Quasisymmetric functions for nestohedra

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Abstract

For a generalized permutohedron Q the enumerator $F(Q)$ of positive lattice points in interiors of maximal cones of the normal fan Σ_Q is a quasisymmetric function. We describe this function for the class of nestohedra as a Hopf algebra morphism from a combinatorial Hopf algebra of building sets. For the class of graph-associahedra the corresponding quasisymmetric function is a new isomorphism invariant of graphs. The obtained invariant is quite natural as it is the generating function of ordered colorings of graphs and satisfies the recurrence relation with respect to deletions of vertices.

Keywords: Hopf algebra, nestohedron, graph, quasisymmetric function, P-partition

1 Introduction

Let Q be a convex polytope. The *normal fan* Σ_Q is the set of cones over the faces of the polar polytope Q^*. The polytope Q is simple if and only if the normal fan Σ_Q is simplicial. The polytope Q is a *Delzant* polytope if its normal fan Σ_Q is regular, i.e. the generators of the normal cone σ_v at any vertex $v \in Q$ can be chosen to form an integer basis of $\mathbb{Z}^n$.

The permutohedron Pe^{n-1} is a $(n-1)$-dimensional polytope which is the convex hull $Pe^{n-1} = \mathrm{Conv}\{x_\omega \mid \omega \in S_n\}$, where $x \in \mathbb{R}^n$ is a point with strictly increasing coordinates $x_1 < \cdots < x_n$ and $x_\omega = (x_{\omega(1)}, \ldots, x_{\omega(n)})$ for a permutation $\omega \in S_n$. The normal fan $\Sigma_{Pe^{n-1}}$ of the permutohedron Pe^{n-1} is the braid arrangement fan. A *generalized permutohedron* Q is a polytope whose normal fan Σ_Q is refined by the braid arrangement fan $\Sigma_{Pe^{n-1}}$. The generalized permutohedra, introduced by Postnikov in [13], include some interesting classes of polytopes, such as matroid polytopes, graphic zonotopes, nestohedra and graph-associahedra.

Let Q be a generalized permutohedron in $\mathbb{R}^n$. A function $f : [n] \to \mathbb{N}$ is Q-*generic* if it lies in the interior of the normal cone σ_v for some vertex $v \in Q$.

Thus a Q-generic function f, as an element of $(\mathbb{R}^n)^*$, uniquely maximizes over Q at a vertex. Let $F(Q)$ be the generating function of all Q-generic functions

$$F(Q) = \sum_{f:\ Q-\text{generic}} \mathbf{x}_f = \sum_{v\in Q}\sum_{f\in\sigma_v} \mathbf{x}_f,$$

where $\mathbf{x}_f = x_{f(1)}\cdots x_{f(n)}$. This power series is introduced and its main properties are derived by Billera, Jia and Reiner in ([2], Section 9). It is a homogeneous quasisymmetric function of degree n. Consider its expansion in the monomial basis of quasisymmetric functions

$$F(Q) = \sum_{\alpha\models n} \zeta_\alpha(Q) M_\alpha,$$

where $M_\alpha = \sum_{i_1<\cdots<i_k} x_{i_1}^{a_1}\cdots x_{i_k}^{a_k}$ for a composition $\alpha = (a_1,\ldots,a_k) \models n$ of the integer n.

If $Q = Z_\Gamma$ is a graphic zonotope the function $F(Z_\Gamma)$ is easily seen to be Stanley's chromatic symmetric function X_Γ of the graph Γ [17]. For the matroid base polytope $Q = P_M$ the quasisymmetric function $F(P_M)$ is an isomorphism invariant of a matroid M introduced by Billera, Jia and Reiner in [2]. The unifying principle of these two examples is a construction of certain combinatorial Hopf algebras such that prescribed invariants are obtained by the universal morphism to quasisymmetric functions. The theory of combinatorial Hopf algebras is developed by Aguiar, Bergeron and Sotille in [1]. We particularly respond to [2, Problem 9.3] and study the quasisymmetric functions $F(Q)$ for the class of nestohedra.

The nestohedron $Q = P_B$ is a simple polytope obtained from a simplex by a sequence of face truncations. The family of faces by which truncations are performed is encoded by a building set B, which is a subset of the face lattice of the simplex. The ground sets of connected subgraphs of a graph Γ produce the graphical building set $B(\Gamma)$. The class of polytopes $P_{B(\Gamma)}$ is called graph-associahedra. It contains an important series of polytopes such as associahedra or Stasheff polytopes, cyclohedra or Bott-Taubes polytopes, stellohedra and permutohedra. For the class of nestohedra we describe coefficients $\zeta_\alpha(P_B)$ in terms of underlying building set B. We construct a certain combinatorial Hopf algebra of building sets $\mathcal{B}$ and show that the canonical morphism maps a building set B precisely to the generating function $F(P_B)$ of the corresponding nestohedron P_B. The Hopf algebra $\mathcal{B}$ is not cocommutative which explains why the function $F(P_B)$ is quasisymmetric rather then symmetric.

After Stanley's chromatic symmetric function of a graph appeared, some of its generalizations were introduced, as a quasisymmetric chromatic function [10] and a noncommutative chromatic symmetric function [7]. We introduce a new quasisymmetric function invariant F_Γ associated to a graph Γ which has independent combinatorial and algebraic descriptions as

1) the enumerator function of lattice points $F_\Gamma = F(P_{B(\Gamma)})$,
2) the Hopf morphism from certain combinatorial Hopf algebra of graphs,
3) the enumerator function of *ordered colorings* of Γ.

We say a coloring of a graph is ordered if colors are linearly ordered and monochromatic vertices are not connected by paths colored by smaller colors. In addition the function F_Γ satisfies the recurrence relation with respect to deletions of vertices

$$F_\Gamma = \sum_{v \in V} (F_{\Gamma \setminus v})_1,$$

where $F \mapsto (F)_1$ is a certain shifting operator on quasisymmetric functions.

The paper is organized as follows. In section 2 we review the necessary facts about nestohedra. In section 3 we review weak orders and preorders and their connections with combinatorics of the permutohedron. In section 4 we construct the combinatorial Hopf algebra $\mathcal{B}$ and prove that the assignment $B \mapsto F(P_B)$ comes from the universal Hopf algebra morphism to quasisymmetric functions. In section 5 the function $F(P_B)$ is related with the multiset of unlabelled rooted trees associated to vertices of P_B. In section 6 the theory of P-partitions is used to determine the expansion of $F(P_B)$ in the fundamental basis and the action of the antipode on $F(P_B)$. In section 7 we give a graph theoretic interpretation of the invariant $F(P_{B(\Gamma)})$. We prove the recurrence relation for F_Γ with respect to deletions of vertices of a graph which serves as the main computational tool. As an application we compute $F(Q)$ for Q be a permutohedron, associahedron, ciclohedron or stellohedron. As the conclusion some open problems concerning the graph invariant F_Γ and Hopf algebra $\mathcal{B}$ are posed.

2 Nestohedra

In this section we review the necessary definitions and facts about nestohedra. This class of polytopes is introduced and studied in [5], [13], [14], [19].

