# PRODUCTS OF BINOMIAL COEFFICIENTS AND UNREDUCED FAREY FRACTIONS 

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#### Abstract

This paper studies the product $\bar{G}_{n}$ of the binomial coefficients in the $n$-th row of Pascal's triangle, which equals the reciprocal of the product of all the reduced and unreduced Farey fractions of order $n$. It studies its size as a real number, measured by $\log \left(\bar{G}_{n}\right)$, and its prime factorization, measured by the order of divisibility $\nu_{p}\left(\bar{G}_{n}\right)=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ by a fixed prime $p$, each viewed as a function of $n$. It derives three formulas for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$, two of which relate it to base $p$ radix expansions of integers up to $n$, and which display different facets of its behavior. These formulas are used to determine the maximal growth rate of each $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ and to explain structure of the fluctuations of these functions. It also defines analogous functions $\nu_{b}\left(\bar{G}_{n}\right)$ for all integer bases $b \geq 2$ using base $b$ radix expansions replacing base $p$-expansions. A final topic relates factorizations of $\bar{G}_{n}$ to Chebyshev-type prime-counting estimates and the prime number theorem.


## 1. Introduction

The complete products of binomial coefficients of order $n$ are the integers

$$
\bar{G}_{n}:=\prod_{k=0}^{n}\binom{n}{k} .
$$

This integer sequence begins $\bar{G}_{1}=1, \bar{G}_{2}=2, \bar{G}_{3}=9, \bar{G}_{4}=96, \bar{G}_{5}=2500, \bar{G}_{6}=$ 162000, and $\bar{G}_{7}=26471025$, and appears as A001142 in OEIS 48. The integer $\bar{G}_{n}$ is the reciprocal of the product $G_{n}$ of all nonzero unreduced Farey fractions of order $n$, as we describe in Section 2. We encountered unreduced Farey products $G_{n}$ while investigating the products $F_{n}$ of all nonzero (reduced) Farey fractions. The connections with Farey fractions and their relations to prime number theory motivated this work.

We study the size of the integers $\bar{G}_{n}$ viewed as real numbers and the behavior of their prime factorizations, as functions of $n$. Since the $\bar{G}_{n}$ grow exponentially fast we measure their size in terms of the rescaled function

$$
\begin{equation*}
\nu_{\infty}\left(\bar{G}_{n}\right):=\log \left(\bar{G}_{n}\right) \tag{1.1}
\end{equation*}
$$

It is easy to show that $\log \left(\bar{G}_{n}\right)$ has smooth growth, given by an asymptotic expansion having leading term $\frac{1}{2} n^{2}$. We derive the first few terms of its asymptotic expansion in Section 3, which are obtainable using Stirling's formula. We observe that from the Farey fraction viewpoint this asymptotic estimate has an analogy with a formulation of the Riemann hypothesis for Farey fractions due to Mikolás

[^0]43. The function $\log \left(\bar{G}_{n}\right)$ actually has a complete asymptotic expansion in negative powers $\frac{1}{n^{k}}$ valid to all orders after its first few lead terms. This full asymptotic expansion is derived in Appendix A, in which we make use of known asymptotics for the Barnes $G$-function.

The relations between primes encoded in the factorizations of binomial products $\bar{G}_{n}$ seem to be of deep arithmetic significance. These factorizations are described by the functions

$$
\begin{equation*}
\nu_{p}\left(\bar{G}_{n}\right):=\operatorname{ord}_{p}\left(\bar{G}_{n}\right) \tag{1.2}
\end{equation*}
$$

with $p^{\operatorname{ord}_{p}\left(\bar{G}_{n}\right)}$ denoting the maximal power of $p$ dividing $\bar{G}_{n}$. The prime factorizations of the first few $\bar{G}_{n}$ are $\bar{G}_{1}=1, \bar{G}_{2}=2, \bar{G}_{3}=3^{2}, \bar{G}_{4}=2^{5} \cdot 3, \bar{G}_{5}=2^{2} \cdot 5^{4}, \bar{G}_{6}=$ $2^{4} \cdot 3^{4} \cdot 5^{3}$ and $\bar{G}_{7}=3^{2} \cdot 5^{2} \cdot 7^{6}$. These initial values already exhibit visible oscillations in $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$, and each function $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ separately has a somewhat complicated structure of oscillations. Figure 1.1 plots values of $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$ for $1 \leq n \leq 1023$. This plot exhibits significant structure in the behavior of $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$, visible as a set of stripes in intervals between successive powers of 2 .


Figure 1.1. $\nu_{2}(n):=\operatorname{ord}_{2}\left(\bar{G}_{n}\right), 1 \leq n \leq 1023=2^{10}-1$.
The behavior of the prime factorizations of $\bar{G}_{n}$ is the main focus of this paper. We derive three different formulas for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$, given in Sections 4, 5 and 6 , respectively. Each of the formulas encodes different information about ord ${ }_{p}\left(\bar{G}_{n}\right)$. The first of these formulas follows from the unreduced Farey product interpretation. The second of these formulas relates $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ to the base $p$ expansion of $n$, which relates to values of the Riemann zeta function $\zeta(s)$ on the line $\operatorname{Re}(s)=0$ through a result of Delange [11. The third of these formulas directly involves the base $p$ radix expansion of $n$, and is linear and bilinear in the radix expansion digits.

