element $y_T \in \mathbb{C}[\mathcal{S}_n]$, called the **Young symmetrizer**. y_T has a key role: It is an idempotent, and its principal right ideal

$$y_T \mathbb{C}[\mathcal{S}_n]$$

is an irreducible module of \mathcal{S}_n . All irreducible modules are generated, up to isomorphism, by Young symmetrizers and two modules, which are generated by the Young symmetrizers of two SYT are isomorphic if and only if these SYT have same shape. The irreducible characters of the symmetric group \mathcal{S}_n over \mathbb{C} are, thus, parameterized by the integer partitions of n .

Proposition 12.1. *The degree of the character indexed by $\lambda \vdash n$ is equal to f^λ , the number of SYT of the ordinary shape λ .*

This phenomenon extends to skew and shifted shapes. The number of SYT of skew shape λ/μ , $f^{\lambda/\mu}$, is equal to the degree of the decomposable module generated by a Young symmetrizer of a SYT of shape λ/μ . Projective representations are indexed by shifted shapes; the number of SYT of shifted shape λ , g^λ , is equal to the degree of the associated projective representation.

Most of the results in this survey have representation theoretic proofs or interpretations. A few examples will be given here.

Proof sketch of Proposition 3.3. The symmetric group \mathcal{S}_n acts naturally on subsets of size k . The associated character, $\mu^{(n-k,k)}$, is multiplicity free; its decomposition into irreducibles is

$$\mu^{(n-k,k)} = \sum_{i=0}^k \chi^{(n-i,i)}. \quad (6)$$

Hence

$$\chi^{(n-k,k)} = \mu^{(n-k,k)} - \mu^{(n-k+1,k-1)}.$$

The degrees thus satisfy

$$f^{(n-k,k)} = \chi^{(n-k,k)}(1) = \mu^{(n-k,k)}(1) - \mu^{(n-k+1,k-1)}(1) = \binom{n}{k} - \binom{n}{k-1}.$$

□

This argumentation may be generalized to prove Theorem 5.4. First, notice that (6) is a special case of the Young rule for decomposing permutation modules. The Young rule implies the determinantal Jacobi-Trudi formula for expressing an irreducible module as an alternating sum of permutation modules, see e.g. [53]. Evaluation of the characters at the identity permutation implies Theorem 5.4.

Next proceed to identities which involve sums of f^λ -s.

Proof of Corollary 4.12(1). Recall that for every finite group, the sum of squares of the degrees of the irreducibles is equal to the size of the group. This fact together with the interpretation of the f^λ -s as degrees of the irreducibles of the symmetric group \mathcal{S}_n (Proposition 12.1) completes the proof. □

The same proof yields Theorem 9.19(1).

The Frobenius-Schur indicator theorem implies that for every finite group, which may be represented over \mathbb{R} , the sum of degrees of the irreducibles is equal to the number of involutions in the group, implying Corollary 4.12(2). The proof of Theorem 9.19(2) is similar, see e.g. [18, 2].

Proof sketch of Corollary 4.14. The permutation module, defined by the action of \mathcal{S}_{2n} on the cosets of $B_n = \mathbb{Z}_2 \wr \mathcal{S}_n$ is isomorphic to a multiplicity free sum of all \mathcal{S}_{2n} -irreducible modules indexed by partitions with all parts even [69, §VII (2.4)]. Comparison of the dimensions completes the proof. □

12.2 Characters and q -enumeration

The Murnaghan-Nakayama rule is a formula for computing values of irreducible \mathcal{S}_n -characters as signed enumerations of rim hook tableaux. Here is an example of special interest.

Proposition 12.2. *For every $\lambda \vdash rn$, the value of the irreducible character χ^λ at a conjugacy class of cycle type r^n is equal to the number of r -rim hook tableaux; namely,*

$$\chi_{(r,\dots,r)}^\lambda = f_r^\lambda.$$

Another interpretation of f_r^λ is as the degree of an irreducible module of the wreath product $\mathbb{Z}_r \wr \mathcal{S}_n$.

An equivalent formula for the irreducible character values is by weighted counts of all SYT of a given shape by their descents; q -enumeration then amounts to a computation of the corresponding Hecke algebra characters.

These character formulas may be applied to counting SYT by descents. Here is a simple example.

