

MULTIPLE DIVISOR FUNCTIONS AND MULTIPLE ZETA VALUES AT LEVEL N

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ABSTRACT. Multiple zeta values (MZVs) are generalizations of Riemann zeta values at positive integers to multiple variable setting. These values can be further generalized to level N multiple polylog values by evaluating multiple polylogs at N -th roots of unity. In this paper, we consider another level N generalization by restricting the indices in the iterated sums defining MZVs to congruences classes modulo N , which we call the MZVs at level N . The goals of this paper are two-fold. First, we shall lay down the theoretical foundations of these values such as their regularizations and double shuffle relations. Second, we will generalize the multiple divisor functions (MDFs) defined by Bachman and Kühn to arbitrary level N and study their relations to MZVs at level N . These functions are all q -series and similar to MZVs, they have both weight and depth filtrations. But unlike that of MZVs, the product of MDFs usually has mixed weights; however, after projecting to the highest weight we can obtain an algebra homomorphism from MDFs to MZVs. Moreover, the image of the derivation $\mathfrak{D} = q \frac{d}{dq}$ on MDFs vanishes on the MZV side, which gives rise to many nontrivial \mathbb{Q} -linear relations. In a sequel to this paper, we plan to investigate the nature of these relations.

1. INTRODUCTION

In the 1730s, long before Riemann, Euler started to study the infinite sums of the following form

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

though he only considered s as a real number. Later, in a series of correspondences with Goldbach in 1740s he also investigated the infinite double sums

$$1 + \frac{1}{2^s} \left(1 + \frac{1}{2^t}\right) + \frac{1}{3^s} \left(1 + \frac{1}{2^t} + \frac{1}{3^t}\right) + \cdots .$$

In modern notation this is $\zeta(s, t) + \zeta(s + t)$ where $\zeta(m, n)$ is called a double zeta value. In the 1990s, Zagier [21] and Hoffman [11, 12] independently extended these objects further to the multiple zeta values (MZVs for short) which are defined as follows. For

$\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$

$$\zeta(\mathbf{s}) := \sum_{k_1 > \dots > k_d > 0} \frac{1}{k_1^{s_1} \dots k_d^{s_d}}. \quad (1)$$

Clearly this series converges if and only if $s_1 \geq 2$. The number d is called the *depth* (or *length*), denoted by $\text{dp}(\mathbf{s})$, and $s_1 + \dots + s_d$ the *weight*, denoted by $|\mathbf{s}|$.

MZVs have been found to play important roles in many areas of mathematics as well as in physics such as in the computation of certain Feynman integrals (see [6]). Moreover, it is noticed that in fact one needs some generalizations of these numbers to multiple polylog values at roots of unity (see for e.g., [5, 6]). Recall that for any positive integers n_1, \dots, n_d , the multiple polylog in complex variables x_1, \dots, x_d is defined as follows:

$$Li_{n_1, \dots, n_d}(x_1, \dots, x_d) = \sum_{k_1 > \dots > k_d > 0} \frac{x_1^{k_1} \dots x_d^{k_d}}{k_1^{n_1} \dots k_d^{n_d}}, \quad (2)$$

where $|x_1 \dots x_j| < 1$ for $j = 1, \dots, d$. It can be analytically continued to a multi-valued meromorphic function on \mathbb{C}^d (see [23]). Now fix an N -th root of unity $\eta = \eta_N := \exp(2\pi\sqrt{-1}/N)$. A multiple polylog value (MPV for short) at level N has the form

$$\mathcal{L}_N(s_1, \dots, s_d; \alpha_1, \dots, \alpha_d) := \text{Li}_{s_1, \dots, s_d}(\eta^{\alpha_1}, \dots, \eta^{\alpha_d}) \quad (3)$$

for some $s_1, \dots, s_d \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_d \in \mathbb{Z}/N\mathbb{Z}$.

Extending the works of [10, 15, 20], we can provide another generalization of MZV to level N by restricting the summation indices in (1) to certain fixed congruence classes modulo N . Suppose $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}/N\mathbb{Z})^d$. The MZVs at level N with *colors* $\boldsymbol{\alpha}$ is

$$\zeta_N(\mathbf{s}; \boldsymbol{\alpha}) = \sum_{\substack{k_1 > \dots > k_d > 0 \\ k_j \equiv \alpha_j \pmod{N} \quad \forall 1 \leq j \leq d}} \frac{1}{k_1^{s_1} \dots k_d^{s_d}}. \quad (4)$$

In section 2 we will investigate the relation between these values and MPVs defined by (3). Then we will develop the general theory of regularizations of these values in section 3. This will enable us to study their underlying algebra structure.

In [1] Bachmann and Kühn considered the multiple divisor functions (MDFs) and applied their results to the study of MZVs and multiple Eisenstein series. In this paper, we generalize their theory to arbitrary levels and study the level $N = 2$ case in some detail. Note that the case $N = d = 2$ were considered by Kaneko and Tasaka [15] and Nakayama and Tasaka [16]. We will see that the correct objects corresponding to MZVs at arbitrary level N are exactly the MZVs at level N .

As for MZVs (see, for e.g., [12]) an extremely effective way to study MDFs is to consider their underlying algebra structure. In section 5, by generalizing Bachmann and Kühn’s work [1] at level 1 we will define the bifiltered algebra \mathcal{MD}_N at arbitrary level N by formalizing the stuffle product of MDFs. Here, the main complication results from the fact that these stuffle products often produce MDFs of lower weights whose coefficients, unlike $N = 1$ case, are generally complex numbers lying in the cyclotomic field $\mathbb{Q}(\eta_N)$. This forces us to study \mathcal{MD}_N as a bifiltered $\mathbb{Q}(\eta_N)$ -algebra. However, all is not lost since when we restrict to the weight-graded pieces we can safely descend to a \mathbb{Q} -algebra (see Theorem 6.11).

In some sense, the MDFs are easier to handle than the MZVs at level N because there is a derivation \mathfrak{D} on \mathcal{MD}_N which, when passing to the MZVs’ side provides many double shuffle relations which are supposed to be difficult to find since one would need two ways to regularize the MZVs at level N first. We will define and compute \mathfrak{D} in section 7. We know that for MZVs at higher levels, double shuffle relations are not sufficient to provide all their \mathbb{Q} -linear relations. In fact, we proved in [27] the distribution relations and the weight one relations are needed to produce the so-called standard relations and, furthermore, non-standard relations do exist. To us, the most interesting question is whether one can produce other relations besides the double shuffle type using the derivation \mathfrak{D} . If not, are there any other relations among MDFs corresponding to the other standard and non-standard relations among MZVs at level N .

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2. RELATIONS BETWEEN MPVs AND MZVs AT LEVEL N

By definition (3) when $N = 1$ MPVs become MZVs. One of the central themes in the study of MPVs is to determine the \mathbb{Q} -linear relations between them. By using motivic mechanism and higher algebraic K -theory Deligne and Goncharov [7] obtained the following result.

Theorem 2.1. *Let $\mathcal{MPV}_{\mathbb{Q}}(w, N)$ be the \mathbb{Q} -vector space spanned by all the MPVs of weight w and level N . Then we have $d(w, N) := \dim_{\mathbb{Q}} \mathcal{MPV}_{\mathbb{Q}}(w, N) \leq DG(w, N)$*

where $1 + \sum_{w=1}^{\infty} DG(w, N)t^w$ is the formal power series

$$\begin{cases} \frac{1}{1-t^2-t^3}, & \text{if } N=1; \\ \frac{1}{1-t-t^2}, & \text{if } N=2; \\ \frac{1}{1-a(N)t+b(N)t^2}, & \text{if } N \geq 3, \end{cases}$$

where $a(N) = \frac{\varphi(N)}{2} + \nu(N)$, $b(N) = \nu(N) - 1$, φ is the Euler's totient function and $\nu(N)$ is the number of distinct prime divisors of N .

For MZVs (i.e. $N = 1$) it is conjectured by Zagier [21] that the upper bound $DG(w, 1)$ is sharp and it is widely believed that all the \mathbb{Q} -linear relations are consequences of the so-called double shuffle relations. At higher levels, however, one needs some extra relations (see [18, 27]). In particular, at level $N = 2$ some comprehensive computation has been carried out in [3], and at level $N = 4$ one needs certain non-standard relations (see [24]).

Another direction of generalizations of MZVs was presented in [20] which was motivated by [10, 15]. There we defined the *double* zeta values at level N by restricting the summation indices to congruence classes modulo N . However, it is straight-forward to define MZVs at level N of *multiple* integer variables as in (4). Here, again, one of the key problems is to determine all the \mathbb{Q} -linear relations among MZVs of fixed weight and level. So we denote by $\mathcal{MZV}_{\mathbb{Q}}(w, N)$ the \mathbb{Q} -vector space spanned by all the MZVs of weight w and level N . It is not hard too see that when $w = 1$ all MZVs at level N diverge. To remedy the problem we will consider their regularized values in section 3. Notice that Racinet described two ways of regularization of MPVs in [18], one by using series representation (we say such regularized values \mathcal{L}^* are ***-regularized) and the other by integral representation (we say such regularized values \mathcal{L}^{III} are *III*-regularized). Let $\mathcal{MPV}_{\mathbb{Q}}^*(w, N)$ be the \mathbb{Q} -vector space generated by all ***-regularized MPVs, and similarly define $\mathcal{MPV}_{\mathbb{Q}}^{\text{III}}(w, N)$, $\mathcal{MZV}_{\mathbb{Q}}^*(w, N)$ and $\mathcal{MZV}_{\mathbb{Q}}^{\text{III}}(w, N)$ (see section 3).

Conjecture 2.2. *For all weight w and level N we have*

$$\dim_{\mathbb{Q}} \mathcal{MZV}_{\mathbb{Q}}^*(w, N) = \dim_{\mathbb{Q}} \mathcal{MZV}_{\mathbb{Q}}^{\text{III}}(w, N) = \dim_{\mathbb{Q}} \mathcal{MPV}_{\mathbb{Q}}^*(w, N) = \dim_{\mathbb{Q}} \mathcal{MPV}_{\mathbb{Q}}^{\text{III}}(w, N).$$

Notice that at level $N = 2$ the MPVs become the so called alternating Euler sums. As usual, instead of writing $\mathcal{L}_2(s_1, \dots, s_d; a_1, \dots, a_d)$ we will write $\zeta(s_1, \dots, s_d)$ and then add bars over s_j 's for all $a_j = 1$. For example, the alternating series $\mathcal{L}_2(s, t; 1, 0)$ is

denoted by $\zeta(\bar{s}, t)$. The following beautiful identity involving alternating Euler sums was first proved in [4] for $n = 1$ and in [26] for arbitrary n :

$$\zeta(\{3\}^n) = 8^n \zeta(\{\bar{2}, 1\}^n).$$

3. REGULARIZATIONS OF MZVS AT LEVEL N

It turns out that it is beneficial to us to have divergent MZVs at level N at our disposal. Similar to MZVs treated in [14], or more generally, MPVs at level N in [18], one can use two ways to regularize divergent MZVs at level N . When $N = d = 2$ this was done in [20]. In this section, we deal with the general case.

Definition 3.1. We define $\Gamma_0 = 0$ and for all $1 \leq \beta < N$

$$\Gamma_\beta = \Gamma_\beta(\emptyset; \emptyset) = -\frac{1}{\beta} + \sum_{n=1}^{\infty} \left(\frac{1}{nN} - \frac{1}{\beta + nN} \right).$$

Let $d > 0$, $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}/N\mathbb{Z})^d$. Define $\Gamma_0(\mathbf{s}; \boldsymbol{\alpha}) := 0$. Set $\alpha_0 = 0$ and for all $1 \leq \beta < N$ define

$$\Gamma_\beta(\mathbf{s}; \boldsymbol{\alpha}) := \begin{cases} \sum_{\substack{n_0 > n_1 > \dots > n_d > 0 \\ n_j \equiv \alpha_j \pmod{N} \ \forall j}} \left(\frac{1}{n_0} - \frac{1}{\beta + n_0} \right) \frac{1}{n_1^{s_1} \dots n_d^{s_d}}, & \text{if } \beta < \alpha_1 \\ & \text{or } d = 0; \\ \sum_{\substack{n_0 > n_1 > \dots > n_d > 0 \\ n_j \equiv \alpha_j \pmod{N} \ \forall j}} \left(\frac{1}{n_0} - \frac{1}{\beta - N + n_0} \right) \frac{1}{n_1^{s_1} \dots n_d^{s_d}}, & \text{otherwise.} \end{cases} \quad (5)$$

Straightforward computation shows that all these values are finite. Let \mathcal{MZV}_N be the \mathbb{Q} -vector space generated by all convergent MZVs of level N together with all the values $\Gamma_\beta(\mathbf{s}; \boldsymbol{\alpha})$ including $(\mathbf{s}; \boldsymbol{\alpha}) = (\emptyset; \emptyset)$.

Definition 3.2. We define the $*$ -regularized version of $\zeta_N(1; 0)$ as a polynomial in T of degree 1:

$$\zeta_N^*(1; 0) := \frac{1}{N} \left(T + \sum_{\beta=0}^{N-1} \Gamma_\beta \right) \in \mathcal{MZV}_N[T]$$

and for all $1 \leq \beta < N$ we define

$$\zeta_N^*(1; \beta) := \zeta_N^*(1; 0) - \Gamma_\beta \in \mathcal{MZV}_N[T].$$

If $s_1 > 1$ then we define the $*$ -regularized version of $\zeta_N(1, \mathbf{s}; \beta, \boldsymbol{\alpha})$ as a polynomial in T of degree 1 by using the stuffle relations:

$$\begin{aligned} \zeta_N^*(1, \mathbf{s}; \beta, \boldsymbol{\alpha}) &:= \zeta_N^*(1; \beta) \zeta_N(\mathbf{s}; \boldsymbol{\alpha}) - \sum_{j=1}^d \delta_{\alpha_j, \beta} \zeta_N(s_1, \dots, s_j + 1, s_{j+1}, \dots, s_d; \boldsymbol{\alpha}) \\ &\quad - \sum_{j=1}^d \zeta_N(s_1, \dots, s_j, 1, s_{j+1}, \dots, s_d; \alpha_1, \dots, \alpha_j, \beta, \alpha_{j+1}, \dots, \alpha_d) \in \mathcal{MZV}_N[T]. \end{aligned}$$

Now suppose $m \geq 1$, $d \geq 0$ and suppose $\zeta_N^*(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}, \boldsymbol{\alpha})$ have been defined as a polynomial in $\mathcal{MZV}_N[T]$ of degree at most m (in fact exactly m , but we leave this to the interested reader to check) for all $\boldsymbol{\beta}' = (\beta_1, \dots, \beta_m) \in (\mathbb{Z}/N\mathbb{Z})^m$, $\boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ with $s_1 > 1$. In the following we use the stuffle relations to define $\zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \boldsymbol{\beta}, \boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} = (\beta_0, \dots, \beta_m) \in (\mathbb{Z}/N\mathbb{Z})^{m+1}$ by some linear algebra.

First we write down the equation

$$T \cdot \zeta_N^*(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha}) = \sum_{\gamma=0}^{N-1} \zeta_N^*(1; \gamma) \zeta_N^*(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha}).$$

Expanding the right hand side using stuffle relations we see that

$$\sum_{\ell=0}^m \sum_{\gamma=0}^{N-1} \zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \beta_1, \dots, \beta_\ell, \gamma, \beta_{\ell+1}, \dots, \beta_m, \boldsymbol{\alpha})$$

is in $\mathcal{MZV}_N[T]$ of degree at most $m+1$. On the other hand we write down the equations

$$\sum_{\gamma=0}^{N-1} \Gamma_\gamma(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha}) = N \zeta_N^*(\{1\}^{m+1}, \mathbf{s}; 0, \boldsymbol{\beta}', \boldsymbol{\alpha}) - \sum_{\gamma=0}^{N-1} \zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \gamma, \boldsymbol{\beta}', \boldsymbol{\alpha}).$$

Adding up the above two equations we get the following element in $\mathcal{MD}_N[T]$

$$N \zeta_N^*(\{1\}^{m+1}, \mathbf{s}; 0, \boldsymbol{\beta}', \boldsymbol{\alpha}) + \sum_{\ell=1}^m \sum_{\gamma=0}^{N-1} \zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \beta_1, \dots, \beta_\ell, \gamma, \beta_{\ell+1}, \dots, \beta_m, \boldsymbol{\alpha}).$$

Subtracting $N \Gamma_{\beta_0}(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha})$ we see that for all $\boldsymbol{\beta} = (\beta_0, \dots, \beta_m) \in (\mathbb{Z}/N\mathbb{Z})^{m+1}$

$$N \zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \boldsymbol{\beta}, \boldsymbol{\alpha}) + \sum_{\ell=1}^m \sum_{\gamma=0}^{N-1} \zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \beta_1, \dots, \beta_\ell, \gamma, \beta_{\ell+1}, \dots, \beta_m, \boldsymbol{\alpha}) = b(\boldsymbol{\beta}),$$

for some $b(\boldsymbol{\beta}) \in \mathcal{MD}_N[T]$ of degree at most $m+1$. Regarding $\zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \boldsymbol{\beta}, \boldsymbol{\alpha})$ as N^{m+1} variables as $\boldsymbol{\beta}$ varies we now only need to show the corresponding $N^{m+1} \times N^{m+1}$ coefficient matrix $M(N, m)$ is nonsingular. Moreover, the inverse of $M(N, m)$ has entries

in \mathbb{Q} . Thus $M(N, m)^{-1}[b(\boldsymbol{\beta})]_{\boldsymbol{\beta} \in (\mathbb{Z}/N\mathbb{Z})^{m+1}}$ provides the definition of $\zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \boldsymbol{\beta}, \boldsymbol{\alpha})$ for all $\boldsymbol{\beta} \in (\mathbb{Z}/N\mathbb{Z})^{m+1}$ at the same time.