The second and third formulas for $\nu_{p}\left(\bar{G}_{n}\right)$ generalize to notions attached to radix expansions to an arbitrary integer base. For each $b \geq 2$ we define integer-valued functions $\nu_{b}\left(\bar{G}_{n}\right)$ (resp. $\nu_{b}^{*}\left(\bar{G}_{n}\right)$ ) for $n \geq 1$, which for primes $p$ satisfy $\nu_{p}\left(\bar{G}_{n}\right)=$ $\nu_{p}^{*}\left(\bar{G}_{n}\right)=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ for all $n \geq 1$. In Appendix B we prove these definitions agree in general: for all $b \geq 2$,

$$
\begin{equation*}
\nu_{b}\left(\bar{G}_{n}\right)=\nu_{b}^{*}\left(\bar{G}_{n}\right) \quad \text { for all } \quad n \geq 1 \tag{1.3}
\end{equation*}
$$

The functions $\nu_{b}\left(\bar{G}_{n}\right)$ for composite $b$ can no longer be interpreted as specifying the amount of "divisibility by $b$ " of the integer $\bar{G}_{n}$. It is an interesting problem to determine what arithmetic information about $\bar{G}_{n}$ the functions $\nu_{b}\left(\bar{G}_{n}\right)$ might encode.

From the formulas obtained for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ we deduce results on its size and the behavior of its fluctuations. We show that

$$
0 \leq \operatorname{ord}_{p}\left(\bar{G}_{n}\right)<n \log _{p} n
$$

and that

$$
\limsup _{n \rightarrow \infty} \frac{\operatorname{ord}_{p}\left(\bar{G}_{n}\right)}{n \log _{p} n}=1
$$

It follows that $n \log _{p} n$ is the correct scale of growth for this function. We also show that each function $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ oscillates infinitely many times between the upper and lower bounds as $n \rightarrow \infty$.

In Section 7 we compare the three formulas for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$. We show that between them they account for much of the structure visible in the picture in Figure 1.1.

In Section 8 we present direct connections between individual binomial products $\bar{G}_{n}$ and the distribution of prime numbers. There is a tension between the smooth asymptotic growth of $\bar{G}_{n}$ and the oscillatory nature of the divisibility of $\bar{G}_{n}$ by individual primes. This tension encodes a great deal of information about the structure of prime numbers. Our results yield a Chebyshev-type estimate for $\pi(x)$ and suggest the possibility of a approach to the prime number theorem via radix expansion properties of $n$ to prime bases. In another direction, a connection of the $\bar{G}_{n}$ to the Riemann hypothesis may exist via their relation to products of Farey fractions, see 36.

## 2. Unreduced Farey Fractions

The Farey sequence $\mathcal{F}_{n}$ of order $n$ is the sequence of reduced fractions $\frac{h}{k}$ between 0 and 1 (including 0 and 1 ) which, when in lowest terms, have denominators less than or equal to $n$, arranged in order of increasing size. It is the set

$$
\mathcal{F}_{n}:=\left\{\frac{h}{k}: 0 \leq h \leq k \leq n: \operatorname{gcd}(h, k)=1 .\right\}
$$

The Farey sequences encode deep arithmetic properties of the integers and are important in Diophantine approximation, e.g. [26, Chap. III].) The distribution of the Farey fractions approaches the uniform distribution on $[0,1]$ as $n \rightarrow \infty$ in the sense of measure theory, and the rate at which it approaches the uniform distribution as a function of $n$ is related to the Riemann hypothesis by a theorem of Franel 21. Extensions of Franel's result are given in many later works, including Landau [37], 38, Mikolás [42, 43], and Huxley [29, Chap. 9 ].

The Farey sequences have a simpler cousin, the unreduced Farey sequence $\mathcal{G}_{n}$, which is the ordered sequence of all reduced and unreduced fractions between 0 and 1 with denominator of size at most $n$. We define the positive unreduced Farey sequence by omitting the value 0 , obtaining

$$
\mathcal{G}_{n}^{*}:=\left\{\frac{h}{k}: 1 \leq h \leq k \leq n\right\}
$$

We let $\Phi^{*}(n)=\left|\mathcal{G}_{n}^{*}\right|$ denote the number of positive unreduced Farey fractions, and clearly

$$
\begin{equation*}
\Phi^{*}(n)=\binom{n+1}{2}=\frac{1}{2} n(n+1) . \tag{2.1}
\end{equation*}
$$

We order these unreduced fractions in increasing order, breaking ties between equal fractions ordering them by increasing denominator. For example, we have

$$
\mathcal{G}_{4}^{*}:=\left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{4}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}\right\}
$$

We label the fractions in $\mathcal{G}_{n}^{*}$ in this order as $\rho_{r}^{*}=\rho_{r, n}^{*}$, and write

$$
\mathcal{G}_{n}^{*}=\left\{\rho_{r}^{*}=\rho_{r, n}^{*}: 1 \leq r \leq \Phi^{*}(n)\right\},
$$

