

Subsemigroup, ideal and congruence growth of free semigroups

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Abstract

Using Rees index, the subsemigroup growth of free semigroups is investigated. Lower and upper bounds for the sequence are given and it is shown to have superexponential growth of strict type n^n for finite free rank greater than 1. It is also shown that free semigroups have the fastest subsemigroup growth of all finitely generated semigroups. Ideal growth is shown to be exponential with strict type 2^n and congruence growth is shown to be at least exponential. In addition we consider the case when the index is fixed and rank increasing, proving that for subsemigroups and ideals this sequence fits a polynomial of degree the index, whereas for congruences this fits an exponential equation of base the index. We use these results to describe an algorithm for computing values of these sequences and give a table of results for low rank and index.

1 Introduction

The concept of word growth of a finitely generated group has been a central research topic connecting differential geometry, geometric and combinatorial group theory for the past 50 years¹. Given a finitely generated group, take the sequence that counts the number of elements of the group of length at most n (with respect to some finite generating set). In 1981, answering a question of Milnor, Gromov proved that a finitely generated group is virtually nilpotent if and only if this sequence has polynomial growth [17]. This powerful result indicated the strong connections between a groups algebraic properties and its asymptotic behaviour.

This result inspired the definition of subgroup growth for a group, see for instance [20]. Given a finitely generated group, take the sequence that counts the number of index n subgroups of the group. In 1993 Lubotzky, Mann and Segal proved that a finitely generated residually finite group is virtually solvable of finite rank if and only if this sequence has polynomial growth [19]. This area has now been extended to many other mathematical objects, e.g. representation growth and subring growth [21].

In this paper we consider the situation for semigroups. In order to define subsemigroup growth of a semigroup we first need to define index. At first, it is not clear what the correct definition should be, and there have been many different attempts depending on the types of semigroups considered. Ideally you want the notion of index to generalise group index, but perhaps more importantly, you want finite index subsemigroups to preserve important properties

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¹For a history, see the introduction to [16].

of the semigroup. For example, Grigorchuk gave a definition of index for a subsemigroup which generalised group index and impressively extended Gromov's polynomial growth theorem to cancellative semigroups. However, his definition of index does not preserve even the simplest property of finite generation [15].

In this paper we choose Rees index, which is defined simply to be the cardinality of the complement. This clearly does not generalise group index, but finite Rees index subsemigroups do preserve a very large number of important properties (e.g. being finitely generated/presented, residually finite, solvable word problem, automatic etc.) See the survey article [9] for more details. The very nature of Rees index makes this an inherently combinatorial problem. For example this paper includes results relating to: binomial coefficients, Stirling numbers (of the first and second kind), Bell numbers, Catalan numbers (and the generalised 'Fuss-Catalan' numbers) and Fibonacci numbers.

This work, and the techniques involved were inspired by the rank one situation: finite Rees index subsemigroups of the free monogenic semigroup are well studied in the literature under the name of numerical semigroups [8, 7, 23, 12, 5, 22]. Increasing interest is being shown in numerical semigroups, with applications arising in commutative algebra and algebraic geometry [4]. It has recently been proved that the numerical semigroups of genus n , that is, the Rees index n subsemigroups of the free monogenic semigroup, have Fibonacci-like growth [22], answering in the positive a conjecture of Bras-Amorós [6].

Recall the following results from the subgroup growth literature (where group index is used). Every free group F_r with free rank $r \geq 2$ has:

1. subgroup growth of strict type n^n [20, Cor 2.2],
2. subnormal subgroup growth of strict type $2^n = n^{n/\log(n)}$ [20, Cor 2.4],
3. normal subgroup growth of strict type $n^{\log(n)}$ [20, Cor 2.8].

Firstly we recall the definitions from the numerical semigroup literature that we adapt to the higher free rank case, then we prove some basic results about generating sets and how to construct semigroups of both higher and lower index from a given subsemigroup.

Using a generalisation of the subsemigroup tree of [8] we then illustrate upper and lower bounds for the number of (Rees) index n subsemigroups of the free semigroups FS_r and conclude that, similar to groups, FS_r has subsemigroup growth of strict type n^n for $r \geq 2$. In Section 5 we consider the situation when the index is fixed and the free rank varies, in which case we show it fits an exact polynomial of degree the index.

We then consider the question of counting just the ideals, after giving upper and lower bounds, we show that free semigroups of rank greater than 1 have exponential ideal growth of strict type $2^n = n^{n/\log(n)}$, and for fixed index they also satisfy a polynomial of degree the index.

Finally congruence growth is considered, where we count the number of congruences with n classes, and we show this to have at least exponential growth. We conjecture that it is in fact exponential. This is analagous to counting normal subgroups of free groups which have intermediate growth, so in some sense free semigroups have 'more' quotients than free groups. We also consider the sequence for a fixed number of classes n as the free rank r increases and show this satisfies an exponential equation of base n .

Part of this project was computational. We use some of our results to describe algorithms which were implemented to calculate values of the sequences for low rank and index. The code is available online [1] and tables of the results are presented at the end of the paper as appendices.

2 Preliminaries

Let S be a semigroup, the **Rees index** of a subsemigroup T of S is defined to be $|S \setminus T|$, and we say T has finite index in S if $|S \setminus T| < \infty$.

Let $X_r = \{g_1, \dots, g_r\}$ be a finite set of symbols, and $FS_r = X_r^+$ denote the free semigroup of rank r on the set X_r , that is, all non-empty words over the alphabet X_r .

Let $\Lambda \subseteq FS_r$ be a finite (Rees) index subsemigroup of FS_r . Following terminology from numerical semigroups, we call $G(\Lambda) = FS_r \setminus \Lambda$ the set of **gaps** of Λ , and we remark that the index of Λ is equal to $|G(\Lambda)|$ (this is usually called the genus). We use $|w|$ to denote the length of any word $w \in FS_r$. We now define a total order on FS_r , usually called the **shortlex order**, where we order first by word length, and then lexicographically. Using the shortlex order we call $f(\Lambda) = \max\{|w| \mid w \in G(\Lambda)\}$ the **Frobenius** of Λ , and $m(\Lambda) = \min\{|w| \mid w \in \Lambda\}$ the **multiplicity** of Λ .

Given any word $w \in FS_r$ and any $1 \leq i \leq |w| - 1$, let $w_{\text{pre}(i)}$ denote the **prefix** of w of length i , and $w_{\text{suf}(i)}$ denote the **suffix** of w of length i .

Lemma 2.1. *For any index n subsemigroup Λ of FS_r , $|f(\Lambda)| \leq 2n - 1$.*

Proof. Let $\Lambda \subseteq FS_r$ with $|G(\Lambda)| = n$, and assume $|f(\Lambda)| = k \geq 2n$. For every $1 \leq i \leq n$, either $f_{\text{pre}(i)} \in G(\Lambda)$ or $f_{\text{suf}(k-i)} \in G(\Lambda)$ as otherwise $f \notin G(\Lambda)$. Therefore, along with $f \in G(\Lambda)$, there must be at least $n + 1$ distinct words in $G(\Lambda)$ which is a contradiction. \square

This immediately gives the following result:

Corollary 2.2. *There are only finitely many index n subsemigroups of FS_r .*

Proof. Let $s = r + r^2 + \dots + r^{2n-1}$ be the number of words of FS_r of length less than $2n$, then there are at most $\binom{s}{n}$ possible choices for the set of gaps. \square

Another basic result that will be used frequently is the following:

Lemma 2.3. *Every finite index subsemigroup of FS_r has a finite unique minimal generating set.*

Proof. Let $\Lambda \subseteq FS_r$ be an index n subsemigroup of FS_r . It is clear that Λ is finitely generated as every word of length at least $4n$ can be written as a product of two words of length at least $2n$ which are all in Λ by Lemma 2.1. Therefore there is a minimal size, say $k \in \mathbb{N}$, for any generating set of Λ . Take two minimal generating sets $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ of Λ and assume $X \neq Y$. Let $y_i \in Y \setminus X$, then $y_i = x_{i_1} x_{i_2} \dots x_{i_l}$ for some $x_{i_1}, x_{i_2}, \dots, x_{i_l} \in X$ with $l \geq 2$. Therefore $|y_i| > |x_{i_j}|$ for each $1 \leq j \leq l$. Now since Y is a generating set, every x_{i_j} is a product of elements from $Y \setminus \{y_i\}$, and so y_i is a product of elements from $Y \setminus \{y_i\}$ contradicting the minimality of Y , hence $X = Y$. \square

Given any finitely generated semigroup S , let $a_n(S)$ denote the number of index n subsemigroups of S , and let $s_n(S) = \sum_{i=1}^n a_i(S)$ be the partial sums. That $a_n(S)$ is always finite is implied by Corollary 2.2 and the next result which essentially says that free semigroups have the fastest subsemigroup growth.

Proposition 2.4. *If S can be generated by r elements, then $a_n(S) \leq a_n(FS_r)$ for all n .*

Proof. Let $S = \langle s_1, \dots, s_r \rangle$, then there exists an epimorphism $\phi : FS_r \rightarrow S$. Given any $s \in S$ there exists a minimal $\bar{s} \in FS_r$ with respect to the shortlex order such that $\phi(\bar{s}) = s$. Let T be an index n subsemigroup of S with $S \setminus T = \{w_1, \dots, w_n\}$ and let $G = \{\bar{w}_1, \dots, \bar{w}_n\}$. The result

follows if we can show that $\Lambda = FS_r \setminus G$ is a subsemigroup of FS_r . Assume that there exists $\overline{w}_i \in G$, and $x, y \in \Lambda$ such that $\overline{w}_i = xy$, then $w_i = \phi(\overline{w}_i) = \phi(x)\phi(y)$. Since T is a subsemigroup, either $\phi(x)$ or $\phi(y)$ is in $S \setminus T$. Without loss of generality, assume $\phi(x) = w_j \in S \setminus T$. Since $\phi(\overline{w}_j) = \phi(x)$ and $x \notin G$, then $\overline{w}_j < x$ by the minimality of \overline{w}_j . However, that implies $\overline{w}_j y < \overline{w}_i$ but $\phi(\overline{w}_j y) = \phi(\overline{w}_j)\phi(y) = \phi(x)\phi(y) = w_i$ which contradicts the minimality of \overline{w}_i and so Λ is a subsemigroup. \square

Recall that given two sequences $f(n), g(n)$, we say that:

- $f(n) = O(g(n))$ if there exists a constant $C > 0$ such that $f(n) \leq C \cdot g(n)$ for all large n .
- $f(n) \asymp g(n)$ if $f(n) = O(g(n))$ and $g(n) = O(f(n))$.
- $f(n) \sim g(n)$ if $f(n)/g(n) \rightarrow 1$ for large n .

We say that a semigroup S has **subsemigroup growth of strict type** $f(n)$ if

$$\log(s_n(S)) \asymp \log(f(n)).$$

Note that, unless it has a subscript, \log is always base 2.

3 Minimal generators

In this section we show how the minimal generators of a finite index subsemigroup of FS_r are connected to its set of gaps. In particular we show the different forms that a generator can take and use this information to outline an algorithm for calculating the set of minimal generators from the set of gaps.

Given a finite index subsemigroup $\Lambda \subseteq FS_r$, let $MG(\Lambda)$ denote the set of minimal generators of Λ .

Remark 3.1. Note that $h \in MG(\Lambda)$ if and only if $h \in \Lambda$ and $\{h_{\text{pre}(i)}, h_{\text{suf}(|h|-i)}\} \cap G(\Lambda) \neq \emptyset$ for all $1 \leq i \leq |h| - 1$. This remark will be used ubiquitously without reference throughout this paper.

Given any index n subsemigroup $\Lambda \subseteq FS_r$ we can construct new semigroups in the following ways:

1. Given $f = f(\Lambda)$, let Λ^f denote the set $\Lambda \cup \{f\}$, which is an index $n - 1$ subsemigroup as every product wf or fw with $w \in \Lambda$, is larger than f in shortlex order (and hence an element of Λ^f).
2. Given any $h \in MG(\Lambda)$, let Λ_h denote the set $\Lambda \setminus \{h\}$. This is a subsemigroup of index $n + 1$, as no pair of elements $x_1, x_2 \in \Lambda$ satisfy $x_1 x_2 = h$. This is the content of Remark 3.1.