Using the N -adic system to order the variables by assigning the position number $\sum_{r=0}^{m-1} \beta_r N^r$ to $\zeta_N^*(\{1\}^{m+1}, \mathbf{s}; \beta_m, \dots, \beta_0, \boldsymbol{\alpha})$ we see that

$$M(N, m) = NI_{N^{m+1}} + \sum_{r=0}^{m-1} \left(E_{i,j}^{(m,r)} \right)_{0 \leq i, j < N^{m+1}},$$

where $I_{N^{m+1}}$ is the identity matrix of size N^{m+1} and $E^{(m,r)}$ is an $N^{m+1} \times N^{m+1}$ matrix whose (i, j) -th entry is given by

$$E_{i,j}^{(m,r)} = \begin{cases} 1, & \text{if } N^r | (j - \bar{i}N) \text{ and } 0 \leq j - \bar{i}N < N^r(N-1); \\ 0, & \text{otherwise,} \end{cases}$$

where $0 \leq \bar{i} < N^m$ such that $\bar{i} \equiv i \pmod{N^m}$. We now prove that

$$\det M(N, m) = N^{N^{m+1}} \cdot (m+1) \cdot \prod_{j=2}^m j^{N^{m-j}(N-1)}. \quad (6)$$

Partition the matrix $M(N, m)$ into N rows and N columns of smaller square blocks of size $N^m \times N^m$. Subtracting the first (0th, strictly speaking) row of these blocks from all the other rows we get the matrix of the form

$$\begin{pmatrix} * & * & * & \cdots & * \\ -NI_{N^m} & NI_{N^m} & 0 & \cdots & 0 \\ -NI_{N^m} & 0 & NI_{N^m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -NI_{N^m} & 0 & 0 & \cdots & NI_{N^m} \end{pmatrix}.$$

Now adding all the columns of these blocks to the first we get

$$\begin{pmatrix} M_1 & * & * & \cdots & * \\ 0 & NI_{N^m} & 0 & \cdots & 0 \\ 0 & 0 & NI_{N^m} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & NI_{N^m} \end{pmatrix},$$

where M_1 is $N^m \times N^m$ matrix with the form

$$M_1 = NI_{N^m} + \sum_{r=0}^{m-1} \left(E_{i,j}^{(m-1,r)} \right)_{0 \leq i, j < N^m}.$$

So $\det M(N, m) = N^{N^m(N-1)} \det M_1$. Applying the same trick on M_1 we can transform it to the $N^m \times N^m$ matrix of the form

$$\begin{pmatrix} M_2 & * & * & \cdots & * \\ 0 & NI_{N^{m-1}} & 0 & \cdots & 0 \\ 0 & 0 & NI_{N^{m-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & NI_{N^{m-1}} \end{pmatrix}.$$

So $\det M(N, m) = N^{N^{m+1}-N^{m-1}} \det M_2$. Repeatedly using this idea and we get for $j = 2, \dots, m$

$$\det M(N, m) = N^{N^{m+1}-N^{m-1}} (2N)^{N^{m-2}(N-1)} \cdot ((j-1)N)^{N^{m-j+1}(N-1)} \det M_j$$

where M_j is $N^{m-j+1} \times N^{m-j+1}$ matrix with the form

$$M_j = (jN)I_{N^{m-j+1}} + \sum_{r=0}^{m-j} \left(E_{i,j}^{(m-j,r)} \right)_{0 \leq i,j < N^{m-j+1}}$$

This can be proved by induction on j . Hence when $j = m$ we get

$$\det M(N, m) = N^{N^{m+1}-N^{m-1}} (2N)^{N^{m-2}(N-1)} \cdot ((m-1)N)^{N(N-1)} \det M_m$$

where M_m is the $N \times N$ matrix with the following form

$$\begin{pmatrix} mN+1 & 1 & 1 & \cdots & 1 \\ 1 & mN+1 & 1 & \cdots & 1 \\ 1 & 1 & mN+1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & mN+1 \end{pmatrix}.$$

Adding all the rows to the first row we get $\det M_m = (m+1)N \det M'_m$ where

$$M'_m = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & mN+1 & 1 & \cdots & 1 \\ 1 & 1 & mN+1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & mN+1 \end{pmatrix}.$$

Subtracting the first row from all the other rows we get $\det M'_m = \det M''_m$ where

$$M''_m = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & mN & 0 & \cdots & 0 \\ 0 & 0 & mN & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & mN \end{pmatrix}.$$

Thus

$$\det M(N, m) = N^{N^{m+1} - N^{m-1}} (2N)^{N^{m-2}(N-1)} ((m-1)N)^{N(N-1)} (mN)^{N-1} (m+1)N.$$

Simplifying we arrive at (6).

Example 3.3. When $N = 2$ and $m = 1$ we have $\Gamma_1 = -\ln(2)$ and

$$\zeta_2^*(1; 0) := \frac{1}{2}(T - \ln(2)), \quad \zeta_2^*(1; 1) := \frac{1}{2}(T + \ln(2)).$$

When $m = 2$ we first write down

$$T \cdot \zeta_2^*(1; 0) = 2\zeta_2^*(1, 1; 0, 0) + \zeta_2^*(1, 1; 0, 1) + \zeta_2^*(1, 1; 1, 0) + \zeta_2(2; 0), \quad (A_{1,0})$$

$$T \cdot \zeta_2^*(1; 1) = \zeta_2^*(1, 1; 0, 1) + \zeta_2^*(1, 1; 1, 0) + 2\zeta_2^*(1, 1; 1, 1) + \zeta_2^*(2; 1). \quad (A_{1,1})$$

Then we write down

$$\Gamma_1(1; 0) = \zeta_2^*(1, 1; 0, 0) - \zeta_2^*(1, 1; 1, 0), \quad (B_{1,0})$$

$$\Gamma_1(1; 1) = \zeta_2^*(1, 1; 0, 1) - \zeta_2^*(1, 1; 1, 1). \quad (B_{1,1})$$

Computing $(A_{1,0}) \pm (B_{1,0})$ and $(A_{1,1}) \pm (B_{1,1})$ we get

$$\begin{aligned} 3\zeta_2^*(1, 1; 0, 0) + \zeta_2^*(1, 1; 0, 1) &= T \cdot \zeta_2^*(1; 0) + \Gamma_1(1; 0) - \zeta_2(2; 0), \\ 2\zeta_2^*(1, 1; 0, 1) + \zeta_2^*(1, 1; 1, 0) + \zeta_2^*(1, 1; 1, 1) &= T \cdot \zeta_2^*(1; 1) + \Gamma_1(1; 1) - \zeta_2(2; 1), \\ \zeta_2^*(1, 1; 0, 0) + 2\zeta_2^*(1, 1; 1, 0) + \zeta_2^*(1, 1; 0, 1) &= T \cdot \zeta_2^*(1; 0) - \Gamma_1(1; 0) - \zeta_2(2; 0), \\ 3\zeta_2^*(1, 1; 1, 1) + \zeta_2^*(1, 1; 1, 0) &= T \cdot \zeta_2^*(1; 1) - \Gamma_1(1; 1) - \zeta_2(2; 1). \end{aligned}$$

Thus

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \zeta_2^*(1, 1; 0, 0) \\ \zeta_2^*(1, 1; 0, 1) \\ \zeta_2^*(1, 1; 1, 0) \\ \zeta_2^*(1, 1; 1, 1) \end{bmatrix} = \begin{bmatrix} T \cdot \zeta_2^*(1; 0) + \Gamma_1(1; 0) - \zeta_2(2; 0) \\ T \cdot \zeta_2^*(1; 1) + \Gamma_1(1; 1) - \zeta_2(2; 1) \\ T \cdot \zeta_2^*(1; 0) - \Gamma_1(1; 0) - \zeta_2(2; 0) \\ T \cdot \zeta_2^*(1; 1) - \Gamma_1(1; 1) - \zeta_2(2; 1) \end{bmatrix}.$$

Hence

$$\begin{aligned} \begin{bmatrix} \zeta_2^*(1, 1; 0, 0) \\ \zeta_2^*(1, 1; 0, 1) \\ \zeta_2^*(1, 1; 1, 0) \\ \zeta_2^*(1, 1; 1, 1) \end{bmatrix} &= \frac{1}{16} \begin{bmatrix} 5 & -3 & 1 & 1 \\ 1 & 9 & -3 & -3 \\ -3 & -3 & 9 & 1 \\ 1 & 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} \frac{T}{2}(T + \Gamma_1) + \Gamma_1(1; 0) - \zeta_2(2; 0) \\ \frac{T}{2}(T - \Gamma_1) + \Gamma_1(1; 1) - \zeta_2(2; 1) \\ \frac{T}{2}(T + \Gamma_1) - \Gamma_1(1; 0) - \zeta_2(2; 0) \\ \frac{T}{2}(T - \Gamma_1) - \Gamma_1(1; 1) - \zeta_2(2; 1) \end{bmatrix} \\ &= \frac{1}{16} \begin{bmatrix} 2T^2 + 4\Gamma_1 T + 4\Gamma_1(1; 0) - 4\Gamma_1(1; 1) - 6\zeta_2(2; 0) + 2\zeta_2(2; 1) \\ 2T^2 - 4\Gamma_1 T + 4\Gamma_1(1; 0) + 12\Gamma_1(1; 1) + 2\zeta_2(2; 0) - 6\zeta_2(2; 1) \\ 2T^2 + 4\Gamma_1 T - 12\Gamma_1(1; 0) - 4\Gamma_1(1; 1) - 6\zeta_2(2; 0) + 2\zeta_2(2; 1) \\ 2T^2 - 4\Gamma_1 T + 4\Gamma_1(1; 0) - 4\Gamma_1(1; 1) + 2\zeta_2(2; 0) - 6\zeta_2(2; 1) \end{bmatrix}. \end{aligned}$$

By comparing with the stuffle relation

$$\frac{1}{4}(T + \Gamma_1)^2 = \zeta_2^*(1; 0)^2 = 2\zeta_2^*(1, 1; 0, 0) + \zeta_2(2; 0)$$

we get a relation

$$\Gamma_1^2 + 2\Gamma_1(1; 1) - 2\Gamma_1(1; 0) = \zeta_2(2; 1) + \zeta_2(2; 0) = \zeta(2),$$

which can also be proved by the fact that $\Gamma_1^2 = \zeta(\bar{1})^2 = 2\zeta(\bar{1}, \bar{1}) + \zeta(2)$, $2\Gamma_1(1; 0) = \zeta(\bar{1}, 1) + \zeta(\bar{1}, \bar{1})$ and $2\Gamma_1(1; 1) = \zeta(\bar{1}, 1) - \zeta(\bar{1}, \bar{1})$.

We leave the interested reader to check that the above Definition 3.2 is consistent with the *-regularized values of MPVs given by Racinet in [18]. Namely, (8) can be extended to these regularized values:

$$\zeta_N^*(\mathbf{s}; \boldsymbol{\alpha}) = \frac{1}{N^d} \sum_{a_1=1}^N \cdots \sum_{a_d=1}^N \eta^{-(a_1\alpha_1 + \cdots + a_d\alpha_d)} \mathcal{L}_N^*(\mathbf{s}; a_1, \dots, a_d) \quad (7)$$

for all $\boldsymbol{\alpha} \in \mathbb{Z}/N\mathbb{Z}$ and $\mathbf{s} \in \mathbb{N}^d$ with $s_1 = 1$.

Notice that Conjecture 2.2 becomes easy to verify if we extend the scalars to the cyclotomic field $\mathbb{Q}(\eta_N)$. Conjecturally, every linear relations between MPVs over $\overline{\mathbb{Q}}$ (the algebraic closure of \mathbb{Q}) must be a consequence of some linear relations over \mathbb{Q} (the same should be true for MZVs at level N). This is the motivation for our conjecture in the first place. To show the $\mathbb{Q}(\eta_N)$ -version of the conjecture, using the identity

$$\sum_{j=0}^{N-1} \eta^{Mj} = \begin{cases} N, & \text{if } N|M; \\ 0, & \text{otherwise,} \end{cases}$$

we can easily find

$$\begin{aligned}\zeta_N(\mathbf{s}; \boldsymbol{\alpha}) &= \frac{1}{N^d} \sum_{k_1 > \dots > k_d > 0} \sum_{a_1=1}^N \eta^{a_1(k_1 - \alpha_1)} \dots \sum_{a_d=1}^N \eta^{a_d(k_d - \alpha_d)} \frac{1}{k_1^{s_1} \dots k_d^{s_d}} \\ &= \frac{1}{N^d} \sum_{a_1=1}^N \dots \sum_{a_d=1}^N \eta^{-(a_1\alpha_1 + \dots + a_d\alpha_d)} \mathcal{L}_N(\mathbf{s}; a_1, \dots, a_d).\end{aligned}\tag{8}$$

Let $h = \eta^{-1}$. It is not difficult to see that the determinant of the $N^d \times N^d$ matrix

$$\left[h^{a_1\alpha_1 + \dots + a_d\alpha_d} \right]_{1 \leq a_1, \dots, a_d, \alpha_1, \dots, \alpha_d \leq N}$$

is given by $\det(V)^{N^{2d-2}}$ where V is the Vandermonde matrix $[h^{a\alpha}]_{1 \leq a, \alpha \leq N}$. Set $f(x) = x^N - 1$. Then

$$(-1)^{\binom{N}{2}} \det(V)^2 = \prod_{1 \leq a \neq b \leq N} (h^a - h^b) = \prod_{a=1}^N f'(h^a) = N^N \left(\prod_{a=1}^N h^a \right)^{N-1} = (-1)^{N-1} N^N.$$

Therefore $\det(V)^2 = \pm N^N$ where we take $+$ for $N \equiv 2 \pmod{4}$ and $-$ otherwise. Consequently, for every fixed $(a_1, \dots, a_d) \in (\mathbb{Z}/N\mathbb{Z})^d$ the MPV $\mathcal{L}_N(\mathbf{s}; a_1, \dots, a_d)$ can be expressed as a $\mathbb{Q}(\eta_N)$ -linear combinations of $\zeta_N(\mathbf{s}; \boldsymbol{\alpha})$ ($\boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d$). The above argument proves the following important fact.

Theorem 3.4. *Let $\mathbb{Q}_N = \mathbb{Q}(\eta_N)$ be the N -th cyclotomic field. As \mathbb{Q}_N -vector spaces we have*

$$\dim_{\mathbb{Q}_N} \mathcal{MZV}_{\mathbb{Q}}^*(w, N) \otimes \mathbb{Q}_N = \dim_{\mathbb{Q}_N} \mathcal{MPV}_{\mathbb{Q}}^*(w, N) \otimes \mathbb{Q}_N.$$

The \mathfrak{m} -regularized version of $\zeta(1; \alpha)$ is defined as follows.

