SUBSTITUTION RULES FOR HIGHER-DIMENSIONAL PAPERFOLDING STRUCTURES

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Abstract. We present a general scheme how to construct a substitution rule for generating d-dimensional analogues of the paperfolding structures. This substitution is proven to be primitive, so that the translation action on the hull forms a strictly ergodic dynamical system. The substitution admits a coincidence in the sense of Dekking, which implies that the dynamical system has pure point spectrum. The same then holds true also for the diffraction spectrum. The substitution also allows us to give estimates on the complexity of the paperfolding structures, and to determine topological invariants like the Čech cohomology groups of the hull for dimensions $d \leq 2$.

1. Introduction

The paperfolding sequences are a classical and well known example of aperiodic sequences studied in mathematics, physics and crystallography. Roughly speaking, they are obtained by repeatedly folding an arbitrarily long strip of paper in the middle, and then unfold it to obtain a sequence of creases, which are of two types – valleys and crests. The one sided paperfolding sequences, `A014577` in [15], are now obtained by reading off the creases starting from the left of the paper, and the two sided version is obtained by reading off from the centre, outwards in both directions. The structure and properties of the paperfolding sequence have been studied in many articles [1, 2, 3, 8, 10, 11, 13] (just to mention a few). In particular, Dekking et al. [11] present a substitution rule on a four letter alphabet for the classical 1-dimensional paperfolding sequence.

Ben-Abraham et al. [7] recently gave a generalisation of the paperfolding sequence to higher dimensions. Using a recursive procedure, they construct paperfolding structures in d dimensions for arbitrary d. Unfortunately, this recursive construction, which we review in Section 2, is not a substitution. In this paper, we construct a primitive substitution which reproduces the higherdimensional paperfolding structures. The existence of such a substitution has a number of immediate consequences on the properties of the paperfolding patterns, and its knowledge makes available a wealth of machinery to study further properties.

In particular, having a substitution allows us to study the *hull* of the pattern, and *dynamical systems* given by certain group actions on the hull. The hull of a pattern is given by the closure of the translation orbit of the pattern under a local topology. Under mild conditions, the hull is a compact space, and the translations act continuously on it. For primitive substitution structures, the translation action on the hull is known to be minimal and uniquely ergodic, which is related to the fact that primitive substitution patterns are repetitive and have unique patch frequencies. For background information on primitive substitution patterns and their dynamical systems, we refer to the recent monograph [6].

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As a first step in studying paperfolding structures in d dimensions, we therefore construct a primitive substitution producing them:

Theorem 1. *For any $d \geq 1$, there exists a primitive substitution μ_d which produces the paperfolding structures introduced by Ben-Abraham et al.* [7].

Using this substitution, we then prove that the d-dimensional paperfolding structures all admit a coincidence in the sense of Dekking [9], which immediately implies the following:

Theorem 2. *For any $d \geq 1$, the d-dimensional paperfolding structures have pure point diffraction spectrum, and the dynamical system of the translation action on the hull has pure point dynamical spectrum.*

The existence of a primitive substitution also gives immediate bounds on the complexity the d-dimensional paperfolding structures:

Theorem 3. *The number of distinct cubic subpatterns of linear size n in a d-dimensional paperfolding structure grows at most as $const \cdot n^d$.*

By means of the Anderson-Putnam [4] method, a primitive substitution also allows to compute topological invariants of the hull, such as Čech cohomology groups. We have determined these cohomology groups for dimensions 1 and 2:

Theorem 4. *The hull of the classical 1-dimensional paperfolding structures has Čech cohomology groups*

$$\check{H}^0 = \mathbb{Z}, \quad \check{H}^1 = \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}.$$

The hull of the 2-dimensional generalised paperfolding structures has Čech cohomology groups

$$\check{H}^0 = \mathbb{Z}, \quad \check{H}^1 = \mathbb{Z}[\tfrac{1}{2}]^2, \quad \check{H}^2 = \mathbb{Z}[\tfrac{1}{4}] \oplus \mathbb{Z}[\tfrac{1}{2}]^2 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}_2.$$

The outline of the paper is as follows. In Section 2, we review the recursive construction of Ben-Abraham et al. [7], and also fix the notation, which is mostly adopted from [7]. In Section 3 we then present the general construction of the paperfolding substitution μ_d in d dimensions. Before we proceed to analysing the properties of these substitutions, we illustrate them with pictures in dimensions 1 and 2 (Section 4). In Section 5 we then prove the results announced above.

2. Recursion

Here, we review the general recursion for the d-dimensional paperfolding structures, as outlined by Ben-Abraham et al. [7]. We also adopt most of their notation.