Observe that in case (1), f becomes a minimal generator of Λ^f , and in case (2), h is no longer a minimal generator. In general these constructions will make changes to the minimal generating set. Minimal generators of Λ that are no longer minimal generators of Λ^f when f is added are said to **turn off**. Similarly, elements of Λ that become new minimal generators of Λ_h are said to **turn on**.

The remainder of this section is considering the interplay of these two constructions and their effect on the minimal generating set.

Lemma 3.2. *Given a finite index subsemigroup Λ of FS_r , let $h \in MG(\Lambda)$ and let $Y = MG(\Lambda_h) \setminus (MG(\Lambda) \setminus \{h\})$ be the set of new generators turned on when we remove h from Λ . Then $x \in Y$ only if it has one of the following three forms:*

1. $x = hw$ with $w \in MG(\Lambda)$; or
2. $x = wh$ with $w \in MG(\Lambda)$; or
3. $x = hwh$ with $w \in MG(\Lambda)$.

Proof. Let $x \in Y$ and note that $G(\Lambda_h) = G(\Lambda) \cup \{h\}$. Then $x \in \Lambda_h$ and $\{x_{\text{pre}(i)}, x_{\text{suf}(|x|-i)}\} \cap (G(\Lambda) \cup \{h\}) \neq \emptyset$ for all $1 \leq i \leq |x| - 1$, but $x \notin MG(\Lambda)$ and so there exists some $x_1, x_2 \in \Lambda$ such that $x = x_1x_2$. Therefore either $x_1 = h$ or $x_2 = h$. We consider each case separately:

1. Let $x_1 = h$. If $x_2 \in MG(\Lambda)$ then x is of form 1. So assume $x_2 \notin MG(\Lambda)$ and $x_2 = a_1 \cdots a_k$, where $a_1, \dots, a_k \in MG(\Lambda)$ and $k \geq 2$. Since $x \in MG(\Lambda_h)$ then either $ha_1 \in G(\Lambda) \cup \{h\}$ or $a_2 \cdots a_k \in G(\Lambda) \cup \{h\}$. Now if $ha_1 \in G(\Lambda) \cup \{h\}$, then $ha_1 \in G(\Lambda)$ which is a contradiction as both $h, a_1 \in \Lambda$. Therefore $a_2 \cdots a_k \in G(\Lambda) \cup \{h\}$ which implies $a_2 \cdots a_k = h$. If $k \geq 3$ then we get a contradiction as h is a minimal generator in Λ . Hence $k = 2$, $a_2 = h$ and x satisfies form 3.
2. Let $x_2 = h$. If $x_1 \in MG(\Lambda)$ then it is of form 2. Otherwise the proof is similar to the previous case and x is of form 3. \square

Lemma 3.3. *Given any finite index subsemigroup Λ of FS_r , we have $h \in MG(\Lambda)$ only if it has one of the following four forms:*

1. $h = g_i$ where $g_i \in X_r$; or
2. $h = xg_i$ where $x \in G(\Lambda)$, $g_i \in X_r$; or
3. $h = g_ix$ where $x \in G(\Lambda)$, $g_i \in X_r$; or
4. $h = xg_iy$, where $x, y \in G(\Lambda)$, $g_i \in X_r$.

Proof. We prove this by induction on the index. Let $|G(\Lambda)| = 0$, then x must be of form 1. Assume the statement is true for all subsemigroups of index n . Given any subsemigroup Λ with index $n + 1$, we can construct a subsemigroup of index n by considering Λ^f , where $f = f(\Lambda)$. If $h \in MG(\Lambda^f)$, then by assumption h has the correct form. Suppose that $h \notin MG(\Lambda^f)$ then h is turned on when we remove f from Λ^f , so by Lemma 3.2 it has one of the forms wf , fw or fwf where $w \in MG(\Lambda^f)$. We consider each case separately:

1. Let $h = fw$ where $w \in MG(\Lambda^f)$, then either $|w| = 1$ and h is of form 2, or $|w| \geq 2$ and $w = g_iw'$ for some $g_i \in X_r$, $w' \in FS_r$. As f is the Frobenius of Λ , we know that $fg_i \in \Lambda$. Now $h \in MG(\Lambda)$, so we must have $w' \in G(\Lambda)$ and h is of form 4.
2. Let $h = wf$ where $w \in MG(\Lambda^f)$, then either $|w| = 1$ and h is of form 3, or similarly to the previous case, h is of form 4.
3. Let $h = fwf$, then either $|w| = 1$ and h is of form 4, or $|w| \geq 2$ and $fwf = (fx_1)(x_2f)$ is not a minimal generator of Λ . \square

We can now use Lemma 3.3 to outline an algorithm that will calculate the set $MG(\Lambda)$ from the set $G(\Lambda)$. Let ϵ be the empty word.

Algorithm 1 Find minimal generators from set of gaps

Require: $G(\Lambda)$ the set of gaps of a finite index subsemigroup Λ of FS_r , and X_r