Definition 3.5.

$$\zeta^{\mathfrak{m}}(1; \alpha) = \frac{1}{N} \left(T + \sum_{n=1}^{N-1} \eta^{-n\alpha} Li_1(\eta^n) \right).$$

Now let $m \geq 1$, $d \geq 0$ and suppose $\mathcal{L}_N^{\mathfrak{m}}(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha})$ have been defined as a polynomial in $\mathcal{MZV}_N[T]$ of degree at most m for all $\boldsymbol{\beta}' = (\beta_1, \dots, \beta_m) \in (\mathbb{Z}/N\mathbb{Z})^m$, $\boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d$ and $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ with $s_1 > 1$. Using the partial fractions

$$\frac{t^{a-1}}{1-t^N} = \frac{1}{N} \sum_{m=1}^N \frac{\eta^{-(a-1)m}}{1-\eta^m t},$$

we obtain

$$\begin{aligned}\zeta_N^{\text{III}}(\mathbf{s}; \boldsymbol{\alpha}) &= \int_0^1 \omega^{s_1-1} \frac{t^{\alpha_1-\alpha_2-1} dt}{1-t^N} \cdots \omega^{s_d-1} \frac{t^{\alpha_d-1} dt}{1-t^N} \\ &= \frac{1}{N^d} \sum_{m_1, \dots, m_d=1}^N \eta^{-\sum_{i=1}^{d-1} m_i(\alpha_i-\alpha_{i+1}-1) - (\alpha_d-1)m_d} \mathcal{L}_N^{\text{III}}(\mathbf{s}; \eta^{\alpha_1}, \dots, \eta^{\alpha_d}),\end{aligned}$$

for $\mathbf{s} \in \mathbb{N}^n$ and $\boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^n$. Thus for $\mathbf{s} = (\{1\}^{m+1}, \mathbf{s}')$ with $\mathbf{s}' = (s_1, \dots, s_d) \in \mathbb{N}^d$ and $s_1 > 1$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}/N\mathbb{Z})^d$ and $\boldsymbol{\beta} = (\beta_0, \dots, \beta_m) \in (\mathbb{Z}/N\mathbb{Z})^{m+1}$,

$$\begin{aligned}\zeta_N^{\text{III}}(\{1\}^{m+1}, \mathbf{s}'; \mathbf{b}, \boldsymbol{\alpha}) &= \frac{1}{N^d} \sum_{m_1, \dots, m_d=1}^N \eta^{-\sum_{i=1}^{d-1} m_i(\alpha_i-\alpha_{i+1}-1) - (\alpha_d-1)m_d} \\ &\quad \cdot \mathcal{L}_N^{\text{III}}(\{1\}^{m+1}, \mathbf{s}'; \eta^{\beta_0}, \dots, \eta^{\beta_m}, \eta^{\alpha_1}, \dots, \eta^{\alpha_d}),\end{aligned}$$

where

$$\begin{aligned}\mathcal{L}_N^{\text{III}}(\{1\}^{m+1}, \mathbf{s}'; \eta^{\beta_0}, \dots, \eta^{\beta_m}, \eta^{\alpha_1}, \dots, \eta^{\alpha_d}) &= \frac{1}{m+1} [Li_1^{\text{III}}(\eta^{\beta_0}) \cdot \mathcal{L}_N^{\text{III}}(\{1\}^m, \mathbf{s}'; \eta^{\beta_1}, \dots, \eta^{\beta_m}, \eta^{\alpha_1}, \dots, \eta^{\alpha_d}) \\ &\quad - \sum_{j=1}^{d-m-1} \sum_{t_j=2}^{s_j} \mathcal{L}_N^{\text{III}}(\{1\}^m, s_1, \dots, t_j, s_j+1-t_j, \dots, s_{d-m-1}; \eta^{\beta_0}, \dots, \eta^{\beta_m}, \eta^{\alpha_1}, \dots, \eta^{\alpha_d}) \\ &\quad - \mathcal{L}_N^{\text{III}}(\{1\}^m, \mathbf{s}', 1; \eta^{\beta_0}, \dots, \eta^{\beta_m}, \eta^{\alpha_1}, \dots, \eta^{\alpha_d})]\end{aligned}$$

$$\text{with } Li_1^{\text{III}}(\eta^{\alpha_1}) = \begin{cases} T, & \text{if } \alpha_1 \equiv 0 \pmod{N}; \\ Li_1(\eta^{\alpha_1}), & \text{if } \alpha_1 \not\equiv 0 \pmod{N}. \end{cases}$$

Next we want to study the relation between the above two different regularizations. Our goal is to show that both regularizations still satisfy the same relation obtained by Ihara, Kaneko, and Zagier (see [14, Theorem 1]). By (7), it suffices to prove the same relation for MPV $\mathcal{L}_N(\mathbf{s}; a_1, \dots, a_d)$, which has been done by Racinet [18].

We now recall some of the results from [18]. For convenience we use Racinet's notation. For a set Z and a field k , let $k\langle Z \rangle$ (resp. $\mathfrak{Lib}_k(Z)$) be the free (noncommutative) associative algebra (resp. the free Lie algebra) on Z with coefficients in k . We denote by $k\langle\langle Z \rangle\rangle$ the set of all formal series on Z over the field k . The algebra $k\langle Z \rangle$, when viewed as the enveloping bigebra of $\mathfrak{Lib}_k(Z)$, has the coproduct Δ defined by $\Delta z = 1 \otimes z + z \otimes 1$, for $\forall z \in Z$.

Let Γ be a multiplicative finite group of \mathbb{C}^\times . In this paper, one may take Γ to be the group of N -th roots of unity. Then one defines the alphabet $X := \{x_\sigma : \sigma \in \Gamma \cup \{0\}\}$. For $(n, \nu) \in \mathbb{N} \times \Gamma$, let $y_{n,\nu} = x_0^{n-1}x_\nu$. We denote by Y the set of all $y_{n,\nu}$ for $(n, \nu) \in \mathbb{N} \times \Gamma$. The subalgebra of $k\langle X \rangle$ generated by Y is denoted by $k\langle Y \rangle$ and is free on Y . As a vector space, it is generated by the words on X not ending by x_0 thus it could be identified with the quotient $k\langle X \rangle/I_0$, where $I_0 = k\langle X \rangle x_0$. The corresponding projection is denoted by π_Y . The vector subspace $k\langle X \rangle_{cv}$ of $k\langle X \rangle$ generated by words on X that are not ending by x_0 and not starting by x_1 is also a graded algebra. It may be identified with the quotient $k\langle X \rangle/I$, where $I = (x_1 k\langle X \rangle + k\langle X \rangle x_0)$. We denote by π_{cv} the corresponding projection. The algebra $k\langle X \rangle_{cv}$ is same as the subalgebra of $k\langle Y \rangle$ generated by words on Y that are not starting by $y_{1,1} = x_1$. Thus it can be identified with the quotient $k\langle Y \rangle/y_{1,1}k\langle Y \rangle$, which is denoted by $k\langle Y \rangle_{cv}$. Again its corresponding projection is denoted by π_{cv} . Similarly one defines $k\langle\langle X \rangle\rangle_{cv}$ and $k\langle\langle Y \rangle\rangle_{cv}$.

We extend the definition of $y_{n,\nu}$ to the case where $n = 0$

$$y_{0,\nu} = \begin{cases} 1, & \text{if } \nu = 1; \\ 0, & \text{if } \nu \neq 1. \end{cases}$$

Then define the coproduct

$$\Delta_*(y_{n,\nu}) = \sum_{\substack{k+l=n \\ \kappa+\lambda=\nu}} y_{k,\kappa} \otimes y_{l,\lambda}.$$

It's easy to check (see [18]) that $(k\langle Y \rangle, \cdot, \Delta_*)$ is a bialgebra. Similarly, one can check that the associative algebra $k\langle\langle X \rangle\rangle$ is a bialgebra (see [19]). So are $k\langle\langle Y \rangle\rangle, k\langle\langle X \rangle\rangle_{cv}$ and $k\langle\langle Y \rangle\rangle_{cv}$. Recall that an element x in any coalgebra (C, ϵ, Δ) is called *diagonal* if $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$.

For any word $y_{s_1,\sigma_1}y_{s_2,\sigma_2} \cdots y_{s_r,\sigma_r}$ on Y , we define the endomorphism \mathbf{q} of $k\langle Y \rangle$ as

$$\mathbf{q}(y_{s_1,\sigma_1}y_{s_2,\sigma_2} \cdots y_{s_r,\sigma_r}) = y_{s_1,\sigma_1}y_{s_2,\sigma_2\sigma_1^{-1}} \cdots y_{s_r,\sigma_r\sigma_{r-1}^{-1}}.$$

Recall the multiple polylog $Li_{s_1,\dots,s_d}(z_1, \dots, z_d)$ is defined by

$$Li_{s_1,\dots,s_d}(z_1, \dots, z_d) = \sum_{k_1 > \dots > k_d > 0} \frac{z_1^{k_1} \cdots z_d^{k_d}}{k_1^{s_1} \cdots k_d^{s_d}}, \quad (9)$$

Let

$$I_{[0,1]}(\sigma_1, \dots, \sigma_r) = \int_{1 \geq t_1 \geq \dots \geq t_r \geq 0} \bigwedge_{i=1}^r \omega_{\sigma_i}(t_i),$$

where

$$\omega_{\sigma_i}(t_i) = \begin{cases} \frac{dt_i}{\sigma_i^{-1} - t_i}, & \text{if } \sigma_i \neq 0; \\ \frac{dt_i}{t_i}, & \text{if } \sigma_i = 0. \end{cases}$$

Then

$$Li_{s_1, \dots, s_r}(1, \dots, 1) = \zeta(s_1, \dots, s_r) = I_{[0,1]}(\{0\}^{s_1-1}, 1, \dots, \{0\}^{s_r-1}, 1).$$

The generating series of $Li_{s_1, \dots, s_r}(z_1, \dots, z_r)$ and $I_{[0,1]}(\sigma_1, \dots, \sigma_r)$ are defined as follows:

$$\begin{aligned} \mathcal{L}_{cv} &:= \sum_{\substack{s_1, \dots, s_r \in \mathbb{N}, \sigma_1, \dots, \sigma_r \in \Gamma \\ (s_1, \sigma_1) \neq (1, 1)}} Li_{s_1, \dots, s_r}(\sigma_1, \dots, \sigma_r) y_{s_1, \sigma_1} \cdots y_{s_r, \sigma_r}, \\ \mathcal{I}_{cv} &:= \sum_{\sigma_1, \dots, \sigma_{r-1} \in \Gamma \cup \{0\}, \sigma_r \in \Gamma} I_{[0,1]}(\sigma_1, \dots, \sigma_r) x_{\sigma_1} \cdots x_{\sigma_r}. \end{aligned}$$

It is clear one can think of them as two elements in $\mathbb{C}\langle\langle Y \rangle\rangle_{cv}$ and $\mathbb{C}\langle\langle X \rangle\rangle_{cv}$, respectively. Racinet further shows

Proposition 3.6. *The series \mathcal{L}_{cv} and \mathcal{I}_{cv} are diagonal elements in $\mathbb{C}\langle\langle Y \rangle\rangle_{cv}$ and $\mathbb{C}\langle\langle X \rangle\rangle_{cv}$, respectively.*

By straight-forward computation one finds the following crucial relation between the above two elements.

Proposition 3.7. *One has*

$$\mathcal{L}_{cv} = \mathbf{q}(\mathcal{I}_{cv}).$$

Remark 3.8. The map \mathbf{q} is added due to the formula

$$I_{[0,1]}(\{0\}^{s_1-1}, \sigma_1, \dots, \{0\}^{s_r-1}, \sigma_r) = Li_{s_1, \dots, s_r}(\sigma_1, \sigma_2 \sigma_1^{-1}, \dots, \sigma_r \sigma_{r-1}^{-1}).$$

Racinet proves the following fundamental result (see [18, Corollary 2.4.4]) by Poincaré-Birkhoff-Witt Theorem.

Theorem 3.9. *Any diagonal series Φ_{cv} of $(\mathbb{C}\langle\langle Y \rangle\rangle_{cv}, \Delta_*)$ is the image of a diagonal element of $(\mathbb{C}\langle\langle Y \rangle\rangle, \Delta)$. Two such elements Φ_1 and Φ_2 are related by*

$$\Phi_2 = \exp((\lambda_2 - \lambda_1)y_{1,1})\Phi_1,$$

where λ_1 and λ_2 are the coefficients of $y_{1,1}$ in Φ_1 and Φ_2 respectively.

With the above theorem available, he defines

Definition 3.10. Let \mathcal{L} (resp. \mathcal{I}) be the unique diagonal element in $(\mathbb{C}\langle\langle Y \rangle\rangle, \Delta_*)$ (resp. $(\mathbb{C}\langle\langle X \rangle\rangle, \Delta)$) such that $\pi_{cv}(\mathcal{L}) = \mathcal{L}_{cv}$ (resp. $\pi_{cv}(\mathcal{I}) = \mathcal{I}_{cv}$) and in which the coefficients of $y_{1,1}$ is 0 (resp. in which the coefficients of x_0 and x_1 are 0).

Remark 3.11. The coefficients of \mathcal{L} (resp. \mathcal{I}) may be viewed as regularized values of the series $Li_{s_1, \dots, s_r}(\sigma_1, \dots, \sigma_r)$ (resp. the iterated integral $I_{[0,1]}(\sigma_1, \dots, \sigma_r)$). Note that Racinet's choice is different from that given by Ihara, Kaneko, and Zagier in which the coefficient of $y_{1,1}$ is T instead of 0.

Theorem 3.12. *The relation between \mathcal{L} and \mathcal{I} is given by*

$$\mathcal{L} = \mathbf{S}(y_{1,1}) \cdot \mathbf{q}\pi_Y(\mathcal{I}),$$

$$\text{where } \mathbf{S}(u) = \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1} \zeta(n)}{n} u^n\right).$$

Let's recall the linear map defined in [14]:

$$\begin{aligned} \rho : \mathbb{R}[T] &\rightarrow \mathbb{R}[T], \\ \rho(e^{Tu}) &\mapsto A(u)e^{Tu}, \end{aligned}$$

where $A(u) = \mathbf{S}(u)^{-1}$. Let $\mathcal{L}' := \exp(Ty_{1,1}) \cdot \mathcal{L}$ and $\mathcal{I}' := \exp(Tx_1) \cdot \mathcal{I}$. By Theorem 3.9, the coefficients of $y_{1,1}$ and x_1 in \mathcal{L}' and \mathcal{I}' are both equal to T . Note that $y_{1,1} = x_1$. We denote by $Li_{s_1, \dots, s_r}^*(\sigma_1, \dots, \sigma_r)$ (resp. $I_{[0,1]}^{\text{III}}(\sigma_1, \dots, \sigma_r)$) the coefficient of $y_{s_1, \sigma_1} \cdots y_{s_r, \sigma_r}$ (resp. $x_{\sigma_1} \cdots x_{\sigma_r}$) in \mathcal{L}' (resp. \mathcal{I}'). Finally we show that the above relation in the sense of Racinet is equivalent to that given in [14].

Theorem 3.13. *For any indices $\mathbf{s} = (s_1, \dots, s_r)$ and $\sigma = (\sigma_1, \dots, \sigma_r)$, we have*

$$\rho(Li_{s_1, \dots, s_r}^*(\sigma_1, \sigma_2 \sigma_1^{-1}, \dots, \sigma_r \sigma_{r-1}^{-1})) = I_{[0,1]}^{\text{III}}(\{0\}^{s_1-1}, \sigma_1, \dots, \{0\}^{s_r-1}, \sigma_r).$$

Proof. Dividing the both sides of the equation in Theorem 3.12 by $\mathbf{S}(y_{1,1})$, we obtain

$$\begin{aligned} A(y_{1,1}) \cdot \mathcal{L}' &= \exp(Ty_{1,1}) \cdot A(y_{1,1}) \cdot \mathcal{L} = \exp(Ty_{1,1}) \cdot \mathbf{q}\pi_Y(\mathcal{I}) \\ &= \mathbf{q}\pi_Y(\exp(Tx_1)\mathcal{I}) = \mathbf{q}\pi_Y(\mathcal{I}'). \end{aligned}$$

Hence

$$\rho(\mathcal{L}') = \rho(\exp(Ty_{1,1}) \cdot \mathcal{L}) = A(y_{1,1}) \cdot \exp(Ty_{1,1}) \cdot \mathcal{L} = \mathbf{q}\pi_Y(\mathcal{I}').$$

The statement follows by comparing the coefficient of $y_{s_1, \sigma_1} y_{s_1, \sigma_2 \sigma_1^{-1}} \cdots y_{s_r, \sigma_r \sigma_{r-1}^{-1}}$ on both sides of the equation. \square

4. SOME GENERATING FUNCTIONS

For all $\alpha \in \mathbb{Z}/N\mathbb{Z}$ put

$$\frac{1}{\eta^\alpha e^x - 1} = \frac{\delta_{\alpha,0}}{x} + \sum_{n=0}^{\infty} \frac{\omega_{n;\alpha}^N}{n!} x^n.$$

In this section, we will provide explicit expressions of ω 's at lower levels. In the next few lemmas we first present some facts that will be useful in our future computations.

Lemma 4.1. *For all $n \geq 0$ and for all N we have*

$$\omega_{n;0}^N = \frac{B_{n+1}}{n+1}.$$

For all $n \in \mathbb{N}$ and $\alpha \in \mathbb{Z}/N\mathbb{Z}$ we have

$$\omega_{n;\alpha}^N = -(-1)^n \omega_{n;-\alpha}^N. \quad (10)$$

Proof. The first equation is straightforward from the definition. For the second, we have

$$\sum_{n=0}^{\infty} \frac{\omega_{n;\alpha}^N + (-1)^n \omega_{n;-\alpha}^N}{n!} x^n = \frac{1}{\eta^\alpha e^x - 1} + \frac{1}{\eta^{-\alpha} e^{-x} - 1} = \frac{1 - \eta^\alpha e^x}{\eta^\alpha e^x - 1} = -1.$$

The lemma follows immediately. \square

Lemma 4.2. *For all $n \in \mathbb{N}$ and $\alpha \in \mathbb{Z}/N\mathbb{Z}$ $\omega_{2n-1;\alpha}^N$ is real and $\omega_{2n;\alpha}^N$ is pure imaginary. They both lie in the cyclotomic field \mathbb{Q}_N . Further*

$$\sum_{\alpha=1}^{N-1} \omega_{n;\alpha}^N = \frac{(N^{n+1} - 1)B_{n+1}}{n+1}. \quad (11)$$

Proof. Set

$$f_\alpha(x) = \frac{1}{\eta^\alpha e^x - 1}.$$

Then

$$f_\alpha(x) + \overline{f_\alpha(-x)} = f_\alpha(x) + f_{N-\alpha}(-x) = \frac{1}{\eta^\alpha e^x - 1} - \frac{\eta^\alpha e^x}{\eta^\alpha e^x - 1} = -1.$$

where $\overline{f_1}$ is the complex conjugation of f_1 . Thus

$$\omega_{2n-1;\alpha}^N - \overline{\omega_{2n-1;\alpha}^N} = \omega_{2n;\alpha}^N + \overline{\omega_{2n;\alpha}^N} = 0 \quad \forall n \geq 1.$$

This proves the first sentence of the lemma. Observe that $F(t) := t^N - 1 = \prod_{j=0}^{N-1} (t - \eta^j)$.