Each paperfolding structure is obtained by a sequence of 1-dimensional folds, each in a hyperplane perpendicular to one of the main coordinate axes. We apply a fold in each of the d directions, before we fold in the same direction again, so that we can combine d 1-dimensional folds into what we call a d-fold. By this we mean a sequence of d 1-dimensional folds of a quadratic (or cubic, hypercubic) "paper", which are edge-to-edge, from the negative to the positive x_i-axis, in the order $i = 1, 2, \ldots, d$. In each fold, the paper is bent "upwards", producing a valley crease in the previously unfolded paper stack. Let $S_d(n)$ be the d-dimensional paperfolding structure after n d-folds. That is, $S_d(n)$ can be seen as a cubic d-dimensional paper that has been folded n times in each direction, and then was unfolded again. We let the origin be in the centre of the paper. The resulting crease pattern, with valleys and crests, is what we call the paperfolding structure.

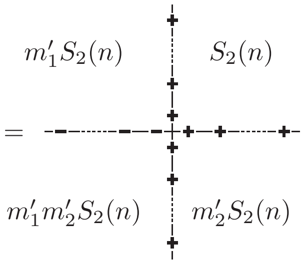
The 1-dimensional paperfolding sequence is a sequence defined on the alphabet $A = \{+, -\}$ ($+$ for valley and $-$ for crest) by the recursion

$$S_1(n+1) = m' S_1(n) \ + \ S_1(n),$$

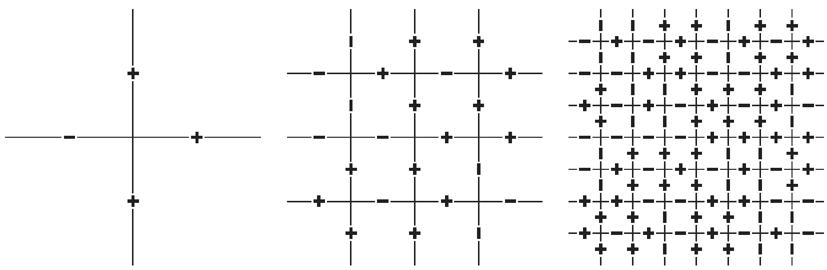
with the initial condition $S_1(0) = \emptyset$, and where m' means the reflected sequence, with valleys and crests exchanged. By *reflection* we shall from here on mean the reverse of the sequence of creases, followed by the swapping their signs.

In the 2-dimensional case the recursion becomes

$S_2(n+1) =$



with the initial condition $S_2(0) = \emptyset$ (the unfolded paper). Here we use the notation m'_a for the reflection in x_a-axis direction. See Figure 1 for a visualisation of S_2 after the first folds. Note that reflecting twice along different axes, $m'_a m'_b$, results in just a rotation by π.



$S_2(1)$ $S_2(2)$ $S_2(3)$

FIGURE 1. The first three generations of the 2-dimensional paperfolding structure.

For the general d-dimensional recursion we need to describe the first d-fold, $S_d(1)$. The creases in $S_d(1)$ can be labelled by vectors $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_d) \in \{-1, 0, 1\}^d$, which have precisely one component 0. We denote the set of admissible crease labels $\boldsymbol{\sigma}$ by C_d. The position k of the component 0 identifies the orientation of the hyperplane containing the fold (perpendicular to the k-axis), and the remaining components $\sigma_i = \pm 1$ together specify one of 2^{d-1} sectors within that hyperplane, via the conditions $\sigma_i x_i > 0$. In each such sector, the crease sign is constant. Therefore, to each crease label $\boldsymbol{\sigma} \in C_d$ there corresponds a unique crease sign, which we denote by $c(\boldsymbol{\sigma})$. In order to simplify the notation, in the following we denote by $\boldsymbol{\sigma}^k$ any crease label having the single component 0 at position k, and by σ^k_i its i-th component.

The first 1-fold of the paper, along $x_1 = 0$, gives rise to the creases labelled by $\boldsymbol{\sigma}^1$, and all of them have the same sign $+$, so that

$$c(\boldsymbol{\sigma}^1) = +,$$

whatever the components σ^1_i are. The second 1-fold of the paper, along $x_2 = 0$, gives rise to the creases $\boldsymbol{\sigma}^2$. Here we have to notice that the creases in the region $x_1 < 0$ must have the opposite sign. This gives

$$c(\boldsymbol{\sigma}^2) = \begin{cases} + & \text{if } \sigma_1^2 > 0 \\ - & \text{if } \sigma_1^2 < 0 \end{cases}$$