A *hypergraph* B on the finite set $[n] = \{1, \ldots, n\}$ is a collection of nonempty subsets of $[n]$. For convenience we suppose that $\{i\} \in B, i \in [n]$. For a subset $I \subset [n]$, let $B \mid_I = \{J \subset I \mid J \in B\}$ be the induced subhypergraphs. The *contraction* of $I \subset V$ from B is the hypergraph $B/I = \{J \subset [n] \setminus I \mid J \in B \text{ or } I' \cup J \in B \text{ for some } I' \subset I\}$.

Let $\Delta_{[n]} = \mathrm{Conv}\{e_1, \ldots, e_n\}$ be the standard coordinate simplex in $\mathbb{R}^n$. To a subset $I \subset [n]$ corresponds the face $\Delta_I = \mathrm{Conv}\{e_i \mid i \in I\} \subset \Delta_{[n]}$. For a hypergraph B on $[n]$ define the polytope P_B as the Minkowski sum of simpleces

$$P_B = \sum_{I \in B} \Delta_I = \sum_{I \in B} \mathrm{Conv}\{e_i \mid i \in I\} = \mathrm{Conv} \sum_{I \in B} \{e_i \mid i \in I\}.$$

The polytope P_B is simple if additionally the hypergraph B satisfies the following condition:

$\diamond$ If $I, J \in B$ and $I \cap J \neq \emptyset$ then $I \cup J \in B$.

In that case B is called a *building set* and the polytope P_B is called a *nestohedron.*

Example 2.1. Given a simple graph Γ on the vertex set $[n]$, the graphical building set $B(\Gamma)$ is defined as the collection of all $I \subset [n]$ such that induced graphs $\Gamma \mid_I$ are connected. For the graph Γ and a subset $I \subset [n]$, the contraction

Γ/I is a graph on the vertex set $[n]\backslash I$ with two vertices u and v connected by the edge if either $\{u,v\}$ is an edge of Γ or there is a path $u, w_1, \ldots, w_k, v$ in Γ with $w_1, \ldots, w_k \in I$. Then it is immediate that $B(\Gamma \mid_I) = B(\Gamma) \mid_I$ and $B(\Gamma/I) = B(\Gamma)/I$. The polytope $P_{B(\Gamma)}$ is called a *graph-associahedron*. For instance the series $Pe^{n-1}, As^{n-1}, Cy^{n-1}, St^{n-1}, n > 2$ of permutohedra, associahedra, ciclohedra and stellohedra correspond respectively to complete graphs K_n, path graphs L_n, cycle graphs C_n and star graphs $K_{1,n-1}$ on n vertices.

Let $B_{\max}$ be the collection of maximal by inclusion elements of a building set B. We say that a building set B is *connected* if $[n] \in B$. Since the Minkowski sum is the product for polytopes which are contained in the complementary subspaces, we have

$$P_B = \sum_{I \in B_{\max}} \sum_{J \in B|_I} \Delta_J = \prod_{I \in B_{\max}} P_{B|_I}.$$

Thus we may restrict ourselves to connected building sets. The realization of nestohedra is given by the following proposition.

Proposition 2.2 ([5], Proposition 3.12)**.** *Let B be a connected building set on the finite set $[n]$ and $\mu(B)$ be the number of elements of B. The nestohedron P_B can be described as the intersection of the hyperplane $H_{[n]}$ with the halfspaces $H_{I,\geq}$ corresponding to all $I \in B \setminus \{[n]\}$, where*

$$H_{[n]} = \{x \in \mathbb{R}^n \mid \sum_{i \in [n]} x_i = \mu(B)\},$$

$$H_{I,\geq} = \{x \in \mathbb{R}^n \mid \sum_{i \in I} x_i \geq \mu(B \mid_I)\}.$$

As a consequence we obtain that for a connected building set B the nestohedron P_B can be obtained by a sequence of face truncations from the simplex $\Delta = H_{[n]} \cap \cap_{i=1}^n H_{\{i\},\geq}$. Let $H_I = \partial H_{I,\geq}$ be the hyperplane corresponding to $I \subset [n]$. We index the face lattice of Δ by $\Delta_I = \Delta \cap \cap_{i \in I} H_{\{i\}}, I \subset [n]$. Then perform the face truncations $\Delta \cap H_{I,\geq}$ prescribed by non-singleton sets $I \in B$ in any reverse order. It follows that facets of the nestohedron P_B are indexed by the elements $I \in B \setminus \{[n]\}$. A facet $F_I \subset P_B$ is isomorphic to the product $P_{B|_I} \times P_{B/I}$. The condition of connectedness of the building set B is important since the same procedure for a disconnected building set $B_1 \sqcup B_2$ does not lead to $P_{B_1 \sqcup B_2} = P_{B_1} \times P_{B_2}$.

Example 2.3. The permutohedron P_n is obtained by truncations along all faces of the simplex Δ in reverse order. Each facet of P_n is of the form $P_k \times P_{n-k}$, for some $1 \leq k \leq n-1$.

The face lattice of P_B is described by the following proposition.

Proposition 2.4 ([5], Theorem 3.14; [13], Theorem 7.4)**.** *Given a connected building set B on $[n]$, let $\{F_I \mid I \in B \setminus \{[n]\}\}$ be the set of facets of the nestohedron P_B. The intersection $F_{I_1} \cap \ldots \cap F_{I_k}, k \geq 2$ is a nonempty face of P_B if and only if*

(N1) $I_i \subset I_j$ *or* $I_j \subset I_i$ *or* $I_i \cap I_j = \emptyset$ *for any* $1 \leq i < j \leq k$.

(N2) $I_{j_1} \cup \cdots \cup I_{j_p} \notin B$ *for any pairwise disjoint sets* $I_{j_1}, \ldots, I_{j_p}$.

A subcollection $\{I_1, \ldots, I_k\} \subset B$ that satisfies the conditions (N1) and (N2) is called a *nested set*. The collection N_B of all nested sets form a simplicial complex called the *nested set complex*. The face poset of N_B is opposite to the face poset of P_B. Therefore N_B may be realized as a simplicial polytope which is polar to P_B.

The Proposition 2.4 implies that vertices of P_B correspond to maximal nested sets. We denote this correspondence by $v \mapsto N_v$. To a vertex $v \in P_B$ associate the poset $(N_v \cup \{[n]\}, \subset)$. For $I \in N_v \cup \{[n]\}$ let $i_I \in [n]$ be the element such that $\{i_I\} = I \setminus \cup \{J \in N_v \mid J \subsetneq I\}$. The correspondence $I \mapsto i_I$ is a well defined bijection by the characterization of maximal nested sets ([13], Proposition 7.6). It defines the partial order $\leq_v$ on $[n]$ by $i_I \leq_v i_J$ if and only if $I \subset J$ in $N_v \cup \{[n]\}$. Denote this poset on $[n]$ by P_v. The Hasse diagram T_v of the poset P_v for $v \in P_B$ is called a *B-tree* [14, Definition 8.1]. So $(i, j) \in T_v$ if and only if $i \lessdot_{P_v} j$ is a covering relation in the poset P_v. The root of T_v is the maximal element of P_v.