Then we can define the unreduced Farey product as

$$
\begin{equation*}
G_{n}:=\prod_{r=1}^{\Phi^{*}(n)} \rho_{r, n}^{*}=\frac{N_{n}^{*}}{D_{n}^{*}}, \tag{2.2}
\end{equation*}
$$

in which $N_{n}^{*}$ (resp. $D_{n}^{*}$ ) denotes the product of the numerators (resp. denominators) of all $\rho_{r, n}^{*}$. The numerator function

$$
N_{n}^{*}=\prod_{k=1}^{n} k!
$$

has been called the superfactorial function and appears as sequence A000178 in OEIS 48. The denominator function

$$
D_{n}^{*}=\prod_{k=1}^{n} k^{k}
$$

has been called the hyperfactorial function, and appears as sequence A002109 in OEIS [48. The hyperfactorial $D_{n}^{*}$ in expressible in terms of factorials as

$$
\begin{equation*}
D_{n}^{*}=\frac{\prod_{k=1}^{n} k^{n}}{1^{n-1} 2^{n-2} \cdots(n-1)^{1} \cdot n^{0}}=\frac{(n!)^{n}}{(n-1)!\cdots 1!}=\frac{(n!)^{n}}{N_{n-1}^{*}} . \tag{2.3}
\end{equation*}
$$

It was studied by Glaisher [22], [23], starting in 1878. The integers $D_{n}^{*}$ were later found to be the sequence of discriminants of the Hermite polynomials ${ }^{1}$ in the probabilist's normalization $H e_{n}(x)=2^{-n / 2} H_{n}\left(\frac{x}{\sqrt{2}}\right)$.

The unreduced Farey products $G_{n}$ have their reciprocal $\bar{G}_{n}=1 / G_{n}$ expressible in terms of binomial coefficients.

Theorem 2.1. The unreduced Farey product $G_{n}$ has its reciprocal $\bar{G}_{n}=1 / G_{n}$ given by the product of binomial coefficients

$$
\begin{equation*}
\bar{G}_{n}=\prod_{j=0}^{n}\binom{n}{j} \tag{2.4}
\end{equation*}
$$

Thus $1 / G_{n}$ is always an integer.

[^1]Proof. Enumerating the unreduced Farey fractions in order of fixed $k$, as $\frac{j}{k}$ with $1 \leq j \leq k \leq n$, we have $\bar{G}_{n}=\frac{D_{n}^{*}}{N_{n}^{*}}$, in which $D_{n}^{*}=\prod_{k=1}^{n} k^{k}$ and $N_{n}^{*}=\prod_{j=1}^{n} j!=$ $\prod_{j=1}^{n} j^{n-j+1}$. Therefore, setting $0!=1$, we have

$$
\bar{G}_{n}=\frac{1^{1} \cdot 2^{2} \cdot 3^{3} \cdots n^{n}}{1^{n} \cdot 2^{n-1} \cdot 3^{n-2} \cdots n}=\frac{\left(\frac{n!}{0!} \cdot \frac{n!}{1!} \cdots \frac{n!}{(n-1)!}\right)}{1!2!\cdots(n-1)!n!}=\prod_{t=1}^{n} \frac{n!}{t!(n-t)!}=\prod_{t=1}^{n}\binom{n}{t}
$$

The last product also equals $\prod_{t=0}^{n}\binom{n}{t}$, as required.
Remark 2.2. Products of binomial coefficients $\bar{G}_{n}$ appear as normalizing constants $c_{n+1}$ associated to the density $z \mapsto \frac{n+1}{\pi}\left(1+|z|^{2}\right)^{-n}$ on $\mathbb{C}$, see Lyons [39, Sec.3.8]. This density is associated with a particular Gaussian orthogonal polynomial ensemble, the $(n+1)$-st spherical ensemble, which is the (randomly ordered) set of eigenvalues of $M_{1}^{-1} M_{2}$ where $M_{i}$ are independent $(n+1) \times(n+1)$ matrices whose entries are independent standard complex Gaussians. This eigenvalue interpretation of the spherical ensemble is due to Krishnapur [32, see [27].

Remark 2.3. The reciprocal $\bar{F}_{n}=1 / F_{n}$ of the product $F_{n}$ of all nonzero Farey fractions of order $n$ is a quantity analogous to $\bar{G}_{n}$. It encodes interesting arithmetic information, but is usually not an integer. The Riemann hypothesis is encoded in its asymptotic behavior, as discussed in Remark 3.4 below. The quantities $\bar{F}_{n}$ and $\bar{G}_{n}$ are related by the identity $\bar{G}_{n}=\prod_{k=1}^{n} \bar{F}_{\lfloor n / k\rfloor}$, which under a form of Möbius inversion yields $\bar{F}_{n}=\prod_{k=1}^{n}\left(\bar{G}_{\lfloor n / k\rfloor}\right)^{\mu(k)}$. Our study of $\bar{G}_{n}$ was motivated in part for its potential to obtain useful information about $\bar{F}_{n}$.