Continuing this way, we see that when making the 1-fold along $x_j = 0$, we have to take into account that all creases in the region $x_{j-1} < 0$ have to have the opposite sign. This leads to the recursion

$$c(\boldsymbol{\sigma}^j) = \begin{cases} c(\boldsymbol{\sigma}^{j-1}) & \text{if } \sigma_{j-1}^j > 0, \\ -c(\boldsymbol{\sigma}^{j-1}) & \text{if } \sigma_{j-1}^j < 0, \end{cases}$$

where $-c$ denotes the opposite sign of c, $\sigma_i^{j-1} = \sigma_i^j$ for $i < j-1$, and $\sigma_i^{j-1} = 1$ for $i > j-1$. By induction it now easily follows that

$$c(\boldsymbol{\sigma}^k) = \operatorname{sign}\Big(\prod_{i<k} \sigma_i^k\Big) \tag{1}$$

The collection of the creases given by the function c on C_d now allows to make the identification

$$S_d(1) \quad \leftrightarrow \quad \bigcup_{\boldsymbol{\sigma} \in C_d} c(\boldsymbol{\sigma}).$$

We will return to the properties of the function c and the sign of the creases of $S_d(1)$ in Section 5.

To describe $S_d(n+1)$, we start from $S_d(1)$ in the centre, and in each of the 2^d orthants we place a reflected copy of $S_d(n)$. The reflections can be described by

$$M(\boldsymbol{\phi}) = \prod_{\{i \,:\, \phi_i = -1\}} m_i'$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_d) \in \{1, -1\}^d$ labels the orthants $\phi_i x_i > 0$. We can now formulate the general recursion by the identification

$$S_d(n+1) \quad \leftrightarrow \quad S_d(1) \cup \Big(\bigcup_{\boldsymbol{\phi} \in \{\pm 1\}^d} M(\boldsymbol{\phi}) S_d(n) \Big) \tag{2}$$

with the initial condition $S_d(0) = \emptyset$.

3. Substitution rule

We present here a general scheme for creating a substitution for the d-dimensional paperfolding structure. Recall that in the recursion described above, a finite paperfolding pattern is extended by appending reflected copies of itself. A substitution works differently. Here, the extra creases that are inserted by the next d-fold in the interior of a cube are determined locally. A cube is first expanded by a factor 2, and then the new creases are inserted so that it gets divided into a block of 2^d new cubes, which have again the original size.

As it turns out, the creases that are inserted into a cube depend not so much on the local crease pattern around the cube, but rather on the parity of the position of the cube. We begin by discussing this in dimension 1. The generalisation to higher dimensions is then immediate.

Lemma 5. *Let s be a unit interval, with left end point in x, in the crease pattern of $S_1(n)$, for $n \geq 2$. When folding the paper n times together, s is facing downwards in the pile if and only if x is even.*

Proof. We give a proof by induction on n. By straight forward inspection, it is easy to check that the Lemma holds for $n = 2$. Suppose now that it holds for some $n \geq 2$, and let s' be an interval of length 2 in the crease pattern $S_1(n + 1)$, with a left end point x that is an even integer. When folding up n times, we get a pile of intervals of length 2. If x is twice an odd integer, s' has its face up on that pile. When folding the pile once more, the left half of the pile, including the left half of s', originally located at the even integer x, gets reflected once more, whereas the right half of s', originally located at the odd position $x + 1$, remains as it is. In this case, the Lemma thus holds also for $n + 1$. On the other hand, if x is twice an even integer, s' is face down on the pile, and the left and right halves of s' have reversed order on the pile. So, the right half of s' now gets reflected again, but not the left half, so that the Lemma holds for $n + 1$ also in this case. □

In the same spirit as in Lemma 5 we can argue that a similar result holds in dimension d. For this generalisation, let s be a unit cube in the (unfolded) crease pattern of $S_d(n)$, $n \geq 2$. We associate a reference point $\boldsymbol{x} \in \mathbb{Z}^d$ to s, as the point in s with the smallest coordinate values in all coordinate directions. To each cube s, we attach an orientation vector $v \in \{1, -1\}^d$ pointing from the reference point of s to the opposite corner. When we talk about the orientation of s, we actually mean its orientation vector. In the unfolded pattern, all orientations are the same. However, reference point and orientation vector are attached to the cube. When folding up the paper, cubes get reflected, and at the same time, their reference points and orientation vectors are mapped to their mirror images. By the notation $m'_i s$ we mean the cube s reflected along the x_i-axis. During this reflection, its orientation vector gets a sign change in its i-th component. It is not hard to see that the reflections fulfil

$$m'_i m'_j s = m'_j m'_i s \qquad \text{and} \qquad m'_i m'_i s = s. \tag{3}$$

In other words, the reflections m'_i are involutions and commute. As we are interested only in the orientation of s, when folding the paper together, it is enough to look at the chain of reflections $m'_{a_1} m'_{a_2} \dots m'_{a_k}$ due to the n d-folds, and keep track of the orientation of s. By the properties (3), this chain can be reduced to a chain of at most d reflections, and with at most one reflection along each axis. The commutativity of the reflections implies that we may consider them as a product of 1-dimensional reflections. By Lemma 5 it now follows that the resulting orientation of s is only dependent on the parity of the coordinates in $\boldsymbol{x}$, as we can consider one axis at a time, and the i-th coordinate $\boldsymbol{x}$ only affects the sign of the i-th coordinate in v. To summarise, we have the following result.