The following proposition, which is a consequence of Proposition 2.2, describes the coordinates and normal cones at vertices of P_B. Note that any nested set $\{I_1, \ldots, I_k\} \subset B$ is ordered by inclusion of sets. The usual covering relations is denoted by $J \lessdot I$.

Proposition 2.5. *Let $v \in P_B$ be a vertex of the nestohedron P_B and $N_v \in N_B$ be the corresponding maximal nested set.*

(i) The coordinates of the vertex v are given by

$$x_{i_I} = \mu(B\mid_I) - \sum_{J \in N_v : J \lessdot I} \mu(B\mid_J), I \in N_v \cup \{[n]\}.$$

(ii) The normal cone σ_v at the vertex v is determined by the inequalities

$$x_{i_J} < x_{i_I}, \text{ for all } J \lessdot I \text{ in } N_v.$$

3 Preorders, Weak orders and Permutohedra

A binary relation $\precsim$ is called a *preorder* on the finite set $V = \{v_1, \ldots, v_n\}$ if it is reflexive and transitive. If it is in addition total, i.e. $u \precsim v$ or $v \precsim u$ for all $u, v \in V$, the preorder $\precsim$ is called a *weak order* or a total preorder. The preorder defines an equivalence relation by $u \sim v$ if and only if $u \precsim v$ and $v \precsim u$. The relation $\precsim / \sim$ is a partial order on the set of equivalence classes $V/\sim$. If $\precsim$ is a weak order on V then $\precsim / \sim$ is a total order on $V/\sim$. Any weak order is represented as an ordered partition of V, i.e. as the ordered family $(V_1, \ldots, V_k)$ of nonempty disjoint subsets which covers V. The relation is recovered by $u \precsim v$ if and only if $u \in V_i$ and $v \in V_j$ for some $1 \leq i \leq j \leq k$. The type of a weak order $\precsim$ is the corresponding composition type$(\precsim) = (|V_1|, \dots, |V_k|) \models n$ and k is its length. Any function $f : V \to \mathbb{N}$ determines a weak order on V by $u \precsim_f v$ if $f(u) \leq f(v)$, for all $u, v \in V$. For any strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ we have $\precsim_f = \precsim_{g\circ f}$. To a weak order $\precsim$ on V is associated the monomial quasisymmetric function

$$M_{\mathrm{type}(\precsim)} = \sum_{\precsim_f = \precsim} x_{f(1)} \cdots x_{f(n)}.$$

Let $\mathbf{WO}(n) = \cup_{k=1,n} \mathbf{WO}_k(n)$ be the set of all weak orders of the set V graded by the lengths. To an ordered partition $(V_1, \dots, V_k)$ is associated the flag $\emptyset \subset V_1 \subset V_1 \cup V_2 \subset \ldots \subset V_1 \cup \ldots \cup V_{k-1} \subset V$. This is one-to-one correspondence of ordered partitions and flags on V. Therefore the set of all weak orders $\mathbf{WO}(n)$ is modelled as the simplicial complex $\Delta[n]^{(1)}$ the first barycentric subdivision of the simplex on V. The simplicial complex $\Delta[n]^{(1)}$ is combinatorially equivalent to the convex simplicial polytope whose polar polytope is the permutohedron Pe^{n-1} (see [12]). Thus k-faces of Pe^{n-1} are labelled by ordered partitions $(V_1, \dots, V_{n-k})$ or equivalently by $(n-k)$-weak orders on V. Accordingly, to any face $F \subset Pe^{n-1}$ is associated the monomial quasisymmetric function M_F, where

$$M_F = M_{\mathrm{type}(\precsim)},$$

for the weak order $\precsim$ on Vcorresponding to the face F. Specially, facets correspond to pairs $(A, V \setminus A)$, for proper subsets $A \subset V$ and the associated monomial quasissymetric functions are of the form $M_{(k,n-k)}$ for $1 \leq k \leq n$. Vertices correspond to linear orders $v_{i_1} < \ldots < v_{i_n}$ on V with associated monomial quasisymmetric functions equal to $M_{(1,\dots,1)}$.

By Proposition 2.5 the $(n-1)$-permutohedron is realized in the hyperplane $H_{[n]} = \{x_1 + \cdots + x_n = 2^n - 1\}$ as the convex hull of vertices

$$Pe^{n-1} = \mathrm{Conv}\{(2^{\pi^{-1}(i_1)-1}, \dots, 2^{\pi^{-1}(i_n)-1}) \mid \pi \in S_n\}.$$

The normal cone at the vertex $v \in Pe^{n-1}$ that corresponds to a permutation $\pi_v = (i_1, \dots, i_n)$ is the Weyl chamber

$$\sigma_v = C_{\pi_v} : x_{i_1} < \cdots < x_{i_n}.$$

The braid arrangement $\mathcal{A}_n$ is the arrangement of hyperplanes

$$\mathcal{A}_n : x_i = x_j, 1 \leq i, j \leq n$$

in the quatient space $\mathbb{R}^n/\mathbb{R} \cdot (1, \dots, 1) \cong \mathbb{R}^{n-1}$. The normal fan $\Sigma_{Pe^{n-1}}$ of the permutohedron is the simplicial fan defined by $\mathcal{A}_n$. A braid cone is the polyhedral cone given by the conjuction of inequalities of the form $x_i \leq x_j$. There is an obvious bijection between preorders $\precsim$ on $[n]$ and braid cones determined by equivalency $x_i \leq x_j$ if and only if $i \precsim j$. The correspondence and properties of preorders and braid cones are given in [14, Proposition 3.5]. We remark that partial orders on $[n]$ correspond to full-dimensional braid cones. The monomial quasisymmetric function M_F is precisely the enumerator for all positive lattice points in the interior of the normal cone associated to the face $F \subset Pe^{n-1}$.

For each generalized permutohedron Q there is a map $\Psi_Q : S_n \to \text{Vertices}(Q)$ defined by $\Psi(\pi) = v$ if and only if the normal cone σ_v of Q at v contains the Weyl chamber C_π or equivalently the permutation $\pi \in S_n$ is a linear extension of the poset determined by the normal cone at v [14, Corollary 3.9].

4 Hopf algebra morphism

The goal of this section is to show that the assignment of quasisymmetric function $F(P_B)$ to a building set B is a Hopf algebra morphism. We construct a Hopf algebra associated with the species of building sets in the sense of [15]. Let $\mathcal{B}$ be the graded vector space generated by the set of all isomorphism classes of building sets. The grading is defined by the number of vertices. Define the multiplication and comultiplication by

$$B_1 \cdot B_2 = B_1 \sqcup B_2 \quad \text{and} \quad \Delta(B) = \sum_{I \subset V} B\mid_I \otimes B/I.$$

The unit is the building set $B_\emptyset$ on the empty set and the counit is defined by $\epsilon(B_\emptyset) = 1$ and zero otherwise.