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1: procedure GENERATORS( $G(\Lambda), X_r$ )
2:   Gens  $\leftarrow$  {}
3:   for all  $x \in G(\Lambda) \cup \{\epsilon\}$  do
4:     for all  $y \in G(\Lambda) \cup \{\epsilon\}$  do
5:       for all  $g \in X_r$  do
6:         if MINGEN( $xgy, G(\Lambda)$ ) then
7:           Gens  $\leftarrow$  Gens  $\cup$  { $xgy$ }
8:         end if
9:       end for
10:    end for
11:  end for
12:  return Gens
13: end procedure

```

Where MINGEN($w, G(\Lambda)$) determines whether or not a word $w \in \Lambda$ is a minimal generator of Λ .

Algorithm 2 Check if a word is a minimal generator

Require: A word $w \in \Lambda$ and the set of gaps $G(\Lambda)$

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1: procedure MINGEN( $w, G(\Lambda)$ )
2:   Pass  $\leftarrow$  FALSE
3:   if  $w \notin G(\Lambda)$  then
4:     Pass  $\leftarrow$  TRUE
5:      $n \leftarrow$  length( $w$ )
6:      $i \leftarrow 1$ 
7:     while Pass and  $i < n$  do
8:       if  $w_{\text{pre}(i)} \notin G(\Lambda)$  and  $w_{\text{suf}(n-i)} \notin G(\Lambda)$  then
9:         Pass  $\leftarrow$  FALSE
10:      end if
11:       $i \leftarrow i + 1$ 
12:    end while
13:  end if
14:  return Pass
15: end procedure

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4 Subsemigroup growth

The sequence $a_n(FS_r)$ has been extensively studied for when $r = 1$, this is precisely the number of numerical semigroups of genus n . It has recently been proved [22] that $a_n(FS_1)$ has ‘Fibonacci like’ growth, that is, $a_n(FS_1) \sim K\phi^n$, where K is a constant and $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ration. In this section we give lower and upper bounds for $a_n(FS_r)$ and in particular show that for $r \geq 2$, $a_n(FS_r)$ grows superexponentially in n with strict growth type n^n . We imitate the methods used in [8] by constructing a tree of all finite index subsemigroups of FS_r .

4.1 Subsemigroup tree

It is clear that every subsemigroup $\Lambda \subseteq FS_r$ of index $n + 1$ gives rise to a subsemigroup Λ^f of index n . Therefore, every index n subsemigroup can be obtained from a subsemigroup of index $n - 1$ by removing a minimal generator larger than the Frobenius (using the shortlex order). Given a subsemigroup Λ of finite index, any subsemigroup Λ_h that is obtained by removing a minimal generator h larger than the Frobenius $f(\Lambda)$ is a descendant. This gives a method for obtaining a tree of all finite index subsemigroups of FS_r .

For example: given the index 1 subsemigroup $\{a\}^c$ of $FS_2 = \langle a, b \rangle$, the minimal generators of $\{a\}^c$ are $\{b, a^2, ab, ba, a^3, aba\}$ and they are all bigger than the Frobenius, the minimal generators of $\{b\}^c$ are $\{a, ab, ba, b^2, bab, b^3\}$ but a is not bigger than the Frobenius and so $\{a, b\}^c$ is a descendant of $\{a\}^c$ but not of $\{b\}^c$.

So the beginning of the tree of all index n subsemigroups of FS_2 looks like Figure 1.

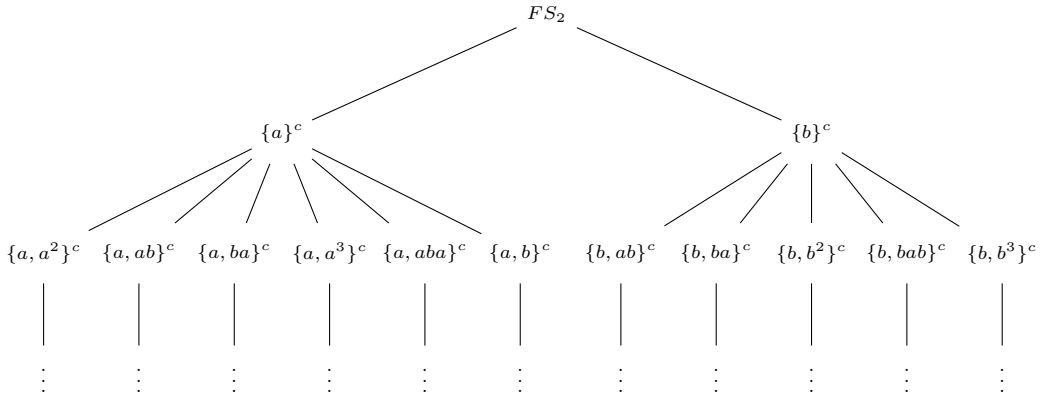


Figure 1: Subsemigroup tree of FS_2

It is clear that $a_n(FS_r)$ is the number of nodes on the n^{th} level of this tree.

Inspired by the numerical semigroup situation [7, 8], we say that a finite index subsemigroup $\Lambda \subseteq FS_r$ is **ordinary** if $f(\Lambda) < m(\Lambda)$ in the shortlex order, that is, all the gaps are ‘at the beginning’. For a given rank r , clearly there is one ordinary subsemigroup for each index n which we denote by $O_r(n)$.

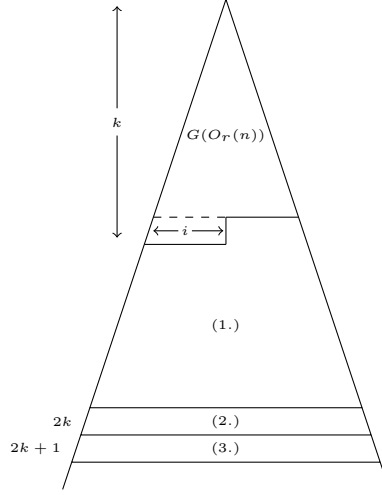
Lemma 4.1. *In this tree the subsemigroup $O_r(n)$ has $(r - 1)n^2 + (2r - 1)n + r$ descendants.*

Proof. Let $O_r(n)$ be the ordinary index n subsemigroup of FS_r and let $k = |f(O_r(n))|$. Figure 2 represents the set of words of $G(O_r(n))$ within the set of all words in FS_r :

Let $i \geq 1$ be the number of words of length k that belong to the set of gaps $G(O_r(n))$. Observe that $n = (\sum_{j=1}^{k-1} r^j) + i$ where $1 \leq i \leq r^k$.

We now consider, in three separate cases, which of the words $w \in O_r(n)$ are minimal generators. These cases correspond to the numbered regions in Figure 2.

1. Let $|w| \leq 2k - 1$. Then $w \in MG(O_r(n))$ if and only if $w \in O_r(n)$ as w cannot be written as a product of two words in $O_r(n)$. Hence there are $(\sum_{j=k}^{2k-1} r^j) - i$ such w that are minimal generators.

Figure 2: Set of gaps $G(O_r(n))$ within FS_r .

2. Let $|w| = 2k$. Then $w \in MG(O_r(n))$ if and only if $w = w_k w'_k$ where $|w_k| = |w'_k| = k$ and at least one of w_k, w'_k are in $G(O_r(n))$. Being careful not to double count the cases where $w_k = w'_k$ we have that there are $\sum_{j=0}^{i-1} (2(r^k - j) - 1) = 2ir^k - i^2$ such $w \in O_r(n)$.
3. Let $|w| \geq 2k + 1$. Clearly no word of length at least $2k + 2$ is a minimal generator as every such word is a product of two words of length at least $k + 1$, all of which are in $O_r(n)$. This leaves the words of length $2k + 1$, which are minimal generators if and only if $w_{\text{pre}(k)} \in G(\Lambda)$ and $w_{\text{suf}(k)} \in G(\Lambda)$, with the middle letter of w being any of the r generators of FS_r . There are therefore ri^2 such words.

Hence

$$|MG(O_r(n))| = \left(\binom{2k-1}{\sum_{j=k}^{2k-1} r^j} - i \right) + (2ir^k - i^2) + (ri^2)$$

After some manipulation, we deduce that:

$$\begin{aligned} |MG(O_r(n))| &= (r-1) \left(\binom{k-1}{\sum_{j=1}^{k-1} r^j} + i \right)^2 + (2r-1) \left(\binom{k-1}{\sum_{j=1}^{k-1} r^j} + i \right) + r \\ &= (r-1)n^2 + (2r-1)n + r. \end{aligned}$$

Since every word in $O_r(n)$ is bigger than the Frobenius, every minimal generator gives rise to a descendant and the result follows. \square

4.2 Lower bound

We now describe how to construct a lower bound for the sequence $a_n(FS_r)$ by constructing a subtree of the subsemigroup tree.

Following [7], we construct a subtree of the subsemigroup tree using the ordinary subsemigroups. We describe it as follows: beginning with the single vertex representing FS_r we attach

to this root the subsemigroups of index 1 obtained by removing each generator. We now proceed inductively. Consider a node (that represents a subsemigroup Λ):

1. If the subsemigroup Λ is ordinary then attach, for every $h \in MG(\Lambda)$, a node for Λ_h .
2. If the subsemigroup Λ is not ordinary, then it is obtained by removing minimal generators from some ordinary subsemigroup of a lower index. Add nodes for each of the minimal generators h of Λ that are bigger than the Frobenius $f(\Lambda)$ and are also minimal generators of this ordinary subsemigroup.

This is a subtree of the subsemigroup tree that has one infinite branch consisting of ordinary subsemigroups of FS_r . The number of nodes on the n^{th} level of the tree is a lower bound for $a_n(FS_r)$. Recalling the result from Lemma 4.1, if we let $p(n, r) = (r - 1)n^2 + (2r - 1)n + r$ be the number of descendants of $O_r(n)$, then Figure 3 illustrates the beginning of this subtree for FS_2 .

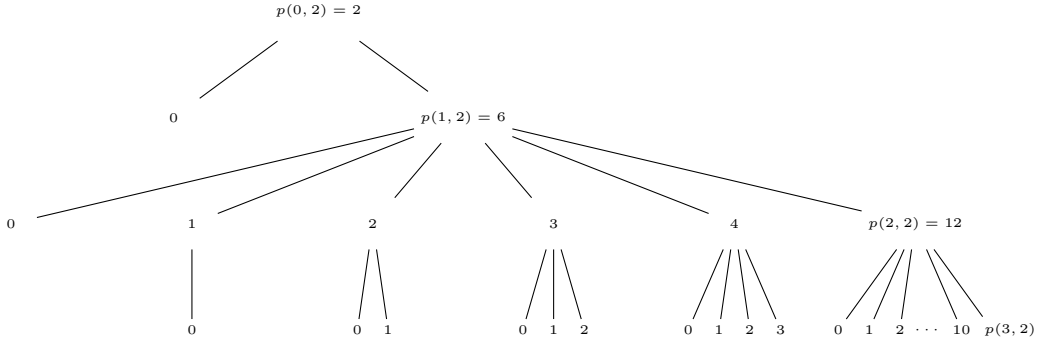


Figure 3: Subtree of subsemigroup tree for FS_2

Observe that every node belongs to a branch starting from some ordinary subsemigroup $O_r(n)$, and corresponds to choosing a subset of $MG(O_r(n))$. Therefore it is clear that the number of nodes on the n^{th} level of this tree is

$$L(n, r) := \sum_{i=0}^{J(n, r)} \binom{p(n - i, r) - 1}{i}.$$

where

$$J(n, r) = \begin{cases} n/2 & \text{for } r = 1 \\ \left\lfloor (n - 1) + \frac{(2r-1) - \sqrt{(r-1)n + (2r-1)}}{r-1} \right\rfloor & \text{for } r > 1 \end{cases}$$

where $J(n, r)$ is obtained from the inequality $p(n - i, r) - 1 \geq i$. Hence we have proved the following Theorem:

Theorem 4.2. $a_n(FS_r) \geq L(n, r)$ for all r, n .

Note that $L(n, 1) = F_{n+1}$ the Fibonacci numbers, which is a good lower bound for the numerical semigroup case, since $a_n(FS_1) \sim C \cdot L(n, 1)$ for some constant C [22]. It seems reasonable to conjecture that for each $r \geq 1$, $a_n(FS_r) \sim C_r \cdot L(n, r)$ for some constant C_r .

The proof in [22] that $a_n(FS_1) \sim C \cdot L(n, 1)$ relied on the fact that almost all numerical semigroups satisfy $f < 3m$, where m is the multiplicity and f is the frobenius. It would be of interest to know whether this proof can be extended to the higher rank case, that is, do almost all finite index subsemigroups $\Lambda \subseteq FS_r$ satisfy $|f(\Lambda)| < 3 \cdot |m(\Lambda)|$?

Theorem 4.3. *For $r \geq 2$, $\log(n^n) = O(\log(s_n(FS_r)))$.*

Proof. First note that $n/2 \leq J(n, r)$ so by considering the term $i = n/2$ we have

$$s_n(FS_r) \geq a_n(FS_r) \geq L(n, r) \geq \binom{p(n - n/2, r) - 1}{n/2}.$$

Note, we can continuously extend the binomial coefficients using Gamma functions, so we need not worry whether n is even. When $r \geq 2$, $p(n, r) \geq n^2 + 1$ and so

$$s_n(FS_r) \geq \binom{(n/2)^2}{n/2} \geq \left(\frac{(n/2)^2}{n/2}\right)^{n/2} = (n/2)^{n/2}.$$