Lemma 6. *Let s be a unit cube in the crease pattern of $S_d(n)$, with $n \geq 2$ and $d \geq 1$. When folding the paper n times together, the orientation of s only depends on the parity of the coordinates of the reference point $\boldsymbol{x} \in \mathbb{Z}^d$ of s.*

An immediate consequence of Lemma 6 is that for a unit cube s in the crease pattern of $S_d(n)$, the creases that will occur inside s, when folding the paper one additional time, do not depend on the bounding creases of s, nor on the general position of s, but only on the parity of the coordinates of its reference point $\boldsymbol{x}$. Moreover, the crease pattern that will occur inside s is just a scaled and reflected copy of $S_d(1)$. Its precise orientation depends, again, only on the parity of the coordinates of $\boldsymbol{x}$. As we can regard a d-fold as a sequence of d 1-folds, Lemma 5 implies the following result.

Lemma 7. *Let s be a unit cube in the crease pattern of $S_d(n)$, with $n \geq 2$ and $d \geq 1$, and let $\boldsymbol{x} \in \mathbb{Z}^d$ be the reference point of s. When folding the paper $n + 1$ times, the folds that will* *appear inside s are*

$$\Big(\prod_{\{i\,:\,x_i \text{ even}\}} m_i'\Big)\, S_d(1). \tag{4}$$

From here on we can now argue how to construct a substitution rule, μ_d, that generates the d-dimensional paperfolding structure. In Section 4 we shall illustrate this construction by concrete examples. The crease pattern of a paperfolding structure divides space into unit cubes, whose faces contain the creases, and where each crease has a sign (valley or crest). In order to attribute each crease to a unique cube, we introduce the concept of a *semi-cube*, which is a half-open cube, containing only one of each pair of parallel faces, namely the one with smaller coordinates in the direction perpendicular to the face. These are the faces which contain the reference point of the cube. Each of these d faces may contain a crease of either sign, so that we have at most 2^d crease configurations on a semi-cube. Furthermore, the reference point of s may have 2^d different parity values, on which the fold introduced in the interior of the cube upon substitution will depend. Taking all together, we need at most $2^d \cdot 2^d$ tile types to define a substitution, where our tiles are semi-cubes carrying parity information and a crease pattern on their faces. Upon substitution, each semi-cube s is now mapped to a block of 2^d semi-cubes. The creases on the outer faces of the block are just copied from the faces of s with the same orientation, whereas the faces in the interior are a reflected copy of $S_d(1)$, as given by (4).

In the spirit of the construction by recursion, we can write the substitution as

$$\mu_d \,:\, s' \mapsto 2\cdot s' \cup \Big(\prod_{\{i\,:\,x_i \text{ even}\}} m_i'\Big)\, S_d(1) \tag{5}$$

where s' is a semi-cube with the reference point $\boldsymbol{x}$, and $2\cdot s'$ is the semi-cube scaled by a factor 2. As a seed for generating the d-dimensional paperfolding structure with the help of μ_d, we take the central 2^d-block of semi-cubes in $S_d(2)$. Note that the substitution μ_d generates the paperfolding structure from the centre outwards.

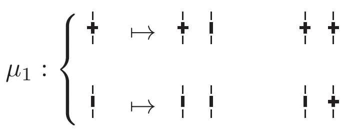
4. Examples

Here we illustrate the general construction of the paperfolding substitution presented in Section 3 by giving a more concrete, detailed representation for the 1- and 2-dimensional cases. In particular, we give a visual representation of the the map given in (5).