Proposition 4.1. *The vector space $\mathcal{B}$ with the above defined operations is a graded commutative and non-cocommutative connected bialgebra.*

Proof. The only nontrivial parts of the statement are the coassociativity and the compatibility of operations, which follows from the straightforward identities $(B/I)\mid_J = (B\mid_{I\sqcup J})/I, (B/I)/J = B/(I \sqcup J)$ for any disjoint $I, J \subset V$ and $(B_1 \cdot B_2)\mid_{I_1 \sqcup I_2} = B_1\mid_{I_1} \cdot B_2\mid_{I_2}, (B_1 \cdot B_2)/(I_1 \sqcup I_2) = B_1/I_1 \cdot B_2/I_2$ for all $I_1 \subset V_1, I_2 \subset V_2$. □

The antipode of $\mathcal{B}$ is determined by general Takeuchi's formula for the antipode of a graded connected bialgebra ([18, Lemma 14], see also [6, Proposition 1.44])

$$S(B) = \sum_{k \geq 1} (-1)^k \sum_{\mathcal{L}_k} \prod_{j=1,k} (B\mid_{I_j})/I_{j-1},$$

where the inner sum goes over all chains of subsets $\mathcal{L}_k : \emptyset = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k = V$.

Remark 4.2. The algebra $\mathcal{B}$ has an additional structure of a differential algebra introduced in [3]. The derivation is determined by

$$d(B) = \sum_{I \in B \setminus \{[n]\}} B\mid_I \cdot B/I$$

for connected building set on $[n]$ and extended by Leibnitz law $d(B_1B_2) = d(B_1)B_2 + B_1d(B_2)$.

Another Hopf algebra of building set $BSet$, which is a Hopf subalgebra of the chromatic Hopf algebra of hypergraphs is studied in [8], [9]. As algebras $\mathcal{B}$ and $BSet$ are the same but the coalgebra structures are different.

Definition 4.3. Given a composition $\alpha = (a_1, \ldots, a_k) \models n$, we say that the chain $\mathcal{L} : \emptyset = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k = V$ is a *splitting chain* of the type $\mathrm{type}(\mathcal{L}) = \alpha$ of a building set B if $(B\mid_{I_j})/I_{j-1}$ is discrete and $|I_j \setminus I_{j-1}| = a_j$ for all $1 \leq j \leq k$. A splitting chain $\mathcal{L}$ determines the weak order $\preceq_{\mathcal{L}}= (I_1, I_2 \setminus I_1, \ldots, I_k \setminus I_{k-1})$ of the same type.

Proposition 4.4. *For a connected building set B the generating function $F(P_B)$ has the following expansion*

$$F(P_B) = \sum_{\alpha \models n} \zeta_\alpha(B) M_\alpha,$$

where $\zeta_\alpha(B)$ is the total number of splitting chains of the type α.

Proof. Let $\mathcal{L}$ be a splitting chain of the length k. The sets $I_j \setminus I_{j-1}, 1 \leq j \leq k$ decompose the set of vertices $V = [n]$. Define the level of a vertex $i \in V$ by $l(i) = j$ if $i \in I_j \setminus I_{j-1}$. Let $S_i = \{i\} \cup \max\{J \subset I_{l(i)-1} \mid \{i\} \cup J \in B\}$ for $i \in V$. Since B is connected and B/I_{k-1} is discrete it follows that $|I_k \setminus I_{k-1}| = 1$, i.e. $S_i = V$ for the unique $i \in V$. Let $N(\mathcal{L}) = \{S_i \mid i \in V\} \setminus \{V\}$.

Claim: The collection $N(\mathcal{L})$ is a maximal nested set.

(N1) Suppose that $S_i \cap S_j \neq \emptyset$ for some $i, j \in V$. If $l = l(i) = l(j)$ then $S_i \cup S_j \in B$ and $\{i, j\} \in (B\mid_{I_l})/I_{l-1}$. If $l(j) < l(i)$ then $i \in S_i \cup S_j \in B$ which implies $S_j \subset S_i$.

(N2) If $S = S_{i_1} \cup \ldots \cup S_{i_p} \in B$ then $S = S_{i_j}$ for a vertex $i_j \in V$ with the maximal level $l = \max\{l(i_1), \ldots, l(i_p)\}$. Therefore $S_{i_1}, \ldots, S_{i_p}$ is not a disjoint collection.

Denote by $v(\mathcal{L})$ the vertex of P_B which corresponds to $N(\mathcal{L})$. It defines the map $g : \mathcal{L} \mapsto v(\mathcal{L}) \in P_B$. We show the following identity

$$\sum_{f \in \sigma_v} \mathbf{x}_f = \sum_{\mathcal{L} \in g^{-1}(v)} M_{\mathrm{type}(\mathcal{L})}.$$

Let $\mathcal{L} \in g^{-1}(v)$ be a splitting chain. Then $N(\mathcal{L}) = N_v$ and the associated level function $i \mapsto l(i)$ satisfies $l(i) < l(j), S_i \lessdot S_j$ in $N(\mathcal{L})$. By Proposition 2.5 (ii) we have $l \in \sigma_v$ which shows that the monomial quasisymmetric function $M_{\mathrm{type}(\mathcal{L})}$ is a summand of $\sum_{f \in \sigma_v} \mathbf{x}_f$. On the other hand, for $f \in \sigma_v$ with the set of values $i_1 < \cdots < i_k$, define the decomposition of the set V by $I_j = f^{-1}(\{i_j\}), 1 \leq j \leq k$. Then $\mathcal{L} : I_1 \subset I_1 \cup I_2 \subset \cdots \subset I_1 \cup \cdots \cup I_k = V$ is a splitting chain of B and $N(\mathcal{L}) = N_v$. The statement of theorem follows from identities

$$F(P_B) = \sum_{v \in P_B} \sum_{f \in \sigma_v} \mathbf{x}_f = \sum_{v \in P_B} \sum_{\mathcal{L} \in g^{-1}(v)} M_{\mathrm{type}(\mathcal{L})} = \sum_{\alpha \models n} \zeta_\alpha(B) M_\alpha.$$

$\square$

Theorem 4.5. *The map $F : \mathcal{B} \to QSym$, defined by $F(B) = F(P_B)$, is a morphism of combinatorial Hopf algebras.*

Proof. Define a character $\zeta : \mathcal{B} \to k$ by $\zeta(B) = 1$ if B is discrete and zero otherwise. There is a unique morphism of combinatorial Hopf algebras $\Psi : (\mathcal{B}, \zeta) \to (QSym, \zeta_Q)$, where $\zeta_Q : QSym \to k$ is the canonical character defined on the monomial basis by $\zeta_Q(M_\alpha) = 1$ for $\alpha = ()$ or $\alpha = (n)$ and zero otherwise ([1], Theorem 4.1). Let $p_j : \mathcal{B} \to \mathcal{B}_j$ be the projection on the homogeneous part of degree j. The morphism Ψ is defined by

$$\Psi(B) = \sum_{\alpha \models n} p_\alpha(B) M_\alpha,$$

where $p_\alpha = p_{(a_1,\ldots,a_k)} = p_{a_1} * \ldots * p_{a_k} = m^{k-1} \circ (p_{a_1} \otimes \ldots \otimes p_{a_k}) \circ \Delta^{k-1}$ is the convolution product of projections. It is straightforward to convince that $p_\alpha(B) = \zeta_\alpha(B)$ for any composition $\alpha \models n$, so by Proposition 4.4 the morphism Ψ coincides with the map F. $\square$