## 3. Growth of $\bar{G}_{n}$

We estimate the growth of $\bar{G}_{n}$ using its connection to superfactorials $N_{n}^{*}$ and hyperfactorials $D_{n}^{*}$. One can derive a complete asymptotic expansion for each of $\log \left(N_{n}^{*}\right), \log \left(D_{n}^{*}\right)$ and $\log \left(\bar{G}_{n}\right)$, using the Barnes G-function, which we present in Appendix A. Here we derive the first few leading terms, for which Stirling's formula suffices, and which permit

Theorem 3.1. For $n \geq 2$ the superfactorials $N_{n}^{*}$ and hyperfactorials $D_{n}^{*}$ satisfy

$$
\begin{aligned}
& \log \left(D_{n}^{*}\right)=\frac{1}{2} n^{2} \log n-\frac{1}{4} n^{2}+\frac{1}{2} n \log n+\quad+O(\log n) \\
& \log \left(N_{n}^{*}\right)=\frac{1}{2} n^{2} \log n-\frac{3}{4} n^{2}+n \log n+\left(\frac{1}{2} \log (2 \pi)-1\right) n+O(\log n)
\end{aligned}
$$

Proof. We will apply Stirling's formula in the truncated form

$$
\log (n!)=n \log n-n+\frac{1}{2} \log n+\frac{1}{2} \log (2 \pi)+O\left(\frac{1}{n}\right)
$$

valid for all $n \geq 1$.
For the denominator term, we have

$$
\log \left(D_{n}^{*}\right)=\sum_{k=1}^{n} k \log k=\sum_{j=1}^{n}\left(\sum_{k=j}^{n} \log k\right)=\sum_{j=1}^{n}(\log (n!)-\log (j-1)!) .
$$

Applying Stirling's formula on the right side (and shifting $j$ by 1) yields

$$
\sum_{k=1}^{n} k \log k=n \log (n!)-\sum_{j=1}^{n-1}\left(j \log j-j+\frac{1}{2} \log j+\frac{1}{2} \log (2 \pi)+O\left(\frac{1}{j}\right)\right)
$$

We move the term $\sum_{j=1}^{n-1} j \log j$ to the left side and obtain
$2\left(\sum_{k=1}^{n} k \log k\right)=n \log (n!)+n \log n+\frac{n(n-1)}{2}-\frac{1}{2} \log (n!)-\left(\frac{1}{2} \log (2 \pi)\right) n+O(\log n)$.
Applying Stirling's formula again on the right and simplifying yields

$$
\begin{equation*}
\sum_{k=1}^{n} k \log k=\frac{1}{2} n^{2} \log n-\frac{1}{4} n^{2}+\frac{1}{2} n \log n+O(\log n) \tag{3.1}
\end{equation*}
$$

which gives the asymptotic formula for $\log \left(D_{n}^{*}\right)$ above.
For the numerator term we have

$$
\begin{aligned}
\log \left(N_{n}^{*}\right) & =\sum_{k=1}^{n} \log (k!)=\sum_{k=1}^{n}\left(k \log k-k+\frac{1}{2} \log k+\frac{1}{2} \log (2 \pi)+O\left(\frac{1}{k}\right)\right) \\
& =\frac{1}{2} n^{2} \log n-\frac{3}{4} n^{2}+\frac{1}{2} n \log n+\frac{1}{2}(\log (2 \pi)-1) n+O(\log n) .
\end{aligned}
$$

The second line used $\sum_{k=1}^{n} \log k=\log (n!)$ with Stirling's formula, plus the asymptotic formula (3.1).

We single out for special emphasis the initial terms in the asymptotic expansion for $\log \left(\bar{G}_{n}\right)$.

Theorem 3.2. The function $\nu_{\infty}\left(\bar{G}_{n}\right):=\log \left(\bar{G}_{n}\right)$ satisfies for $n \geq 2$ the estimate

$$
\begin{equation*}
\log \left(\bar{G}_{n}\right)=\frac{1}{2} n^{2}-\frac{1}{2} n \log n+\left(1-\frac{1}{2} \log (2 \pi)\right) n+O(\log n) \tag{3.2}
\end{equation*}
$$

Here $1-\frac{1}{2} \log (2 \pi) \approx 0.08106$.
Proof. The asymptotic formula for $\log \left(\bar{G}_{n}\right)=\log \left(D_{n}^{*}\right)-\log \left(N_{n}^{*}\right)$ follows immediately from Theorem 9.2

Remark 3.3. This expansion captures a connection to density of primes and has a further analogy with the Riemann hypothesis, given the next remark. It shows that the function $\log \left(\bar{G}_{n}\right)$ is asymptotic to $\frac{1}{2} n^{2}$, which is smaller by a logarithmic factor than either of $\log \left(D_{n}^{*}\right)$ or $\log \left(N_{n}^{*}\right)$ separately. That is, the top terms in the asymptotic expansions of $\log \left(D_{n}^{*}\right)$ or $\log \left(N_{n}^{*}\right)$ cancel. This savings of a logarithmic factor in the main term of the asymptotic formula is directly related to primes having density $O\left(\frac{n}{\log n}\right)$, and to obtaining Chebyshev-type bounds for $\pi(x)$, see Section 8 ,