By the Leibniz rule

$$\sum_{j=0}^{N-1} \frac{t}{t - \eta^j} = \frac{tF'(t)}{F(t)} = \frac{Nt^N}{t^N - 1}.$$

Using the substitution $t = e^{-x}$ we get

$$\sum_{j=0}^{N-1} \frac{1}{\eta^j e^x - 1} = \frac{N}{e^{Nx} - 1}. \quad (12)$$

Now (11) quickly follows from this. \square

Corollary 4.3. *Let $N = 2$. Then for all $n \geq 0$ we have*

$$\omega_{n;1}^2 = \frac{(2^{n+1} - 1)B_{n+1}}{n + 1}. \quad (13)$$

Proof. This directly follows from (11) when $N = 2$. \square

Lemma 4.4. *Let $N = 3$ and $\eta = e^{2\pi\sqrt{-1}/3}$. Then $\omega_{0;1}^3 = (\eta^2 - 1)/3$ and for all $n \geq 1$*

$$\omega_{2n-1;1}^3 = \omega_{2n-1;2}^3 = \frac{(3^{2n} - 1)B_{2n}}{4n}, \quad (14)$$

$$\omega_{2n;1}^3 = -\omega_{2n;2}^3 = -\frac{\sqrt{-3}}{6(2n+1)} \sum_{j=0}^{2n} 3^{2n-j} (2^{j+1} - 1) \binom{2n+1}{2n-j} B_{2n-j}. \quad (15)$$

Proof. Clearly

$$\omega_{0;1}^3 = \frac{1}{\eta - 1} = \frac{\eta^2 - 1}{3}.$$

By Lemma 4.2 we see that $\omega_{2n-1;1}^3$ are real and $\omega_{2n;1}^3$ are pure imaginary for all $n \geq 1$. Further, by (11)

$$\omega_{n;1}^N + \overline{\omega_{n;1}^N} = \omega_{n;1}^N + \omega_{n;2}^N = \frac{(3^{n+1} - 1)B_{n+1}}{n + 1}.$$

Hence we have

$$2\omega_{2n-1;1}^3 = \frac{(3^{2n} - 1)B_{2n}}{2n}$$

which quickly yields (14). On the other hand

$$\begin{aligned} f_1(x) - f_2(x) &= -\frac{\sqrt{-3}e^x}{e^{2x} + e^x + 1} = -\frac{\sqrt{-3}(e^{2x} - e^x)}{e^{3x} - 1} \\ &= -\frac{\sqrt{-3}}{3} \sum_{n=0}^{\infty} \frac{2^{n+1} - 1}{(n+1)!} x^n \cdot \sum_{m=0}^{\infty} \frac{3^m B_m}{m!} x^m. \end{aligned}$$

Hence (15) follows immediately. \square

Lemma 4.5. *Let $N = 4$ and $n \geq 0$. Then $\omega_{n;2}^4 = \omega_{n;1}^2$ is given by (13). Further we have $\omega_{n;3}^4 = \overline{\omega_{n;1}^4}$ where*

$$\omega_{n;1}^4 = \begin{cases} \frac{-1 - \sqrt{-1}}{2}, & \text{if } n = 0; \\ \frac{2^n(2^{n+1} - 1)B_{n+1}}{n+1}, & \text{if } n \text{ is odd}; \\ -\frac{\sqrt{-1}}{2}E_n, & \text{if } n \geq 2 \text{ is even,} \end{cases} \quad (16)$$

where E_n are Euler numbers defined by the generating function

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n.$$

Proof. The relation $\omega_{n;2}^4 = \omega_{n;1}^2$ is trivial. Let $\eta = \sqrt{-1}$. Clearly

$$\omega_{0;1}^4 = \frac{1}{\eta - 1} = \frac{-1 - \sqrt{-1}}{2}.$$

By Lemma 4.2 we see that $\omega_{2n-1;1}^4$ are real and $\omega_{2n;1}^4$ are pure imaginary for all $n \geq 1$. Further

$$\begin{aligned} f_1(x) + \overline{f_1}(x) &= f_1(x) + f_3(x) = \frac{-2}{e^{2x} + 1} \\ &= \frac{4}{e^{4x} - 1} - \frac{2}{e^{2x} - 1} = \sum_{n=0}^{\infty} \frac{2^{n+1}(2^{n+1} - 1)B_{n+1}}{(n+1)!} x^n. \end{aligned}$$

This proves the odd n case in (16). On the other hand

$$f_1(x) - \overline{f_1}(x) = f_1(x) - f_3(x) = \frac{-2\eta e^x}{e^{2x} + 1} = \frac{-2\eta}{e^x + e^{-x}} = -\eta \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n.$$

This proves the even n case in (16). □

5. MULTIPLE DIVISOR FUNCTIONS

Recall that $\eta = \exp(2\pi\sqrt{-1}/N)$ is the N -th root of unity. Recall $\mathbb{Q}_N = \mathbb{Q}(\eta_N)$ is the N -th cyclotomic field. As a generalization of the classical divisor sums we define for integers $s_1, \dots, s_d \geq 0$ and $\alpha_1, \dots, \alpha_d \in \mathbb{Z}/N\mathbb{Z}$ the *multiple divisor sum at level N* by

$$\sigma_{s_1, \dots, s_d}^{\alpha_1, \dots, \alpha_d}(n) = \sum_{\substack{u_1 v_1 + \dots + u_d v_d = n \\ u_1 > \dots > u_d > 0}} \eta^{\alpha_1 v_1 + \dots + \alpha_d v_d} v_1^{s_1} \dots v_d^{s_d}. \quad (17)$$

In general, these sums are complex numbers lying in \mathbb{Q}_N .

Suppose $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}/N\mathbb{Z})^d$. We set $\mathbf{s} - \mathbf{1} = (s_1 - 1, \dots, s_d - 1)$. Then the *multiple divisor function* at level N is the generating q -series of the multiple divisor sum $\sigma_{\mathbf{s}}^{\boldsymbol{\alpha}}$ given by

$$[\mathbf{s}; \boldsymbol{\alpha}]_N(q) := \frac{1}{(s_1 - 1)! \dots (s_d - 1)!} \sum_{n>0} \sigma_{\mathbf{s}-\mathbf{1}}^{\boldsymbol{\alpha}}(n) q^n \in \mathbb{Q}_N[[q]].$$

Here and in what follows we will simply write $[\mathbf{s}; \boldsymbol{\alpha}]_N$ or even $[\mathbf{s}; \boldsymbol{\alpha}]$ instead of $[\mathbf{s}; \boldsymbol{\alpha}]_N(q)$. Similar to MZVs we call $|\mathbf{s}| := s_1 + \dots + s_d$ the *weight* and $\text{dp}(\mathbf{s}) := d$ the *depth*. At level $N = 2$ we use the special notation by putting the letter o (for “odd”) on top of s_j if and only if the corresponding color $\alpha_j = 1$ is odd.

Example 5.1. We give a few examples at level $N = 2$:

$$\begin{aligned} [2] &= [2; 1]_2 = -q + q^2 - 4q^3 + 5q^4 - 6q^5 + 4q^6 - 8q^7 + 13q^8 + \dots, \\ [2, 1] &= [2, 1; 0, 1]_2 = -q^3 - 4q^5 + q^6 - 9q^7 + 4q^8 - 17q^9 + 8q^{10} - 25q^{11}, \end{aligned}$$

Recall that in [1] a *normalized polylogarithm* is defined by

$$\tilde{\text{Li}}_s(z) := \frac{\text{Li}_{1-s}(z)}{\Gamma(s)},$$

where for $s \in \mathbb{N}$, $z \in \mathbb{C}$, $|z| < 1$ the polylogarithm $\text{Li}_s(z) = \sum_{n \geq 1} \frac{z^n}{n^s}$ of weight s . By [9] we see that $\tilde{\text{Li}}_s(z)$ is a rational function in z and is holomorphic in $|z| < 1$.

Proposition 5.2. *For $q \in \mathbb{C}$ with $|q| < 1$ and for all $s_1, \dots, s_d \in \mathbb{N}$ we can write the MDFs as*

$$[s_1, \dots, s_d; \alpha_1, \dots, \alpha_d] = \sum_{n_1 > \dots > n_d > 0} \tilde{\text{Li}}_{s_1}(\eta^{\alpha_1} q^{n_1}) \dots \tilde{\text{Li}}_{s_d}(\eta^{\alpha_d} q^{n_d}).$$

Proof. We leave the straightforward computation to the interested reader. \square

Lemma 5.3. *For $\alpha, \beta \in \mathbb{Z}/N\mathbb{Z}$,*

$$\tilde{\text{Li}}_a(\eta^\alpha z) \cdot \tilde{\text{Li}}_b(\eta^\beta z) = \sum_{j=1}^a \lambda_{a,b;\alpha-\beta}^{j;N} \tilde{\text{Li}}_j(\eta^\alpha z) + \sum_{j=1}^b \lambda_{b,a;\beta-\alpha}^{j;N} \tilde{\text{Li}}_j(\eta^\beta z) + \delta_{\alpha,\beta} \tilde{\text{Li}}_{a+b}(\eta^\alpha z)$$

where the coefficients $\lambda_{a,b;\alpha}^{j;N} \in \mathbb{Q}(\eta)$ are given by

$$\lambda_{a,b;\alpha}^{j;N} = (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{\omega_{a+b-j-1;\alpha}^N}{(a+b-j-1)!}. \quad (18)$$

It satisfies that

$$\lambda_{a,b;\alpha}^{j;N} = (-1)^{a+b} \lambda_{b,a;\alpha}^{j;N} = (-1)^{a+b+j} \lambda_{a,b;-\alpha}^{j;N} = (-1)^j \lambda_{b,a;-\alpha}^{j;N}. \quad (19)$$

Proof. First we observe that (19) follows from (10).

Now we consider the generating series

$$L_\alpha(x) = \sum_{k_1=1}^{\infty} \tilde{\text{Li}}_{k_1}(\eta^\alpha z) x^{k_1-1} \quad \text{and} \quad L_\beta(y) = \sum_{k_2=1}^{\infty} \tilde{\text{Li}}_{k_2}(\eta^\beta z) y^{k_2-1}. \quad (20)$$

A simple computation shows that

$$L_\alpha(x) \cdot L_\beta(y) = \frac{1}{\eta^{\alpha-\beta} e^{x-y} - 1} L_\alpha(x) + \frac{1}{\eta^{\beta-\alpha} e^{y-x} - 1} L_\beta(y).$$

We now consider two cases: (1) $\alpha = \beta$ and (2) $\alpha \neq \beta$.

(1). If $\alpha = \beta$, then the proof is exactly the same as that of [1, Lemma 2.6].

(2). If $\alpha \neq \beta$, then we have

$$\begin{aligned} L_\alpha(x) \cdot L_\beta(y) &= \frac{e^x \eta^\alpha z}{1 - e^x \eta^\beta z} \cdot \frac{e^y \eta^\alpha z}{1 - e^y \eta^\beta z} \\ &= \frac{1}{\eta^{\alpha-\beta} e^{x-y} - 1} L_\alpha(x) + \frac{1}{\eta^{\beta-\alpha} e^{y-x} - 1} L_\beta(y) \\ &= \sum_{n>0} \frac{\omega_{n;\alpha-\beta}^N}{n!} (x-y)^n L_\alpha(x) + \sum_{n>0} \frac{\omega_{n;\beta-\alpha}^N}{n!} (y-x)^n L_\beta(y). \end{aligned}$$

By definition (20) this equals

$$\begin{aligned} &\sum_{n>0} \frac{\omega_{n;\alpha-\beta}^N}{n!} (x-y)^n \sum_{j \geq 1} \tilde{\text{Li}}_j(\eta^\alpha z) x^{j-1} + \sum_{n>0} \frac{\omega_{n;\beta-\alpha}^N}{n!} (y-x)^n \sum_{j \geq 1} \tilde{\text{Li}}_j(\eta^\beta z) y^{j-1} \\ &= \sum_{b \geq 1} \sum_{n=1}^{b-1} \sum_{j \geq 1} \frac{\omega_{n;\alpha-\beta}^N}{n!} \binom{n}{b-1} (-1)^{b-1} x^{n-b+j} y^{b-1} \tilde{\text{Li}}_j(\eta^\alpha z) \\ &+ \sum_{a \geq 1} \sum_{n=1}^{a-1} \sum_{j \geq 1} \frac{\omega_{n;\beta-\alpha}^N}{n!} \binom{n}{a-1} (-1)^{a-1} x^{a-1} y^{n-a+j} \tilde{\text{Li}}_j(\eta^\beta z) \\ &= \sum_{a \geq 1} \sum_{b \geq 1} \sum_{j=1}^a \frac{\omega_{a+b-j-1;\alpha-\beta}^N}{(a+b-j-1)!} \binom{a+b-j-1}{b-1} (-1)^{b-1} \tilde{\text{Li}}_j(\eta^\alpha z) x^{a-1} y^{b-1} \\ &+ \sum_{a \geq 1} \sum_{b \geq 1} \sum_{j=1}^b \frac{\omega_{a+b-j-1;\beta-\alpha}^N}{(a+b-j-1)!} \binom{a+b-j-1}{a-1} (-1)^{a-1} \tilde{\text{Li}}_j(\eta^\beta z) x^{a-1} y^{b-1}. \end{aligned}$$

The lemma now follows from a comparison of coefficients. \square

Corollary 5.4. *Let $N = 2$. For $a, b \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$ we have*

$$\tilde{\text{Li}}_a(\eta^\alpha z) \cdot \tilde{\text{Li}}_b(\eta^\beta z) = \sum_{j=1}^a \lambda_{a,b;\alpha-\beta}^{j;2} \tilde{\text{Li}}_j(\eta^\alpha z) + \sum_{j=1}^b \lambda_{b,a;\beta-\alpha}^{j;2} \tilde{\text{Li}}_j(\eta^\beta z) + \delta_{\alpha,\beta} \tilde{\text{Li}}_{a+b}(\eta^\alpha z)$$

where the coefficients $\lambda_{a,b;\alpha}^{j;2} \in \mathbb{Q}(\eta)$ are given by

$$\begin{aligned} \lambda_{a,b;0}^{j;2} &= (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!}, \\ \lambda_{a,b;1}^{j;2} &= (2^{a+b-j} - 1) \lambda_{a,b;0}^{j;2}. \end{aligned}$$

Proof. This follows from Corollary 4.3 easily. □

Example 5.5. Let $N = 2$. We have $\lambda_{1,1;0}^{1;2} = \lambda_{1,1;1}^{1;2} = B_1 = -\frac{1}{2}$ and thus

$$\tilde{\text{Li}}_1(z) \cdot \tilde{\text{Li}}_1(z) = \tilde{\text{Li}}_2(z) - \tilde{\text{Li}}_1(z), \quad \tilde{\text{Li}}_1(z) \cdot \tilde{\text{Li}}_1(-z) = -\frac{1}{2} \tilde{\text{Li}}_1(z) - \frac{1}{2} \tilde{\text{Li}}_2(-z).$$

Therefore the product $[1; 0] \cdot [1; 1]$ is given by (22).

Proposition 5.6. *For $\alpha, \beta \in \mathbb{Z}/N\mathbb{Z}$, we have*

$$\begin{aligned} [s; \alpha] \cdot [t; \beta] &= [s, t; \alpha, \beta] + [t, s; \beta, \alpha] + \delta_{\alpha,\beta} [s+t; \alpha] \\ &\quad + \sum_{j=1}^s \lambda_{s,t;\alpha-\beta}^{j;N} [j; \alpha] + \sum_{j=1}^t \lambda_{t,s;\beta-\alpha}^{j;N} [j; \beta]. \end{aligned}$$

Proof. By Proposition 5.2 we have

$$\begin{aligned} [s; \alpha] \cdot [t; \beta] &= \left(\sum_{n_1 > n_2 > 0} + \sum_{n_2 > n_1 > 0} + \sum_{n_1 = n_2} \right) \tilde{\text{Li}}_s(\eta^{\alpha n_1} q^{n_1}) \tilde{\text{Li}}_t(\eta^{\beta n_2} q^{n_2}) \\ &= [s, t; \alpha, \beta] + [t, s; \beta, \alpha] + \sum_{n=1}^{\infty} \tilde{\text{Li}}_s(\eta^{\alpha n} q^n) \tilde{\text{Li}}_t(\eta^{\beta n} q^n). \end{aligned}$$

The proposition now follows from Lemma 5.3. □

Example 5.7. The first nontrivial product at any level N is given by

$$[1; 1] \cdot [1; 1] = 2[1, 1; 1, 1] + [2; 1] - [1; 1] \tag{21}$$

since $\lambda_{1,1;0}^{1;N} = B_1 = -\frac{1}{2}$ for all N .

Example 5.8. The first products of MDFs at level $N = 2$ are given by

$$[1] \cdot [\overset{\circ}{1}] = [1, \overset{\circ}{1}] + [\overset{\circ}{1}, 1] - \frac{1}{2}[1] - \frac{1}{2}[\overset{\circ}{1}], \quad (22)$$

$$[\overset{\circ}{2}] \cdot [\overset{\circ}{1}] = [2, \overset{\circ}{1}] + [1, \overset{\circ}{2}] - \frac{1}{2}[\overset{\circ}{2}] + [\overset{\circ}{3}], \quad (23)$$

$$[2] \cdot [\overset{\circ}{1}] = [2, \overset{\circ}{1}] + [\overset{\circ}{1}, 2] + \frac{1}{4}([1] - 2[2] - [\overset{\circ}{1}]), \quad (24)$$

$$[2] \cdot [\overset{\circ}{2}] = [2, \overset{\circ}{2}] + [\overset{\circ}{2}, 2] - \frac{1}{4}([2] + [\overset{\circ}{2}]). \quad (25)$$

Here we have used the letter o (for “odd”) on top of s_j if the corresponding color $\alpha_j = 1$ is odd.