As a consequence we obtain the following identities for the function F:

$$F(P_{B_1} \times P_{B_2}) = F(P_{B_1}) F(P_{B_2}),$$
$$\Delta(F(P_B)) = \sum_{I \subset V} F(P_{B|_I}) \otimes F(P_{B/I}).$$

Remark 4.6. The function $F(P_B)$ is not a combinatorial invariant of nestohedra. For example, the building sets $B_1 = \{1, 2, 3, 4, 12, 123\}$ and $B_2 = \{1, 2, 3, 4, 12, 34\}$ on the four element set $V = [4]$ have P_{B_1} and P_{B_2} combinatorially equivalent to the 3-cube, but $F(B_1) \neq F(B_2)$.

5 Unlabelled rooted trees

Let T be an unlabelled rooted tree on the set of vertices $V = \{v_1, \ldots, v_n\}$. It defines a poset $(V, \leq_T)$ with $v_i \leq v_j$ if and only if v_j is the node on the unique path from v_i to the root. We do not make a difference between the rooted tree T and the corresponding Hasse diagram of the poset $(V, \leq_T)$.

Remark 5.1. Let $\mathcal{T}_n$ be the set of all unlabelled rooted trees on n nodes and $r(n)$ be the total number of elements of $\mathcal{T}_n$. In Neil Sloan's OEIS the sequence $\{r(n)\}_{n \in \mathbb{N}}$ is numerated by A000081.

We need some basic notions from Stanley's theory of P-partitions. A detailed survey of the theory can be found in [16], [6]. A function $f : T \to \mathbb{N}$ on vertices of a rooted tree T is called T-*partition* if $f(v_i) < f(v_j)$ for any oriented edge $v_i \to v_j \in T$. Write $\mathcal{A}(T)$ for the set of all T-partitions. Let $F(T)$ be the quasisymmetric enumerator

$$F(T) = \sum_{f \in \mathcal{A}(T)} \mathbf{x}_f.$$

Example 5.2. There are four unlabelled rooted trees on 4 vertices. They are depicted in the Figure 1 with corresponding enumerators $F(T)$ in the monomial basis.

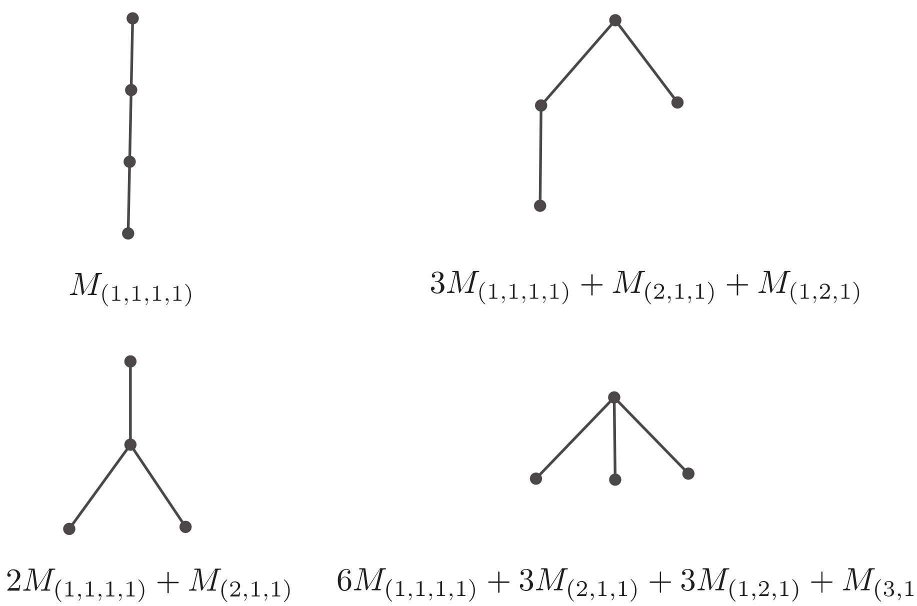

Figure 1: The unlabelled rooted trees $\mathcal{T}_4$

The quasisymmetric function $F(T)$ can be determined recursively. To each vertex $v \in V$ define $T_{\leq v}$ as the complete subtree on the set $\{u \in V \mid u \leq v\}$ of predecessors of v. The leaf is a vertex $v \in V$ for which $T_{\leq v} = \{v\}$. For a rooted forest $T = \sqcup_{i=1,k} T_i$ which is a finite collection of rooted trees we extend multiplicatively definition of T-partitions enumerators

$$F(\sqcup_{i=1,k} T_i) = F(T_1) \cdots F(T_k).$$

Definition 5.3. A shifting operator $F \mapsto (F)_1$ on quasisymmetric functions is the linear extension of the map defined on the monomial basis by $(M_\alpha)_1 = M_{(\alpha,1)}$, for each composition α.

Theorem 5.4. *Given an unlabelled rooted tree T on the set of vertices V with the root $v_0 \in V$. Let $T_1, \ldots, T_k$ be connected components of the forest $T \setminus \{v_0\}$. Then*

$$F(T) = (\prod_{i=1,k} F(T_i))_1 = F(T \setminus \{v_0\})_1.$$

Proof. Denote by $v_1, \ldots, v_k$ the neighbors in T of the root v_0. Then $T_i = T_{\leq v_i}$ for $i = 1, \ldots, k$. A function $f : T \to \mathbb{N}$ is a T-partition if and only if its restrictions $f|_{T_i} : T_i \to \mathbb{N}$ are T_i-partitions for all $i = 1, \ldots, k$ and $f(v) < f(v_0)$ for each $v \neq v_0$. $\square$