Remark 3.4. The analogy of the asymptotic formula (3.2) with the Riemann hypothesis arises from its interpretation in terms of products of unreduced Farey fractions and concerns its remainder term $O(\log n)$. We can rewrite it in terms of the number $\Phi^{*}(n)=\binom{n+1}{2}$ of unreduced Farey fractions as

$$
\begin{equation*}
\left.\log \left(\bar{G}_{n}\right)=\Phi^{*}(n)-\frac{1}{2} n \log n+\left(\frac{1}{2}-\frac{1}{2} \log (2 \pi)\right)\right) n+O(\log n) \tag{3.3}
\end{equation*}
$$

with $\frac{1}{2}-\frac{1}{2} \log (2 \pi) \approx-041894$. This expression is directly comparable with an expression for the logarithm of (inverse) Farey products $\log \left(\bar{F}_{n}\right)$ having the form

$$
\begin{equation*}
\log \left(\bar{F}_{n}\right)=\Phi(n)-\frac{1}{2} n+R(n) \tag{3.4}
\end{equation*}
$$

in which $\Phi(n)$ counts the number of Farey fractions and $R(n)$ is a remainder term defined by the equality (3.4). The function $\Phi(n)=\sum_{k=1}^{n} \phi(k)$ is the summatory function for the Euler $\phi$-function, and satisfies $\Phi(n) \sim \frac{3}{\pi^{2}} n^{2}$ as $n \rightarrow \infty$. In 1951 Mikolás [43, Theorem 1], showed that the Riemann hypothesis is equivalent to the assertion that the remainder term is small, satisfying

$$
R(n)=O\left(n^{\frac{1}{2}+\epsilon}\right)
$$

for each $\epsilon>0$ for $n \geq 2$. In fact he showed that estimates of form $R(n)=O\left(x^{\theta+\epsilon}\right)$ for fixed $1 / 2 \leq \theta<1$ and for all $\epsilon>0$ were equivalent to a zero-free region of the Riemann zeta function for $\operatorname{Re}(s)>\theta$. We can therefore view (3.3) by analogy as an "unreduced Farey fraction Riemann hypothesis" in view of its small error term $O(\log n)$. A Riemann hypothesis type estimate would require only an error term of form $O\left(n^{1 / 2+\epsilon}\right)$.

See [36, Section 3] for a further discussion of Mikolas's results, which include unconditional error bounds for $R(n)$. The true subtlety in the Mikoläs formula seems to resolve around oscillations in the function $\Phi(x)$ of magnitude at least $\Omega(x \sqrt{\log \log x})$ which themselves are related to zeta zeros.

## 4. Prime-power divisibility of $\bar{G}_{n}$ : Formulas using integer parts

We study the divisibility of $\bar{G}_{n}$ by powers of a fixed prime. We obtain three distinct formulas for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$, in this section and in the following two sections, respectively.

The first formula simply encodes the Farey product decomposition.
Theorem 4.1. For $p$ a prime, the function $\nu_{p}\left(G_{n}\right):=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ satisfies

$$
\begin{equation*}
\nu_{p}\left(\bar{G}_{n}\right)=\operatorname{ord}_{p}\left(D_{n}^{*}\right)-\operatorname{ord}_{p}\left(N_{n}^{*}\right) \tag{4.1}
\end{equation*}
$$

where $D_{n}^{*}=\prod_{k=1}^{n} k^{k}$ and $N_{n}^{*}=\prod_{k=1}^{n} k!$.
Proof. This formula follows directly by applying $\operatorname{ord}_{p}(\cdot)$ to both sides of the decomposition $\bar{G}_{n}=\frac{1}{G_{n}}=\frac{D_{n}^{*}}{N_{n}^{*}}$.

The formula 4.1 has several interesting features.
(i) This formula expresses $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ as a difference of two positive terms,

$$
\mathrm{S}_{p, 1}^{+}(n):=\operatorname{ord}_{p}\left(D_{n}^{*}\right)=\sum_{k=1}^{n} \operatorname{ord}_{p}\left(k^{k}\right)
$$

and

$$
\mathrm{S}_{p, 1}^{-}(n):=\operatorname{ord}_{p}\left(N_{n}^{*}\right)=\sum_{k=1}^{n} \operatorname{ord}_{p}(k!)
$$

Both terms are nondecreasing in $n$, that is,

$$
\Delta\left(\mathrm{S}_{p, 1}^{ \pm}\right)(n):=\mathrm{S}_{p, 1}^{ \pm}(n)-\mathrm{S}_{p, 1}^{ \pm}(n-1)
$$

are nonnegative functions. Furthermore the difference term $\Delta\left(\mathrm{S}_{p, 1}^{-}\right)(n)$ is nondecreasing in $n$.
(ii) There is a race in size between the terms $\mathrm{S}_{p, 1}^{+}(n)$ and $\mathrm{S}_{p, 1}^{-}(n)$, as $n$ varies. The first term $\mathrm{S}_{p, 1}^{+}(n)$ jumps only when $p$ divides $n$ and makes large jumps at these values. In contrast, the second term $\mathrm{S}_{p, 1}^{-}(n)$ changes in smaller nonzero increments, making a positive contribution whenever $p \nmid n$ and $n>p$. In consequence: For $n \geq p, \operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ increases going from $n-1$ to $n$ when $p \mid n$, and strictly decreases when $p \nmid n$.
We may re-express the terms in formula 4.1) using the floor function (greatest integer part function). We start with de Polignac's formula (attributed to Legendre by Dickson [16, p. 263]), which states that

$$
\operatorname{ord}_{p}(n!)=\sum_{j=1}^{\infty}\left\lfloor\frac{n}{p^{j}}\right\rfloor
$$