Thus $\log(s_n(FS_r)) \geq \frac{n}{2}(\log(n) - \log(2))$ and so $n \log(n) = O(\log(s_n(FS_r)))$. \square

4.3 Upper bound

We now construct an upper bound for the sequence $a_n(FS_r)$. In order to do this we show that, similar to the situation for numerical semigroups (see [7]), ordinary subsemigroups have the maximum number of descendants. To show this we require the following Proposition:

Proposition 4.4. *Let Λ be a finite index non-ordinary subsemigroup of FS_r , then $|MG(\Lambda)| \leq |MG(\Lambda_m^f)|$.*

Proof. We prove the statement by showing that every minimal generator of Λ turned off by adding f gives rise to a unique new minimal generator of Λ_m^f turned on by removing m from Λ^f . First note that the minimal generators of Λ that are turned off by adding f are precisely the new minimal generators of Λ turned on by removing the minimal generator f from Λ^f . By Lemma 3.2, such a minimal generator $x \in D = MG(\Lambda) \setminus (MG(\Lambda^f) \setminus \{f\})$ has one of three possible forms fw, wf or fwf where $w \in MG(\Lambda^f)$. Note that x could have more than one of these forms. We now partition D in to nine distinct (possibly empty) sets.

$$\begin{aligned} D_1 &:= \{fwf \in D \mid w \in MG(\Lambda^f)\} \\ D_2 &:= \{fyf \in D \mid yf \text{ or } fy \in MG(\Lambda^f), my \notin \Lambda^f\} \\ D_3 &:= \{fyf \in D \mid yf \text{ or } fy \in MG(\Lambda^f), my \in \Lambda^f, ym \notin \Lambda^f\} \\ D_4 &:= \{fyf \in D \mid yf \text{ or } fy \in MG(\Lambda^f), my, ym \in \Lambda^f\} \\ D_5 &:= \{fw \in D \mid w \in MG(\Lambda^f), m \neq w \neq f, fw \notin D_1 \cup D_2 \cup D_3 \cup D_4\} \\ D_6 &:= \{wf \in D \mid w \in MG(\Lambda^f), m \neq w \neq f, wf \notin D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5\} \\ D_7 &:= \{fm \in D \mid fm \notin D_6\} \\ D_8 &:= \{mf \in D \mid mf \notin D_5 \cup D_7\} \\ D_9 &:= \{f^2 \in D\} \end{aligned}$$

It is a straightforward matter to check that D_1, \dots, D_9 are indeed disjoint and that their union is D (in particular D_2 is disjoint from D_1 as fy or $yf \in MG(\Lambda^f)$ implies $y \in G(\Lambda)$, so $y \notin MG(\Lambda)$). Now for each $x \in D$, we are going to assign some $x \in C = MG(\Lambda_m^f) \setminus (MG(\Lambda^f) \setminus \{m\})$. All the proofs here use Remark 3.1.

1. Let $x = fwf \in D_1$. We intend to show that $mwm \in C$. Firstly, $|w| = 1$ as otherwise $w = x_1x_2$ for some $x_1, x_2 \in FS_r$ and $x = (fx_1)(x_2f) \notin MG(\Lambda)$ as $fx_1, x_2f \in \Lambda$. Also, since m and everything smaller than m is in $G(\Lambda_m^f)$, we have $mwm_{\text{pre}(i)} \in G(\Lambda_m^f)$ for all $1 \leq i \leq |m|$ and $mwm_{\text{suf}(|mwm|-i)} \in G(\Lambda_m^f)$ for all $|m| + 1 = |mw| \leq i \leq |mwm| - 1$. Now since $w, m \in \Lambda^f$ we see that $mwm \in \Lambda^f$, and $mwm \in MG(\Lambda_m^f)$ but $(mw)(m) \notin MG(\Lambda^f)$, so $mwm \in C$.
2. Let $x = fyf \in D_2$. Firstly, as before, $|y| = 1$ as otherwise $fyf \notin MG(\Lambda)$. Secondly, since $my \notin \Lambda^f$, then $my \notin \Lambda_m^f$. Now, since m and everything smaller than m is not in Λ_m^f we have $myf_{\text{pre}(i)} \in G(\Lambda_m^f)$ for all $1 \leq i \leq |m|$, and $myf_{\text{pre}(|myf|-(|m|+1))} = my \in G(\Lambda_m^f)$ and since $x \in MG(\Lambda)$ and $fy \in \Lambda$, we also have $myf_{\text{suf}(|myf|-i)} \in G(\Lambda_m^f)$ for all $|m| + 2 \leq i \leq |myf| - 1$. Therefore since $myf \in \Lambda_m^f$ we have $myf \in MG(\Lambda_m^f)$ but $(m)(yf) \notin MG(\Lambda^f)$, and so $myf \in C$.
3. Let $x = fyf \in D_3$. Similarly to the previous case, $fym \in C$.
4. Let $x = fym \in D_4$. Again, $|y| = 1$ as otherwise $fym \notin MG(\Lambda)$. Since m and everything smaller than m is not in Λ_m^f , we have $mym_{\text{pre}(i)} \in G(\Lambda_m^f)$ for all $1 \leq i \leq |m|$ and $mym_{\text{suf}(|mym|-i)} \in G(\Lambda_m^f)$ for all $|m| + 1 = |my| \leq i \leq |mym| - 1$. Since m and $my \in \Lambda^f$ we see $m \neq mym \in \Lambda_m^f$ and $mym \in MG(\Lambda_m^f)$ but $(my)(m) \notin MG(\Lambda^f)$, and so $mym \in C$.
5. Let $x = fw \in D_5$. Firstly, since $fw \notin D_1 \cup D_2 \cup D_3 \cup D_4$, then w does not have f as a proper suffix. Secondly, since m and $w \in \Lambda^f$, then $mw \notin MG(\Lambda^f)$. Now since $fw \in MG(\Lambda)$ and anything bigger than f is in Λ_m^f and w does not have f as a proper suffix, we have $fw_{\text{suf}(|fw|-i)} \in G(\Lambda_m^f)$ for all $|f| + 1 \leq i \leq |fw| - 1$. This implies $mw_{\text{suf}(|mw|-i)} \in G(\Lambda_m^f)$ for all $|m| + 1 \leq i \leq |mw| - 1$ and since m and everything smaller than m is in $G(\Lambda_m^f)$ we also have $mw_{\text{pre}(i)} \in G(\Lambda_m^f)$ for all $1 \leq i \leq |m|$. Finally, since $m, mw \in \Lambda^f$, then $m \neq mw \in \Lambda_m^f$ and $mw \in MG(\Lambda_m^f)$, so $mw \in C$.
6. Let $x = wf \in D_6$. Similarly to the previous case, this implies $wm \in C$.
7. Let $x = fm \in D_7$. Firstly, recall that since $fm \in MG(\Lambda)$ we have $fm \in \Lambda$ and $\{fm_{\text{pre}(i)}, fm_{\text{suf}(|fm|-i)}\} \cap G(\Lambda) \neq \emptyset$ for all $1 \leq i \leq |fm| - 1$. We wish to show that fm is also in $MG(\Lambda_m^f)$. It is enough to check the cases when f is a prefix or suffix of fm . Obviously f is a prefix, but in that case $m \in G(\Lambda_m^f)$. Assume fm has f as a suffix, that is $fm = wf$ for some $w \in FS_r$, not equal to f . Then $w \notin MG(\Lambda^f)$ as $fm \notin D_6$, but $|w| = |m|$ implies that $w \in MG(\Lambda^f)$. Hence, fm does not have f as a suffix. Observe also that $fm \in \Lambda_m^f$. Finally, $(f)(m) \notin MG(\Lambda^f)$ and so $fm \in C$.
8. Let $x = mf \in D_8$. Similarly to $x \in D_7$, we have $mf \in C$.
9. Let $x = f^2$. It is clear that m^2 is always a minimal generator of Λ_m^f as firstly, $m \neq m^2 \in \Lambda_m^f$ and secondly, m and everything smaller than m is in $G(\Lambda_m^f)$ and so $m^2_{\text{pre}(i)} \in G(\Lambda_m^f)$ for all $1 \leq i \leq |m|$ and $m^2_{\text{suf}(|m^2|-i)} \in G(\Lambda_m^f)$ for all $|m| \leq i \leq |m^2| - 1$. Clearly also $m^2 = mm \notin MG(\Lambda^f)$ and so $m^2 \in C$.

Hence we can construct a function

$$\Phi : D \rightarrow C$$

$$x \mapsto \begin{cases} mwm & \text{if } x = fwf \in D_1, \\ myf & \text{if } x = fyf \in D_2, \\ fym & \text{if } x = fyf \in D_3, \\ mym & \text{if } x = fyf \in D_4, \\ mw & \text{if } x = fw \in D_5, \\ wm & \text{if } x = wf \in D_6, \\ fm & \text{if } x = fm \in D_7, \\ mf & \text{if } x = mf \in D_8, \\ m^2 & \text{if } x = f^2 \in D_9. \end{cases}$$

Since D is the disjoint union of D_1, \dots, D_9 , this function is well-defined. Since FS_r is cancellative it is clear that if $x_1, x_2 \in D_i$ for some $1 \leq i \leq 9$, then $\Phi(x_1) = \Phi(x_2)$ implies $x_1 = x_2$. Therefore to show that Φ is injective, it is enough to show that $\Phi(x_1)$ can never equal $\Phi(x_2)$ whenever $x_1 \in D_i, x_2 \in D_j$ and $i \neq j$. There are 36 different cases to check, which are straightforward but tedious. We consider two of the cases and leave the rest to the reader, as they are either trivial or identical in nature to the ones presented.

1. Let $x_1 \in D_9$ and $x_2 \in D_1$, if $\Phi(x_1) = \Phi(x_2)$ then $m^2 = mwm$ which is an immediate contradiction as $|w| \geq 1$.
2. Let $x_1 \in D_6$ and $x_2 \in D_5$. If $\Phi(x_1) = \Phi(x_2)$ then $wm = mw'$, hence $w = mz$ for some $z \in FS_r$. This implies $wf = m(zf) \notin MG(\Lambda)$ since $m, zf \in \Lambda$ which is a contradiction.

Therefore Φ is injective and we can extend it to an injective function

$$\Psi : MG(\Lambda) \rightarrow MG(\Lambda_m^f)$$

$$x \mapsto \begin{cases} \Phi(x) & \text{if } x \in D \\ x & \text{if } x \in MG(\Lambda) \cap MG(\Lambda^f), x \neq m \\ f & \text{if } x = m \end{cases}$$

and so $|MG(\Lambda)| \leq |MG(\Lambda_m^f)|$. □

Corollary 4.5. *For a fixed index, ordinary subsemigroups of FS_r have the maximum number of descendants in the subsemigroup tree.*

Proof. Firstly note that ordinary subsemigroups have the maximum number of minimal generators. In fact, given any non-ordinary finite index subsemigroup $\Lambda \subseteq FS_r$, by Proposition 4.4, Λ_m^f has no less minimal generators than Λ so apply Proposition 4.4 to Λ finitely many times until it is ordinary. Since every minimal generator of an ordinary subsemigroup is bigger than the Frobenius, each gives rise to a descendant and the result follows. □

Now we can prove the following Theorem using the information from Lemma 4.1

Theorem 4.6. *For $r \geq 2$, $a_n(FS_r) \leq (r-1)^n(n+1)(n!)^2$.*

Proof. By Corollary 4.5, the ordinary subsemigroups have the maximum number of descendants, which by Lemma 4.1, is $(r-1)n^2 + (2r-1)n + r$. So let us assume every subsemigroup has this number of descendants to construct an upper bound. Then

$$\begin{aligned}
a_n(FS_r) &\geq \prod_{k=0}^{n-1} ((r-1)k^2 + (2r-1)k + r) \\
&= \prod_{k=0}^{n-1} ((r-1)(k+1)^2 + (k+1)) \\