Proof sketch of Proposition 10.23. By the Murnaghan-Nakayama rule, the character of χ^λ at a transposition $s_i = (i, i+1)$ is equal to

$$|\{T \in \text{SYT}(\lambda) : i \notin \text{Des}(T)\}| - |\{T \in \text{SYT}(\lambda) : i \in \text{Des}(T)\}| = f^\lambda - 2|\{T \in \text{SYT}(\lambda) : i \in \text{Des}(T)\}|.$$

Combining this with the explicit formula for this character [52]

$$\chi_{(2, 1^{n-2})}^\lambda = \frac{\sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2}}{\binom{n}{2}} f^\lambda.$$

completes the proof. □

Finally, we quote two classical results, which apply enumeration by major index.

Theorem 12.3. (Kraśkiewicz-Weyman, in a widely circulated manuscript finally published as [56]) *Let ω be a primitive 1-dimensional character on the cyclic group C_n of order n . Then, for any partition λ of n , the multiplicity of χ^λ in the induced character $\text{Ind}_{C_n}^{S_n} \omega$, which is also the character of the S_n action on the multilinear part of the free Lie algebra on n generators (and of many other actions on combinatorial objects) is equal to the number of $T \in \text{SYT}(\lambda)$ with $\text{maj}(T) \equiv 1 \pmod{n}$.*

This result may actually be deduced from the following one.

Theorem 12.4. (Lusztig-Stanley) *For any partition λ of n and any $0 \leq k \leq \binom{n}{2}$, the multiplicity of χ^λ in the character of the S_n action on the k -th homogeneous component of the coinvariant algebra is equal to the number of $T \in \text{SYT}(\lambda)$ with $\text{maj}(T) = k$.*

A parallel powerful language is that of symmetric functions. The interested reader is referred to the excellent textbooks [114, Ch. 7] and [69].

13 Appendix 2: Asymptotics and probabilistic aspects

An asymptotic formula is sometimes available when a simple explicit formula is not known. Sometimes, such formulas do lead to the discovery of surprising explicit formulas. A number of important asymptotic results will be given in this appendix.

Recall the exact formulas (Corollary 3.4, Theorem 4.15 and Theorem 4.17) for the total number of SYT of ordinary shapes with small height. An asymptotic formula for the total number of SYT of bounded height was given by Regev [87]; see also [13][117][89].

Theorem 13.1. [87] *Fix a positive integer k and a positive real number α . Then, asymptotically as $n \rightarrow \infty$,*

$$F_{k,\alpha}(n) := \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq k}} (f^\lambda)^{2\alpha} \sim k^{2\alpha n} \cdot n^{-\frac{1}{2}(k-1)(\alpha k + 2\alpha - 1)} \cdot c(k, \alpha),$$

where

$$c(k, \alpha) := k^{\frac{1}{2}k(\alpha k + \alpha - 1)} (2\alpha)^{-\frac{1}{2}(k-1)(\alpha k + 1)} (2\pi)^{-\frac{1}{2}(k-1)(2\alpha - 1)} \prod_{i=1}^k \frac{\Gamma(i\alpha)}{\Gamma(\alpha)}.$$

In particular, $\lim_{n \rightarrow \infty} F_{k,\alpha}(n)^{1/n} = k^{2\alpha}$. Important special cases are: $\alpha = 1$, which gives (by Theorem 4.9 and Proposition 4.11) the asymptotics for the number of permutations in \mathcal{S}_n which do not contain a decreasing subsequence of length $k+1$; and $\alpha = 1/2$, which gives (by Corollary 4.12(3)) the asymptotics for the number of involutions in \mathcal{S}_n with the same property. See [87] for many other applications.

The proof of Theorem 13.1 uses the hook length formula for f^λ , factoring out the dominant terms from the sum and interpreting what remains (in the limit $n \rightarrow \infty$) as a k -dimensional integral. An explicit evaluation of this integral, conjectured by Mehta and Dyson [75, 74], has been proved using Selberg's integral formula [107].