The sum on the right is always finite, with largest nonzero term $j=\left\lfloor\log _{p} n\right\rfloor$, with $p^{j} \leq n<p^{j+1}$. We obtain

$$
\operatorname{ord}_{p}\left(N_{n}^{*}\right)=\sum_{k=1}^{n}\left(\sum_{j=1}^{\infty}\left\lfloor\frac{k}{p^{j}}\right\rfloor\right)
$$

and, using 2.3),

$$
\operatorname{ord}_{p}\left(D_{n}^{*}\right)=n\left(\sum_{j=1}^{\infty}\left\lfloor\frac{n}{p^{j}}\right\rfloor\right)-\operatorname{ord}_{p}\left(N_{n-1}^{*}\right)
$$

We next obtain asymptotic estimates with error term for $\operatorname{ord}_{p}\left(N_{n}^{*}\right)$ and $\operatorname{ord}_{p}\left(D_{n}^{*}\right)$, and use these estimates to upper bound the size of $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$.

Theorem 4.2. For $p$ a prime, and all $n \geq 2$,

$$
\operatorname{ord}_{p}\left(N_{n}^{*}\right)=\frac{1}{2(p-1)} n^{2}+O\left(n \log _{p} n\right)
$$

and

$$
\operatorname{ord}_{p}\left(D_{n}^{*}\right)=\frac{1}{2(p-1)} n^{2}+O\left(n \log _{p} n\right)
$$

It follows that,

$$
\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=O\left(n \log _{p} n\right)
$$

for all $n \geq 1$.
Proof. We rewrite de Polignac's formula using the identity $\frac{n}{p^{j}}=\left\lfloor\frac{n}{p^{j}}\right\rfloor+\left\{\frac{n}{p^{j}}\right\}$, with the fractional part function $\{x\}:=x-\lfloor x\rfloor$, to obtain

$$
\begin{equation*}
\operatorname{ord}_{p}(n!)=\frac{n}{p-1}-\sum_{j=1}^{\infty}\left\{\frac{n}{p^{j}}\right\} \tag{4.2}
\end{equation*}
$$

For $j>\log _{p} n$ we have $\left\{\frac{n}{p^{j}}\right\}=\frac{n}{p^{j}}$ so the series becomes a geometric series past this point and can be summed. One obtains the estimate

$$
\operatorname{ord}_{p}(n!)=\frac{n}{p-1}+O\left(1+\log _{p} n\right)
$$

with an $O$-constant independent of $p$.

For $N_{n}^{*}$ we have $\operatorname{ord}_{p}\left(N_{n}^{*}\right)=\sum_{k=1}^{N} \operatorname{ord}_{p}(k!)$, and applying 4.2 yields

$$
\begin{aligned}
\operatorname{ord}_{p}\left(N_{n}^{*}\right) & =\sum_{k=1}^{n} \frac{k}{p-1}-\sum_{k=1}^{n}\left(\sum_{j=1}^{\infty}\left\{\frac{k}{p^{j}}\right\}\right) \\
& =\frac{n^{2}+n}{2(p-1)}+O\left(n\left(1+\log _{p} n\right)\right)
\end{aligned}
$$

The result follows by shifting $\frac{1}{2(p-1)} n$ to the remainder term.
For $D_{n}^{*}$ we have, using (2.3), that

$$
\begin{align*}
\operatorname{ord}_{p}\left(D_{n}^{*}\right) & =n \operatorname{ord}_{p}(n!)-\operatorname{ord}_{p}\left(N_{n-1}^{*}\right) \\
& =\left(n\left(\frac{n}{p-1}\right)+O\left(1+\log _{p} n\right)\right) \\
& -\left(\frac{1}{2(p-1)}(n-1)^{2}+O\left(n\left(1+\log _{p} n\right)\right)\right.  \tag{4.3}\\
& =\frac{1}{2(p-1)} n^{2}+O\left(n \log _{p} n\right)
\end{align*}
$$

For $\bar{G}_{n}^{*}$ the result follows from $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=\operatorname{ord}_{p}\left(D_{n}^{*}\right)-\operatorname{ord}_{p}\left(N_{n}^{*}\right)$.
The bound $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=O\left(n \log _{p} n\right)$, valid for all $n \geq p$, quantifies the smaller size of in size of $\log \left(\bar{G}_{n}\right)$ compared to either $\log \left(N_{n}^{*}\right)$ and $\log \left(D_{n}^{*}\right)^{6}$. In the situation here the smaller size is by almost a square root factor. The bound on $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ above is the correct order of magnitude, and we obtain a sharp constant in Theorem 6.8 below.
5. Prime-power divisibility of $\bar{G}_{n}$ : Formulas using base $p$ digit sums

We obtain a second formula for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$, one expressed directly in terms of base $p$ digit sums, and draw consequences. We start from

$$
\begin{equation*}
\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=\sum_{k=0}^{n} \operatorname{ord}_{p}\binom{n}{k} . \tag{5.1}
\end{equation*}
$$

The divisibility of binomial coefficients by prime powers $p^{k}$ has been studied for over 150 years, see the extensive survey of Granville [25]. Divisibility properties are well known to be related to the coefficients $a_{j}$ of the the base $p$ radix expansion of $n$, written as

$$
n=\sum_{j=0}^{k} a_{j} p^{j}, \quad 0 \leq a_{j} \leq p-1
$$

with $k=\left\lfloor\log _{p} n\right\rfloor$.
5.1. Prime-power divisibility of $\bar{G}_{n}$ : digit summation form. We derive a formula for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ that expresses it in terms of summatory functions of base $p$ digit sums.

