

SEARCHING FOR A COUNTEREXAMPLE TO KUREPA'S CONJECTURE

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ABSTRACT. Kurepa's conjecture states that there is no odd prime p that divides $!p = 0! + 1! + \dots + (p-1)!$. We search for a counterexample to this conjecture for all $p < 2^{34}$. We introduce new optimization techniques and perform the computation using graphics processing units. Additionally, we consider the generalized Kurepa's left factorial given by $!^k n = (0!)^k + (1!)^k + \dots + ((n-1)!)^k$, and show that for all integers $1 < k < 100$ there exists an odd prime p such that $p \mid !^k p$.

1. INTRODUCTION

Đuro Kurepa defines an arithmetic function $!n$, known as the left factorial, by $!n = 0! + 1! + \dots + (n-1)!$ [14]. He conjectures that $\text{GCD}(!n, n!) = 2$ holds for all integers $n > 1$. This conjecture, originally introduced in 1971 [14], is also listed in [10, Section B44] and remains an open problem. For additional information, the reader can consult the expository article [12].

It is easy to see that the statement above is equivalent to the statement that $!n$ is not divisible by $n > 2$, which can be reduced to primes. Thus, Kurepa's conjecture states that there is no odd prime p such that p divides $!p$.

There have been numerous attempts to solve this problem, mostly by searching for a counterexample. The results of these attempts are listed in the accompanying table.

Upper Bound	Author	Year
$p < 3 \cdot 10^5$	Ž. Mijačlović [16]	1990
$p < 10^6$	G. Gogić [9]	1991
$p < 3 \cdot 10^6$	B. Malešević [15]	1998
$p < 2^{23}$	M. Živković [21]	1999
$p < 2^{26}$	Y. Gallot [8]	2000
$p < 1.44 \cdot 10^8$	P. Jobling [13]	2004

In 2004, Barsky and Benzaghoul published a proof of Kurepa's conjecture [3], but owing to irreparable calculation errors the proof was retracted [4]. Belief that

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the problem was solved may explain why there have been no recent attempts to find a counterexample.

In this article, we extend the search to all primes $p < 2^{34}$. The number of arithmetic operations required to search for a counterexample in this interval is greater by a factor of approximately 11300 compared to the latest published results. To achieve this, we improve the algorithm by reducing the number of required modular reductions by a factor of 2 and perform our calculations on graphics processing units (GPUs).

We have calculated and recorded the residues $r_p = !p \pmod p$ for all primes $p < 2^{34}$. We confirm all r_p given in results from previous searches and find no counterexample ($r_p = 0$). It is worth noting that $r_p = 0$ is not the only interesting residue. For example, $r_p = 1$ is also mentioned in [10, Section B44], with two solutions $p = 3$ and $p = 11$, as well as the remark that there are no other solutions for $p < 10^6$. Our search verifies that there are no new solutions to $p \mid 1! + 2! + \dots + (p-1)!$ for $p < 2^{34}$. However, we managed to find a new solution to $r_p = 2$, which means that $p \mid 2! + \dots + (p-1)!$ for $p = 6855730873$. This is the first such prime after $p = 31$ and $p = 373$.

Finally, for a positive integer k , we consider the generalized Kurepa's left factorial

$$!^k n = (0!)^k + (1!)^k + \dots + ((n-1)!)^k,$$

where $!^1 k = !k$. Similarly, we can ask whether some odd prime p divides $!^k p$. Such considerations are known from Brown's MathPages [5], where the analog of Kurepa's conjecture for $k = 5$ is introduced as an open problem. We find a counterexample, $p \mid !^5 p$ for $p = 9632267$, and present further examples where $p \mid !^k p$ for all $1 < k < 100$.

