COUNTING FRIEZES IN TYPE D_n

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Abstract. We prove that there is an finite number of friezes in type D_n, and we provide a formula to count them. As a corollary, we obtain formulas to count the number of friezes in types B_n, C_n and G_2. We conjecture finiteness (and precise numbers) for other Dynkin types.

1. Introduction

Friezes of type A_n were defined by Coxeter [4] and studied by Conway and Coxeter [3] in the early '70's. An observation credited to Caldero in [1] is that Fomin and Zelevinsky's cluster algebras [7] allow for a huge generalization of the original definition. In this paper, we are interested in friezes of Dynkin types.

One way to define friezes is to say that they are ring homomorphisms from a cluster algebra to the ring of integers such that all cluster variables are sent to positive integers. In Dynkin types, a cluster-free definition may be given as follows [1, Section 3]. Let $C = (C_{i,j})_{n\times n}$ be a Cartan matrix of Dynkin type Δ, and assume that we have an acyclic orientation of the associated Dynkin diagram. Then a *frieze of type* Δ is a collection of positive integers $a(j,m)$, with $j \in \{1,\dots,n\}$ and $m \in \mathbb{Z}$, such that

$$a(j,m)a(j,m+1) = 1 + \Big(\prod_{j\to i} a(i,m)^{|C_{i,j}|}\Big)\Big(\prod_{i\to j} a(i,m+1)^{|C_{i,j}|}\Big).$$

This is conveniently represented as in Figure 1. For friezes of type D_n, there is a model developed by Schiffler [11] (see also [2] and [6]) involving tagged arcs in a punctured polygon. We recall this model in section 2.1.

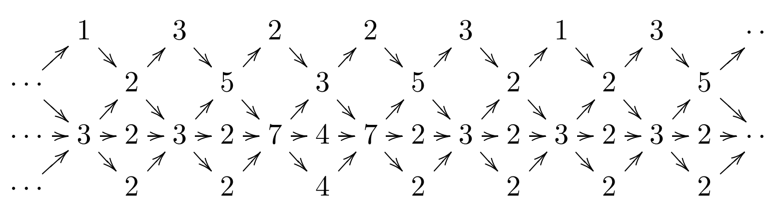

Figure 1. A frieze in type D_5.

Conway and Coxeter proved in [3] that in type A_n, there is only a finite number of friezes, and that this number is the $(n+1)$-st Catalan number. In type D_4, Morier-Genoud, Ovsienko and Tabachnikov [9] proved that there are 51 friezes, a result conjectured by Propp [10] (in fact, they were working with 2-friezes, which in these small cases are related to friezes). In this paper, we extend these results to arbitrary D_n types:

Theorem 1.1 (2.9)**.** *The number of D_n friezes is $\sum_{m=1}^{n} d(m) \begin{pmatrix} 2n-m-1 \\ n-m \end{pmatrix}$, where $d(m)$ is the number of divisors of m.*

As a corollary to this and to the results in [3], we can count friezes in types B_n, C_n and G_2 by folding Dynkin diagrams:

Corollary 1.2 (3.2, 3.3, 3.4)**.** *The number of friezes in type B_n, C_n and G_2 is $\sum_{m\leq\sqrt{n+1}} \begin{pmatrix} 2n-m^2+1 \\ n \end{pmatrix}$, $\begin{pmatrix} 2n \\ n \end{pmatrix}$ and 9, respectively.*

It is worth noting that in types B_n, D_n and G_2, the number of friezes is strictly greater than the number of clusters. The sequences of numbers of friezes in types D_n and B_n make up two new entries in the On-Line Encyclopedia of Integer Sequences [12] [13].

For the other Dynkin types, we propose the following

Conjecture 1.3 (3.6)**.** *The number of friezes of type E_6, E_7, E_8 and F_4 is 868, 4400, 26592 and 112, respectively.*

Note that the number for type E_6 was conjectured already by Propp [10], and evidence for this number was further obtained by Morier-Genoud, Ovsienko and Tabachnikov [9].