&= \prod_{k=1}^n ((r-1)k^2 + k) \leq \prod_{k=1}^n ((r-1)k^2 + (r-1)k) \\
&= (r-1)^n \prod_{k=1}^n (k(k+1)) \\
&= (r-1)^n (n+1)(n!)^2. \quad \square
\end{aligned}$$

Theorem 4.7. For $r \geq 2$, $\log(s_n(FS_r)) = O(\log(n^n))$.

Proof. Let $U(n, r) = (r-1)^n(n+1)(n!)^2$. Since $a_n(FS_r) \leq U(n, r)$ and since $U(n, r)$ is non-decreasing, we have $s_n(FS_r) \leq n \cdot U(n, r)$. Since $n! < n^n$, we have

$$\log(s_n(FS_r)) \leq \log(n) + n \log(r-1) + \log(n+1) + 2n \log(n) = O(n \log(n)). \quad \square$$

Corollary 4.8. For $r \geq 2$, FS_r has subsemigroup growth of strict type n^n .

Proof. By Theorems 4.3 and 4.7. □

5 Subsemigroup growth for a fixed index

In this section we consider the growth of $a_n(FS_r)$ when n is fixed and the rank r varies. In this case the sequence fits an explicit polynomial. In order to prove this, we need some preliminary remarks.

Given any word $w = g_{\alpha(1)} \dots g_{\alpha(m)} \in FS_r$, with $g_{\alpha(i)} \in X_r$, and any permutation $\sigma \in \text{Sym}(X_r)$, let $\sigma(w)$ denote the word $\sigma(g_{\alpha(1)}) \dots \sigma(g_{\alpha(m)})$. Given Λ a finite index subsemigroup of FS_r with $G(\Lambda) = \{w_1, \dots, w_n\}$, then let $\sigma(G(\Lambda))$ denote the set $\{\sigma(w_1), \sigma(w_2), \dots, \sigma(w_n)\}$. It is clear that $FS_r \setminus \sigma(G(\Lambda))$ is also a finite index subsemigroup of FS_r isomorphic to Λ .

For any index n subsemigroup $\Lambda \subseteq FS_r$ we wish to think of the set of gaps of Λ as a ‘pattern’ by forgetting the labels of the generators of FS_r . To make this idea precise, let Fin be the set of all finite index subsemigroups of finite rank free semigroups.

We now define an equivalence relation \sim on Fin . Let $\Lambda_1, \Lambda_2 \in Fin$ be two finite index subsemigroups with $\Lambda_1 \subseteq FS_p$ and $\Lambda_2 \subseteq FS_q$ say. We say $\Lambda_1 \sim \Lambda_2$ if there exists $\sigma \in \text{Sym}(X_{\max\{p,q\}})$ such that $G(\Lambda_1) = \sigma(G(\Lambda_2))$. If $\Lambda_1 \sim \Lambda_2$ then we say that Λ_1 and Λ_2 have the same **gap pattern**.

Let $w \in FS_r$. Define the **support** of w to be the set of minimal generators that make up w . That is, if $w = g_{\alpha(1)} g_{\alpha(2)} \dots g_{\alpha(m)}$ where $g_{\alpha(i)} \in X_r$, then $\text{supp}(w) := \{g_{\alpha(1)}, \dots, g_{\alpha(m)}\}$. Given $\Lambda \in Fin$, with $G(\Lambda) = \{w_1, w_2, \dots, w_n\}$, we define $\text{supp}(G(\Lambda))$ to be $\bigcup_{i=1}^n \text{supp}(w_i)$.

Let $\Lambda \in Fin$ with $|\text{supp}(G(\Lambda))| = k$. Then Λ has a minimal representative $\Lambda_{min} \subseteq FS_k$ of the \sim -class of Λ so that $\text{supp}(\Lambda_{min}) = X_k$ and the elements of the support ‘first appear in order’. More formally, let $g_j = \max\{g_i \mid g_i \in \text{supp}(\Lambda)\}$ be the largest generator in the gaps of

Λ using the standard lexicographical order. If we let $y = w_1 \dots w_n$ be the concatenation of the set of gaps of Λ , then we can totally order the $\sigma \in \text{Sym}(X_j)$ using the lexicographical order on $\sigma(y)$. Let σ_{min} be the smallest permutation with respect to this total order, then Λ_{min} is the complement of $\sigma_{min}(G(\Lambda))$ in FS_k .

Now, for a fixed $\Lambda \in Fin$ let the **orbit** of Λ be defined as $\text{Orb}(\Lambda) := \{\sigma(\Lambda) \mid \sigma \in \text{Sym}(\text{supp}(\Lambda))\}$. This is the set of all elements of Fin with the same gap pattern and the same support as Λ . By the orbit-stabilizer theorem, $|\text{Orb}(\Lambda)|$ is a divisor of $|\text{Sym}(\text{supp}(\Lambda))| = |\text{supp}(\Lambda)|!$.

For each $n \geq 1$, let $Z(n)$ be the set of minimal representatives of \sim -classes of index n subsemigroups. For each $k \geq 1$, let $Z(n, k) := \{P \in Z(n) \mid |\text{supp}(P)| = k\}$ and for each $i \mid k!$, let $Z(n, k, i) := \{P \in Z(n, k) \mid |\text{Orb}(P)| = i\}$. Note, we have that

$$Z(n, k) = \bigsqcup_{i \mid k!} Z(n, k, i). \quad (1)$$

Observe that each of the sets $Z(n, k)$ are finite. In fact, as each P is minimal $|Z(n, k)|$ is no bigger than the number of index n subsemigroups of FS_k , which by Corollary 2.2, is finite.

Now given any r and n , we wish to determine the number of index n subsemigroups of FS_r . For each $P \in Z(n, k)$ there are k possible generators we could choose from r for the support of P , and there are $|\text{Orb}(P)|$ different subsemigroups of FS_r with the same gap pattern and the same support as P . Therefore, there are $|\text{Orb}(P)| \cdot \binom{r}{k}$ index n subsemigroups of FS_r with the same gap pattern as P , and we have the following equation:

$$a_n(FS_r) = \sum_{k=1}^r \sum_{P \in Z(n, k)} |\text{Orb}(P)| \cdot \binom{r}{k}.$$

Lemma 5.1. $Z(n, k) = \emptyset$ for all $k > n$.

Proof. This is equivalent to proving that every index n subsemigroup Λ has $|\text{supp}(G(\Lambda))| \leq n$. We prove by induction on the index. Let $n = 1$ and given any index 1 subsemigroup Λ with $G(\Lambda) = \{w_1\}$, then w_1 must be in X_r and so $|\text{supp}(\Lambda)| = 1$. Now assume the statement is true for $n \geq 1$. Given any subsemigroup with $|G(\Lambda)| = n+1$, let $f = f(\Lambda)$ and consider the semigroup Λ^f which has $|G(\Lambda^f)| = n$. If $G(\Lambda^f) = \{w_1, \dots, w_n\}$ then by the assumption $|\text{supp}(G(\Lambda^f))| \leq n$. Since f is a minimal generator of Λ^f , by Lemma 3.3, f is of the form $w_i g$, $g w_i$ or $w_i g w_j$ for some $w_i, w_j \in G(\Lambda^f)$, $g \in X_r$ and therefore $\text{supp}(G(\Lambda^f)) = \text{supp}(G(\Lambda)) \cup \text{supp}(g)$ and $|\text{supp}(\Lambda)| \leq n+1$. \square

Therefore we can refine our range of summation slightly to get:

$$\begin{aligned} a_n(FS_r) &= \sum_{k=1}^n \sum_{P \in Z(n, k)} |\text{Orb}(P)| \cdot \binom{r}{k} \\ &= \sum_{k=1}^n \sum_{i \mid k!} \sum_{P \in Z(n, k, i)} i \binom{r}{k} \quad \text{by (1)} \\ &= \sum_{k=1}^n \sum_{i \mid k!} |Z(n, k, i)| \cdot i \binom{r}{k} \\ &= \sum_{k=1}^n \sum_{i \mid k!} |Z(n, k, i)| \frac{i}{k!} \prod_{j=0}^{k-1} (r-j) \end{aligned}$$

So if we let

$$c(n, k) := \sum_{i|k!} |Z(n, k, i)| \cdot i$$

and $s(n, k)$ be the (signed) Stirling numbers of the first kind, then

$$\begin{aligned} a_n(FS_r) &= \sum_{k=1}^n \frac{c(n, k)}{k!} \sum_{j=0}^n s(n, j) r^j \\ &= \sum_{k=1}^n \left(\left(\sum_{j=k}^n \frac{c(n, j)}{j!} s(n, k) \right) r^k \right) \end{aligned}$$

We remark that the ordinary subsemigroup $O_n(n)$ has $|\text{supp}(O_n(n))| = n$ and so $c(n, n) \neq 0$. Therefore we have proved:

Theorem 5.2. $a_n(FS_r)$ is a polynomial in r of degree n with no constant term.

We now use this result to describe an algorithm (making use of the previous algorithms) for inductively computing the sets $Z(n, k, i)$ and therefore the values $a_n(FS_r)$. See Algorithm 3.

Using this algorithm, we computed the polynomials and hence the values of $a_n(FS_r)$ for $1 \leq n \leq 9$ which are presented in Appendix A. This was implemented on the Iridis 4 compute cluster [2] using C++ code which is available for download [1]. It took 2 hours 30 minutes running on 64 x Intel Xeon E5-2670 processor cores, equivalent to approximately one week of computation on a standard desktop computer.

6 Ideal growth

Recall that a subsemigroup I of a semigroup S is called a left (resp. right) ideal of S if $SI \subseteq I$ (resp. $IS \subseteq I$), and a (two-sided) ideal if it is both a left ideal and a right ideal. Let $a_n^{LI}(FS_r)$ denote the number of (Rees) index n left ideals of FS_r , let $a_n^{RI}(FS_r)$ denote the number of index n right ideals and $a_n^I(FS_r)$ denote the number of index n two-sided ideals. Note that the number of left ideals is equal to the number of right ideals as $FS_r \rightarrow FS_r, w \mapsto \text{rev}(w)$ is an anti-isomorphism, and since every ideal is also a subsemigroup it is clear that

$$a_n^I(FS_r) \leq a_n^{LI}(FS_r) = a_n^{RI}(FS_r) \leq a_n(FS_r).$$

We show that both $a_n^{LI}(FS_r) = a_n^{RI}(FS_r)$ and $a_n^I(FS_r)$ have exponential growth, that is, strict growth type $2^n = n^{n/\log(n)}$ where strict growth type is defined as in subsemigroup growth.

6.1 One-sided ideals

We now make an observation and give an exact formula for the number of index n one-sided ideals of FS_r .

Let Λ be an index n right ideal of FS_r , then it is clear that given any $w \in G(\Lambda)$, we must also have $w_{\text{pre}(|w|-1)} \in G(\Lambda)$. Therefore considering the right multiplication tree, the set of gaps (including the empty word) corresponds to a rooted r -ary tree with $n+1$ vertices. The number of such is precisely the ‘Fuss-Catalan’ numbers:

$$a_n^{RI}(FS_r) = \frac{1}{(r-1)(n+1)+1} \binom{r(n+1)}{n+1}.$$

Algorithm 3 Find the sets $Z(n, k, i)$ for all i , from the sets $Z(n-1, k, i)$ and $Z(n-1, k-1, i)$

Require: Index n , support k , the sets $Z(n-1, k, i)$ and $Z(n-1, k-1, i)$ for all i