Example 5.9. At level $N = 3$ we have

$$\begin{aligned} [1; 1] \cdot [1; 2] &= [1, 1; 1, 2] + [1, 1; 2, 1] + \frac{\sqrt{-3}}{6}([1; 2] - [1; 1]), \\ [1; 1] \cdot [2; 0] &= [2, 1; 0, 1] + [1, 2; 1, 0] + \frac{1}{2}([1; 1] - [1; 0]) - \frac{\sqrt{-3}}{6}[2; 0], \\ [2; 1] \cdot [1; 2] &= [2, 1; 1, 2] + [1, 2; 2, 1] + \frac{1}{2}([1; 2] - [1; 1]) - \frac{\sqrt{-3}}{6}[2; 1], \\ [2; 1] \cdot [2; 2] &= [2, 2; 1, 2] + [2, 2; 2, 1] + \frac{1}{2}([2; 1] + [2; 2]) + \frac{\sqrt{-3}}{9}([1; 2] - [1; 1]), \\ [3; 1] \cdot [2; 2] &= [3, 2; 1, 2] + [2, 3; 2, 1] + \frac{1}{2}[3; 1] - \frac{\sqrt{-3}}{9}[2; 1] - \frac{\sqrt{-3}}{18}[2; 2]. \end{aligned} \quad (26)$$

Example 5.10. At level $N = 4$ we have

$$\begin{aligned} [1; 1] \cdot [1; 2] &= [1, 1; 1, 2] + [1, 1; 2, 1] + \frac{\sqrt{-1}}{2}([1; 1] - [1; 2]), \\ [1; 1] \cdot [1; 3] &= [1, 1; 1, 3] + [1, 1; 3, 1] - \frac{1}{2}([1; 1] + [1; 3]), \\ [1; 1] \cdot [2; 0] &= [2, 1; 0, 1] + [1, 2; 1, 0] + \frac{1}{2}([1; 0] - [1; 1] + \sqrt{-1}[2; 0]), \\ [1; 1] \cdot [2; 2] &= [1, 2; 1, 2] + [2, 1; 2, 1] + \frac{1}{2}([1; 2] - [1; 1] - \sqrt{-1}[2; 2]), \\ [1; 1] \cdot [2; 3] &= [1, 2; 1, 3] + [2, 1; 3, 1] + \frac{1}{4}([1; 3] - [1; 1] - 2[2; 3]), \\ [3; 1] \cdot [2; 2] &= [3, 2; 1, 2] + [2, 3; 2, 1] + \frac{1}{2}([1; 1] - [3; 1] - [1, 2] + \sqrt{-1}[2; 1]) + \frac{\sqrt{-1}}{4}[2, 2]. \end{aligned} \quad (27)$$

6. THE ALGEBRA OF MDFs

In this section we will extend Proposition 5.6 to the general case by studying the underlying algebra structure of the MDFs.

Definition 6.1. We define \mathcal{MD}_N to be the \mathbb{Q}_N -vector space generated by $[\emptyset] = 1 \in \mathbb{Q}_N[[q]]$ and all MDFs $[\mathbf{s}; \boldsymbol{\alpha}]$ for $\mathbf{s} \in \mathbb{N}^d$ and $\boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d$. On \mathcal{MD}_N we have the increasing filtration $\text{Fil}_\bullet^{\mathbf{W};N}$ given by the weight and the increasing filtration $\text{Fil}_\bullet^{\mathbf{L};N}$ given by the depth, i.e.,

$$\begin{aligned} \text{Fil}_k^{\mathbf{W};N}(\mathcal{MD}_N) &:= \langle [\mathbf{s}; \boldsymbol{\alpha}] \mid \mathbf{s} \in \mathbb{N}^d, \boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d, d \in \mathbb{N}, |\mathbf{s}| \leq k \rangle_{\mathbb{Q}_N}, \\ \text{Fil}_d^{\mathbf{L};N}(\mathcal{MD}_N) &:= \langle [\mathbf{s}; \boldsymbol{\alpha}] \mid \mathbf{s} \in \mathbb{N}^r, \boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^r, r \leq d \rangle_{\mathbb{Q}_N}. \end{aligned}$$

When considering the depth and weight filtrations at the same time we write $\text{Fil}_{k,d}^{\mathbf{W},\mathbf{L};N} := \text{Fil}_k^{\mathbf{W};N} \text{Fil}_d^{\mathbf{L};N}$. As usual we define the graded pieces by

$$\begin{aligned} \text{gr}_k^{\mathbf{W};N}(\mathcal{MD}_N) &:= \text{Fil}_k^{\mathbf{W};N}(\mathcal{MD}_N) / \text{Fil}_{k-1}^{\mathbf{W};N}(\mathcal{MD}_N) \\ \text{gr}_d^{\mathbf{L};N}(\mathcal{MD}_N) &:= \text{Fil}_d^{\mathbf{L};N}(\mathcal{MD}_N) / \text{Fil}_{d-1}^{\mathbf{L};N}(\mathcal{MD}_N). \end{aligned}$$

and $\text{gr}_{k,d}^{\mathbf{W},\mathbf{L};N} := \text{gr}_k^{\mathbf{W};N} \text{gr}_d^{\mathbf{L};N}$.

Definition 6.2. Let $G_N = \text{Gal}(\mathbb{Q}_N/\mathbb{Q})$. Let $q\mathcal{MZ}_N$ be the \mathbb{Q}_N -vector space generated by

$$S_N := \left\{ [\mathbf{s}; \boldsymbol{\alpha}] : s_1 > 1 \right\} \cup \left\{ g_\beta(\mathbf{s}; \boldsymbol{\alpha}) : 1 \leq \beta < N, \mathbf{s} \in \mathbb{N}^d, \boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d, d \geq 0 \right\}, \quad (28)$$

where $g_\beta(\mathbf{s}; \boldsymbol{\alpha}) := [1, \mathbf{s}; 0, \boldsymbol{\alpha}] - [1, \mathbf{s}; \beta, \boldsymbol{\alpha}]$ is defined precisely as follows. If $\beta < \alpha_1$ or $\mathbf{s} = \emptyset$ then

$$\begin{aligned} g_\beta(\mathbf{s}; \boldsymbol{\alpha}) &:= \sum_{n > n_1 > \dots > n_d > 0} \left(\tilde{\text{Li}}_1(q^n) - \tilde{\text{Li}}_1(\eta^\beta q^{n+\beta}) \right) \tilde{\text{Li}}_{s_1}(\eta^{\alpha_1} q^{n_1}) \dots \tilde{\text{Li}}_{s_d}(\eta^{\alpha_d} q^{n_d}) \\ &\quad - \sum_{\ell=1}^{\beta} \sum_{n_1 > \dots > n_d > 0} \tilde{\text{Li}}_1(\eta^\beta q^{n_1+\ell}) \tilde{\text{Li}}_{s_1}(\eta^{\alpha_1} q^{n_1}) \dots \tilde{\text{Li}}_{s_d}(\eta^{\alpha_d} q^{n_d}), \quad (29) \end{aligned}$$

where $n_1 = 0$ if $\mathbf{s} = \emptyset$. If $d \geq 1$ and $\beta \geq \alpha_1$ then

$$\begin{aligned} g_\beta(\mathbf{s}; \boldsymbol{\alpha}) &:= \sum_{n > n_1 > \dots > n_d > 0} \left(\tilde{\text{Li}}_1(q^n) - \tilde{\text{Li}}_1(\eta^\beta q^{n+\beta-N}) \right) \tilde{\text{Li}}_{s_1}(\eta^{\alpha_1} q^{n_1}) \dots \tilde{\text{Li}}_{s_d}(\eta^{\alpha_d} q^{n_d}) \\ &\quad + \sum_{\ell=\beta+1}^N \sum_{n_1 > \dots > n_d > 0} \tilde{\text{Li}}_1(\eta^\beta q^{n_1-N+\ell}) \tilde{\text{Li}}_{s_1}(\eta^{\alpha_1} q^{n_1}) \dots \tilde{\text{Li}}_{s_d}(\eta^{\alpha_d} q^{n_d}). \quad (30) \end{aligned}$$

Let $q\mathcal{MZ}_N(\mathbb{Q})$ be the \mathbb{Q} -vector space generated by S_N given by (28). Define

$$\begin{aligned}\overline{q\mathcal{MZ}_N} &= \bigoplus_k \text{gr}_k^{W;N} q\mathcal{MZ}_N, \\ \overline{q\mathcal{MZ}_N}(\mathbb{Q}) &= \bigoplus_k (\text{gr}_k^{W;N} q\mathcal{MZ}_N)^{G_N} = \bigoplus_k \text{gr}_k^{W;N} q\mathcal{MZ}_N(\mathbb{Q}).\end{aligned}$$

Further we define \mathcal{MZ}_N to be the \mathbb{Q}_N -vector space generated by

$$\left\{ \zeta_N(\mathbf{s}; \boldsymbol{\alpha}) : s_1 > 1 \right\} \cup \left\{ \Gamma_\beta(\mathbf{s}; \boldsymbol{\alpha}) : 1 \leq \beta < N, \mathbf{s} \in \mathbb{N}^d, \boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d, d \geq 0 \right\},$$

where $\Gamma_\beta(\mathbf{s}; \boldsymbol{\alpha}) = \zeta_N^*(1, \mathbf{s}; 0, \boldsymbol{\alpha}) - \zeta_N^*(1, \mathbf{s}; j, \boldsymbol{\alpha})$ are define by (5) in section 3. Similarly, $\mathcal{MZ}_N(\mathbb{Q})$ is the corresponding \mathbb{Q} -vector space.

Proposition 5.6 is a special case of Theorem 6.5 given below. To prove the theorem in general we now consider the associated quasi-shuffle algebras (cf. [13]). Let \mathbb{F} be a field of characteristic 0 (we will take \mathbb{F} to be either \mathbb{Q} or \mathbb{Q}_N later). Let

$$A = \{z_{j;\alpha} : j \in \mathbb{N}, \alpha \in \mathbb{Z}/N\mathbb{Z}\}$$

be the alphabet, $\mathbb{F}A$ the \mathbb{F} -vector space generated by letters in A and $\mathbb{F}\langle A \rangle$ the non-commutative polynomial algebra over \mathbb{F} generated by words with letters in A . For a commutative and associative product \diamond on $\mathbb{F}A$, $a, b \in A$ and $w, v \in \mathbb{F}\langle A \rangle$ we define on $\mathbb{F}\langle A \rangle$ recursively a product by $1 * w = w * 1 = w$ and

$$aw * bv := a(w * bv) + b(aw * v) + (a \diamond b)(w * v). \quad (31)$$

Equipped with this product one has the

Proposition 6.3. *The vector space $\mathbb{F}\langle A \rangle$ with the product $*$ is a commutative \mathbb{F} -algebra.*

Proof. See [13, Theorem 2.1]. □

Motivated by the product expression of the polylogarithms in Lemma 5.3 we define the product \diamond on $\mathbb{F}A$ by

$$z_{a;\alpha} \diamond z_{b;\beta} = \sum_{j=1}^a \lambda_{a,b;\alpha-\beta}^{j;N} z_{j;\alpha} + \sum_{j=1}^b \lambda_{b,a;\beta-\alpha}^{j;N} z_{j;\alpha} + \delta_{\alpha,\beta} z_{a+b;\alpha}. \quad (32)$$

This is an commutative and associative product on $\mathbb{F}A$, because it arises from the product of the pairwise linearly independent polylogarithms $\tilde{\text{Li}}_t(\eta^\alpha z)$ in Lemma 5.3, and therefore $(\mathbb{F}\langle A \rangle, *)$ is a commutative \mathbb{F} -algebra by Proposition 6.3 above. Theorem 6.5 now follows from the next proposition.

Proposition 6.4. *Let $[\] : (\mathbb{Q}_N\langle A \rangle, *) \longrightarrow (\mathcal{MD}_N, \cdot)$ be the \mathbb{Q}_N -linear map such that $[z_{s_1; \alpha_1} \cdots z_{s_d; \alpha_d}] := [s_1, \dots, s_d; \alpha_1, \dots, \alpha_d]$. Then we have*

$$[w * v] = [w] \cdot [v] \quad \forall w, v \in \mathbb{Q}_N\langle A \rangle.$$

Therefore (\mathcal{MD}_N, \cdot) is a \mathbb{Q}_N -algebra and $[\]$ a \mathbb{Q}_N -algebra homomorphism.

Proof. This can be proved by the same argument as in the MZV case (see, for e.g., [12, Theorem 3.2]) by using induction on the depth of the words w and v together with Lemma 5.3. \square

Theorem 6.5. *The \mathbb{Q}_N -vector space \mathcal{MD}_N has the structure of a bifiltered \mathbb{Q}_N -Algebra $(\mathcal{MD}_N, \cdot, \text{Fil}_{\bullet}^{\text{W};N}, \text{Fil}_{\bullet}^{\text{L};N})$, where the multiplication is the natural multiplication of formal power series and the filtrations $\text{Fil}_{\bullet}^{\text{W};N}$ and $\text{Fil}_{\bullet}^{\text{L};N}$ are induced by the weight and depth, in particular*

$$\text{Fil}_{k_1, d_1}^{\text{W}, \text{L}; N}(\mathcal{MD}_N) \cdot \text{Fil}_{k_2, d_2}^{\text{W}, \text{L}; N}(\mathcal{MD}_N) \subset \text{Fil}_{k_1+k_2, d_1+d_2}^{\text{W}, \text{L}; N}(\mathcal{MD}_N).$$

Proof. This follows easily from Proposition 6.4, the definition in (31) and (32). \square

Example 6.6. For $a, b, c \in \mathbb{N}$ and $\alpha, \beta, \gamma \in \mathbb{Z}/N\mathbb{Z}$ we have

$$\begin{aligned} [a; \alpha] \cdot [b, c; \beta, \gamma] &= [z_{a; \alpha} * (z_{b; \beta} z_{c; \gamma})] = [z_{a; \alpha} z_{b; \beta} z_{c; \gamma} + z_{b; \beta} z_{a; \alpha} z_{c; \gamma} + z_{b; \beta} z_{c; \gamma} z_{a; \alpha}] \\ &\quad + [z_{b; \beta} (z_{a; \alpha} \diamond z_{c; \gamma}) + (z_{a; \alpha} \diamond z_{b; \beta}) z_{c; \gamma}] \\ &= [a, b, c; \alpha, \beta, \gamma] + [b, a, c; \beta, \alpha, \gamma] + [b, c, a; \beta, \gamma, \alpha] \\ &\quad + \delta_{\alpha, \gamma} [b, a + c; \beta, \alpha] + \delta_{\alpha, \beta} [a + b, c; \gamma, \alpha] \\ &\quad + \sum_{j=1}^a \lambda_{a, c; \alpha - \gamma}^j [b, j; \beta, \alpha] + \sum_{j=1}^c \lambda_{c, a; \gamma - \alpha}^j [b, j; \beta, \gamma] \\ &\quad + \sum_{j=1}^a \lambda_{a, b; \alpha - \beta}^j [j, c; \alpha, \gamma] + \sum_{j=1}^b \lambda_{b, a; \beta - \alpha}^j [j, c; \beta, \gamma]. \end{aligned}$$

Lemma 6.7. *For all $a, b, s_1, \dots, s_d \in \mathbb{N}$ and $\alpha, \beta, \alpha_1, \dots, \alpha_d \in \mathbb{Z}/N\mathbb{Z}$ we have*

$$[(z_{a; \alpha} \diamond z_{b; \beta}) z_{s_1; \alpha_1} \cdots z_{s_d; \alpha_d}] \in \text{qMZ}_N.$$

Proof. By the definition (32)

$$z_{a; \alpha} \diamond z_{b; \beta} = \sum_{j=1}^a \lambda_{a, b; \alpha - \beta}^{j; N} z_{j; \alpha} + \sum_{j=1}^b \lambda_{b, a; \beta - \alpha}^{j; N} z_{j; \beta} + \delta_{\alpha, \beta} z_{a+b; \alpha}.$$

By the definition of λ 's in Proposition 5.6 it is easy to see that

$$\begin{aligned}\lambda_{a,b;\alpha-\beta}^{1;N} z_{1;\alpha} + \lambda_{b,a;\beta-\alpha}^{1;N} z_{1;\beta} &= \frac{(-1)^{a-1} \omega_{a+b-2;\alpha-\beta}^N z_{1;\alpha} + (-1)^{b-1} \omega_{a+b-2;\beta-\alpha}^N z_{1;\beta}}{(a-1)!(b-1)!} \\ &= \frac{(-1)^{a-1} \omega_{a+b-2;\alpha-\beta}^N}{(a-1)!(b-1)!} (z_{1;\alpha} - z_{1;\beta}).\end{aligned}$$

by Lemma 4.1. Hence

$$[(z_{1;\alpha} - z_{1;\beta}) z_{s_1;\alpha_1} \cdots z_{s_d;\alpha_d}] = g_\beta(\mathbf{s}; \boldsymbol{\alpha}) - g_\alpha(\mathbf{s}; \boldsymbol{\alpha}) \in \mathfrak{qMZ}_N.$$

This finishes the proof of the lemma. \square

Theorem 6.8. *The vector space \mathfrak{qMZ}_N is a subalgebra of \mathcal{MD}_N .*

Proof. It suffices to show that \mathfrak{qMZ}_N is closed under multiplication. We need to consider three cases:

- (i) Let $f = [a, \dots; \alpha, \dots]$ and $g = [b, \dots; \beta, \dots]$, $a > 1$ and $b > 1$,
- (ii) Let $f = [a, \dots; \alpha, \dots]$ and $g = g_\beta(\mathbf{s}; \boldsymbol{\alpha})$, $a > 1$ and $1 \leq \beta < N$,
- (iii) Let $f = g_\gamma(\mathbf{s}; \boldsymbol{\alpha})$ and $g = g_\beta(\mathbf{t}; \boldsymbol{\beta})$, $1 \leq \beta, \gamma < N$.