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2. Friezes of type D_n

2.1. **Triangulations of the punctured polygon.** Recall that we defined a frieze as an evaluation of all cluster variables, where each variable is a positive integer. The following geometric model is due to Schiffler:

Theorem 2.1 ([11])**.** *The cluster variables of D_n correspond to the (tagged) arcs in a once punctured n-gon. Moreover, the exchange relations are those of Figure 2.*

Thus, as is noted in [2], a D_n frieze is simply a choice of positive integer weight for each (tagged) arc in the punctured disk model, satisfying the relations of Figure 2. In the rest of the paper, this is the point of view from which we will view friezes.

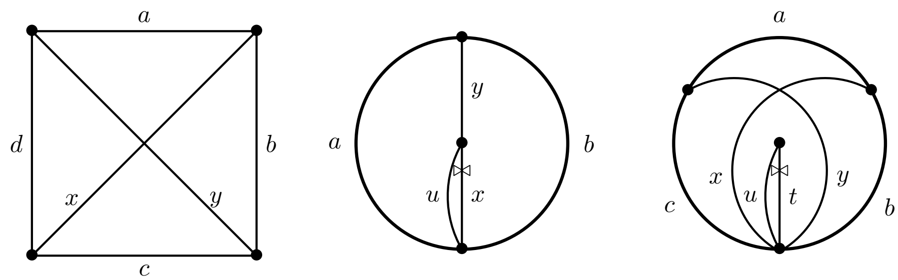

FIGURE 2. Ptolemy relation $xy = ac + bd$ (left) and other relations $xy = a + b$ (middle) and $xy = bc + atu$ (right). The middle relation also holds if u and y are tagged and x is untagged, and the left one does for any tagging.

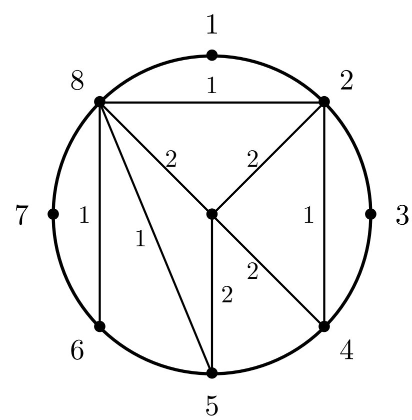

FIGURE 3. An example of a triangulation as in Proposition 2.2.

2.2. **Description of all friezes.** We will prove the following proposition, which describes the friezes in type D_n and ensures that there is only a finite number of them.

Proposition 2.2. *From any frieze of type D_n can be extracted a unique tagged triangulation T of the punctured n-gon in such a way that*

(1) *T contains all arcs of weight 1;*
(2) *all arcs of T connecting marked points on the boundary have weight 1;*
(3) *the m arcs of T incident with the puncture are untagged and all have the same weight, which can be any divisor of m.*

In particular, there is only a finite number of friezes of type D_n.

Figure 3 gives an example of a triangulation satisfying (1), (2) and (3). If such a triangulation exists for a given frieze, then its uniqueness is clear.

We prove its existence in several steps. First, we show that there is indeed a triangulation containing all the arcs of weight 1 of the frieze:

Lemma 2.3. *Two arcs of weight* 1 *in a frieze of type* D_n *cannot cross.*

Proof. When two arcs cross, then their weights, say x and y, have to satisfy a relation of the form $xy = m_1 + m_2$, where m_1 and m_2 are monomials in the weights of the other arcs (see Figure 2). In particular, xy is at least 2, so x and y cannot both be equal to 1. □

Let T_0 be the triangulation of the punctured n-gon defined thus: add to T_0 all arcs of weight 1, and then add untagged arcs from the boundary to the puncture whenever it is possible to do so without crossings. Then T_0 is a uniquely defined triangulation which satisfies conditions (1) and (2) of Proposition 2.2.

The following statement was proved by Hugh Thomas in an appendix to [2, Proposition A.2].