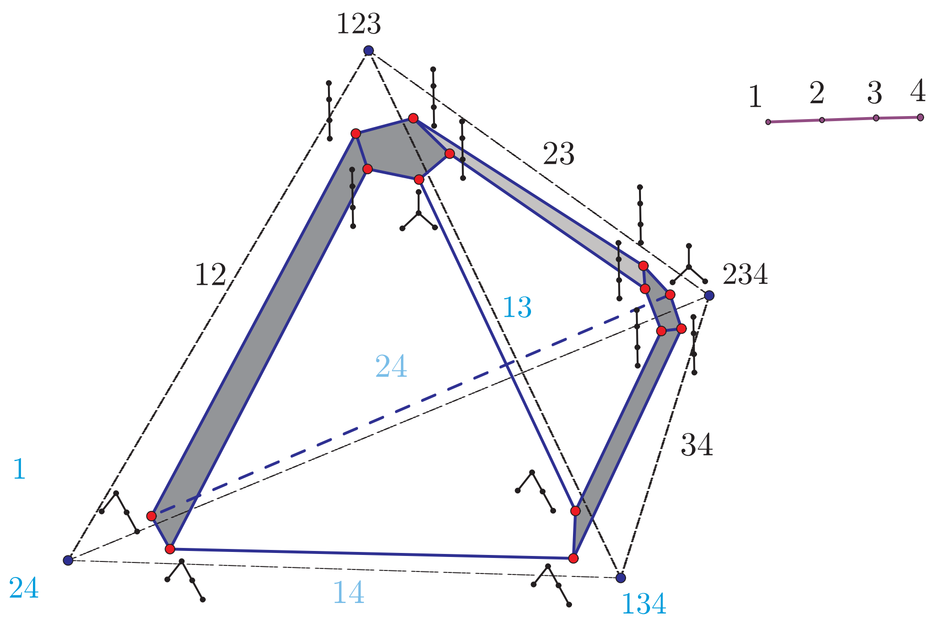

Figure 2: Associahedron As^3

Given a connected building set B, recall that to each vertex $v \in P_B$ is associated the rooted tree T_v, called B-tree, which is the Hasse diagram of the poset P_v. Let $T(B) = \{T_v \mid v \in P_B\}$ be the multiset of the corresponding unlabelled rooted trees. The following expansion is a special case of [2, Theorem 9.2] which is formulated without proof for generalized permutohedra.

Theorem 5.5. *For a building set B the quasisymmetric enumerator $F(P_B)$ is the sum of T-partitions enumerators corresponding to vertices of P_B*

$$F(P_B) = \sum_{T \in T(B)} F(T).$$

Proof. It is sufficient to show the identity $F(T_v) = \sum_{f\in\sigma_v} \mathbf{x}_f$ which follows from the description of the normal cone σ_v at a vertex $v \in P_B$, see Proposition 2.5 (ii). $\square$

Corollary 5.6. *The quasisymmetric function $F(P_B)$ depends only on the multiset $T(B)$ of unlabelled rooted trees T_v corresponding to the vertices $v \in P_B$.*

Question 5.7. In what extent the multiset $T(B)$ determines a building set B?

Example 5.8. The 3-dimensional associahedron As^3 is realized as the graph-associahedron $P_{B(L_4)}$ corresponding to the path graph L_4 on the set of vertices $\{1,2,3,4\}$. The determining building set is $B(L_4) = \{1,2,3,4,12,23,34,123,234,1234\}$. At Figure 2 is indicated the correspondence of vertices $v \in As^3$ and unlabelled rooted trees T_v. By Theorem 5.5 we find

$$F(As^3) = 24M_{(1,1,1,1)} + 6M_{(2,1,1)} + 4M_{(1,2,1)}.$$

Each T-partition $f : T \to \mathbb{N}$ takes the maximal value at the root of T. Therefore each monomial function M_α in the expansion of $F(T)$ in the monomial basis is indexed by the composition α whose the last coefficient is 1. Since $r(n) > 2^{n-2} = \dim(QSym_{n-1})$ for $n > 4$, we proved the following

Proposition 5.9. *The quasisymmetric functions $\{F(T)\}_{T\in\mathcal{T}_n}$ are linearly dependent for each $n > 4$.*

Example 5.10. We have $r(5) = 9$ and $\dim(QSym_4)_1 = 8$. The unique linear dependence relation is presented on Figure 3.

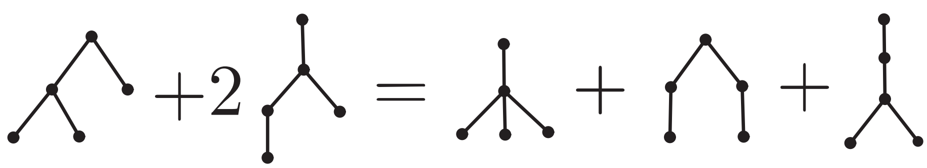

Figure 3: Linear dependence relation in $\mathcal{T}_5$

6 Expansions in the fundamental basis and the antipode

The expansion of the lattice points enumerator $F(Q)$ in the fundamental basis and the action of the antipode on it is determined for a general class of generalized permutohedra in [2, Theorem 9.2]. We consider these formula for a special class of nestohedra.

The fundamental basis $\{L_\alpha\}_{\alpha\models n, n\in\mathbb{N}}$ of $QSym$ is defined by $L_\alpha = \sum_{\alpha\preceq\beta} M_\beta$, where $\alpha \preceq \beta$ if and only if β refines α.

For a rooted tree T let $\omega : V \to [n]$ be a labelling of vertices such that $\omega(v_i) > \omega(v_j)$ whenever $v_i \to v_j \in T$. Denote by $\mathcal{L}(T)$ the set of linear extensions of the induced poset on the set of labels $[n]$. Any linear extension of the poset on $[n]$ can be regarded as the permutation $\pi \in S_n$. Let $\mathrm{des}(\pi)$ be the *descent composition* of a permutation $\pi \in S_n$ whose components are given by the lengths of consecutive maximal increasing subsequences of π. For instance $\mathrm{des}(24153) = (2, 2, 1)$.

The following expansion is fundamental in the theory of P-partitions ([16, Corollary 7.19.5], [6, Proposition 5.19])

$$F(T) = \sum_{\pi \in \mathcal{L}(T)} L_{\mathrm{des}(\pi)}.$$

By Theorem 5.5 it follows

$$F(P_B) = \sum_{T \in T(B)} F(T) = \sum_{T \in T(B)} \sum_{\pi \in \mathcal{L}(T)} L_{\mathrm{des}(\pi)},$$

which shows the positivity of $F(P_B)$ in the fundamental basis.

We determine how the antipode S on quasisymmetric functions acts on the function $F(P_B)$. The formula for antipode in monomial and fundamental basis are obtained independently in [11, Corollary 2.3], [4, Proposition 3.4], see also [6, Proposition 5.26]. We have

$$S(L_{\mathrm{des}(\pi)}) = (-1)^{|\pi|} L_{\mathrm{des}(\overline{\pi})},$$

where $\overline{\pi}$ is the opposite permutation to π defined as $\overline{\pi} = \pi_0 \circ \pi$ for $\pi_0 = (n, n-1, \ldots, 2, 1)$. Therefore

$$S(F(P_B)) = (-1)^n \sum_{T \in T(B)} \sum_{\pi \in \mathcal{L}(T)} L_{\mathrm{des}(\overline{\pi})}.$$

The quasisymmetric function $F^*(P_B) = S(F(P_B))$ has a combinatorial interpretation as the enumerator function

$$F^*(P_B) = \sum_{v \in P_B} \sum_{f \in \overline{\sigma}_v} \mathbf{x}_f,$$

where $\overline{\sigma}_v$ is the closer of the normal cone σ_v at the vertex $v \in P_B$.