We will consider digit sums more generally for radix expansions to an arbitrary integer base $b \geq 2$. Write a positive integer $n$ in base $b \geq 2$ as

$$
n:=\sum_{i=0}^{k} a_{i} b^{i}, \text { for } b^{k} \leq n<b^{k+1}
$$

with digits $0 \leq a_{i} \leq b-1$. Here $k=\left\lfloor\log _{b} n\right\rfloor$. and $a_{i}:=a_{i}(n)$ with $a_{k}(n) \geq 1$.
(1) The sum of digits function $d_{b}(n)$ (to base $b$ ) of $n$ is

$$
\begin{equation*}
d_{b}(n):=\sum_{i=0}^{k} a_{i}(n) \tag{5.2}
\end{equation*}
$$

with $k=\left\lfloor\log _{b} n\right\rfloor$.
(2) The running digit sum function $S_{b}(n)$ (to base b) is

$$
\begin{equation*}
S_{b}(n):=\sum_{j=0}^{n-1} d_{b}(j) \tag{5.3}
\end{equation*}
$$

Our second formula for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ is given in terms of these quantities; we defer its proof to the end of this subsection.

Theorem 5.1. Let the prime $p$ be fixed. Then for all $n \geq 1$,

$$
\begin{equation*}
\nu_{p}\left(\bar{G}_{n}\right):=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=\frac{1}{p-1}\left(2 S_{p}(n)-(n-1) d_{p}(n)\right) . \tag{5.4}
\end{equation*}
$$

The formula (5.4 has several interesting features.
(i) This formula expresses $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ as a difference of two positive terms,

$$
\begin{equation*}
\mathrm{S}_{p, 2}^{+}(n):=\frac{2}{p-1} S_{p}(n) \tag{5.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{S}_{p, 2}^{-}(n):=\frac{n-1}{p-1} d_{p}(n) \tag{5.6}
\end{equation*}
$$

The two functions, $S_{p}(n)$ and $d_{p}(n)$ have been extensively studied in the literature. They exhibit very different behaviors as $n$ varies: $S_{p}(n)$ grows rather smoothly while $d_{p}(n)$ exhibits large abrupt variations in size.
(ii) The function $S_{p}(n)$ has smooth variation and obeys the asymptotic estimate

$$
S_{p}(n)=\left(\frac{p-1}{2}\right) n \log _{p} n+O(n)
$$

see Theorem 5.6. In consequence $\mathrm{S}_{p, 2}^{+}(n)=n \log _{p} n+O(n)$. Furthermore $S_{p}(n)$ can itself be written as a difference of two positive functions using the identity 5.12 below, and noting the second term is nonpositive, by Theorem 5.8 (1).
(iii) The function $d_{p}(n)$ is known to have average size $\frac{p-1}{2} \log _{p}(n)$ but is oscillatory. For most $n$ it is rather close to its average size, however it varies from 1 to a value as large as $(p-1) \log _{p} n$ infinitely often as $n \rightarrow \infty$, as given by the distribution of $d_{p}(n)$ as $n$ varies. If one takes $n=p^{k}$ and samples $m$ uniformly on the range $\left[1, p^{k}\right]$, then $d_{p}(m)$ it is a sum of $k$ identically distributed independent random variables, and as $k \rightarrow \infty$ will obey a central limit theorem. One can show that it has size sharply concentrated around $\left(\frac{p-1}{2}\right) \log _{p} n$ with a spread on the order of $C_{p} \sqrt{k}\left(\frac{p-1}{2}\right)$. In consequence, the second term $\mathrm{S}_{p, 2}^{-}(n)=\frac{n+1}{p-1} d_{p}(n)$ is positive and has average size $\frac{1}{2} n \log _{p} n+O(n)$, which is in magnitude half that of the first term. It has large variations in size, between being twice its average size and being $o(n \log n)$.
(iv) The function $d_{p}(n)$ is highly correlated between successive values of $n$. It exhibits an "odometer" behavior where it has increases by one at most steps, but has jumps downward of size about $p^{k}$ at values of $n$ that $p^{k}$ exactly divides.
To derive Theorem 5.1, we make use of the following elegant formula for $\operatorname{ord}_{p}\binom{n}{t}$ noted by Granville [25, 25, equation following (18)].
Proposition 5.2. For $n \geq 1$ and $0 \leq t \leq n$,

$$
\begin{equation*}
\operatorname{ord}_{p}\binom{n}{t}=\frac{1}{p-1}\left(d_{p}(t)+d_{p}(n-t)-d_{p}(n)\right) \tag{5.7}
\end{equation*}
$$