```

1: procedure FINDNEXTSETS( $n, k$ )
2:   Input sets  $Z(n-1, k, i), Z(n-1, k-1, i)$  for all  $i$ .
3:    $Z(n-1, k) \leftarrow \cup_i Z(n-1, k, i)$ 
4:    $Z(n-1, k-1) \leftarrow \cup_i Z(n-1, k-1, i)$ 
5:   Descendants  $\leftarrow \{\}$ 
6:   for all  $\Lambda \in Z(n-1, k)$  do
7:      $M \leftarrow \text{GENERATORS}(G(\Lambda), X_k)$ 
8:     for all  $h \in M$  do
9:       if  $h > f(\Lambda)$  then
10:        Descendants  $\leftarrow$  Descendants  $\cup \{G(\Lambda) \cup \{h\}\}$ 
11:       end if
12:     end for
13:   end for
14:   for all  $G(\Lambda) \in Z(n-1, k-1)$  do
15:      $M \leftarrow \text{GENERATORS}(G(\Lambda), \{g_k\})$ 
16:     for all  $h \in M$  do
17:       if  $h > f(\Lambda)$  then
18:        Descendants  $\leftarrow$  Descendants  $\cup \{G(\Lambda) \cup \{h\}\}$ 
19:       end if
20:     end for
21:   end for
22:   for all  $G(\Lambda) \in \text{Descendants}$  do
23:     Orbit  $\leftarrow \{\}$ 
24:     for all  $\sigma \in \text{Sym}(X_k)$  do
25:       Orbit  $\leftarrow$  Orbit  $\cup \{\sigma(\Lambda)\}$ 
26:     end for
27:     MinRep  $\leftarrow$  minimal representative of Orbit
28:      $Z(n, k, |\text{Orbit}|) \leftarrow Z(n, k, |\text{Orbit}|) \cup \{\text{MinRep}\}$ 
29:   end for
30:   Output sets  $Z(n, k, i)$  for all  $i$ .
31: end procedure

```

Note that when $r = 2$ this reduces to the standard Catalan numbers and when $r = 1$, $a_n^{RI}(FS_r) = a_n^{LI}(FS_r) = a_n^I(FS_r) = 1$ as the only ideals of $(\mathbb{N}, +)$ are the ordinary subsemigroups.

Theorem 6.1. *For $r \geq 2$, FS_r has one-sided ideal growth of strict type $2^n = n^{n/\log(n)}$.*

Proof. This follows from the fact that $r^{n+1} \leq \binom{r(n+1)}{(n+1)} \leq (er)^{n+1}$. \square

For a fixed index n , with some basic manipulation the formula above yields a polynomial in r of degree n with no constant term:

$$a_n^{RI}(FS_r) = \sum_{k=0}^n \left(\binom{s(n+1, k)}{(n+1)!} (n+1)^k r^j \right)$$

where $s(n, k)$ are the (signed) Stirling numbers of the first kind.

6.2 Two-sided ideals

Recalling the subsemigroup tree in Section 4 we consider the subtree of all (two-sided) ideals. That this really is a tree follows from the fact that given any ideal $\Lambda \subseteq FS_r$, then Λ^f is also an ideal.

Remark 6.2. Note that given any ideal $\Lambda \subseteq FS_r$ in this subtree, then Λ_h is a descendant of Λ if and only if $h > f(\Lambda)$ and $h_{\text{pre}(|h|-1)}, h_{\text{suf}(|h|-1)} \in G(\Lambda)$.

The ordinary subsemigroups $O_r(n)$ are clearly ideals. Let $\text{desc}(O_r(n))$ denote the set of ideals that are descendants of $O_r(n)$ in this tree, and let $D(O_r(n)) = \{h \in MG(O_r(n)) \mid O_r(n)_h \in \text{desc}(O_r(n))\}$. Clearly the sets $D(O_r(n))$ and $\text{desc}(O_r(n))$ are in bijection. We now construct a lower bound on the size of these sets. First we need a technical lemma.

Lemma 6.3. $g_1 w_{\text{pre}(|w|-1)} \leq w$ for all $w \in FS_r$.

Proof. Let $w = g_{\alpha(1)} \cdots g_{\alpha(m)}$ and assume $g_1 g_{\alpha(1)} \cdots g_{\alpha(m-1)} > g_{\alpha(1)} \cdots g_{\alpha(m)}$. Then $g_1 \geq g_{\alpha(1)} \geq g_{\alpha(2)} \geq \cdots \geq g_{\alpha(m-1)} \geq g_{\alpha(m)}$ and so $g_{\alpha(i)} = g_1$ for all $1 \leq i \leq m$ in which case $g_1^m < g_1^m$ which is a contradiction. \square

Lemma 6.4. $|\text{desc}(O_r(n))| \leq |\text{desc}(O_r(n+1))|$

Proof. Let $f = f(O_r(n+1))$, then it is clear that whenever $O_r(n)_h$ is an ideal, $O_r(n+1)_h$ is also an ideal for all $h \neq f$. So we need only show that $D(O_r(n+1))$ contains at least one element that $D(O_r(n))$ does not. Let $h = g_1 f$, then by Lemma 6.3, $h_{\text{pre}(|h|-1)} \leq f \in G(O_r(n+1))$ and clearly $h_{\text{suf}(|h|-1)} = f \in G(O_r(n+1))$ but is not in $G(O_r(n))$. Therefore $O_r(n+1)_h$ is an ideal but $O_r(n)_h$ is not an ideal. \square

Proposition 6.5. $|\text{desc}(O_r(n))| \geq r^m$ where $m = \lfloor \log_r((r-1)n+r) \rfloor$.

Proof. Consider the case when $n = r + r^2 + \cdots + r^{m-1} = \frac{r^m - r}{r-1}$ for some $m \geq 1$, then $G(O_r(n)) = \{w \in FS_r \mid |w| \leq r^{m-1}\}$ and every word w of length m satisfies $w_{\text{pre}(|w|-1)}, w_{\text{suf}(|w|-1)} \in G(O_r(n))$ and no longer words do. Hence $|\text{desc}(O_r(n))| = r^m$. Otherwise $n = \frac{r^m - r}{r-1} + i$ for some $m \geq 1$ and some $1 \leq i \leq r^m - 1$. By inductively applying Lemma 6.4, we have $|\text{desc}(O_r(n))| \geq |\text{desc}(O_r(n-i))| = r^m$. \square

This underestimate has the largest error when $n = r + r^2 + \dots + r^m - 1 = \frac{r^{m+1}-r}{r-1} - 1$. So take this as a lower bound and for $r \geq 2$, it is always true that

$$h(n, r) := \frac{r-1}{r}n + \frac{2r-1}{r} = \frac{r-1}{r}(n+1) + 1 = r^m \leq |\text{desc}(O_r(n))|.$$

So if we let

$$L^I(n, r) = \sum_{i=0}^{K(n, r)} \binom{h(n-i, r) - 1}{i}$$

where $K(n, r) = \left\lfloor \frac{r-1}{2r-1}(n+1) \right\rfloor$ is obtained from the inequality $h(n-i, r) - 1 \geq i$. Then similar to the argument in Section 4 for the lower bound, we have proved the following:

Theorem 6.6. For $r \geq 2$, $a_n^I(FS_r) \geq L^I(n, r)$.

As a consequence we can see that two-sided ideal growth is bounded below by an exponential:

Theorem 6.7. For $r \geq 2$, $\log(2^n) = O(\log(s_n^I(FS_r)))$.

Proof. First note that for $r \geq 2$, $n/4 \leq K(n, r)$ so by considering the term $i = n/4$ we have

$$s_n^I(FS_r) \geq a_n^I(FS_r) \geq L^I(n, r) \geq \left(\frac{3n(r-1)/4r + (2r-1)/r}{n/4} \right)^{n/4} \sim \left(\left(\frac{3(r-1)}{r} \right)^{\frac{1}{4}} \right)^n$$

Thus $\log(2^n) = n = O(\log(s_n^I(FS_r)))$. □

Since $a_n^I(FS_r) \leq a_n^{RI}(FS_r)$, we immediately deduce from Theorem 6.1, $\log(s_n^I(FS_r)) = O(\log(2^n))$ for $r \geq 2$. Hence,

Theorem 6.8. For $r \geq 2$, FS_r has ideal growth of strict type $2^n = n^{n/\log(n)}$.

It can be checked that the argument in Section 5 is valid for ideals also, and so for a fixed index n , we also have the following:

Theorem 6.9. $a_n^I(FS_r)$ is a polynomial in r of degree n with no constant term.

We can easily adapt our previous algorithm to use Remark 6.2 instead of Remark 3.1. This is computationally a much easier condition to check. This algorithm was implemented using C++ code which is available for download [1]. The polynomials and hence the values of $a_n^I(FS_r)$ for $1 \leq n \leq 12$ were calculated and are presented in Appendix B. It took less than an hour to calculate running on a single Intel Xeon E5-2670 processor.

We observed an interesting connection between $a_n^I(FS_2)$ and the central binomial coefficients $\binom{n}{\lfloor n/2 \rfloor}$ which we are unable to explain (see Appendix D). They agree for the first 6 values and then the central binomial coefficients seem to be an upper bound. It would be of interest to know whether they are an asymptotic upper bound.

Note that, given any finite index ideal $I \subseteq FS_r$ and any word in the set of gaps $G(I)$, then every prefix and suffix of that word is also in the set of gaps, hence $\text{supp}(I) \subseteq G(I)$. This implies that there is only one index n ideal I with $|\text{supp}(I)| = n$, namely the ordinary subsemigroup, and hence using the notation from Section 5, $c(n, n) = 1$. Consequently,

Theorem 6.10. For a fixed index n

$$a_n^I(FS_r) \sim \frac{n^r}{n!}.$$

7 Congruence growth

Let $\text{Cong}_n(FS_r)$ denote the set of (two-sided) congruences on FS_r with precisely n nonempty congruence classes, and let $a_n^C(FS_r) = |\text{Cong}_n(FS_r)|$. We say that a semigroup S is at most r generated if there exists an r -element subset of S that generates the semigroup. It is clear that given any at most r generated semigroup S of order n , there exists a congruence $\rho \in \text{Cong}_n(FS_r)$ such that $FS_r/\rho \cong S$. The number of congruences in $\text{Cong}_n(FS_r)$ that quotient to give S is precisely the number of distinct epimorphisms from FS_r to S . There are at most n^r such epimorphisms since the map is determined entirely by the image of the generators X_r . Hence

$$f(n, r) \leq a_n^C(FS_r) \leq n^r \cdot f(n, r)$$

where $f(n, r)$ is the number of non-isomorphic at most r generated semigroups of order n .

Proposition 7.1. *For $r \geq 2$, $\log(2^n) = O(\log(s_n^C(FS_r)))$.*

Proof. Every index n ideal I of FS_r gives rise to a distinct congruence $\rho_I \in \text{Cong}_{n+1}(FS_r)$ where $\rho_I = ((FS_r \setminus I) \times (FS_r \setminus I)) \cup \text{id}_I$ is usually called the Rees congruence on FS_r modulo I . Hence $a_n^I(FS_r) \leq a_{n+1}^C(FS_r)$ and the result follows by Theorem 6.7. \square

Theorem 7.2. *FS_r has congruence growth of the same strict type as $f(n, r)$.*

Proof. For a fixed r , it is clear that $f(n, r)$ is non-decreasing and so by the observation above, $s_n^C(FS_r) \leq n^{r+1} \cdot f(n, r)$. Therefore $\log(s_n^C(FS_r)) \leq (r+1) \cdot \log(n) + \log(f(n, r))$ and so by the previous proposition, $\log(s_n(FS_r)) = O(\log(f(n, r)))$. Again, by the observation above $f(n, r) \leq a_n^C(FS_r) \leq s_n^C(FS_r)$ and so we have $\log(s_n^C(FS_r)) \asymp \log(f(n, r))$. \square

We conclude from this that for $r \geq 2$, the number of non-isomorphic at most r -generated semigroups of order n is at least exponential of strict type $2^n = n^{n/\log(n)}$ (whereas the number of at most r -generated groups of order n is sub-exponential with strict growth type $n^{\log(n)}$).

Question 7.3. *Can $f(n, r)$ be bounded above by an exponential?*

If we can also show that $f(n, r)$ grows at most exponentially then we will have proved that FS_r has congruence growth of strict type 2^n . In answering the above question, one may be tempted to consider 3-nilpotent semigroups as almost all finite semigroups are 3-nilpotent (see [18] and [11] for example). However it turns out almost none of the r -generated semigroups are 3-nilpotent, that is, there are no r -generated 3-nilpotent semigroups of order greater than $r^2 + r + 1$. The Smallsemi [10] data library in GAP [13] tells us that for n from 2 through 8, $f(n, 2)$ equals: 5, 17, 68, 217, 670, 1937, 5686. This seems to be exponential, and we conjecture that the above question is true.

7.1 Ascendingly generated tables

When counting all semigroups of order n , a natural question is whether to count order n semigroups up to isomorphism, which we will denote $f(n)$, or whether to count ‘up to equality’ all n -element Cayley tables, that is, all binary operations on an n -element set, which we will denote $m(n)$. There are at most $n!$ distinct Cayley tables of any semigroup up to isomorphism (precisely when the semigroup has trivial automorphism group). Hence $m(n) = O(n! \cdot f(n))$. As an aside, it is generally believed (although still an open question) that almost all semigroups have trivial automorphism group, in which case not only does $m(n) = O(n! \cdot f(n))$ but $m(n) \sim n! \cdot f(n)$.

We here introduce something between the two which turns out to be important for counting congruences, namely what we have called ascendingly generated tables. Heuristically we are

counting the generators $\{w_1, \dots, w_k\}$ up to equality and the non-generators $\{w_{k+1}, \dots, w_n\}$ up to isomorphism and it satisfies $O(n^k \cdot f(n, k))$.