Let us start with the 1-dimensional case, for which a substitution is already well known [3, 11]. We discuss it here to illustrate the general construction. Any tile corresponds to a semi-cube s in the crease pattern of $S_1(n)$, that is a unit interval, with reference point $\boldsymbol{x} = (x_1)$ at the left end point. The right end point is removed, so that we have a half-open unit interval. Upon substitution, each such semi-cube is doubled in size, with left end point unaffected by the substitution, and with a reflected copy of $S_1(1)$ added to the interior, according to Lemma 7. Hence, we may define μ_1 as follows:

x_1 even $\quad x_1$ odd

(6)



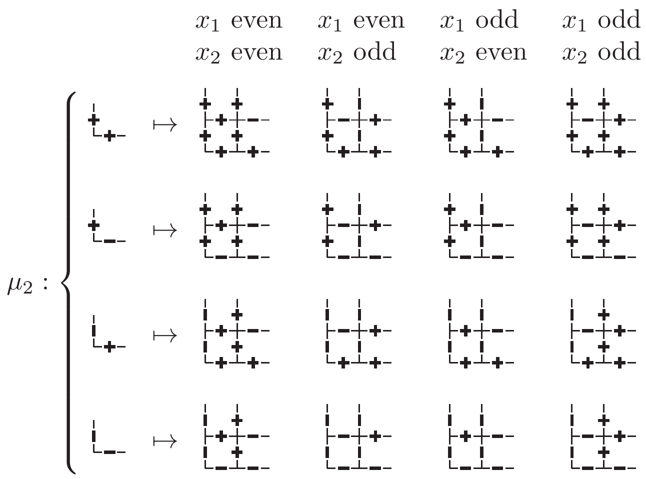
For the seed we take the pattern , which covers the half-open interval $[-1, 1)$, where the crease is at the centre. The substitution μ_1 can also be written as a substitution on the 4 letter alphabet $A = \{a_{ij} : 0 \leq i, j \leq 1\}$, where the first index indicates the type of crease at the left end point of the semi-cube, and the second index the parity of its position. The assignment of the indices is as given in the rows, resp. columns of the table in Eq. (6). Symbolically, the substitution can thus be written as

$$\nu_1 : \begin{cases} a_{00} \mapsto a_{00}\ a_{11}, & a_{01} \mapsto a_{00}\ a_{01}, \\ a_{10} \mapsto a_{10}\ a_{11}, & a_{11} \mapsto a_{10}\ a_{01}, \end{cases}$$

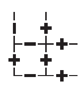
with seed $a_{11}a_{00}$.

Similarly, we can define a substitution μ_2 for the 2-dimensional paperfolding structure. Again, we start by considering a unit square s in the crease pattern of $S_2(n)$, with lower left corner at $\boldsymbol{x} = (x_1, x_2)$. The crease pattern in the interior of the inflated tiles follows from Lemma 7, resulting in a substitution μ_2 as follows:

(7)



As a seed to generate the 2-dimensional paperfolding structure with the help of μ_2, we may take



where we put the origin at the centre.

As in the 1-dimensional case, we can write the substitution μ_2 symbolically, as a block substitution on a 16 letter alphabet $B = \{b_{ij} : 0 \leq i, j \leq 3\}$. Here again, the first index indicates the crease pattern on the boundaries of the semi-cube, and the second index the parities of the reference point, in the order of the rows, resp. columns of the table in Eq. (7).

Each letter is replaced by a block of letters as follows:

$$\nu_2 : \begin{cases} b_{00} \mapsto \begin{smallmatrix} b_{01} & b_{13} \\ b_{00} & b_{02} \end{smallmatrix}, \quad b_{01} \mapsto \begin{smallmatrix} b_{11} & b_{23} \\ b_{00} & b_{22} \end{smallmatrix}, \quad b_{02} \mapsto \begin{smallmatrix} b_{01} & b_{33} \\ b_{00} & b_{22} \end{smallmatrix}, \quad b_{03} \mapsto \begin{smallmatrix} b_{11} & b_{03} \\ b_{00} & b_{02} \end{smallmatrix}, \\ b_{10} \mapsto \begin{smallmatrix} b_{01} & b_{13} \\ b_{10} & b_{12} \end{smallmatrix}, \quad b_{11} \mapsto \begin{smallmatrix} b_{11} & b_{23} \\ b_{10} & b_{32} \end{smallmatrix}, \quad b_{12} \mapsto \begin{smallmatrix} b_{01} & b_{33} \\ b_{10} & b_{32} \end{smallmatrix}, \quad b_{13} \mapsto \begin{smallmatrix} b_{11} & b_{03} \\ b_{10} & b_{12} \end{smallmatrix}, \\ b_{20} \mapsto \begin{smallmatrix} b_{21} & b_{13} \\ b_{20} & b_{02} \end{smallmatrix}, \quad b_{21} \mapsto \begin{smallmatrix} b_{31} & b_{23} \\ b_{20} & b_{22} \end{smallmatrix}, \quad b_{22} \mapsto \begin{smallmatrix} b_{21} & b_{33} \\ b_{20} & b_{22} \end{smallmatrix}, \quad b_{23} \mapsto \begin{smallmatrix} b_{31} & b_{03} \\ b_{20} & b_{02} \end{smallmatrix}, \\ b_{30} \mapsto \begin{smallmatrix} b_{21} & b_{13} \\ b_{30} & b_{12} \end{smallmatrix}, \quad b_{31} \mapsto \begin{smallmatrix} b_{31} & b_{23} \\ b_{30} & b_{32} \end{smallmatrix}, \quad b_{32} \mapsto \begin{smallmatrix} b_{21} & b_{33} \\ b_{30} & b_{32} \end{smallmatrix}, \quad b_{33} \mapsto \begin{smallmatrix} b_{31} & b_{03} \\ b_{30} & b_{12} \end{smallmatrix}, \end{cases}$$