2. ALGORITHMIC CONSIDERATIONS

Wilson's theorem states that $(p-1)! \equiv -1 \pmod p$ for all primes p . It is easy to show that

$$(2.1) \quad (p-k)!(k-1)! \equiv (-1)^k \pmod p,$$

for all primes p and all $1 \leq k \leq p$. Using (2.1), we can obtain

$$(2.2) \quad !p \equiv \sum_{i=0}^{p-1} (-1)^i \frac{(p-1)!}{i!} \pmod p,$$

or, using the theory of derangements,

$$!p \equiv \left\lfloor \frac{(p-1)!}{e} \right\rfloor + 1 \pmod p.$$

The value r_p can be calculated using recurrent formulas. Regroup the parentheses in the definition of $!p$ to get

$$!p = 1 + 1(1 + 2(1 + 3(1 + \dots(1 + (p-2)(1 + (p-1))) \dots))).$$

This formula shows that if we define A_i by $A_1 = 0$ and $A_i = (1 - iA_{i-1}) \pmod p$ for $i > 0$, then $A_{p-1} = r_p$. Likewise, regrouping (2.2) in the same way shows that if we define B_i by $B_1 = 0$ and $B_i = ((-1)^i + iB_{i-1}) \pmod p$ for $i > 1$, then $B_{p-1} = r_p$. Both formulas appear in [12]. It is worth mentioning that the sequence generated

by the same recurrence as B_i but without modular reduction is a well-known combinatorial sequence representing the number of derangements [18]. Similarly, we can use the definition $D_1 = C_1 = 1$, $D_i = iD_{i-1}$, $C_i = (C_{i-1} + D_{i-1}) \bmod p$ and get $r_p = C_p$, as proved in [8].

The best-known algorithm for computing a single value r_p has time complexity $O(p)$, which implies that the time required to search for a counterexample for all $p < n$ is $O(n^2/\ln n)$. In practice, computer hardware performs integer arithmetic with a fixed precision, so the product of two operands often needs to be reduced modulo p . All previous attempts to resolve r_p required p modular reductions and a few additional multiplications and additions.

Modern hardware is optimized to perform fast multiplication and addition on integers and floats, while the biggest concern is division which is, on average, orders of magnitude slower. Algorithms that compute $a \cdot b \pmod{p}$ quickly, such as Montgomery reduction [17] or inverse multiplication [2], use multiplication and addition to replace division, resulting in execution being several times faster.

Since we tested $p < 2^{34}$, we could not use the 32-bit integer arithmetic. Instead, we used double precision floating point hardware in a novel way to compute with integers up to $h = 2^{51}$. Note that the IEEE 754 Standard for double precision floating point arithmetic specifies a 52-bit significand.

Our algorithm uses a modified approach that allows us to reduce the number of required modular reductions by a factor of 2. While $p < h$, we can avoid modular reduction in cases where we need to perform $p + a$ for any $a < h - p$, or when we multiply p by a constant $a < \frac{h}{p}$.

Let us regroup $!p$ in the following way:

$$!p = \sum_{i=0}^{p-1} i! = 1 + (1! + 2!) + (3! + 4!) + \cdots + ((p-4)! + (p-3)!) + ((p-2)! + (p-1)!).$$

Because $(p-1)! \equiv -1 \pmod{p}$ and $(p-2)! \equiv 1 \pmod{p}$, we have

$$\begin{aligned} !p &\equiv 1 + \sum_{i=1}^{\frac{p-3}{2}} ((p-2i-2)! + (p-2i-1)!) \pmod{p} \\ &\equiv 1 + \sum_{i=1}^{\frac{p-3}{2}} (p-2i-2)!(1 + (p-2i-1)) \pmod{p} \\ &\equiv 1 - 2 \sum_{i=1}^{\frac{p-3}{2}} i(p-2i-2)! \pmod{p}. \end{aligned}$$

From (2.1) we have $(p-2i-2)!(2i+1)! \equiv (p-2)! \pmod{p}$, and therefore

$$(2.3) \quad r_p \equiv 1 - 2 \sum_{i=1}^{\frac{p-3}{2}} i \frac{(p-2)!}{(2i+1)!} \pmod{p}.$$