Lemma 2.4. *If one of the arcs of a frieze of type* D_n *incident with the puncture has weight* 1*, then the frieze contains a triangulation of arcs of weight* 1*. In particular, if one of the arcs of* T_0 *incident with the puncture has weight* 1*, then all arcs of* T_0 *have weight* 1*.*

Proof. Assume that an arc of a frieze of type D_n incident with the puncture has weight 1. Then all the arcs compatible with this one form a frieze of type A_{n-1}; in particular, by [3], there is a triangulation consisting of arcs of weight 1. □

Lemma 2.5. *Assume a frieze of type* D_m *contains no arcs of weight* 1*. Then*

(1) *All untagged arcs incident with the puncture have the same weight. The same is true for tagged arcs.*
(2) *Any arc not incident with the puncture and forming a* $(d+1)$*-gon not containing the puncture has weight* d*.*
(3) *If* x *and* y *are the weights of the tagged and untagged arcs, respectively, incident with the puncture, then* $xy = m$*.*

Proof. Let $a_1, a_2, \dots, a_m$ be the weights of the untagged arcs incident with the puncture, in clockwise order. Without loss of generality, we can assume that $a_1 \geq a_i$ for all $i \in \{1, \dots, m\}$. Let t_i be the weight of the arc forming a triangle with the arcs weighted a_i and a_{i+2}, where the indices are viewed modulo m. Figure 4 illustrates this in an octogon.

For each i, there is a Ptolemy relation $a_{i+1}t_i = a_i + a_{i+2}$. By our assumptions, $t_i \geq 2$, and by maximality of a_1, we get

$$2a_1 \leq a_1 t_m = a_m + a_2 \leq 2a_1,$$

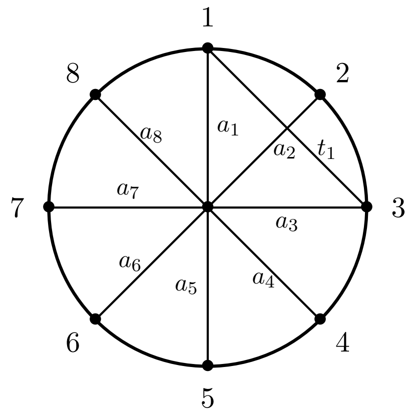

FIGURE 4. Labeling of the weights.

so $2a_1 = a_m + a_2$, which implies that $a_m = a_1 = a_2$, again by maximality of a_1. This argument propagates around the polygon, so by induction, we get that all the a_i's are equal. This proves part (1) for untagged arcs; the proof for tagged arcs is the same.

By the above relations, $a_{i+1}t_i = a_i + a_{i+2} = 2a_{i+1}$, so $t_i = 2$ for all i. Thus all arcs forming a triangle with the boundary have weight 2. We prove (2) by induction from here: assume that for a given d, all arcs forming a $(d+1)$-gon with the boundary have weight d. Let z be the weight of an arc forming a $(d+2)$-gon with the boundary. Then there is a Ptolemy relation of the form $a_{i+d}z = a_i + a_{i+d+1}d$, so $a_{i+d}z = a_{i+d}(1+d)$, and therefore $z = d+1$. Part (2) is proved.

Part (3) follows from part (2) and from the relation on the right in Figure 2. □

It follows from Lemma 2.5 that T_0 satisfies condition (3). Indeed, cutting along all arcs of weight 1, we are left with a smaller punctured polygon whose arcs have weight at least 2 and form a frieze of type D. Thus Lemma 2.5 applies. This finishes the proof of Proposition 2.2.

2.3. **Triangulations of punctured n-gons.** Let $T_{n,m}$ be the number of triangulations of a once-punctured n-gon with exactly m untagged arcs, or spokes, from the outer marked points to the inner puncture.

Theorem 2.6. $T_{n,m} = \begin{pmatrix} 2n-m-1 \\ n-1 \end{pmatrix}$.