with seed $\begin{smallmatrix} b_{32} & b_{00} \\ b_{13} & b_{01} \end{smallmatrix}$.

5. Properties of the Paperfolding Substitution

In this Section we turn to the study of the properties of the paperfolding substitution μ_d. In particular, we must show that it is *primitive*, which is defined as follows.

Definition 8. *Let ρ be a symbolic block substitution on the alphabet $A = \{\alpha_1, \ldots, \alpha_m\}$. We say that ρ is primitive if there is an integer k such that for all i the iteration $\rho^k(\alpha_i)$ contains all α_j.*

In order to prove the primitivity, we must show that all types of semi-cube do occur in the generated structure at the same time. More precisely, in each of the 2^d parity classes of sites, semi-cubes with all 2^d combinations of creases on their faces must occur.

We begin by looking at sites $\boldsymbol{x}$, whose local crease pattern, on the faces containing $\boldsymbol{x}$, is a translated and/or reflected copy of $S_d(1)$. This is the case for all sites whose parities are all even or all odd. The signs of the creases in $S_d(1)$, which is the result of the first d-fold, is given by (1). We shall now analyse the behaviour of $S_d(1)$ under a sequence of reflections, that is, we look at the signs of the creases of

$$m'_{a_1} \ldots m'_{a_r} S_d(1). \tag{8}$$

Let $\boldsymbol{a} = a_1 a_2 \ldots a_r$ with $a_i \in 1, \ldots, d$ be the sequence of reflections of $S_d(1)$ from (8). Recall that the function c is used to denote the crease signs, and let us therefore generalise this notation to $c_{\boldsymbol{a}}$, meaning the signs of the creases of $S_d(1)$ after a sequence of reflections, as in (8).

Let us start by considering the signs of the creases of $S_d(1)$ after one reflection, that is, the signs of $m'_j S_d(1)$. As before let $\boldsymbol{\sigma}^k$ label the creases. Recall (1) that $c(\boldsymbol{\sigma}^k)$ changes the sign under $\sigma_j^k \mapsto -\sigma_j^k$ if $j < k$, and stays the same otherwise. As m'_j reflects at $x_j = 0$ and then reverts all creases, this gives

$$c_j(\boldsymbol{\sigma}^k) = \begin{cases} \operatorname{sign}\Big(-\prod_{i<k} \sigma_i^k \Big) & \text{if } k \le j \\ \operatorname{sign}\Big(\prod_{i<k} \sigma_i^k \Big) & \text{if } j < k \end{cases}$$

For a sequence of reflections this generalises to

$$c_{\boldsymbol{a}}(\boldsymbol{\sigma}^k) = \operatorname{sign}\Big((-1)^{N(\boldsymbol{a},k)} \prod_{i<k} \sigma_i^k\Big) \tag{9}$$

with $N(\boldsymbol{a},k) = \big|\{t : k \le a_t, 1 \le t \le |\boldsymbol{a}|\}\big|$. From (9) we see that we may choose the elements of $\boldsymbol{a}$ to be unique and to be in strictly increasing order, that is, $a_i < a_{i+1}$.

Let us now look at the signs of the creases in the first orthant (the orthant with $x_i > 0$ for all i) of $S_d(1)$ and its reflections. The fold $S_d(1)$ divides the space into 2^d orthants, where the first orthant contains the semi-cube with the origin as reference point. The reflections of $S_d(1)$ will in general contain a different semi-cube in their first orthant. Let $Q(d)$ be the set of all semi-cubes contained in the first orthant of some reflection of $S_d(1)$. Then we have the following result.