For $F \in QSym$ let $\chi(F, m) = \mathrm{ps}_m(F)$ be the principal specialization defined by algebraic extension of $\mathrm{ps}_m(x_i) = 1$ for $1 \leq i \leq m$ and $\mathrm{ps}_m(x_i) = 0$ for $i > m$. Since $\mathrm{ps}_m(M_\alpha) = \binom{m}{k(\alpha)}$ we have

$$\chi(P_B, m) = \sum_{\alpha \models n} \zeta_\alpha(B) \binom{m}{k(\alpha)},$$

which counts the number of P_B-generic functions $f : [n] \to [m]$. It is related with $\chi^*(P_B, m) = \mathrm{ps}_m(F^*(P_B))$ by

$$\chi(P_B, -m) = (-1)^n \chi^*(P_B, m).$$

Specially, for $m = 1$, we obtain the following

Proposition 6.1. *The number of vertices $f_0(P_B)$ of a nestohedron P_B is determined by $\chi(P_B, -1) = \sum_{\alpha \models n} (-1)^{k(\alpha)} \zeta_\alpha(B) = (-1)^n f_0(P_B)$.*

Proof. The statement follows from the identity $\mathrm{ps}_1(F^*(P_B)) = c_{(n)}$, where $c_{(n)}$ is the coefficient by $M_{(n)}$ in the expansion of $F^*(P_B)$ in the monomial basis. □

Example 6.2. Let $B = B(L_4)$ and $As^3 = P_B$ as in Example 5.8. Then in the fundamental basis $F(As^3) = 14L_{(1,1,1,1)} + 6L_{(2,1,1)} + 4L_{(1,2,1)}$ and $F^*(As^3) = 14L_{(4)} + 6L_{(1,3)} + 4L_{(2,2)}$.

7 The graph invariant $F(P_{B(\Gamma)})$

In this section we investigate the quasisymmetric function $F(P_{B(\Gamma)})$ associated to a simple graph Γ.

The vector space $\mathcal{G}$ spanned by all isomorphism classes of simple graphs is endowed with the Hopf algebra structure by operations

$$\Gamma_1 \cdot \Gamma_2 = \Gamma_1 \sqcup \Gamma_2 \text{ and } \Delta(\Gamma) = \sum_{I \subset V} \Gamma \mid_I \otimes \Gamma / I.$$

The map that associates the graphical building set $B(\Gamma)$ to a graph Γ is extended to a Hopf algebra monomorphism $i : \mathcal{G} \to \mathcal{B}$. It follows from Theorem 4.5 that the quasisymmetric function $F(P_{B(\Gamma)})$ is a multiplicative graph invariant. By Proposition 4.4 it may be defined purely in a graph theoretic manner.

Let Γ be a simple graph on n vertices $V = \{v_1, \cdots, v_n\}$ and $\lambda : V \to \mathbb{N}$ be a coloring with the set of colors $\{i_1 < \cdots < i_k\}$. Define a flag $\emptyset = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k = V$ by $I_j = \lambda^{-1}(\{i_1, \cdots, i_j\})$ for $1 \leq j \leq k$. We say that λ is a *ordered coloring* of Γ if the graphs $\Gamma|_{I_j}/I_{j-1}$ are discrete for all $1 \leq j \leq k$. This means that each monochromatic set of vertices is discrete and no two vertices of the same color are connected by a path trough vertices colored by smaller colors. The type of an ordered coloring λ is the composition $\mathrm{co}(\lambda) = (a_1, \cdots, a_k) \models n$, where $a_j = |I_j \setminus I_{j-1}|$ is the number of vertices colored by i_j, for all $1 \leq j \leq k$. Let $Col^{\leq}(\Gamma)$ be the set of all ordered colorings of the graph Γ and F_Γ be the enumerator function

$$F_\Gamma = \sum_{\lambda \in Col^{\leq}(\Gamma)} \mathbf{x}_\lambda.$$

By Proposition 4.4 it coincides with the quasisymmetric function of graph-associahedra $B(\Gamma)$

$$F_\Gamma = F(P_{B(\Gamma)}).$$

Thus in the monomial basis it has the expansion $F_\Gamma = \sum_{\alpha \models n} \zeta_\alpha(\Gamma) M_\alpha$, where $\zeta_\alpha(\Gamma)$ is the number of ordered colorings $\lambda : V \to \{1, \cdots, k(\alpha)\}$ of the type

$\text{co}(\lambda) = \alpha$. The polynomial $\chi(\Gamma, m) = \chi(B(\Gamma), m)$ counts the number of ordered colorings with at most m colors.

Remark 7.1. Stanley's chromatic symmetric function of a graph X_Γ introduced in [17] is the enumerator of proper colorings $\lambda : V(\Gamma) \to \mathbb{N}$. A coloring λ is proper if the graph Γ does not contain a monochromatic edge, i.e. the induced graph on $\lambda^{-1}(\{i\})$ for each color $i \in \mathbb{N}$ is discrete. The sizes of monochromatic parts define the type of the proper coloring which is a partition of the number of vertices of the graph since ordering of colors is inessential. The assignment X_Γ is the canonical morphism from the chromatic Hopf algebra of graphs to symmetric functions, see ([1], Example 4.5). The coefficients $c_\mu(\Gamma)$ in the expansion in the monomial basis of symmetric functions

$$X_\Gamma = \sum_{\mu \vdash n} c_\mu(\Gamma) m_\mu,$$

count the numbers of proper colorings of prescribed types $\mu \vdash n$. Recall that $m_\mu = \sum_{s(\alpha)=\mu} M_\alpha$, where the sum is over all compositions $\alpha \models n$ that can be rearranged to the partition $\mu \vdash n$.

The coefficients $\zeta_\alpha(\Gamma), \alpha \models n$ satisfy the following properties. Recall that a graph Γ is called q-connected if it remains connected after removing any $q-1$ vertices.

Theorem 7.2. *Given a graph Γ on the set of vertices $V = [n]$.*

(a) The coefficients $\zeta_{(k,1^{n-k})}(\Gamma), 1 \leq k \leq n$ determine the $f-$vector of the independence complex $Ind(\Gamma)$ of the graph Γ.

(b) If Γ is q-connected then $\zeta_\alpha(\Gamma) = 0$ for all $\alpha \models n$ with $a_j > 1$ for some $j > k(\alpha) - q$.

(c) If Γ is q-connected then $\zeta_{(1^{n-q-k},k,1^q)}(\Gamma)$ is determined by q-element sets of vertices $S \subset V$ such that $\Gamma\mid_{V\setminus S}$ has k components.

(d) For any pair $\alpha \preceq \beta$ it holds $\zeta_\alpha(\Gamma) \leq \zeta_\beta(\Gamma)$.

(e) $\zeta_\alpha(\Gamma) \leq c_\mu(\Gamma)$ for each composition $\alpha \models n$ such that $s(\alpha) = \mu \vdash n$ and $c_\mu(\Gamma)$ are the coefficients of X_Γ in the monomial basis $\{m_\mu\}_{\mu \vdash n}$ of symmetric functions.

Proof. Recall that the coefficient $\zeta_\alpha(\Gamma)$ counts the number of ordered colorings $\lambda : V \to [k(\alpha)]$ of the type $\alpha \models n$.