In case (i), by Proposition 6.4 we have

$$f \cdot g = [z_{a;\alpha} w] \cdot [z_{b;\beta} v] = [z_{a;\alpha} w * z_{b;\beta} v] \quad \text{for some } w, v \in \mathbb{Q}_N \langle A \rangle.$$

So in order to prove the statement we have to show that $[z_{a;\alpha} w * z_{b;\beta} v] \in \mathfrak{qMZ}_N$. By the definition of the quasi-shuffle product $*$ we have

$$z_{a;\alpha} w * z_{b;\beta} v = z_{a;\alpha}(w * z_{b;\beta} v) + z_{b;\beta}(z_{a;\alpha} w * v) + (z_{a;\alpha} \diamond z_{b;\beta})(w * v).$$

The first two terms obviously lie in \mathfrak{qMZ}_N after applying $[\]$ since $a, b > 1$ and by Lemma 6.7 $[(z_{a;\alpha} \diamond z_{b;\beta})(w * v)] \in \mathfrak{qMZ}_N$. Thus in case (i) the multiplication is closed.

(ii) We assume for some $w, v \in \mathbb{Q}_N \langle A \rangle$

$$f \cdot g = [z_{a;\alpha} w] \cdot [(z_{1;0} - z_{1;\beta}) v] = [z_{a;\alpha} w * z_{1;0} v] - [z_{a;\alpha} w * z_{1;\beta} v].$$

Then

$$\begin{aligned}z_{a;\alpha} w * z_{1;0} v - z_{a;\alpha} w * z_{1;\beta} v &= z_{a;\alpha}(w * z_{1;0} v) - z_{a;\alpha}(w * z_{1;\beta} v) \\ &\quad + (z_{a;\alpha} \diamond z_{1;0})(w * v) - (z_{a;\alpha} \diamond z_{1;\beta})(w * v) + (z_{1;0} - z_{1;\beta})(z_{a;\alpha} w * v).\end{aligned}$$

By similar argument as in case (i) one can handle easily the first four terms on the right hand side. Now assume $[z_{a;\alpha}w * v] = \sum_{i=1}^r [\mathbf{s}_i; \mathbf{a}_i]$ we see that

$$[(z_{1;0} - z_{1;\beta})(z_{a;\alpha}w * v)] = \sum_{i=1}^r g_{\beta}(\mathbf{s}_i; \mathbf{a}_i) \in \mathfrak{qMZ}_N.$$

(iii) Assuming $f = [(z_{1;0} - z_{1;\beta})w]$ and $g = [(z_{1;0} - z_{1;\gamma})v]$ for $w, v \in \mathbb{Q}\langle A \rangle$ we have

$$f \cdot g = [z_{1;0}w * z_{1;0}v] - [z_{1;0}w * z_{1;\gamma}v] - [z_{1;\beta}w * z_{1;0}v] + [z_{1;\beta}w * z_{1;\gamma}v].$$

Then

$$\begin{aligned} & z_{1;0}w * z_{1;0}v - z_{1;0}w * z_{1;\gamma}v - z_{1;\beta}w * z_{1;0}v + z_{1;\beta}w * z_{1;\gamma}v \\ &= (z_{1;0} - z_{1;\beta})(w * z_{1;0}v) - (z_{1;0} - z_{1;\beta})(w * z_{1;\gamma}v) \\ &+ (z_{1;0} \diamond z_{1;0})(w * v) - (z_{1;0} \diamond z_{1;\gamma})(w * v) \\ &+ (z_{1;0} - z_{1;\gamma})(z_{1;0}w * v) - (z_{1;\beta} \diamond z_{1;0})(w * v) \\ &+ (z_{1;\beta} \diamond z_{1;\gamma})(w * v) - (z_{1;0} - z_{1;j})(z_{1;i}w * v). \end{aligned}$$

By the same argument as in (i), all the terms involving \diamond operation lie in \mathfrak{qMZ}_N after applying []. All the other terms have the form $g_{\beta}(\mathbf{s}; \boldsymbol{\alpha})$ or $g_{\gamma}(\mathbf{s}; \boldsymbol{\alpha})$ after applying []. This finishes the proof of the theorem. \square

Theorem 6.9. *For any positive integer N we set $t_N = \sum_{\alpha=0}^{N-1} [1; \alpha]$.*

- (i) *We have $\mathcal{MD}_N = \mathfrak{qMZ}_N[t_N]$.*
- (ii) *The algebra \mathcal{MD}_N is a polynomial ring over \mathfrak{qMZ}_N with indeterminate t_N , i.e. \mathcal{MD}_N is isomorphic to $\mathfrak{qMZ}_N[T]$ by sending t_N to T .*

Proof. (i) We will use the induction approach used by Bachmann and Kühn at level one. But for the higher levels the appearance of different colors brings some complications into our argument.

First we show that any $f \in \text{Fil}_k^{\mathbf{W};N}(\mathcal{MD}_N)$ can be written as a polynomial in t_N . If we show that for a fixed d and $f \in \text{Fil}_{k,d}^{\mathbf{W},\mathbf{L};N}(\mathcal{MD}_N)$ one can find $g_1 \in \text{Fil}_{k,d}^{\mathbf{W},\mathbf{L};N}(\mathfrak{qMZ}_N)$ and $g_2, g_3 \in \text{Fil}_{k,d-1}^{\mathbf{W},\mathbf{L};N}(\mathcal{MD}_N)$ such that f can be written as

$$f = g_1 + t_N \cdot g_2 + g_3, \tag{33}$$

then the claim follows directly by induction on d .

To show (33) it suffices to assume $f = [\{1\}^m, \mathbf{s}; \beta_1, \dots, \beta_m, \boldsymbol{\alpha}]$, with $s_1 > 1$ and $k = m + |\mathbf{s}|$. By induction on m we now prove that every element of such form can be written as in (33). Notice we are using a double induction now with the outside layer

induction on d and the inside on m . If $m = 0$ then clearly $f \in \text{Fil}_{k,d}^{\text{W,L};N}(\text{qMZ}_N)$. If $m = 1$, $s_1 > 1$ and $\beta_1 = 0$ then we have

$$\begin{aligned}
& -N[1, \mathbf{s}; 0, \boldsymbol{\alpha}] + \sum_{\gamma=1}^{N-1} g_\gamma(\mathbf{s}; \boldsymbol{\alpha}) + t_N \cdot [\mathbf{s}; \boldsymbol{\alpha}] \\
&= \sum_{\gamma=0}^{N-1} \left([1; \gamma] \cdot [\mathbf{s}; \boldsymbol{\alpha}] - [1, \mathbf{s}; \gamma, \boldsymbol{\alpha}] \right) \\
&= \sum_{\gamma=0}^{N-1} \left[z_{1;\gamma} * (z_{s_1;\alpha_1} \cdots z_{s_n;\alpha_n}) - z_{1;\gamma} z_{s_1;\alpha_1} \cdots z_{s_n;\alpha_n} \right] \quad (n = d-1) \\
&= \sum_{\gamma=0}^{N-1} \left[z_{s_1;\alpha_1} (z_{1;\gamma} * (z_{s_2;\alpha_2} \cdots z_{s_n;\alpha_n})) - (z_{1;\gamma} \diamond z_{s_1;\alpha_1}) (z_{s_2;\alpha_2} \cdots z_{s_n;\alpha_n}) \right] \in \text{qMZ}_N
\end{aligned}$$

by Lemma 6.7. Hence $[1, \mathbf{s}; 0, \boldsymbol{\alpha}] \in \text{qMZ}_N[t_N]$ and therefore for all $\beta \in \mathbb{Z}/N\mathbb{Z}$

$$[1, \mathbf{s}; \beta, \boldsymbol{\alpha}] = [1, \mathbf{s}; 0, \boldsymbol{\alpha}] - g_\beta(\mathbf{s}; \boldsymbol{\alpha}) \in \text{qMZ}_N[t_N].$$

Now we assume $m \geq 1$ and (33) holds for $f = [\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha}]$ for all $\mathbf{s} \in \mathbb{N}^{d-m}$ with $s_1 > 1$ and all $\boldsymbol{\beta}' = (\beta_1, \dots, \beta_m) \in (\mathbb{Z}/N\mathbb{Z})^m$. Then we have

$$t_N \cdot [\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha}] = \sum_{\gamma=0}^{N-1} \left[z_{1,\gamma} * (z_{1,\beta_1} \cdots z_{1,\beta_m} z_{s_1;\alpha_1} \cdots z_{s_d;\alpha_d}) \right].$$

Thus by induction

$$\sum_{\ell=0}^m \sum_{\gamma=0}^{N-1} [\{1\}^{m+1}, \mathbf{s}; \beta_1, \dots, \beta_\ell, \gamma, \beta_{\ell+1}, \dots, \beta_m, \boldsymbol{\alpha}] \in \text{qMZ}_N[t_N].$$

On the other hand

$$\sum_{\gamma=0}^{N-1} g_\gamma(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha}) = N[\{1\}^{m+1}, \mathbf{s}; 0, \boldsymbol{\beta}', \boldsymbol{\alpha}] - \sum_{\gamma=0}^{N-1} [\{1\}^{m+1}, \mathbf{s}; \gamma, \boldsymbol{\beta}', \boldsymbol{\alpha}].$$

Adding up the above two equations we get the following element in $\text{qMZ}_N[t_N]$

$$N[\{1\}^{m+1}, \mathbf{s}; 0, \boldsymbol{\beta}', \boldsymbol{\alpha}] + \sum_{\ell=1}^m \sum_{\gamma=0}^{N-1} [\{1\}^{m+1}, \mathbf{s}; \beta_1, \dots, \beta_\ell, \gamma, \beta_{\ell+1}, \dots, \beta_m, \boldsymbol{\alpha}].$$

Subtracting $Ng_{\beta_0}(\{1\}^m, \mathbf{s}; \boldsymbol{\beta}', \boldsymbol{\alpha})$ we see that for all $\boldsymbol{\beta} = (\beta_0, \dots, \beta_m) \in (\mathbb{Z}/N\mathbb{Z})^{m+1}$

$$N[\{1\}^{m+1}, \mathbf{s}; \boldsymbol{\beta}, \boldsymbol{\alpha}] + \sum_{\ell=1}^m \sum_{\gamma=0}^{N-1} [\{1\}^{m+1}, \mathbf{s}; \beta_1, \dots, \beta_\ell, \gamma, \beta_{\ell+1}, \dots, \beta_m, \boldsymbol{\alpha}] = b(\boldsymbol{\beta}),$$

for some $b(\boldsymbol{\beta}) \in \mathfrak{qMZ}_N[t_N]$. Regarding $[\{1\}^{m+1}, \mathbf{s}; \boldsymbol{\beta}, \boldsymbol{\alpha}]$ as N^{m+1} variables as $\boldsymbol{\beta}$ varies we now only need to show the corresponding $N^{m+1} \times N^{m+1}$ coefficient matrix $M(N, m)$ is nonsingular, which has been proved by (6).

(ii) We only need to show that t_N is transcendental over \mathfrak{qMZ}_N . By definition

$$\begin{aligned} t_N &= \sum_{\alpha=0}^{N-1} [1; \alpha] = \sum_{\alpha=0}^{N-1} \sum_{n=1}^{\infty} \sum_{d|n} \eta^{d\alpha} q^n \\ &= \sum_{n=1}^{\infty} \sum_{d|n} Nq^{Nn} \approx \frac{-N \log(1 - q^N)}{1 - q^N} \quad (\text{when } q \text{ is close to } \eta^{-1}) \end{aligned}$$

by [17, Lemma 2]. Now it is an easy computation to show that when q is close to η^{-1}

$$\frac{-N \log(1 - q^N)}{1 - q^N} \approx \frac{-\log(1 - \eta q)}{1 - \eta q}.$$

By Proposition 6.12 we see that if $|\mathbf{s}| = k > 1$ we have the approximations $[\mathbf{s}; \boldsymbol{\alpha}] \approx \frac{1}{(1 - \eta q)^k}$ when q close to η^{-1} . Also by Proposition 6.12, if $|\mathbf{s}| = k - 1$ then

$$\lim_{q \rightarrow 1} (1 - q)^k g_{\beta}(\mathbf{s}; \boldsymbol{\alpha}) [q\eta^{-1}] = \Gamma_{\beta}(\mathbf{s}; \boldsymbol{\alpha})$$

which is finite by the definition (5), and therefore when q is close to η^{-1} $g_{\beta}(\mathbf{s}; \boldsymbol{\alpha}) \approx \frac{1}{(1 - \eta q)^k}$. Consequently, t_N is transcendental over $\mathfrak{qMZ}_N[T]$.

We have completed the proof of the theorem. \square

The following result is a simple generalization of [1, Lemma 2.5].

Lemma 6.10. *For $s_1, \dots, s_d \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_d \in \mathbb{Z}/N\mathbb{Z}$ we have*

$$[s_1, \dots, s_d; \alpha_1, \dots, \alpha_d](q) = \frac{1}{(s_1 - 1)! \dots (s_d - 1)!} \sum_{n_1 > \dots > n_d > 0} \prod_{j=1}^d \frac{\eta^{\alpha_j} q^{n_j} P_{s_j-1}(\eta^{\alpha_j} q^{n_j})}{(1 - \eta^{\alpha_j} q^{n_j})^{s_j}},$$

where $P_k(t) = \sum_{n=0}^{k-1} A_{k,n} t^n$ is the k -th Eulerian polynomial with the coefficient $A_{k,n}$ defined by

$$A_{k,n} = \sum_{i=0}^n (-1)^i \binom{k+1}{i} (n+1-i)^k,$$

which is positive.

Proof. By Proposition 5.2 and the properties of Eulerian polynomial (see [8, 9]) we have

$$\sum_{n_1 > \dots > n_d > 0} \prod_{j=1}^d \frac{\eta^{\alpha_j} q^{n_j} P_{s_j-1}(\eta^{\alpha_j} q^{n_j})}{(1 - \eta^{\alpha_j} q^{n_j})^{s_j}} = \sum_{n_1 > \dots > n_d > 0} \prod_{j=1}^d \sum_{v_j=1}^{\infty} \eta^{v_j \alpha_j} v_j^{s_j-1} q^{v_j n_j} = \sum_{n=1}^{\infty} \sigma_{\mathbf{s}-1}^{\boldsymbol{\alpha}}(n) q^n.$$

The positivity of $A_{k,n}$ follows from its combinatorial interpretation as the number of permutations of $\{1, 2, \dots, n\}$ with k ascents (see [8] or search the sequence A008292 on oeis.org). This is also proved by Pupyrev [17, Lemma 1]. The lemma follows at once. \square

Theorem 6.11. (i) *The \mathbb{Q}_N -linear map $Z : \mathfrak{qMZ}_N \rightarrow \mathcal{MZ}_N$ defined by*

$$\begin{aligned} Z([\mathbf{s}; \boldsymbol{\alpha}]) &= \zeta_N(\mathbf{s}; \boldsymbol{\alpha}) \quad \forall s_1 > 1, \\ Z(g_\beta(\mathbf{s}; \boldsymbol{\alpha})) &= \Gamma_\beta(\mathbf{s}; \boldsymbol{\alpha}) \quad \forall \beta \in \mathbb{Z}/N\mathbb{Z} \end{aligned}$$

*is a homomorphism of weight-filtered algebras. It can be factored through the weight-graded object $(\overline{\mathfrak{qMZ}_N}, *)$:*

$$\begin{array}{ccc} \mathfrak{qMZ}_N & \longrightarrow & \overline{\mathfrak{qMZ}_N} \\ & \searrow Z & \downarrow \overline{Z} \\ & & \mathcal{MZ}_N \end{array}$$

where the multiplication $$ of $\overline{\mathfrak{qMZ}_N}$ is inherited from that of \mathfrak{qMZ}_N . Moreover, $(\overline{\mathfrak{qMZ}_N}, *)$ satisfies the stuffle relations and \overline{Z} induces a \mathbb{Q} -algebra homomorphism*

$$\overline{Z}_{\mathbb{Q}} : (\overline{\mathfrak{qMZ}_N}(\mathbb{Q}), *) \longrightarrow (\mathcal{MZ}_N(\mathbb{Q}), \cdot).$$

(ii) *Extending Z by setting*

$$Z([\mathbf{s}; \boldsymbol{\alpha}]_N t_N^r) = \zeta_N(\mathbf{s}; \boldsymbol{\alpha}) T^r$$

we then obtain a \mathbb{Q}_N -algebra homomorphisms

$$\begin{array}{ccc} \mathcal{MD}_N = \mathfrak{qMZ}_N[t_N] & \longrightarrow & \overline{\mathfrak{qMZ}_N}[t_N] \\ & \searrow Z & \downarrow \overline{Z} \\ & & \mathcal{MZ}_N[T], \end{array}$$

such that

$$Z([1, s_2, \dots, s_d; \alpha_1, \dots, \alpha_d]) = \zeta_N^*(1, s_2, \dots, s_d; \alpha_1, \dots, \alpha_d)$$

as defined in section 3. Moreover, \overline{Z} induces a \mathbb{Q} -algebra homomorphism

$$\overline{Z}_{\mathbb{Q}} : \overline{\mathfrak{qMZ}_N}(\mathbb{Q})[t_N] \longrightarrow \mathcal{MZ}_N(\mathbb{Q})[T].$$