Lemma 2.7. *We have* $T_{n,m} = \frac{n}{m}\sum_{i_1+\cdots+i_m=n-m}\prod_j C_{i_j}$, *where* C_n *is the* n*-th Catalan number.*

Proof. Given a triangulation of the punctured n-gon with m spokes, the portion of the triangulation between two adjacent spokes is an honest triangulation of a $k+2$-gon, where k is the number of vertices contained in between the two spokes. The two extra vertices are the end points of the spokes themselves. Thus there are C_k possible triangulations that fit between the two given spokes. The total number of vertices not involved with the spokes is $n-m$, so we partition $n-m$ into m non-negative pieces $i_1+\cdots+i_m = n-m$ with $i_j \geq 0$. Fix one of the spokes as a starting point, then we should see $\sum_{i_1+\cdots+i_m=n-m}\prod_j C_{i_j}$ triangulations. This under counts the true number since rotating a triangulation one step can give a different triangulation. Thus if we multiply by n, the total number of possible rotations, we would count each triangulation at least once. But we are ignoring the fact that we fixed one of the m spokes, so we are now over counting by a factor of m. This leaves us with $T_{n,m} = \frac{n}{m}\sum_{i_1+\cdots+i_m=n-m}\prod_j C_{i_j}$. $\square$

We can now prove Theorem 2.6:

Proof. Recall that $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function for the Catalan numbers. The coefficient of x^n in $(c(x))^k$ is known as the ballot number $B(n,k)$ and has closed form $B(n,k) = \frac{k}{2n+k}\begin{pmatrix} 2n+k \\ n \end{pmatrix}$. But, $T_{n,m} = \frac{n}{m}B(n-m,m) = \begin{pmatrix} 2n-m-1 \\ n-1 \end{pmatrix}$. $\square$

Since the generating function for the k ballot numbers is $(c(x))^k$, then the sum $1+(c(x))y+(c(x))^2y^2+\cdots = \frac{1}{1-yc(x)}$ is a two variable generating function for the ballot numbers. If we examine $\frac{1}{1-xyc(x)}$, then we see that the coefficient for x^ny^m is $B(n-m,m)$.

Lemma 2.8. *The generating function for $T_{n,m}$ is $\frac{1}{(c(x)-2)(1-xyc(x))}$.*

Proof. Note that $\frac{1}{1-xyc(x)}$ is almost a generating function for $T_{n,m}$, it is off by a factor of $\frac{n}{m}$ in term x^ny^m. This can be corrected by integration and differentiation:

$$\int\left(\frac{x}{y}\frac{d}{dx}\left(\frac{1}{1-xyc(x)}\right)\right)dy = \frac{c(x)+xc'(x)}{c(x)(xyc(x)-1)}.$$

One can check that $1+\frac{xc'(x)}{c(x)} = \frac{1}{2-c(x)}$, in which case the generating function becomes $\frac{1}{(2-c(x))(1-xyc(x))}$. $\square$

2.4. **Counting friezes.** We can now prove our main theorem.

Theorem 2.9. *The number of D_n friezes is $\sum_{m=1}^{n} d(m)\begin{pmatrix} 2n-m-1 \\ n-m \end{pmatrix}$.*

Proof. A frieze is determined by its weights on a single cluster, or (tagged) triangulation. This, together with Proposition 2.2, tells us that the number of friezes is $\sum_{m=1}^{n} d(m)T_{n,m}$, where $T_{n,m}$ is as in section 2.3. The result follows from Theorem 2.6.

$\square$

3. Friezes of other Dynkin types

What remains is to discover how many friezes there are in other Dynkin types. In the case of B_n, C_n and G_2 we can use the folding method of [5]. To summarize, if Δ is a Dynkin quiver and G a group of automorphisms, then Δ/G is a valued quiver and the action of G lifts to the cluster algebra $A(\Delta)$. In this case [5, Corollary 5.16] shows that $A(\Delta/G)$ can be identified with a subalgebra of $A(\Delta)/G$. More over [5, Corollary 7.3] gives equality since Δ is Dynkin. The projection $\pi : A(\Delta) \to A(\Delta)/G$ can then be thought of as a surjective ring homomorphism from $A(\Delta)$ to $A(\Delta/G)$, which sends the cluster variables of $A(\Delta)$ to the cluster variables of $A(\Delta/G)$ via a quotient by G.

Lemma 3.1. *Let Δ by a Dynkin quiver and G a group of automorphisms, then each Δ/G frieze gives rise to a Δ frieze. More over, each Δ frieze that is G invariant descends to a Δ/G frieze.*

Proof. For the first part, if we consider a Δ/G frieze to be a ring homomorphism from the cluster algebra $A(\Delta/G)$ to $\mathbb{Z}$, then composing with the map π gives a Δ frieze.