Lemma 9. *All 2^d types of semi-cube occur in the first orthant of some reflection of $S_d(1)$. In other words, $|Q(d)| = 2^d$.*

Proof. As noted before, we have at most 2^d different semi-cubes, so that $|Q(d)| \le 2^d$. To prove equality, note first that the creases in the semi-cube in the first orthant of $S_d(1)$ can be described by $\boldsymbol{\sigma}^k \in \{0,1\}^d$ with precisely one 0. Let $\boldsymbol{a} = a_1 a_2 \ldots a_r$ and $\boldsymbol{b} = b_1 b_2 \ldots b_t$ with $a_i, b_i \in \{1, \ldots, d\}$ be two sequences of reflections. We may, as noted above, assume that both sequences are strictly increasing. Assume that

$$c_{\boldsymbol{a}}(\boldsymbol{\sigma}^k) = c_{\boldsymbol{b}}(\boldsymbol{\sigma}^k) \qquad \text{for all } 1 \le k \le d, \tag{10}$$

that is, two sequences of reflections give rise to the same semi-cube in the first orthant. The strict monotonicity of the sequence $\boldsymbol{a}$ implies that

$$0 \le N(\boldsymbol{a}, k-1) - N(\boldsymbol{a}, k) \le 1 \quad \text{for } k = 2, \ldots, d,$$

and similar for the sequence $\boldsymbol{b}$. This at most one step change on N implies that $N(\boldsymbol{a},k) = N(\boldsymbol{b},k)$ for all $1 \le k \le d$, since otherwise we would, via (9),

$$(-1)^{N(\boldsymbol{a},k)} = (-1)^{N(\boldsymbol{b},k)},$$

contradict the assumption (10). Therefore $\boldsymbol{a} = \boldsymbol{b}$, that is, two sequences of reflections that give rise to the same semi-cube must be the same. □

Consider now the points $(\pm1, \pm1, \ldots, \pm1)$ in the crease pattern $S_d(3)$. They have their parities all odd, and they form an orbit of length 2^d under the reflection group. Likewise, their local crease patterns also form an orbit under the reflection group. As the local crease pattern at $(1,1,\ldots,1)$ is $S_d(1)$, by Lemma 9 we conclude that this orbit has length 2^d, and thus all 2^d semi-cube types occur at this parity class of points. The same argument also applies to the points $(\pm2, \pm2, \ldots, \pm2)$, with parities all even.

Unfortunately, there are also points whose local crease pattern is not a reflection of $S_d(1)$. However, the local crease pattern of any point is obtained by a sequence of d 1-folds in d different directions, in *some* order. These crease patterns are thus obtained from $S_d(1)$ (and its reflections) by a permutation of the main coordinate axes. The local crease patterns located on points in an orbit under the reflection group still form an orbit of full length, so that also on these points, which are all in the same parity class, all types of semi-cubes occur. We therefore obtain the following Theorem:

Theorem 10. *The paperfolding substitution μ_d is primitive.*

Proof. The crease pattern $S_d(3)$ contains points of all parity classes, whose orbit under the reflection group has length 2^d. The semi-cubes with origin in such an orbit of points (which are all in the same parity class) occur with 2^d different crease patterns on their faces. In other words, all semi-cube types occur with reference points in all parity classes. □

We note that, independently of the dimension, three iterations of the substitution are enough to generate all semi-cube types from our standard seed, and four iterations from a single semi-cube of any type.

As indicated in the introduction, the existence of a generating primitive substitution has a number of far reaching consequences for the d-dimensional paperfolding structures. These are all standard for primitive substitution structures, compare [6].

To start, one should first note that, instead of studying an individual structure, it has many advantages if one studies its *hull*. The hull Ω_d^{pf} of the d-dimensional paperfolding structures is given by the closure of the translation orbit of one d-dimensional paperfolding structure, with respect to a topology in which two patterns are ϵ-close if they agree in a ball $B_{1/\epsilon}$ centred at the origin, up to a translation of order ϵ. The hull is a compact space, and consists of all patterns locally indistinguishable from the starting one, i.e., all structures having the same finite subpatterns. Every element of the hull is repetitive, i.e., given any finite subpattern, its translates occurring in the structure form a relatively dense set (a set without holes of arbitrary size). There is a natural translation action by homeomorphisms on the hull. That is, the hull, equipped with this group action, forms a topological dynamical system. Due to the repetitivity of the elements of the hull, this action is minimal (every orbit is dense), and it is uniquely ergodic, admitting a unique translation invariant probability measure. This latter property is related to the fact that any finite subpattern has a well-defined frequency.

An interesting question is whether we can say something on the nature of the diffraction spectrum of d-dimensional paperfolding structures, or, equivalently, on the dynamical spectrum of the translation action on the hull (compare [6]). Having a generating primitive block substitution is of help here, too. For such structures, there is a simple criterion by Dekking [9]. If it is satisfied, the structure has pure point dynamical and diffraction spectrum.