(a) The only condition for a coloring $\lambda : V \to [n-k+1]$ to be ordered with $\text{type}(\lambda) = (k, 1^{n-k})$ is that the set of vertices colored by 1 is k-element and discrete. Hence $\zeta_{(k,1^{n-k})}(\Gamma) = (n-k)! f_{k-1}(Ind(\Gamma))$.

(b) Take $j_0 = \max\{j \mid a_j > 1\}$. Removing the vertices colored by last $k - j_0$ colors disconnects the graph Γ, so $k - j_0 \geq q$. Specially if Γ is connected then $\zeta_\alpha(\Gamma) = 0$ for all $\alpha \models n$ with $a_{k(\alpha)} > 1$.

(c) Let $\lambda : V \to [n-k+1]$ be an ordered coloring with $\mathrm{type}(\lambda) = (1^{n-q-k}, k, 1^q)$. Removing the vertices colored by last q colors disconnects Γ into k parts. On the other hand any choice of q vertices which disconnects the graph into k parts defines $(n-q-k)!q!\prod_{j=1}^{k} m_j$ ordered colorings of the type $(1^{n-q-k}, k, 1^q)$, where $m_j, j = \overline{1,k}$ are the sizes of components.

(d) Suppose that α is obtained from β by combining some of its adjacent parts, i.e. $\alpha = (a_1, \ldots, a_i, \ldots, a_k)$ and $\beta = (a_1, \ldots, a_i^{'}, a_i^{''}, \ldots, a_k)$ with $a_i = a_i^{'} + a_i^{''}$. Then any ordered coloring of the type α defines at least $\binom{a_i}{a_i^{'}}$ ordered colorings of the type β.

(e) It is obvious since any ordered coloring of a type $\alpha \models n$ is the coloring of the type $s(\alpha) \vdash n$.

$\square$

Example 7.3. The invariant F_Γ differs graphs on 5 vertices. Specially graphs given in Stanley's example of graphs with the same chromatic symmetric functions X_Γ are distinguished by F_Γ.

Proposition 7.4. *The invariant F_Γ is not a complete invariant of graphs, i.e. there are non-isomorphic graphs which are not distinguished by F_Γ.*

Proof. The total number γ_n of non-isomorphic graphs on n vertices satisfies $\gamma_n \sim 2^{\binom{n}{2}}/n!, n \to \infty$ and $\gamma_n > 2^{\binom{n}{2}}/n!$. The coefficients of the expansion $F_\Gamma = \sum_{\alpha \models n} c_\alpha L_\alpha$ are in the range $0 \leq c_\alpha \leq n!$. The statement follows from the inequality $2^{\binom{n}{2}}/n! > 2^{n-1}n!$, which holds for $n > 12$. $\square$

The following theorem allows one to define the invariant F_Γ recursively starting with $F_\emptyset = M_{()} = 1$. Recall that $F \mapsto (F)_1$ is the shifting operator, see Definition 5.3.

Theorem 7.5. *For a connected graph Γ on the vertex set $[n]$ it holds*

$$F_\Gamma = \sum_{i \in [n]} (F_{\Gamma \setminus \{i\}})_1.$$

Proof. We arrange the vertices $v \in P_{B(\Gamma)}$ according to the maximal elements of corresponding posets P_v. Let $T(B(\Gamma))_i = \{T_v \mid v \in P_{B(\Gamma)}, \max P_v = i\}$ be the multiset of specified $B(\Gamma)$-trees. Then by Theorem 5.5 we have

$$F_\Gamma = \sum_{i=\overline{1,n}} \sum_{T \in T(B(\Gamma))_i} F(T).$$

The formula follows from the recurrence formula for T-partitions enumerators, see Theorem 5.4

$$\sum_{T\in T(B(\Gamma))_i} F(T) = \sum_{T\in T(B(\Gamma))_i} (F(T\setminus\{\mathrm{root}(T)\}))_1 = (F_{\Gamma\setminus\{i\}})_1.$$

□

As an application of Theorem 7.5 we obtain the recurrence relations satisfied by enumerators $F(Q)$ for $Q = Pe^{n-1}, As^{n-1}, Cy^{n-1}, St^{n-1}$. We assume the realization of Q as a graph-associahedron of the corresponding graph as in Example 2.1. By convention the only (-1)-dimensional polytope is $\emptyset$.

Corollary 7.6. *For $n \geq 1$ the following recurrence relations hold*

$$F(Pe^{n-1}) = n(F(Pe^{n-2}))_1,$$
$$F(As^{n-1}) = (\sum_{k=1}^{n} F(As^{k-2})F(As^{n-k-1}))_1,$$
$$F(Cy^{n-1}) = n(F(As^{n-2}))_1,$$
$$F(St^{n-1}) = ((n-1)F(St^{n-2}) + M_{(1)}^{n-1})_1.$$

From Proposition 6.1 and Corollary 7.6 we recover the recurrence relations satisfied by numbers of vertices of corresponding graph-associahedra. Note that $\chi((F)_1, -1) = -\chi(F, -1)$ which is a consequence of $\chi(M_\alpha, -1) = (-1)^{k(\alpha)}$.

Corollary 7.7. *For $n \geq 1$ we have that the number of vertices $p_n = f_0(Pe^{n-1})$, $a_n = f_0(As^{n-1}), c_n = f_0(Cy^{n-1})$ and $s_n = f_0(St^{n-1})$ satisfy*

$$p_n = np_{n-1},$$
$$a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k},$$
$$c_n = na_{n-1},$$
$$s_n = (n-1)s_{n-1} + 1$$

with $p_1 = a_1 = c_1 = s_1 = 1$. Therefore $p_n = n!, a_n = \frac{1}{n+1}\binom{2n}{n}, c_n = \binom{2n-2}{n-1}$ and $s_n = (n-1)!\sum_{k=0}^{n-1}\frac{1}{k!}$.

8 Conclusion

We conclude with several natural questions in connection with the Hopf algebra $\mathcal{B}$ and the graph invariant F_Γ.

Problem 8.1. In a combinatorial Hopf algebra are defined the generalized Dehn-Sommerville relations which characterize the odd subalgebra (see [1], Section 5). Find a graph or a building set that satisfies the generalized Dehn-Sommerville relations for $\mathcal{B}$. The same problem is resolved in [9] for the chromatic Hopf algebra of hypergraphs, where the whole class of solutions called eulerian hypergraphs are found.

Problem 8.2. In what extent the function F_Γ differs simple graphs? Find two non-isomorphic graphs Γ_1 and Γ_2 such that $F_{\Gamma_1} = F_{\Gamma_2}$. According to Theorem 7.5, two graphs Γ_1 and Γ_2 with the same multisets of vertex-deleted subgraphs satisfy $F_{\Gamma_1} = F_{\Gamma_2}$. But this leads to the famous Reconstruction conjecture in graph theory. One could try to find two such graphs by using linear dependence relations among enumerators of T-partitions, see Example 3. Relate Stanly's chromatic symmetric function X_Γ with F_Γ. Does it hold that $X_{\Gamma_1} \neq X_{\Gamma_2}$ implies $F_{\Gamma_1} \neq F_{\Gamma_2}$?

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