Let $W_n = \{w_1 < w_2 < \dots < w_n\}$ be an n -element set and let $C_n = \{(W_n, \otimes)\}$ be the set of all Cayley tables on W_n . Let $C_{n,k} \subseteq C_n$ be the set of all Cayley tables on (W_n, \otimes) that are generated (not necessarily minimally) by $\{w_1, \dots, w_k\}$, so that for example, $C_{n,n} = C_n$.

Given any $w \in W_n$, $(W_n, \otimes) \in C_{n,k}$, there exists some decomposition (possibly many) of w as a product of elements from $\{w_1, \dots, w_k\}$. Let $\text{dec}(w)$ denote the smallest decomposition with respect to $<_{\otimes}$, the shortlex order on W_n over the alphabet $\{w_1, \dots, w_k\}$. We say that (W_n, \otimes) is **ascendingly generated** by $\{w_1, \dots, w_k\}$ if it is generated by $\{w_1, \dots, w_k\}$ and

$$\text{dec}(w_{k+1}) <_{\otimes} \text{dec}(w_{k+2}) <_{\otimes} \dots <_{\otimes} \text{dec}(w_n).$$

Let $T_{n,k} \subseteq C_{n,k}$ denote the set of all $(W_n, \otimes) \in C_{n,k}$ ascendingly generated by $\{w_1, \dots, w_k\}$.

Lemma 7.4. *We observe the following facts:*

1. $|T_{n,1}| = n$.
2. $|T_{n,n}| = m(n)$.
3. $T_{n,r} \subseteq T_{n,r+1}$.
4. $f(n, r) \leq |T_{n,r}| \leq r! \cdot \binom{n}{r} \cdot f(n, r)$.
5. $|T_{n,r}| = O(n^r \cdot f(n, r))$ for a fixed r .

Proof. 1. Let $(W_n, \otimes) \in T_{n,1}$. Since (W_n, \otimes) is ascendingly generated by $\{w_1\}$ we must have

$$\overbrace{w_1 \otimes \dots \otimes w_1}^i = w_i \quad \text{for all } 1 \leq i \leq n-1$$

and $(w_i)^n$ is allowed to equal any of the n elements.

2. From the definition, every Cayley table on W_n is generated ascendingly by the whole of W_n .
3. Again this follows vacuously from the definition.
4. Given any at most r generated semigroup S we can always label the elements $\{w_1, \dots, w_n\}$ such that $\{w_1, \dots, w_r\}$ generate S and the remaining elements are labelled in the order they are generated so that it is ascendingly generated, hence $f(n, r) \leq |T_{n,r}|$. Given any at most r -generated semigroup S there are at most $\binom{n}{r}$ possible ways of choosing a generating set which we can label in at most $r!$ ways $\{w_1, \dots, w_r\}$, but for the remaining elements we have no choice how to label them if we want S to be generated ascendingly by $\{w_1, \dots, w_r\}$. Hence $|T_{n,r}| \leq r! \cdot \binom{n}{r} \cdot f(n, r)$.
5. This follows immediately from 4. □

Recall that there are $n!$ monogenic Cayley tables of order n , but only n up to isomorphism. So ascendingly generated tables behave like ‘up to isomorphism’ for $k = 1$ but like ‘up to equality’ for $k = n$.

Using the Smallsemi [10] data library in GAP [13] we calculated the number of Cayley tables on $\{w_1, \dots, w_n\}$ generated ascendingly by $\{w_1, \dots, w_k\}$ (see Table 1).

Again recall from Lemma 7.4(2) that the diagonal in Table 1 is equal to $m(n)$, see sequence A023814 in OEIS [3].

k \ n	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2		8	37	145	452	1374	3933
3			113	1257	9020	60826	356023
4				3492	67394	938194	30492722
5					183732	6398792	466578957
6						17061118	3032145644
7							7743056064

Table 1: $|T_{n,k}|$ for $1 \leq k \leq n \leq 7$.

See [1] for the code used to calculate these values. This ran on the Iridis 4 compute cluster [2] and took one hour running on 64 x Intel Xeon E5-2670 processor cores, equivalent to approximately 64 hours on a standard desktop computer.

That ascendingly generated Cayley tables are important is revealed in the next result.

A congruence $\rho \in \text{Cong}_n(FS_r)$ is completely determined by two pieces of information: an ascendingly generated Cayley table and by which congruence classes the generators X_r are assigned to. We formalise this below.

We say a function $f : X_r \rightarrow W_n$ is an **assignment** if $f(g_1) = w_1$ and $f(g_j) \in \{w_1, \dots, w_{\alpha(j)+1}\}$ where $w_{\alpha(j)} = \max_{i < j} \{f(g_i)\}$ for all $2 \leq j \leq n$. Let $A_{r,k}$ be the set of assignments from X_r to W_n such that the image has cardinality k .

We claim that

Proposition 7.5. *There exists a bijection*

$$\phi : \text{Cong}_n(FS_r) \rightarrow \bigsqcup_{1 \leq k \leq r} (T_{n,k} \times A_{r,k}).$$

Proof. Given any $\rho \in \text{Cong}_n(FS_r)$ we choose a subset $W_n \subseteq FS_r$ in the following way: let $w_1 < w_2 < \dots < w_n$ be the minimal representatives of the n ρ -classes with respect to $<$, the shortlex order on FS_r over the alphabet X_r . Given any word $w \in FS_r$, let $[w]_\rho \in W_n$ denote the minimal representative of the class of w . Let $k = |\{[g_1]_\rho, [g_2]_\rho, \dots, [g_r]_\rho\}|$ be the number of classes that the generators X_r traverse.

Let f be defined as $f(g_i) := [g_i]_\rho$ for all $1 \leq i \leq r$. Clearly $f(g_1) = w_1$ and for any $j > 1$, either g_j is in the same ρ -class as some g_i for $i < j$ in which case $f(g_j) = f(g_i)$, or it is in a different class from all g_i with $i < j$ in which case $f(g_j) = w_{\alpha(j)+1}$ where $w_{\alpha(j)} = \max_{i < j} \{f(g_i)\}$. Hence f is indeed an assignment and $f \in A_{r,k}$. We now define a binary operation on W_n , which can be done by setting $w_i \otimes w_j := [w_i w_j]_\rho$ for all $1 \leq i, j \leq n$. This is associative as ρ is a congruence. We now show that (W_n, \otimes) is generated by $\{w_1, \dots, w_k\}$. In fact, given any $w \in W_n$, let $w = g_{\alpha(1)} g_{\alpha(2)} \dots g_{\alpha(m)}$ be its unique decomposition in FS_r , then $w_i = [w_i]_\rho = [g_{\alpha(1)} g_{\alpha(2)} \dots g_{\alpha(m)}]_\rho = [g_{\alpha(1)}]_\rho \otimes [g_{\alpha(2)}]_\rho \otimes \dots \otimes [g_{\alpha(m)}]_\rho$, where $[g_{\alpha(1)}]_\rho, [g_{\alpha(2)}]_\rho, \dots, [g_{\alpha(m)}]_\rho \in \text{im}(f) = \{w_1, \dots, w_k\}$. Hence $(W_n, \otimes) \in C_{n,k}$.

Given any $w_i \in W_n$, let $\text{dec}(w_i) = w_{\gamma(1)} \otimes \dots \otimes w_{\gamma(m)}$ be the minimal decomposition of w_i . As w_i belongs to FS_r , we also have its unique decomposition $w_i = g_{\alpha(1)} \dots g_{\alpha(l)}$ using letters in X_r . We intend to show that $g_{\alpha(1)} \dots g_{\alpha(l)} = w_{\gamma(1)} \dots w_{\gamma(m)}$. To do this, we make two initial observations:

1. $[g_{\alpha(1)}]_\rho \dots [g_{\alpha(l)}]_\rho \leq g_{\alpha(1)} \dots g_{\alpha(l)}$;
2. $[g_{\alpha(1)}]_\rho \dots [g_{\alpha(l)}]_\rho$ is in the same class as w_i , which is minimal in its class.

We can immediately deduce that $[g_{\alpha(1)}]_{\rho} \cdots [g_{\alpha(l)}]_{\rho} = g_{\alpha(1)} \cdots g_{\alpha(l)}$ and $g_{\alpha(1)}, \dots, g_{\alpha(l)} \in W_n$. As $\text{dec}(w_i)$ is the smallest decomposition of w_i in W_n we also have $w_{\gamma(1)} \otimes \cdots \otimes w_{\gamma(m)} \leq_{\otimes} g_{\alpha(1)} \otimes \cdots \otimes g_{\alpha(l)}$. Assume that $w_{\gamma(1)} \cdots w_{\gamma(m)} \neq g_{\alpha(1)} \cdots g_{\alpha(l)}$, then $w_{\gamma(1)} \cdots w_{\gamma(m)} < w_i$ but in the same ρ -class as w_i which is a contradiction. Hence $g_{\alpha(1)} \cdots g_{\alpha(l)} = w_{\gamma(1)} \cdots w_{\gamma(m)}$ and $w_i < w_j$ if and only if $\text{dec}(w_i) <_{\otimes} \text{dec}(w_j)$ thus $(W_n, \otimes) \in T_{n,k}$. Let $\phi(\rho) := ((W_n, \otimes), f)$, it is clear that if $\rho = \sigma \in \text{Cong}_n(FS_r)$ then $\phi(\rho) = \phi(\sigma)$, whence ϕ is a well-defined function.

Now we prove surjectivity of ϕ by constructing a congruence: given any $((W_n, \otimes), f) \in \bigsqcup_{1 \leq k \leq r} (T_{n,k} \times A_{r,k})$ we define ρ as follows: let $(a, b) \in \rho$ if and only if $f(g_{\alpha(1)}) \otimes \cdots \otimes f(g_{\alpha(l)}) = f(g_{\beta(1)}) \otimes \cdots \otimes f(g_{\beta(m)})$ where $a = g_{\alpha(1)} \cdots g_{\alpha(l)}$ and $b = g_{\beta(1)} \cdots g_{\beta(m)}$ are their unique decompositions in FS_r . It is straightforward to check that this is indeed a congruence. We now show that ρ has n congruence classes. The generators X_r clearly traverse $|\text{im}(f)| = k$ classes. For each $w_i \in \{w_{k+1}, \dots, w_n\}$ let $\text{dec}(w_i) = w_{\gamma(1)} \otimes \cdots \otimes w_{\gamma(l)}$. Then $w_{\gamma(1)} \cdots w_{\gamma(k)}$ is in a different class from all $\{w_1, \dots, w_{i-1}\}$ and therefore $\rho \in \text{Cong}_n(FS_r)$.

Let $\phi(\rho) = ((W_n, \oplus), f')$. We intend to show that $((W_n, \otimes), f) = ((W_n, \oplus), f')$. Firstly note that $f(g_1) = w_1 = f'(g_1)$. We now proceed by induction: given any $j > 1$, assume $f(g_i) = f'(g_i)$ for all $i < j$. Then either $f(g_j) = f'(g_j)$ for some $i < j$ in which case $f'(g_j) = [g_j]_{\rho} = [g_i]_{\rho} = f'(g_i) = f(g_i) = f(g_j)$ or, alternatively, $f(g_j)$ is different from all $f(g_i)$ with $i < j$. In which case $f(g_j) = w_{\alpha(j)+1}$ and $f'(g_j) = w_{\beta(j)+1}$ where $w_{\alpha(j)} = \max_{i < j} \{f(g_i)\}$ and $w_{\beta(j)} = \max_{i < j} \{f'(g_i)\}$. By our assumption, $w_{\alpha(j)} = w_{\beta(j)}$ and so $f = f'$.

We now intend to show that $(W_n, \otimes) = (W_n, \oplus)$. Given any $w_i, w_j \in W_n$, with $\text{dec}(w_i) = w_{\gamma(1)} \otimes \cdots \otimes w_{\gamma(l)}$, $\text{dec}(w_j) = w_{\delta(1)} \otimes \cdots \otimes w_{\delta(m)}$, let $w_p = w_i \otimes w_j$ where $\text{dec}(w_p) = w_{\epsilon(1)} \otimes \cdots \otimes w_{\epsilon(q)}$. By the argument above we know that $w_i = w_{\gamma(1)} \cdots w_{\gamma(l)}$, $w_j = w_{\delta(1)} \cdots w_{\delta(m)}$, $w_p = w_{\epsilon(1)} \cdots w_{\epsilon(q)}$ are also their unique decompositions in FS_r as the minimal representatives of the ρ -classes. Hence $w_i \oplus w_j = [w_i w_j]_{\rho} = [w_p]_{\rho} = w_p = w_i \otimes w_j$. So $(W_n, \otimes) = (W_n, \oplus)$ and hence ϕ is surjective.

Finally, we show that ϕ is injective: consider $\phi(\rho) = ((W_n, \otimes), f) = \phi(\sigma)$ for some $\rho, \sigma \in \text{Cong}_n(FS_r)$. Then $(a, b) \in \rho$ if and only if $f(g_{\alpha(1)}) \otimes \cdots \otimes f(g_{\alpha(l)}) = f(g_{\beta(1)}) \otimes \cdots \otimes f(g_{\beta(m)})$ where $a = g_{\alpha(1)} \cdots g_{\alpha(l)}$ and $b = g_{\beta(1)} \cdots g_{\beta(m)}$ if and only if $(a, b) \in \sigma$ and ϕ is a bijection. \square

Note that $|A_{r,k}|$ is precisely the number of ways of partitioning an r -element set into k non-empty subsets, that is $\left\{ \begin{smallmatrix} r \\ k \end{smallmatrix} \right\}$ the Stirling numbers of the second kind. Hence we have proved the following,

Theorem 7.6.

$$a_n^C(FS_r) = \sum_{k=1}^r \left\{ \begin{smallmatrix} r \\ k \end{smallmatrix} \right\} |T_{n,k}|.$$

Hence from the values of $|T_{n,k}|$ in Table 7.1 we can calculate $a_n^C(FS_r)$ for $1 \leq n \leq 7$ which is presented in Appendix C.

Corollary 7.7. For a fixed rank r , $a_n^C(FS_r) \asymp |T_{n,r}|$.

Proof. Clearly $a_n^C(FS_r) \geq |T_{n,r}|$. From Lemma 7.4(3) $|T_{n,k}| \leq |T_{n,r}|$ for all $k \leq r$, hence $a_n^C(FS_r) \leq B(r) \cdot |T_{n,r}|$ where $B(r) = \sum_{k=1}^r \left\{ \begin{smallmatrix} r \\ k \end{smallmatrix} \right\}$ are the Bell numbers. \square

Compare this result to Theorem 7.2 and notice that this is much stronger than saying they have the same strict growth type.

7.2 Congruence growth for a fixed number of classes

Given some fixed n we now prove that $a_n^C(FS_r)$ satisfies an exponential equation with base n .

Theorem 7.8.

$$a_n^C(FS_r) = \sum_{j=1}^n \left(\left(\sum_{k=j}^n (-1)^{k-j} \binom{k}{j} \frac{|T_{n,k}|}{k!} \right) j^r \right).$$

Proof.