Definition 11. *We say that the d-dimensional block substitution ρ admits a coincidence in the sense of Dekking* [9]*, if there are integers k and $t_1, \ldots, t_d$, such that for all i, the iteration $\rho^k(\alpha_i)$ has the same symbol α_j at position $(t_1, \ldots, t_d)$.*

Theorem 12. *The d-dimensional paperfolding structures have pure point diffraction spectrum, and the associated dynamical system given by the translation action on the hull has pure point dynamical spectrum.*

Proof. By [9], it is enough to show that the paperfolding substitutions μ_d admit a coincidence in the sense of Dekking (for the higher-dimensional case, see also [14]). Substituting a semi-cube with reference point at the origin once, we obtain at position $(1, \ldots, 1)$ a semi-cube with parities all odd. Substituting a second time, the semi-cube at $(1, \ldots, 1)$ is mapped to a 2^d block at $(2, \ldots, 2)$. The semi-cube at $(3, \ldots, 3)$ in this block depends only on the parity of the semi-cube at $(1, \ldots, 1)$ after the first substitution, which was all odd. So, the semi-cube at $(3, \ldots, 3)$ is always the same, whatever the starting semi-cube was. □

Having a generating substitution also allows to estimate the complexity of the generated structures. A natural complexity measure is the growth of the number of distinct cubic subpatterns of linear size n. The key observation here is, that there is a hierarchy of semi-cubes of all orders in the structure. The substitution maps a semi-cube to a block of 2^d semi-cubes, which we call a super-semi-cube. This super-semi-cube is mapped to a super-semi-cube of order two, and so on. The arrangement of super-semi-cubes of any order now is, after rescaling, locally indistinguishable from the arrangement of semi-cubes (the structure is self-similar). In particular, the number of distinct 2^d blocks of super-semi-cubes of order k is the same as the number of 2^d blocks of semi-cubes. This allows us to estimate the complexity function.

Theorem 13. *The number of distinct cubic subpatterns of linear size n in a d-dimensional paperfolding structure grows at most as $\mathrm{const} \cdot n^d$.*

Proof. Let k be the smallest natural number such that $2^k \geq n$. Any cubic subpattern of linear size n is now contained in some 2^d block of super-semi-cubes of order k. There is a fixed number c of such blocks (independent of k). Let $\boldsymbol{x}$ be the corner of the n^d block with the lowest coordinates. To count all distinct n^d blocks, it is enough to let $\boldsymbol{x}$ vary within one super-semi-cube of order k. So, the number of distinct n^d subpatterns is at most $c \cdot 2^{kd} \leq c \cdot (2n)^d$. □

Denote by $P_d(n)$ the number of cubic n-patterns in the d-dimensional paperfolding structure, by which we mean semi-cubes with side length n (with interior). By Theorem 13 we know that $P_d(n) = O(n^d)$. In [1] Allouche shows that $P_1(n) = 4n$ for $n \geq 7$, and computer enumerations indicate that the following holds for $n \geq 3$:

$$P_2(n) = 12n^2 - 4 - 16 \cdot 2^{2\lfloor \log_2(n-1)\rfloor} + 24n \cdot 2^{\lfloor \log_2(n-1)\rfloor}.$$

Finally, having a primitive substitution generating the paperfolding structures allows to determine topological invariants of its hull. Anderson and Putnam [4] have shown that a generating primitive substitution allows to construct the hull as an inverse limit of a sequence of finite cell complexes and cellular maps between them. This in turn allows to compute the Čech cohomology as the direct limit of the cohomologies of these approximant cell complexes. We have a computer program that implements this computation for arbitrary block substitutions in dimensions 1 and 2. Applying it to the paperfolding substitutions, we obtain the following results:

Theorem 14. *The hull of the classical 1-dimensional paperfolding structures has Čech cohomology groups*

$$\check{H}^0 = \mathbb{Z}, \quad \check{H}^1 = \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}.$$

The hull of the 2-dimensional generalised paperfolding structures has Čech cohomology groups

$$\check{H}^0 = \mathbb{Z}, \quad \check{H}^1 = \mathbb{Z}[\tfrac{1}{2}]^2, \quad \check{H}^2 = \mathbb{Z}[\tfrac{1}{4}] \oplus \mathbb{Z}[\tfrac{1}{2}]^2 \oplus \mathbb{Z}^3 \oplus \mathbb{Z}_2.$$

6. Concluding Remarks

In our paperfolding structures, we have always folded the half-space with negative coordinates onto the half-space with positive coordinates, first in direction 1, then 2, and so on. Ben-Abraham et al. [7] have chosen the same convention. This could be varied, of course, but the generated structures would be different, in general. For instance, one could change the order of the folding in the different directions, or fold underneath instead of ontop of the positive half-space in some directions. One could even vary the type of d-fold according to a periodic or aperiodic sequence, along the lines of [12]. We suspect that as long as the sequence of folds applied is periodic, a generating substitution may still exist, but the situation with an aperiodic sequence of folds will be considerably more complicated.

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