The sequence

$$s_k = \sum_{i=1}^k \frac{i(2k+1)!}{k(2i+1)!}$$

can be written recursively as

$$s_i = (2i - 2)(2i + 1)s_{i-1} + 1$$

with $s_1 = 1$, because of the induction step

$$\begin{aligned} (2k - 2)(2k + 1)s_{k-1} + 1 &= 2(k - 1)(2k + 1) \sum_{i=1}^{k-1} \frac{i}{k - 1} \frac{(2k - 1)!}{(2i + 1)!} + 1 \\ &= \sum_{i=1}^{k-1} \frac{i}{k} \frac{(2k + 1)!}{(2i + 1)!} + 1 = s_k. \end{aligned}$$

Thus (2.3) gives

$$r_p \equiv 1 - 2^{\frac{p-3}{2}} \sum_{i=1}^{\frac{p-3}{2}} \frac{i}{\frac{p-3}{2}} \frac{(2^{\frac{p-3}{2}} + 1)!}{(2i + 1)!} \equiv 1 + 3s_{\frac{p-3}{2}} \pmod{p}.$$

We provide the final form of s_i for $1 \leq i \leq \frac{p-3}{2}$ using two new sequences:

$$(2.4) \quad s_i = (m_{i-1}s_{i-1} + 1) \pmod{p}, \quad m_i = m_{i-1} + k_{i-1}, \quad k_i = k_{i-1} + 8,$$

with initial values $s_1 = 1, m_1 = 10, k_1 = 18$, which gives $r_p \equiv 1 + 3s_{\frac{p-3}{2}} \pmod{p}$.

To ensure that m_i remains below h , we do not need to perform modular reduction until after a large number of iterations. If we start the loop with $k_i \leq p$, then while k_i increases linearly by 8, m_i will exceed h after approximately $\frac{h}{p}$ iterations. Because we examine primes less than 2^{34} , we can perform approximately 2^{17} iterations before a modular reduction.

Although we were able to reduce the number of operations required to calculate r_p , the time complexity of our algorithm is still $O(n^2/\ln n)$.

3. MACHINE CONSIDERATIONS

In the last few years the processing power of GPUs has increased, making general-purpose computing on these devices possible. Additionally, significant improvements in double-precision floating point performance has broadened the computational opportunities.

In Algorithm 1, we present the pseudo-code for the procedure we used to calculate s_i from (2.4). The procedure is adjusted to run efficiently on a GPU, where a large number of threads execute the same program simultaneously on different sets of data. This program executed on the device is called the kernel. Because division is not natively supported by the GPU, we used inverse multiplication to implement modular reduction. As we ensure that all operands are less than 2^{51} , we avoid code branching and reduce the overall complexity. For the computations, we also used the fused multiply-add (FMA) operation, natively supported by the device, where $a \cdot b + c$ is executed as a single instruction. The big advantage of FMA instructions is that rounding of the product $a \cdot b$ is performed only once, after the addition. This leaves the intermediate results of the multiplication in full precision of 104-bits.

Multiple instances of Algorithm 1 can be executed in parallel, as there are no computational dependencies between the tasks for each p . In our implementation, kernel execution is limited to 10000 iterations after which all operands are reduced modulo p . Each process is repeated until $i > \frac{p-3}{2}$, when all results are returned to

the host program and the final values of r_p are calculated. Long kernel execution keeps the arithmetic units occupied, which effectively masks memory latency. Due to high arithmetic intensity and parallelization capability, we find that Algorithm 1 is suitable for efficient execution in a SIMD environment.

We implemented the algorithm using the OpenCL framework. As the main computation unit, we used two AMD (ATI) Radeon R9 280x GPUs with a total peak double-precision performance of 2 teraFLOPS. This GPU is based on the Graphics Core Next (GCN) architecture and has 32 compute units, each having four 16-wide SIMD units [1]. In OpenCL terminology, a single kernel instance is called a work-item. At the hardware level, 64 work-items are grouped together and executed on an SIMD unit. This minimum execution unit is called a wavefront. In a single compute unit, four SIMD units process instructions from four different wavefronts simultaneously. Each SIMD unit has an instruction buffer for up to 10 wavefronts that can be issued to hide ALU and memory latencies. Overall, this model of GPU can process up to 81920 active work-items. In practice, the number of active wavefronts should be large enough to hide memory latencies and keep all compute units busy. For the effective utilization, we processed 65536 work-items (1024 wavefronts) for kernel execution on a single GPU.