Proof. We only need to show the property on \overline{Z} . But this follows immediately from the fact that the matrix $M(N, m)$ in the proof of Theorem 6.9 has integer entries so that its inverse has rational entries. \square

For any $\mathbf{s} \in \mathbb{N}^d$ and $\boldsymbol{\alpha} \in (\mathbb{Z}/N\mathbb{Z})^d$ we define

$$\begin{aligned} Z_k([\mathbf{s}; \boldsymbol{\alpha}](q)) &= \lim_{q \rightarrow 1^-} (1-q)^k [\mathbf{s}; \boldsymbol{\alpha}](q\eta^{-1}) \quad \forall |\mathbf{s}| \leq k, \\ Z_k(g_\beta(\mathbf{s}; \boldsymbol{\alpha})) &= \lim_{q \rightarrow 1^-} (1-q)^k g_\beta(\mathbf{s}; \boldsymbol{\alpha})(q\eta^{-1}) \quad \forall |\mathbf{s}| \leq k-1. \end{aligned}$$

The next result shows that Z_k is the restriction of Z to $\text{Fil}_k^{\text{W};N}(\text{qMZ}_N)$,

Proposition 6.12. *The map Z_k is \mathbb{Q}_N -linear on $\text{Fil}_k^{\text{W};N}(\text{qMZ})$ and*

$$Z_k([\mathbf{s}; \boldsymbol{\alpha}](q)) = \begin{cases} \zeta_N(\mathbf{s}; \boldsymbol{\alpha}), & \text{if } s_1 > 1 \text{ and } |\mathbf{s}| = k; \\ 0, & \text{if } |\mathbf{s}| < k, \end{cases}$$

and for all $1 \leq \beta < N$

$$Z_k(g_\beta(\mathbf{s}; \boldsymbol{\alpha})) = \begin{cases} \Gamma_\beta(\mathbf{s}; \boldsymbol{\alpha}), & \text{if } |\mathbf{s}| = k-1; \\ 0, & \text{if } |\mathbf{s}| < k-1. \end{cases} \quad (34)$$

Proof. First we assume $s_1 > 1$. Using Lemma 6.10 we get

$$\begin{aligned} & Z_k([s_1, \dots, s_d; \alpha_1, \dots, \alpha_d]) \\ &= \lim_{q \rightarrow 1^-} ((1-q)^k [s_1, \dots, s_d; \alpha_1, \dots, \alpha_d](q\eta^{-1})) \\ &= \lim_{q \rightarrow 1^-} \left((1-q)^k \sum_{n_1 > \dots > n_d > 0} \prod_{j=1}^d \frac{\eta^{\alpha_j - n_j} q^{n_j} P_{s_j-1}(\eta^{\alpha_j - n_j} q^{n_j})}{(s_j - 1)! (1 - \eta^{\alpha_j - n_j} q^{n_j})^{s_j}} \right) \\ &= \sum_{n_1 > \dots > n_d > 0} \prod_{j=1}^d \lim_{q \rightarrow 1^-} \left((1-q)^{s_j} \frac{\eta^{\alpha_j - n_j} q^{n_j} P_{s_j-1}(\eta^{\alpha_j - n_j} q^{n_j})}{(s_j - 1)! (1 - \eta^{\alpha_j - n_j} q^{n_j})^{s_j}} \right) \\ &= \sum_{\substack{n_1 > \dots > n_d > 0 \\ n_j \equiv \alpha_j \pmod{N} \quad \forall 1 \leq j \leq d}} \prod_{j=1}^d \lim_{q \rightarrow 1^-} \left((1-q)^{s_j} \frac{q^{n_j} P_{s_j-1}(q^{n_j})}{(s_j - 1)! (1 - q^{n_j})^{s_j}} \right) \\ &= \zeta_N(s_1, \dots, s_d; \alpha_1, \dots, \alpha_d). \end{aligned} \quad (35)$$

Here we have used the fact that the k -th Eulerian polynomial $P_k(t)$ satisfies $P_k(1) = k!$. We also need to justify the exchange of the order of taking the limit and taking the infinite sum. But by triangle inequality

$$1 \leq |1 - \eta^{\alpha-n} q^n| + |\eta^{\alpha-n} q^n| = |1 - \eta^{\alpha-n} q^n| + q^n.$$

Thus for all $q \in [1/2, 1)$ we have

$$\frac{1}{|1 - \eta^{\alpha-n} q^n|} \leq \frac{1}{1 - q^n}.$$

Moreover, since $P_{s_j-1}(x)$ has positive coefficients by Lemma 6.10 we see that

$$|P_{s_j-1}(\eta^{\alpha_j-n_j}q^n)| \leq P_{s_j-1}(q^{n_j}) \quad \text{for all } j.$$

Hence the series appearing in (35) after the limit symbol converges absolutely:

$$\begin{aligned} & (1-q)^k \sum_{n_1 > \dots > n_d > 0} \prod_{j=1}^d \left| \frac{\eta^{\alpha_j-n_j}q^{n_j}P_{s_j-1}(\eta^{\alpha_j-n_j}q^{n_j})}{(s_j-1)!(1-\eta^{\alpha_j-n_j}q^{n_j})^{s_j}} \right| \\ & \leq \sum_{n_1 > \dots > n_d > 0} \prod_{j=1}^d \frac{(1-q)^{s_j}q^{n_j}P_{s_j-1}(q^{n_j})}{(s_j-1)!(1-q^{n_j})^{s_j}} \leq \zeta(2, 1, \dots, 1) \end{aligned}$$

by [1, Lemma 6.6(ii)].

We now turn to the proof of (34). By definition (29), if $1 \leq \beta < \alpha_1$ then

$$\begin{aligned} Z_k(g_\beta(\mathbf{s}; \boldsymbol{\alpha})) &= \lim_{q \rightarrow 1^-} \sum_{n > n_1 > \dots > n_d > 0} \\ & (1-q)^k \left(\frac{\eta^{-n}q^n}{1-\eta^{-n}q^n} - \frac{\eta^{-n}q^{\beta+n}}{1-\eta^{-n}q^{\beta+n}} \right) \prod_{j=1}^d \frac{\eta^{\alpha_j-n_j}q^{n_j}P_{s_j-1}(\eta^{\alpha_j-n_j}q^{n_j})}{(s_j-1)!(1-\eta^{\alpha_j-n_j}q^{n_j})^{s_j}} \end{aligned} \quad (36)$$

$$- \lim_{q \rightarrow 1^-} (1-q)^k \sum_{\ell=1}^{\beta} \sum_{n_1 > \dots > n_d > 0} \frac{\eta^{\beta-n_1-\ell}q^{n_1+\ell}}{1-\eta^{\beta-n_1-\ell}q^{n_1+\ell}} \prod_{j=1}^d \frac{\eta^{\alpha_j-n_j}q^{n_j}P_{s_j-1}(\eta^{\alpha_j-n_j}q^{n_j})}{(s_j-1)!(1-\eta^{\alpha_j-n_j}q^{n_j})^{s_j}}. \quad (37)$$

It's easy to see that (37) vanishes unless $d > 0$ and there exist n_1 and ℓ such that

$$1 \leq \ell \leq \beta, \quad \alpha_1 \equiv n_1 \quad \text{and} \quad \beta \equiv n_1 + \ell \pmod{N}.$$

But this forces $\ell + \alpha_1 \equiv \beta \pmod{N}$ which is impossible since $\beta < \ell + \alpha_1 < \beta + N$. Hence (37)=0. On the other hand, by the proof of [1, Lemma 6.6(ii)] we have

$$h_n(q) := \frac{(1-q)^2q^n}{(1-q^n)^2} - \frac{1}{n^2} < 0 \quad \forall q \in [1/2, 1).$$

Therefore

$$(1-q) \left| \frac{\eta^{-n}q^n}{1-\eta^{-n}q^n} - \frac{\eta^{-n}q^{\beta+n}}{1-\eta^{-n}q^{\beta+n}} \right| = \left| \frac{(1-q)(1-q^\beta)q^n}{(1-\eta^{-n}q^n)(1-\eta^{\beta-n}q^n)} \right| \leq \frac{\beta(1-q)^2q^n}{(1-q^n)^2} < \frac{\beta}{n^2}.$$

Hence by exchanging the order of the limit and the summation (36) we arrive at (34).

When $\beta \geq \alpha_1$ we can use definition (30) and prove the claim of the proposition in a similar way. We only point out that

$$\left| \frac{1-q^{\beta-N}}{1-q} \right| < 2^{N-\beta}(N-\beta) \quad \forall q \in [1/2, 1)$$

and leave the rest of the details to the interested reader. \square

7. A DERIVATION AND LINEAR RELATIONS IN \mathcal{MD}_N

Our main result in this section is that the \mathbb{Q}_N -linear operator $\mathfrak{D} = q \frac{d}{dq}$ on \mathcal{MD}_N is a derivation and it increases the weight by at most 2 and the depth by at most 1 (see Theorem 7.8). Although this is completely similar to the level 1 case proved in [1], there is some technical complications caused by the appearance of the colors. Our proof is constructive, similar to the level $N = 1$ case, but we will derive the explicit formula for all depths. Since the depth one case is a little different from the general case so it will be treated separately. We will provide some concrete examples at the end of this section.

Lemma 7.1. *The generating series $T_{\alpha_1, \dots, \alpha_d}(x_1, \dots, x_d)$ of MDFs of depth d can be written as*

$$\begin{aligned} T_{\alpha_1, \dots, \alpha_d}(x_1, \dots, x_d) &= \sum_{s_1, \dots, s_d > 0} [s_1, \dots, s_d; \alpha_1, \dots, \alpha_d] x_1^{s_1-1} \dots x_d^{s_d-1} \\ &= \sum_{n_1, \dots, n_d > 0} \prod_{j=1}^d \frac{e^{n_j x_j} \eta^{\alpha_j n_j} q^{n_1 + \dots + n_j}}{1 - q^{n_1 + \dots + n_j}}. \end{aligned}$$

Proof. This can be seen by direct computation. See [1, Lemma 3.1] for details. □

A prototype of the following can be found also in [10].

Lemma 7.2. *We have*

$$\begin{aligned} T_\alpha(x) \cdot T_\beta(y) &= T_{\alpha+\beta, \alpha}(x+y, x) + T_{\alpha+\beta, \beta}(x+y, y) \\ &\quad - T_{\alpha+\beta}(x+y) + R_{\alpha+\beta}^{(1;1)}(x+y), \end{aligned}$$

where

$$R_\alpha^{(1;1)}(x) = \sum_{n>0} e^{nx} \frac{\eta^{\alpha n} q^n}{(1 - q^n)^2} = \sum_{k>0} \frac{\mathfrak{D}[k; \alpha]}{k} x^k + [2; \alpha].$$

Proof. By Lemma 7.2

$$\begin{aligned} T_\alpha(x) T_\beta(y) &= \sum_{n_1, n_2 > 0} \eta^{\alpha n_1 + \beta n_2} e^{n_1 x + n_2 y} \frac{q^{n_1}}{1 - q^{n_1}} \frac{q^{n_2}}{1 - q^{n_2}} \\ &= \sum_{n_1 > n_2 > 0} \dots + \sum_{n_2 > n_1 > 0} \dots + \sum_{n_2 = n_1 > 0} \dots =: F_1 + F_2 + F_3, \end{aligned}$$

where by using the substitution $n_1 = n_2 + n'_1$

$$\begin{aligned} F_1 &= \sum_{n_1 > n_2 > 0} \eta^{\alpha n_1 + \beta n_2} e^{n_1 x + n_2 y} \frac{q^{n_1}}{1 - q^{n_1}} \frac{q^{n_2}}{1 - q^{n_2}} \\ &= \sum_{n'_1, n_2 > 0} \eta^{\alpha n'_1 + (\alpha + \beta) n_2} e^{n'_1 x + n_2 (x + y)} \frac{q^{n_2}}{1 - q^{n_2}} \frac{q^{n'_1 + n_2}}{1 - q^{n'_1 + n_2}} = T_{\alpha + \beta, \alpha}(x + y, x). \end{aligned}$$

By similar argument $F_2 = T_{\alpha + \beta, \beta}(x + y, y)$. Now, since $\left(\frac{q^n}{1 - q^n}\right)^2 = \frac{q^n}{(1 - q^n)^2} - \frac{q^n}{1 - q^n}$

$$\begin{aligned} F_3 &= \sum_{n > 0} \eta^{\alpha n + \beta n} e^{n x + n y} \left(\frac{q^n}{1 - q^n}\right)^2 \\ &= \sum_{n > 0} \eta^{(\alpha + \beta)n} e^{n(x + y)} \frac{q^n}{(1 - q^n)^2} - \sum_{n > 0} \eta^{(\alpha + \beta)n} e^{n(x + y)} \frac{q^n}{1 - q^n} \\ &= R_{\alpha + \beta}^{(1;1)}(x + y) - T_{\alpha + \beta}(x + y). \end{aligned}$$

To express $R_{\alpha}^{(1;1)}(x)$ using the derivation \mathfrak{D} we notice that

$$\sum_{k > 0} \mathfrak{D}[k; \alpha] x^{k-1} = \mathfrak{D} T_{\alpha}(x) = \mathfrak{D} \sum_{n > 0} \eta^{\alpha n} e^{n x} \frac{q^n}{1 - q^n} = \sum_{n > 0} n \eta^{\alpha n} e^{n x} \frac{q^n}{(1 - q^n)^2}.$$

Thus

$$\begin{aligned} \sum_{k > 0} \frac{\mathfrak{D}[k; \alpha]}{k} x^k &= \int_0^x \mathfrak{D} T_{\alpha}(t) dt = \sum_{n > 0} \int_0^x n \eta^{\alpha n} e^{n t} dt \frac{q^n}{(1 - q^n)^2} \\ &= \sum_{n > 0} \eta^{\alpha n} e^{n x} \frac{q^n}{(1 - q^n)^2} - [2; \alpha]. \end{aligned}$$

This completes the proof of the lemma. \square

Proposition 7.3. *For $\alpha, \beta \in \mathbb{Z}/N\mathbb{Z}$, $s_1, s_2 \in \mathbb{N}$ with $s_1 + s_2 > 2$ and $s = s_1 + s_2 - 2$ we have*

$$\begin{aligned} \binom{s}{s_1 - 1} \frac{\mathfrak{D}[s; \alpha + \beta]}{s} &= [s_1; \alpha] \cdot [s_2; \beta] + \binom{s}{s_1 - 1} [s + 1; \alpha + \beta] \\ &\quad - \sum_{a+b=s+2} \left(\binom{a-1}{s_1 - 1} [a, b; \alpha + \beta, \beta] + \binom{a-1}{s_2 - 1} [a, b; \alpha + \beta, \alpha] \right). \end{aligned}$$

Proof. We have

$$\begin{aligned}
 T_{\alpha+\beta,\alpha}(x+y,x) &= \sum_{s_1,s_2>0} \sum_{a=s_2}^{s_1+s_2-1} \binom{a-1}{s_2-1} [a, s_1+s_2-a; \alpha+\beta, \alpha] x^{s_1-1} y^{s_2-1}, \\
 T_{\alpha+\beta,\beta}(x+y,y) &= \sum_{s_1,s_2>0} \sum_{a=s_1}^{s_1+s_2-1} \binom{a-1}{s_1-1} [a, s_1+s_2-a; \alpha+\beta, \beta] x^{s_1-1} y^{s_2-1}, \\
 T_{\alpha+\beta}(x+y) &= \sum_{s_1,s_2>0} \binom{s_1+s_2-2}{s_1-1} [s_1+s_2-1; \alpha+\beta] x^{s_1-1} y^{s_2-1}, \\
 \sum_{k>0} \frac{\mathfrak{D}[k; \alpha+\beta]}{k} (x+y)^k &= \sum_{s_1,s_2>0} \binom{s_1+s_2-2}{s_1-1} \frac{\mathfrak{D}[s_1+s_2-2; \alpha+\beta]}{s_1+s_2-2} x^{s_1-1} y^{s_2-1}.
 \end{aligned}$$

Now the proposition follows immediately from Lemma 7.2. \square

Example 7.4. Taking $\alpha = 0$ and $\beta = 1$ we get for all $N \geq 0$

$$\begin{aligned}
 \mathfrak{D}[1; 1] &= [2; 0] \cdot [1; 1] + [2; 1] - [2, 1; 1, 1] - [2, 1; 1, 0] - [1, 2; 1, 0] \\
 &= [2, 1; 0, 1] - [2, 1; 1, 1] - [2, 1; 1, 0] + [2; 1] - \omega_{0;1}^N [2; 0] + \omega_{1;1}^N ([1; 0] - [1; 1])
 \end{aligned}$$

by Proposition 5.6 and (18). Similarly

$$\begin{aligned}
 \mathfrak{D}[2; 1] &= [2; 0] \cdot [2; 1] + 2[3; 1] - [2, 2; 1, 0] - 2[3, 1; 1, 0] - [2, 2; 1, 1] - 2[3, 1; 1, 1] \\
 &= [2, 2; 0, 1] - [2, 2; 1, 1] - 2[3, 1; 1, 0] - 2[3, 1; 1, 1] + 2[3; 1] \\
 &\quad - \omega_{1;1}^N ([2; 0] + [2; 1]) + \omega_{2;1}^N ([1; 0] - [1; 1]),
 \end{aligned}$$

again by Proposition 5.6 and (18).