Okounkov and Olshanski [77] introduced and studied a non-homogeneous analogue of Schur functions, the **shifted Schur function**. As a combinatorial application, they gave an explicit formula for the number of SYT of skew shape λ/μ . Stanley [115] proved a formula for $f^{\lambda/\mu}$ in terms of values of symmetric group characters. By applying the Vershik-Kerov \mathcal{S}_∞ -theory together with the Okounkov-Olshanski theory of shifted Schur functions, he used that formula to deduce the asymptotics of $f^{\lambda/\mu}$. See also [118]

Asymptotic methods were applied to show that certain distinct ordinary shapes have the same multiset of hook lengths [90]. Bijective and other purely combinatorial proofs were given later [91, 14, 60, 44].

In two seminal papers, Logan and Shepp [67], and independently Vershik and Kerov [133], studied the problem of the **limit shape** of the pair of SYT which correspond, under the RS correspondence, to a permutation chosen uniformly at random from \mathcal{S}_n . In other words, choose each partition λ of n with probability $\mu_n(\lambda) = (f^\lambda)^2/n!$. This probability measure on the set of all partitions of n is called **Plancherel measure**.

It was shown in [67, 133] that, under Plancherel measure, probability concentrates near one asymptotic shape. See also [17].

Theorem 13.2. [67, 133] *Draw a random ordinary diagram of size n in Russian notation (see Subsection 2.4) and scale it down by a factor of $n^{1/2}$. Then, as n tends to infinity, the shape converges in probability, under Plancherel measure, to the following limit shape:*

$$f(x) = \begin{cases} \frac{2}{\pi}(x \arcsin \frac{x}{2} + \sqrt{4-x^2}), & \text{if } |x| \leq 2; \\ |x|, & \text{if } |x| > 2. \end{cases}$$

This deep result had significant impact on mathematics in recent decades [98].

A closely related problem is to find the shape which maximizes f^λ . First, notice that Corollary 4.12 implies that

$$\sqrt{\frac{n!}{p(n)}} \leq \max\{f^\lambda : \lambda \vdash n\} \leq \sqrt{n!},$$

where $p(n)$ is the number of partitions of n .

Theorem 13.3. [134]

(1) *There exist constants $c_1 > c_0 > 0$ such that*

$$e^{-c_1\sqrt{n}}\sqrt{n!} \leq \max\{f^\lambda : \lambda \vdash n\} \leq e^{-c_0\sqrt{n}}\sqrt{n!}.$$

(2) *There exists constants $c'_1 > c'_0 > 0$ such that*

$$\lim_{n \rightarrow \infty} \mu_n \left\{ \lambda \vdash n : c'_0 < -\frac{1}{\sqrt{n}} \ln \frac{f^\lambda}{\sqrt{n!}} < c'_1 \right\} = 1.$$

Similar phenomena occur when Plancherel measure is replaced by other measures. For the uniform measure see, e.g., [83, 84].

Motivated by the limit shape result, Pittel and Romik proved that there exists a limit shape to the two-dimensional surface defined by a uniform random SYT of rectangular shape [85].

Consider a fixed $(i, j) \in \mathbb{Z}_+^2$ and a SYT T chosen according to some probability distribution on the SYT of size n . A natural task is to estimate the probability that $T(i, j)$ has a prescribed value. Regev was the first to give an asymptotic answer to this problem, for some probability measures, using \mathcal{S}_∞ -theory [88]; see also [78]. A combinatorial approach was suggested later by McKay, Morse and Wilf [73]. Here is an interesting special case.

Proposition 13.4. [88][73] *For a random SYT T of order n and a positive integer $k > 1$*

$$\text{Prob}(T(2, 1) = k) \sim \frac{k-1}{k!} + O(n^{-3/2}).$$

In [73], Proposition 13.4 was deduced from the following theorem.

Theorem 13.5. [73] *Let $\mu \vdash k$ be a fixed partition and let T be a fixed SYT of shape μ . Let $n \geq k$ and let $N(n; T)$ denote the number of SYT with n cells that contain T . Then*

$$N(n; T) \sim \frac{t_n f^\lambda}{k!},$$

where t_n denotes the number of involutions in the symmetric group \mathcal{S}_n .

It follows that

$$\sum_{\lambda \vdash n} f^{\lambda/\mu} \sim \frac{t_n f^\lambda}{k!}.$$

Stanley [115], applying techniques of symmetric functions, deduced precise formulas for $N(n; T)$ in the form of finite linear combinations of the t_n -s

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