For the second part, a Δ frieze that is G invariant descends to a ring homomorphism from $A(\Delta)/G$ to $\mathbb{Z}$ and thus gives a Δ/G frieze under the identification of $A(\Delta)/G$ with $A(\Delta/G)$. □

For the case of B_n, C_n and G_2, these are quotients of D_{n+1}, A_{2n-1} and D_4 respectively where the automorphisms we use are the maps swapping the short arms of D_{n+1}, mirroring A_{2n-1} through the middle vertex and the order 3 rotation of D_4.

Theorem 3.2. *The number of C_n friezes is* $\begin{pmatrix} 2n \\ n \end{pmatrix}$.

Proof. Since C_n is a folding of A_{2n-1}, by the above lemma, each C_n frieze can be lifted to a unique A_{2n-1} frieze which is G-invariant. One can check that the action of G on the A_{2n-1} cluster variables is given by the following action on the arcs of a $2n+2$-gon: take an arc and map it to the arc whose end points are diametrically opposed to the originals. Recall from [3] that the set of arcs in the $2n+2$-gon that are labeled 1 must form a triangulation. But the image of each arc labeled 1 under G is also an arc labeled 1, so the triangulation is G-invariant. Thus we have a G-invariant cluster in A_{2n-1} on which the frieze evaluates to 1, but by [5], this descends to a cluster of C_n.

Thus each C_n frieze is determined by fixing one cluster with every variable being 1 and the number of C_n friezes is the number of C_n clusters, $\begin{pmatrix} 2n \\ n \end{pmatrix}$ (see [8, Table 3]). □

Theorem 3.3. *The number of B_n friezes is* $\sum_{m \leq \sqrt{n+1}} \begin{pmatrix} 2n - m^2 + 1 \\ n \end{pmatrix}$.

Proof. Since B_n is a folding of D_{n+1}, each B_n frieze lifts to D_{n+1} frieze which is G-invariant. The two nodes on the end of D_{n+1} which are identified by G correspond to an untagged/tagged pair of parallel arcs in the punctured $n+1$-gon. Thus it follows that the label assigned to each pair is the same. Now as outlined in the calculation of the D_{n+1} friezes, when we decompose a frieze into a partial triangulation of all arcs labeled 1, and a D_m frieze containing no 1's, the D_m contains at least one spoke from the D_{n+1} frieze. Moreover, in Lemma 2.5, we see that the product of an untagged spoke with its parallel tagged spoke in the D_m frieze is m. Thus m must be a perfect square and moreover, the only D_m frieze which is allowed is a frieze with the square root labeling the spokes. Applying this reduction to the D_{n+1} formula results in the given formula. □

Theorem 3.4. *The number of G_2 friezes is* 9*.*

Proof. Since G_2 is a folding of D_4, each G_2 frieze lifts to a D_4 frieze which is G-invariant. Of the 50 D_4 friezes which come from setting a cluster to all 1's, only 8 are G-invariant and thus correspond to the 8 G_2 friezes which also come from setting a cluster to all 1's. The remaining frieze assigns 2 to the outer nodes of D_4 and 3 to the center, and this is also G-invariant, so it descends to the sole remaining G_2 frieze, leaving us with 9 friezes. □

What remains are the sporadic E_6, E_7, E_8 and F_4. We developed an algorithm to enumerate the friezes of each type, but it depends on the following conjecture:

Conjecture 3.5. *The value of a frieze at a node in a Dynkin diagram is less than the maximal value of the node over the set of unitary friezes, that is, friezes obtained by evaluating all cluster variables in a given cluster to* 1*.*

Since the set of unitary friezes is computable (i.e. using Sage for instance), this puts an easily computed maximal bound on the entries in a frieze.

Conjecture 3.6. *The number of E_6, E_7 and E_8 frieze are* 868*,* 4400 *and* 26592 *respectively. Since F_4 is a folding of E_6, the number of F_4 friezes is 112.*

The listing of friezes and the programs used to generate them are available at [14].

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