$$a_n^C(FS_r) = \sum_{k=1}^r \left\{ \begin{matrix} r \\ k \end{matrix} \right\} |T_{n,k}| = \sum_{k=1}^r \left(\sum_{j=1}^k \frac{(-1)^{k-j}}{k!} \binom{k}{j} j^r \right) |T_{n,k}|$$

since $|T_{n,k}| = 0$ for all $k > n$, the result follows by a simple rearrangement. \square

Using Table 7.1, we calculate the exponential equations for $1 \leq n \leq 7$ and present them in Appendix C.

Corollary 7.9. *For a fixed n , we have $a_n^C(FS_r) \sim \frac{m(n)}{n!} n^r$.*

8 Further work

There are many open questions regarding subsemigroup growth. It was shown in [20, Theorem 3.1] that all groups with superexponential subgroup growth are similar to free groups, in that they involve every finite group as an upper section. What necessary conditions are imposed on semigroups? In answering this question, it would be of interest to first investigate other classes of semigroups. What is the subsemigroup growth of free commutative semigroups, free inverse semigroups, the bicyclic monoid etc? For example, if free commutative semigroups have exponential subsemigroup growth, then by an argument similar to Proposition 2.4, non-commutativity would certainly be a necessary condition for superexponential growth.

How does the geometry of the semigroup relate to its subsemigroup growth, for example, is superexponential growth connected to hyperbolicity?

It may also be that another definition of index is more appropriate. One particularly natural definition that might be considered is Green index [14] which agrees with Rees index for free semigroups.

It may also be of interest to consider counting other objects, e.g. right/left congruences (related to counting cyclic acts), maximal sub(semi)groups, classes related to Green's relations etc.

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References

- [1] <http://www.alexbaileyonline.com/software>.
- [2] *The IRIDIS High Performance Computing Facility.*
(<http://www.southampton.ac.uk/isolutions/computing/hpc/iridis>).

- [3] *The On-Line Encyclopedia of Integer Sequences*. published electronically at (<http://oeis.org>).
- [4] Valentina Barucci, David E. Dobbs, and Marco Fontana. Maximality properties in numerical semigroups and applications to one-dimensional analytically irreducible local domains. *Mem. Amer. Math. Soc.*, 125(598):x+78, 1997.
- [5] Víctor Blanco, Pedro A. García-Sánchez, and Justo Puerto. Counting numerical semigroups with short generating functions. *Internat. J. Algebra Comput.*, 21(7):1217–1235, 2011.
- [6] Maria Bras-Amorós. Fibonacci-like behavior of the number of numerical semigroups of a given genus. *Semigroup Forum*, 76(2):379–384, 2008.
- [7] Maria Bras-Amorós. Bounds on the number of numerical semigroups of a given genus. *J. Pure Appl. Algebra*, 213(6):997–1001, 2009.
- [8] Maria Bras-Amorós and Stanislav Bulygin. Towards a better understanding of the semigroup tree. *Semigroup Forum*, 79(3):561–574, 2009.
- [9] Alan J. Cain and Victor Maltcev. *For a few elements more: A survey of finite Rees index*. arXiv:1307.8259.
- [10] A. Distler and J. D. Mitchell. *Smallsemi — A Library of Small Semigroups, A GAP 4 Package [13], Version 0.6.6*, 2013. (<http://tinyurl.com/jdmitchell/smallsemi/>).
- [11] Andreas Distler and J. D. Mitchell. The number of nilpotent semigroups of degree 3. *Electron. J. Combin.*, 19(2):Paper 51, 19, 2012.
- [12] Sergi Elizalde. Improved bounds on the number of numerical semigroups of a given genus. *J. Pure Appl. Algebra*, 214(10):1862–1873, 2010.
- [13] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.6,5*, 2013. (<http://www.gap-system.org>).
- [14] R. Gray and N. Ruškuc. Green index and finiteness conditions for semigroups. *J. Algebra*, 320(8):3145–3164, 2008.
- [15] R. I. Grigorchuk. Semigroups with cancellations of degree growth. *Mat. Zametki*, 43(3):305–319, 428, 1988.
- [16] R. I. Grigorchuk. Milnor’s Problem on the Growth of Groups and its Consequences. <http://arxiv.org/abs/1111.0512>, 2013.
- [17] Mikhael Gromov. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, (53):53–73, 1981.
- [18] Daniel J. Kleitman, Bruce R. Rothschild, and Joel H. Spencer. The number of semigroups of order n . *Proc. Amer. Math. Soc.*, 55(1):227–232, 1976.
- [19] Alexander Lubotzky, Avinoam Mann, and Dan Segal. Finitely generated groups of polynomial subgroup growth. *Israel J. Math.*, 82(1-3):363–371, 1993.
- [20] Alexander Lubotzky and Dan Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.

- [21] Christopher Voll. A newcomer's guide to zeta functions of groups and rings. In *Lectures on profinite topics in group theory*, volume 77 of *London Math. Soc. Stud. Texts*, pages 99–144. Cambridge Univ. Press, Cambridge, 2011.
- [22] Alex Zhai. Fibonacci-like growth of numerical semigroups of a given genus. *Semigroup Forum*, 86(3):634–662, 2013.
- [23] Yufei Zhao. Constructing numerical semigroups of a given genus. *Semigroup Forum*, 80(2):242–254, 2010.

A Values of $a_n(F S_r)$

$r \setminus n$	1	2	3	4	5	6	7	8	9
1	1	2	4	7	12	23	39	67	118
2	2	11	62	382	2562	18413	140968	1142004	9745298
3	3	27	250	2568	28746	347691	4495983	61714968	894242997
4	4	50	644	9209	143416	2415078	43532832	833734416	16863679508
5	5	80	1320	24150	480736	10340800	238120365	5826981430	150609007570
6	6	117	2354	52437	1269738	33192442	928558122	27600653310	866466783828
7	7	161	3822	100317	2859878	87935351	2892046165	101031525714	3726895105059
8	8	212	5800	175238	5746592	203079088	7672012360	307755240801	13032655134280
9	9	270	8364	285849	10596852	423019929	18042714315	816825050010	39027404931886
10	10	335	11590	442000	18274722	813079415	38632533180	1947580054285	103592924112830
11	11	407	15554	654742	29866914	1465238951	76729376515	4261622698733	249671899238553
12	12	486	20332	936327	46708344	2504570454	143291607432	8692072992879	556011110821900

$$\begin{aligned}
 a_1(F S_r) &= r & a_2(F S_r) &= \frac{7}{2}r^2 - \frac{3}{2}r \\
 a_3(F S_r) &= \frac{38}{3}r^3 - 11r^2 + \frac{7}{3}r & a_4(F S_r) &= \frac{1201}{24}r^4 - \frac{239}{4}r^3 + \frac{311}{24}r^2 + \frac{15}{4}r \\
 a_5(F S_r) &= \frac{6389}{30}r^5 - \frac{613}{2}r^4 + \frac{185}{6}r^3 + \frac{255}{2}r^2 - \frac{264}{5}r \\
 a_6(F S_r) &= \frac{696049}{720}r^6 - \frac{72727}{48}r^5 - \frac{58627}{144}r^4 + \frac{33101}{16}r^3 - \frac{509257}{360}r^2 + \frac{973}{3}r \\
 a_7(F S_r) &= \frac{11708603}{2520}r^7 - \frac{87143}{12}r^6 - \frac{146903}{18}r^5 + \frac{54431}{2}r^4 - \frac{9126049}{360}r^3 + \frac{129725}{12}r^2 - \frac{13019}{7}r \\
 a_8(F S_r) &= \frac{947714177}{40320}r^8 - \frac{5336487}{160}r^7 - \frac{55786441}{576}r^6 + \frac{7419257}{24}r^5 - \frac{2105526961}{5760}r^4 + \frac{110385341}{480}r^3 - \frac{52875299}{672}r^2 + \frac{95103}{8}r \\
 a_9(F S_r) &= \frac{5649947729}{45360}r^9 - \frac{78967849}{560}r^8 - \frac{1039050691}{1080}r^7 + \frac{142822454}{45}r^6 - \frac{9770306269}{2160}r^5 + \frac{2708660903}{720}r^4 - \frac{44177206909}{22680}r^3 \\
 &\quad + \frac{378138079}{630}r^2 - \frac{776555}{9}r
 \end{aligned}$$

B Values of $a_n^I(FS_r)$

$r \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	3	6	10	20	35	68	126	242	458	886	1696
3	3	6	16	36	96	237	624	1608	4221	11043	29109	76768
4	4	10	32	89	284	866	2776	8860	28744	93464	305608	1000982
5	5	15	55	180	656	2330	8620	32020	120900	459660	1761230	6779350
6	6	21	86	321	1302	5212	21582	90132	382602	1639917	7096674	30926564
7	7	28	126	525	2331	10297	46796	215012	1003877	4740008	22622985	108914792
8	8	36	176	806	3872	18600	91520	455849	2306152	11808484	61161312	319883860
9	9	45	237	1179	6075	31395	165591	884592	4796848	26337348	146326572	821478540
10	10	55	310	1660	9112	50245	281920	1602175	9236660	53921531	318568940	1902539090

$$\begin{aligned}
 a_1^I(FS_r) &= r & a_2^I(FS_r) &= \frac{1}{2}r^2 + \frac{1}{2}r & a_3^I(FS_r) &= \frac{1}{6}r^3 + \frac{3}{2}r^2 - \frac{2}{3}r \\
 a_4^I(FS_r) &= \frac{1}{24}r^4 + \frac{5}{4}r^3 - \frac{1}{24}r^2 - \frac{1}{4}r & a_5^I(FS_r) &= \frac{1}{120}r^5 + \frac{7}{12}r^4 + \frac{67}{24}r^3 - \frac{43}{12}r^2 + \frac{6}{5}r \\
 a_6^I(FS_r) &= \frac{1}{720}r^6 + \frac{3}{16}r^5 + \frac{461}{144}r^4 - \frac{73}{48}r^3 - \frac{1513}{360}r^2 + \frac{10}{3}r \\
 a_7^I(FS_r) &= \frac{1}{5040}r^7 + \frac{11}{240}r^6 + \frac{263}{144}r^5 + \frac{115}{16}r^4 - \frac{8089}{360}r^3 - \frac{319}{15}r^2 - \frac{48}{7}r \\
 a_8^I(FS_r) &= \frac{1}{40320}r^8 + \frac{13}{1440}r^7 + \frac{1979}{2880}r^6 + \frac{173}{18}r^5 - \frac{66113}{5760}r^4 - \frac{43913}{1440}r^3 + \frac{227777}{3360}r^2 + \frac{281}{-8}r \\
 a_9^I(FS_r) &= \frac{1}{362880}r^9 + \frac{1}{672}r^8 + \frac{1657}{8640}r^7 + \frac{547}{90}r^6 + \frac{377749}{17280}r^5 - \frac{37105}{288}r^4 + \frac{18446699}{90720}r^3 - \frac{294191}{2520}r^2 + \frac{136}{9}r \\
 a_{10}^I(FS_r) &= \frac{1}{3628800}r^{10} + \frac{17}{80640}r^9 + \frac{5129}{120960}r^8 + \frac{14423}{5760}r^7 + \frac{5648053}{172800}r^6 - \frac{789689}{11520}r^5 - \frac{20055283}{90720}r^4 + \frac{18449327}{20160}r^3 - \frac{28177631}{25200}r^2 + \frac{2292}{5}r \\
 a_{11}^I(FS_r) &= \frac{1}{39916800}r^{11} + \frac{19}{725760}r^{10} + \frac{937}{120960}r^9 + \frac{91897}{120960}r^8 + \frac{3771383}{172800}r^7 + \frac{2583703}{34560}r^6 - \frac{133247833}{181440}r^5 \\
 &\quad + \frac{289546877}{181440}r^4 - \frac{5427659}{6300}r^3 - \frac{1158377}{1260}r^2 + \frac{9054}{11}r \\
 a_{12}^I(FS_r) &= \frac{1}{479001600}r^{12} + \frac{1}{345600}r^{11} + \frac{10487}{8709120}r^{10} + \frac{17567}{96768}r^9 + \frac{137379751}{14515200}r^8 + \frac{1567309}{12800}r^7 - \frac{664752743}{1741824}r^6 \\
 &\quad - \frac{22482445}{13824}r^5 + \frac{108858294689}{10886400}r^4 - \frac{11855920577}{604800}r^3 + \frac{5629166951}{332640}r^2 - \frac{21769}{4}r
 \end{aligned}$$

C Values of $a_n^C(FS_r)$

r \ n	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	1	10	40	149	457	1380	3940
3	1	26	227	1696	10381	64954	367829
4	1	58	940	12053	124683	1312774	32656398
5	1	122	3383	68524	1089957	17321988	780465754
6	1	250	11320	344609	7962407	179542398	12045020929
7	1	506	36347	1609696	52053881	1600876052	147519031977
8	1	1018	113860	7172573	316326523	12911778902	1565476753784
9	1	2042	351263	30972244	1828173277	97095768316	15081546028136
10	1	4090	1073200	130896569	10196063247	694127660206	135628506406503

$$\begin{aligned}
 a_1^C(FS_r) &= 1 \\
 a_2^C(FS_r) &= 4 \cdot 2^r - 6 \\
 a_3^C(FS_r) &= \frac{113}{6}3^r - 38 \cdot 2^r + \frac{45}{2} \\
 a_4^C(FS_r) &= \frac{291}{2}4^r - \frac{745}{2}3^r + 317 \cdot 2^r - \frac{189}{2} \\
 a_5^C(FS_r) &= \frac{15311}{10}5^r - \frac{58169}{12}4^r + 5582 \cdot 3^r - \frac{5493}{2}2^r + \frac{2917}{6} \\
 a_6^C(FS_r) &= \frac{8530559}{360}6^r - \frac{444264}{5}5^r + \frac{3069971}{24}4^r - \frac{782245}{9}3^r + \frac{216245}{8}2^r - \frac{43211}{15} \\
 a_7^C(FS_r) &= \frac{161313668}{105}7^r - \frac{235545521}{36}6^r + \frac{261192269}{24}5^r - \frac{210522757}{24}4^r + \frac{122535049}{36}3^r - \frac{7912154}{15}2^r + \frac{458861}{24}
 \end{aligned}$$

D Comparing $a_n^I(FS_2)$ with central binomial coefficients

n	$a_n^I(FS_2)$	$\binom{n}{\lfloor n/2 \rfloor}$	Difference
1	2	2	0
2	3	3	0
3	6	6	0
4	10	10	0
5	20	20	0
6	35	35	0
7	68	70	2
8	126	126	0
9	242	252	10
10	458	462	4
11	886	924	38
12	1696	1716	20
13	3284	3432	148
14	6339	6435	96
15	12302	12870	568
16	23850	24310	460
17	46390	48620	2230
18	90244	92378	2134
19	175940	184756	8816
20	343246	352716	9470
21	670714	705432	34718
22	1311764	1352078	40314
23	2568740	2704156	135416
24	5034652	5200300	165648
25	9877768	10400600	522832