The next result provides us with a way to find \mathbb{Q} -linear relations in $\mathcal{MZV}(w, N)$ systematically.

Theorem 7.5. *For any positive integer $w \geq 3$ and N we have*

$$\text{Fil}_{w-1}^{\mathbb{W};N}(\mathcal{MD}_N) \subseteq \ker(Z_w), \quad \text{and} \quad \mathfrak{D} \text{Fil}_{w-2}^{\mathbb{W};N}(\mathcal{MD}_N) \subseteq \ker(Z_w).$$

Proof. Essentially the same proof for the level $N = 1$ case used by Bachman and Kühn (see [1, Proposition 7.3]) works in general. We leave the details to the interested reader. \square

Example 7.6. Suppose $N = 2$. By Example 7.4

$$\mathfrak{D}[\overset{\circ}{1}] = [2, \overset{\circ}{1}] - [\overset{\circ}{2}, 1] - [\overset{\circ}{2}, \overset{\circ}{1}] + [\overset{\circ}{2}] + \frac{1}{4}([1] - [\overset{\circ}{1}] - 2[2]), \quad (38)$$

$$\mathfrak{D}[\overset{\circ}{2}] = 2[\overset{\circ}{3}] - 2[\overset{\circ}{3}, \overset{\circ}{1}] - [\overset{\circ}{2}, \overset{\circ}{2}] - 2[\overset{\circ}{3}, 1] + [2, \overset{\circ}{2}] - \frac{1}{4}([2] + [\overset{\circ}{2}]). \quad (39)$$

Applying the map Z to (38) and (39) we get two relations at level 2 for the MZVs of weight 3 and 4, respectively:

$$\zeta_2(2, \overset{\circ}{1}) = \zeta_2(\overset{\circ}{2}, 1) + \zeta_2(\overset{\circ}{2}, \overset{\circ}{1}) \quad (40)$$

$$\zeta_2(2, \overset{\circ}{2}) = 2\zeta_2(\overset{\circ}{3}, 1) + \zeta_2(\overset{\circ}{2}, \overset{\circ}{2}) + 2\zeta_2(\overset{\circ}{3}, 1).$$

Notice that

$$\begin{aligned} 4\zeta_2(2, \overset{\circ}{1}) &= \zeta(2, 1) - \zeta(2, \bar{1}) + \zeta(\bar{2}, 1) - \zeta(\bar{2}, \bar{1}), \\ 4\zeta_2(\overset{\circ}{2}, 1) &= \zeta(2, 1) - \zeta(\bar{2}, 1) + \zeta(2, \bar{1}) - \zeta(\bar{2}, \bar{1}), \\ 4\zeta_2(\overset{\circ}{2}, \overset{\circ}{1}) &= \zeta(2, 1) - \zeta(2, \bar{1}) - \zeta(\bar{2}, 1) + \zeta(\bar{2}, \bar{1}), \\ 4\zeta_2(\overset{\circ}{3}, 1) &= \zeta(3, 1) - \zeta(\bar{3}, 1) + \zeta(3, \bar{1}) - \zeta(\bar{3}, \bar{1}), \end{aligned} \quad (41)$$

$$4\zeta_2(\overset{\circ}{3}, \overset{\circ}{1}) = \zeta(3, 1) - \zeta(3, \bar{1}) - \zeta(\bar{3}, 1) + \zeta(\bar{3}, \bar{1}), \quad (42)$$

$$4\zeta_2(2, \overset{\circ}{2}) = \zeta(2, 2) - \zeta(2, \bar{2}) + \zeta(\bar{2}, 2) - \zeta(\bar{2}, \bar{2}), \quad (43)$$

$$4\zeta_2(\overset{\circ}{2}, \overset{\circ}{2}) = \zeta(2, 2) - \zeta(2, \bar{2}) - \zeta(\bar{2}, 2) + \zeta(\bar{2}, \bar{2}). \quad (44)$$

Therefore we get the relations among alternating Euler sums:

$$3\zeta(\bar{2}, 1) = \zeta(2, 1) + \zeta(2, \bar{1}) + \zeta(\bar{2}, \bar{1}),$$

$$2\zeta(3, 1) = 2\zeta(\bar{3}, 1) + \zeta(\bar{2}, 2) - \zeta(\bar{2}, \bar{2}).$$

These two relations can be easily verified using the explicit structural results contained in [25, Proposition 3.3 and 3.5] (unfortunately, these were omitted in the published version [26]).

To deal with the general depth we define for all $\alpha_1, \dots, \alpha_d \in (\mathbb{Z}/N\mathbb{Z})^d$ and $1 \leq j \leq d$

$$R_{\alpha_1, \dots, \alpha_d}^{(j; d)}(x_1, \dots, x_d) := \sum_{n_1, \dots, n_d > 0} \prod_{i=1}^d \frac{\eta^{\alpha_i n_i} e^{n_i x_i} q^{n_1 + \dots + n_i}}{(1 - q^{n_1 + \dots + n_i})^{\delta_{i, j+1}}}.$$

and for $\mathbf{y} = (y_1, \dots, y_d)$

$$R_{\beta; \alpha}^{(d)}(x; \mathbf{y}) = \sum_{j=1}^d R_{\beta+\alpha_1, \dots, \beta+\alpha_j, \alpha_{j+1}, \dots, \alpha_d}^{(j; d)}(x + y_1, \dots, x + y_j, y_{j+1}, \dots, y_d). \quad (45)$$

The operator D_x on functions $f(x, y, \dots)$ is defined by

$$D(f) = \left. \frac{\partial f(x, y, \dots)}{\partial x} \right|_{x=0}.$$

Observe that $D_x(R_{\beta; \alpha}^{(1)}(x; y)) = D_x(R_{\alpha+\beta}^{(1; 1)}(x + y)) = \mathfrak{D} T_{\alpha+\beta}(y)$. This can be partially generalized to arbitrary depths by the following lemma.

Lemma 7.7. *For all $d \geq 2$ and $\alpha_1, \dots, \alpha_d \in (\mathbb{Z}/N\mathbb{Z})^d$ we have*

$$D_x \left(R_{0; \alpha_1, \dots, \alpha_d}^{(d)}(x; y_1, \dots, y_d) \right) = \mathfrak{D} T_{\alpha_1, \dots, \alpha_d}(y_1, \dots, y_d).$$

Proof. For any fixed j we have

$$\begin{aligned} D_x \left(R_{\alpha_1, \dots, \alpha_d}^{(j; d)}(x + y_1, \dots, x + y_j, y_{j+1}, \dots, y_d) \right) = \\ \sum_{n_1, \dots, n_d > 0} (n_1 + \dots + n_j) \eta^{\alpha_1 n_1 + \dots + \alpha_d n_d} e^{n_1 y_1 + \dots + n_d y_d} \prod_{i=1}^d \frac{q^{n_1 + \dots + n_i}}{(1 + q^{n_1 + \dots + n_i})^{\delta_{i, j+1}}}. \end{aligned}$$

Adding all these equations together when j goes from 1 to d we now can prove the lemma easily by using $\mathfrak{D} \frac{q^n}{1-q^n} = \frac{n \cdot q^n}{(1-q^n)^2}$ and the product formula. \square

Theorem 7.8. (i) *The operator $\mathfrak{D} = q \frac{d}{dq}$ is a derivation on \mathcal{MD}_N .*

(ii) *Let $d \geq 2$, $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{Z}/N\mathbb{Z})^d$. For any $j \leq d$ let $\mathbf{e}_j = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^d$ be the j -th standard unit vector and put $\alpha(j) = (\alpha_1, \dots, \alpha_j, \alpha_j, \dots, \alpha_d)$ with only α_j repeated once. Then we have*

$$\begin{aligned} \mathfrak{D}[\mathbf{s}; \alpha] &= [2; 0] \cdot [\mathbf{s}; \alpha] + \sum_{j=1}^d (d - j + 1) s_j [\mathbf{s} + \mathbf{e}_j; \alpha] - \sum_{j=1}^d s_j [\mathbf{s} + \mathbf{e}_j, 1; \alpha, 0] \\ &\quad - [\mathbf{s}, 2; \alpha, 0] - \sum_{j=1}^d \sum_{a+b=s_j+2} (a-1) [s_1, \dots, s_{j-1}, a, b, s_{j+1}, \dots, s_d; \alpha(j)] \\ &\quad - \sum_{j=1}^d \sum_{\ell=1}^{j-1} \sum_{a+b=s_j+1} s_\ell [s_1 + \delta_{\ell, 1}, \dots, s_{j-1} + \delta_{\ell, j-1}, a, b, s_{j+1}, \dots, s_d; \alpha(j)]. \end{aligned}$$

Hence

$$\mathfrak{D} : \text{Fil}_{k, d}^{\mathbf{W}, \mathbf{L}; N}(\mathcal{MD}_N) \longrightarrow \text{Fil}_{k+2, d+1}^{\mathbf{W}, \mathbf{L}; N}(\mathcal{MD}_N).$$

Proof. (i) Clearly \mathfrak{D} is a derivation since for all $m, n \in \mathbb{N}$ we have $\mathfrak{D}(q^{m+n}) = (m+n)q^{m+n} = \mathfrak{D}(q^m)q^n + q^m \mathfrak{D}(q^n)$.

(ii) Put $\mathbf{y} = (y_1, \dots, y_d)$ and $x + \mathbf{y} = (x + y_1, \dots, x + y_d)$. We have

$$\begin{aligned} T_0(x) \cdot T_{\alpha}(\mathbf{y}) &= \sum_{m, n_1, \dots, n_d > 0} e^{mx + n_1 y_1 + \dots + n_d y_d} \frac{q^m}{1 - q^m} \frac{q^{n_1}}{1 - q^{n_1}} \cdots \frac{q^{n_1 + \dots + n_d}}{1 - q^{n_1 + \dots + n_d}} \\ &= T_{\alpha, 0}(x + \mathbf{y}, x) + \sum_{j=1}^d T_{\alpha(j)}(x + y_1, \dots, x + y_j, y_j, \dots, y_d) \\ &\quad + R_{0; \alpha}^{(d)}(x; \mathbf{y}) - \sum_{j=1}^d T_{\alpha}(x + y_1, \dots, x + y_j, y_{j+1}, \dots, y_d). \end{aligned} \quad (46)$$

We obtain the above expansion by breaking up the sum in the first line in the following way: second line is from the sums with $n_1 + \dots + n_d < m$ and $n_1 + \dots + n_{j-1} < m < n_1 + \dots + n_j$ for $j = 1, \dots, d+1$. Setting $m = n_1 + \dots + n_{j-1} + m'$ and $n_j = m' + n'_j$ for these terms it is easy to see that one gets the sum over $m', n_1, \dots, n'_j, \dots, n_d$ which then gives $T_{\alpha(j)}(x + y_1, \dots, x + y_j, y_j, \dots, y_d)$ when $j \leq d$ and $T_{\alpha, 0}(x + \mathbf{y}, x)$ when $j = d+1$. The third line of (46) arises from the sum over $m = n_1 + \dots + n_j$. In this case one uses the definition (45) together with the identity

$$\left(\frac{q^n}{1 - q^n} \right)^2 = \frac{q^n}{(1 - q^n)^2} - \frac{q^n}{1 - q^n}.$$

Finally, applying D_x on both sides of (46) and comparing the coefficient of $y_1^{s_1-1} \dots y_d^{s_d-1}$ one can derive the expression of \mathfrak{D} immediately from Lemma 7.7. This finishes the proof of the theorem. \square

Corollary 7.9. *For all $s, t \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{Z}/N\mathbb{Z}$ we have*

$$\begin{aligned} \mathfrak{D}[s, t; \alpha, \beta] &= [2; 0] \cdot [s, t; \alpha, \beta] + 2s[s + 1, t; \alpha, \beta] + t[s, t + 1; \alpha, \beta] - [s, t, 2; \alpha, \beta, 0] \\ &\quad - s[s + 1, t, 1; \alpha, \beta, 0] - t[s, t + 1, 1; \alpha, \beta, 0] - \sum_{a+b=s+2} (a-1)[a, b, t; \alpha, \alpha, \beta] \\ &\quad - \sum_{a+b=t+2} (a-1)[s, a, b; \alpha, \beta, \beta] - \sum_{a+b=t+1} s[s + 1, a, b; \alpha, \beta, \beta]. \end{aligned}$$

Example 7.10. Suppose $N \geq 2$. Using Corollary 7.9 and the relation obtained from Example 6.6

$$\begin{aligned} \mathfrak{D}[1, 1; 1, 1] &= [2, 1, 1; 0, 1, 1] - [2, 1, 1; 1, 1, 0] - 2[2, 1, 1; 1, 1, 1] + [1, 2, 1; 1, 0, 1] \\ &\quad - [1, 2, 1; 1, 1, 0] - [1, 2, 1; 1, 1, 1] + 2[2, 1; 1, 1] + [1, 2; 1, 1] \\ &\quad + \omega_{1;1}^N ([1, 1; 1, 0] + [1, 1; 0, 1] - 2[1, 1; 1, 1]) - \omega_{0;1}^N ([1, 2; 1, 0] + [2, 1; 0, 1]) \end{aligned}$$

by Proposition 5.6 and (18), we see that by Theorem 7.5 the image of the above under the map Z_4 is

$$\begin{aligned} \zeta_N(2, 1, 1; 0, 1, 1) - \zeta_N(2, 1, 1; 1, 1, 0) - 2\zeta_N(2, 1, 1; 1, 1, 1) \\ = \zeta_N^*(1, 2, 1; 1, 1, 0) + \zeta_N^*(1, 2, 1; 1, 1, 1) - \zeta_N^*(1, 2, 1; 1, 0, 1). \end{aligned}$$

By stuffle relations we must have

$$\begin{aligned} \zeta_N(2, 1, 1; 1, 0, 1) - \zeta_N(2, 1, 1; 0, 1, 1) = \zeta_N(2, 2; 0, 1) - \zeta_N(2, 2; 1, 1) \\ - \zeta_N(3, 1; 1, 0) - \zeta_N(3, 1; 1, 1) \end{aligned} \quad (47)$$

and

$$\zeta_N(2, 1; 1, 0) + \zeta_N(2, 1; 1, 1) = \zeta_N(2, 1; 0, 1). \quad (48)$$

When $N = 2$ (48) is just (40) while (47) also follows from [25, Proposition 3.5], (41) to (44) and the following identities:

$$\begin{aligned} 8\zeta_2(\overset{\circ}{2}, 1, \overset{\circ}{1}) = \zeta(2, 1, 1) - \zeta(\bar{2}, 1, 1) + \zeta(2, \bar{1}, 1) - \zeta(\bar{2}, \bar{1}, 1) \\ - \zeta(2, 1, \bar{1}) + \zeta(\bar{2}, 1, \bar{1}) - \zeta(2, \bar{1}, \bar{1}) + \zeta(\bar{2}, \bar{1}, \bar{1}), \end{aligned} \quad (49)$$

$$\begin{aligned} 8\zeta_2(2, \overset{\circ}{1}, \overset{\circ}{1}) = \zeta(2, 1, 1) + \zeta(\bar{2}, 1, 1) - \zeta(2, \bar{1}, 1) - \zeta(\bar{2}, \bar{1}, 1) \\ - \zeta(2, 1, \bar{1}) - \zeta(\bar{2}, 1, \bar{1}) + \zeta(2, \bar{1}, \bar{1}) + \zeta(\bar{2}, \bar{1}, \bar{1}). \end{aligned} \quad (50)$$

In the next example we show how one can apply the Leibniz rule of \mathfrak{D} to produce some \mathbb{Q} -linear relations in \mathcal{MD}_N .

Example 7.11. Suppose $N \geq 2$. Applying \mathfrak{D} to (21) one gets for all N

$$\mathfrak{D}([1; 1] \cdot [1; 1]) = 2\mathfrak{D}[1, 1; 1, 1] + \mathfrak{D}[2; 1] - \mathfrak{D}[1; 1] = 2[1; 1] \cdot \mathfrak{D}[1; 1].$$

Using Examples 6.6, 7.4, 7.10 and Proposition 5.6 one can finally gets a \mathbb{Q} -linear relation in \mathcal{MD}_N :

$$\begin{aligned} & 2[2, 1, 1; 0, 1, 1] - 2[2, 1, 1; 1, 0, 1] = [2, 2; 1, 1] - [2, 2; 0, 1] + [2, 1; 0, 1] \\ & + \omega_{1;1}^N (2[1, 1; 1, 1] - 2[1, 1; 0, 1] + [2; 1] - [2; 0] - [1; 0] - [1; 1]) - 2(\omega_{0;1}^N)^2 [2; 0] \\ & + \omega_{0;1}^N ([2; 0] + 2[2, 1; 0, 1] + 2[2, 1; 1, 1] - 2[2, 1; 1, 0]) + (\omega_{2;1}^N + 4\omega_{0;1}^N \omega_{1;1}^N) ([1; 0] - [1; 1]). \end{aligned}$$

Applying Z_4 we get

$$2\zeta_N(2, 1, 1; 0, 1, 1) - 2\zeta_N(2, 1, 1; 1, 0, 1) = \zeta_N(2, 2; 1, 1) - \zeta_N(2, 2; 0, 1).$$

When $N = 2$ this can be proved also by (43), (44), (49), (50), and [25, Proposition 3.5].

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