Further, we improved the performance by processing two values of s_i in the same kernel. We boosted the performance by an additional 20% using loop unrolling and instruction pairing. Using the profiler, we confirmed that the GPU resources were efficiently used. In this setup, we were able to resolve a block of 262144 primes in 930 seconds on average, for primes in the region of 2^{30} . This translates to approximately $1.51 \cdot 10^{11}$ iterations of the main loop per second, as given in Algorithm 1. The process took approximately 240 days to resolve r_p for all $p < 2^{34}$.

Comparing this performance to previous implementations run on a single i7 class quad-core CPU, our implementation is more than 150 times faster for $p < 2^{32}$. We did not compare similar procedures on CPUs when p takes values larger than 2^{32} . We believe that because of the limitations of the architecture, possible solutions on CPUs might be even slower.

To verify our results we used two methods. First we selected random values of r_p obtained from the GPU computation, and verified them on a CPU using a slow but precise procedure that operates in 128-bit precision. In this way we verified more than 200000 values. It has been reported that in some states, such as when overclocked, GPUs can generate memory errors [11]. Thus we also reran 20 million randomly selected values of r_p and compared them with the previous results. For both tests, the results matched exactly.

Algorithm 1 Computation kernel for s_i

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1: procedure KUREPA( $s, i, p$ )
2:    $m \leftarrow (4i^2 - 2i - 2) \bmod p$ 
3:    $k \leftarrow (8i + 2) \bmod p$ 
4:    $g \leftarrow 1/p$ 
5:    $p_1 \leftarrow p + 1$ 
6:    $c \leftarrow 2^{51} + 2^{52}$ 
7:    $i_{max} \leftarrow \min \{i + 10000, \frac{p-3}{2}\}$ 
8:   while  $i \leq i_{max}$  do
9:      $u \leftarrow s \cdot m$ 
10:     $b \leftarrow \text{FMA}(u, g, c) - c$ 
11:     $s \leftarrow p_1 + \text{FMA}(s, m, -u) - \text{FMA}(p, b, -u)$ 
12:     $m \leftarrow m + k$ 
13:     $k \leftarrow k + 8$ 
14:     $i \leftarrow i + 1$ 
15:  end while
16:   $s \leftarrow s \bmod p$ 
17:  return  $s$ 
18: end procedure

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4. HEURISTIC CONSIDERATIONS

We strongly believe that there is a counterexample to Kurepa's conjecture. Heuristic considerations suggest that $!p$ is a random number modulo p with uniform distribution, so the probability that r_p has any particular value is approximately $\frac{1}{p}$, and the sum of reciprocals of the primes diverges. The natural question then is whether there are infinitely many such counterexamples. Let us remark that the following considerations are similar to those in [7] and [21].

We might expect the number of counterexamples in an interval $[a, b]$ to be $\sum_{a \leq p \leq b} \frac{1}{p}$. According to Mertens' second theorem, $\sum_{p \leq x} \frac{1}{p} = \ln \ln x + c + O(\frac{1}{\ln x})$, where $c \approx 0.261497\dots$ is the Mertens constant, and therefore $\sum_{a \leq p \leq b} \frac{1}{p} \approx \ln(\frac{\ln b}{\ln a})$.

For example, if we consider the interval $p \in [2^{30}, 2^{34}]$, then the expected number of primes p with $|r_p| < l$ is approximately $(2l - 1) \ln(\frac{17}{15})$. For $l = 100$, this approach predicts 25 such primes whereas the actual number is 23, and for $l = 10000$ it predicts 2503 such primes versus the actual number of 2486.

With such statistical considerations, the probability that there is no counterexample in an interval $[a, b]$ is $\prod_{a \leq p \leq b} (1 - \frac{1}{p})$. According to Mertens' third theorem $\prod_{p \leq x} (1 - \frac{1}{p}) = \frac{e^{-\gamma}}{\ln x} (1 + O(\frac{1}{\ln x}))$, where $\gamma \approx 0.577215\dots$ is Euler's constant, and therefore is $\prod_{a \leq p \leq b} (1 - \frac{1}{p}) \approx \frac{\ln a}{\ln b}$.

These heuristic considerations predict a 50% chance for a counterexample in an interval $[2^t, 2^{2t}]$. For example, it predicts an approximately 20% chance for a counterexample in the new interval $[1.44 \cdot 10^8, 2^{34}]$ covered by this paper. According to this, counterexamples should be rare and hard to verify. Increasing the upper bound to $p < 2^{35}$ gives only a 3% chance of finding a counterexample, while for a 50% chance of finding a counterexample we would need to search $p < 2^{68}$, which cannot be attained using existing hardware and algorithms.

There are many open number theory problems with the same statistical considerations. For example, searches for Wieferich primes ($2^{p-1} \equiv 1 \pmod{p^2}$) have yielded only 1093 and 3511 for $p < 1.4 \cdot 10^{17}$ [20], searches for Wilson primes ($(p-1)! \equiv -1 \pmod{p^2}$) have yielded only 5, 13, and 563 for $p < 2 \cdot 10^{13}$ [6], while no Wall–Sun–Sun primes ($F_{p-\frac{p}{5}} \equiv 0 \pmod{p^2}$) has been found for $p < 2.8 \cdot 10^{16}$ [19]. We note that for the listed problems there are algorithms with better asymptotic time complexity than $O(n^2/\ln n)$, which is the complexity of the best-known algorithms for searching for a counterexample to Kurepa's conjecture.

5. RESULTS

In our search, we calculated and recorded all the residues r_p . It is convenient to use the convention that r_p belongs to an interval $[-\frac{p-1}{2}, \frac{p-1}{2}]$. In the following table, we list all the values $|r_p| < 100$ for primes $1.44 \cdot 10^8 < p < 2^{34}$.

p	r_p	p	r_p	p	r_p
145946963	-49	709692847	-38	3730087171	-69
171707099	52	758909887	-85	4244621567	58
301289203	-57	1023141859	96	4360286653	-48
309016481	92	1167637147	67	6855730873	2
309303529	26	1250341679	15	8413206301	89
345002117	-5	1278568703	-6	9484236149	-64
348245083	-83	1283842181	80	11102372713	54
353077883	6	1330433659	-75	12024036851	-72
441778013	-27	1867557269	-42	14594548571	89
473562253	25	2176568417	-60	16541646029	-88
499509403	-74	2480960057	-57	16650884357	-82
530339209	-24	2649813899	-51	16683898487	91
594153589	14	3113484391	-67		
653214853	78	3122927597	-48		

We also considered the generalized left factorial $!^k p = (0!)^k + (1!)^k + \dots + ((p-1)!)^k$ for small values of k . For even k , we have $!^k 3 \equiv 1 + 1 + 2^k \equiv 0 \pmod{3}$, and therefore $3 \mid !^k 3$. Similarly, $!^k 5 \equiv 1 + 1 + 2^k + 6^k + 24^k \equiv 2 + 2^k + 1 + (-1)^k \pmod{5}$, and therefore $5 \mid !^k 5$ for $k \equiv 0, 3 \pmod{4}$. Thus, we need to check only $k \equiv 1 \pmod{4}$.

We searched for the smallest odd prime p such that $p \nmid !^k p$ for all $1 < k < 100$. The results are listed in the following table, including a counterexample to Brown's open problem ($k = 5$).

k	p	k	p	k	p	k	p
5	9632267	29	14293	53	59	77	7852751
9	29	33	89	57	109	81	229
13	97894723	37	29	61	47	85	61
17	17203	41	487	65	29	89	104207
21	27920293	45	233	69	3307	93	29
25	61	49	859	73	174978647	97	18211

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