Proof. Writing $n=\sum_{i=0}^{k} a_{i} p^{i}$, and applying de Polignac's formula, we have

$$
\begin{aligned}
\operatorname{ord}_{p}(n)= & \frac{n-a_{0}}{p}+\frac{n-\left(a_{1} p+a_{0}\right)}{p^{2}}+\cdots+\frac{n-\left(a_{k-1} p^{k-1}+\cdots+a_{0}\right)}{p^{k}} \\
& +\sum_{i=k+1}^{\infty} \frac{n-\left(a_{k} p^{k}+a_{k-1} p^{k-1}+\cdots+a_{0}\right)}{p^{i}}
\end{aligned}
$$

in which all the terms in the last sum are identically zero. Collecting the terms for $n$ and for each $a_{i}$ separately on the right side of this expression, each forms a geometric progression, yielding

$$
\begin{equation*}
\operatorname{ord}_{p}(n!)=\frac{1}{p-1}\left(n-\left(a_{k}+a_{k-1}+\cdots+a_{0}\right)\right)=\frac{1}{p-1}\left(n-d_{p}(n)\right) \tag{5.8}
\end{equation*}
$$

Writing the binomial coefficient $\binom{n}{t}=\frac{n!}{t!(n-t)!}$ and substituting 5.8) above yields the desired formula.

Proof of Theorem 5.1. Combining Theorem 2.1 with Proposition 5.2 and noting that $d_{p}(0)=0$, we have

$$
\begin{aligned}
\operatorname{ord}_{p}\left(\bar{G}_{n}\right) & =\sum_{t=0}^{n} \operatorname{ord}_{p}\binom{n}{t}=\frac{1}{p-1} \sum_{t=0}^{n}\left(d_{p}(t)+d_{p}(n-t)-d_{p}(n)\right) \\
& =\frac{1}{p-1}\left(2 \sum_{t=0}^{n} d_{p}(t)-(n+1) d_{p}(n)\right) \\
& =\frac{1}{p-1}\left(2 S_{p}(n)-(n-1) d_{p}(n)\right)
\end{aligned}
$$

as required.
5.2. Analogue function $\nu_{b}\left(\bar{G}_{n}\right)$ for a general radix base $b$. The functions of digit sums on the right side of (5.4 make sense for all radix bases $b \geq 2$, which leads us to define general functions $\nu_{b}\left(\bar{G}_{n}\right)$ for $b \geq 2$.
Definition 5.3. For each integer $b \geq 2$ and $n \geq 1$, the generalized order $\nu_{b}\left(\bar{G}_{n}\right)$ of $\bar{G}_{n}$ to base $b$ is

$$
\begin{equation*}
\nu_{b}\left(\bar{G}_{n}\right):=\frac{1}{b-1}\left(2 S_{b}(n)-(n-1) d_{b}(n)\right) . \tag{5.9}
\end{equation*}
$$

Theorem 5.1 shows that for prime $p$ we have $\nu_{p}\left(\bar{G}_{n}\right)=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$. However for composite $b$ the function $\nu_{b}\left(\bar{G}_{n}\right)$ does not always coincide with the largest power of $b$ dividing $\bar{G}_{n}$, even for $b=p^{k}(k \geq 2)$ a prime power, i.e. for composite $b$ $\nu_{b}\left(\bar{G}_{n}\right) \neq \operatorname{ord}_{b}\left(\bar{G}_{n}\right)$ occurs for some $n$.

One may obtain an upper bound for $\nu_{b}(n)$ using an upper bound for the running digit sum function $S_{b}(n)=\sum_{m=0}^{n-1} d_{b}(m)$. In 1952 Drazin and Griffith [17] obtained the following sharp upper bound, as a special case of more general results.

Theorem 5.4. (Drazin and Griffith (1952)) Let $b \geq 2$ be an integer. Then for all $n \geq 1$,

$$
\begin{equation*}
S_{b}(n) \leq \frac{b-1}{2} n \log _{b} n . \tag{5.10}
\end{equation*}
$$

and equality holds if and only if $n=b^{k}$ for $k \geq 0$.
Proof. This result is [17, Theorem 1], taking $t=1$, asserting $\Delta_{1}(b, n) \geq 0$ with equality for $n=b^{k}$. In their notation $\sigma_{1}(b)=(b-1) / 2, F_{1}(b, n)=\frac{b-1}{2} n \log _{b} n$ and $\Delta_{1}(b, n)=\frac{2}{b-1}\left(F_{1}(b, n)-S_{b}(n)\right)$.

We deduce the following upper bound for the generalized order to base $b$.
Theorem 5.5. Let $b \geq 2$ be an integer. Then for all $n \geq 1$,

$$
\begin{equation*}
\nu_{b}\left(\bar{G}_{n}\right) \leq n \log _{b} n-\frac{n-1}{b-1} . \tag{5.11}
\end{equation*}
$$

Proof. Using the definition and Theorem 5.4 we have

$$
\begin{aligned}
\nu_{b}\left(\bar{G}_{n}\right) & =\frac{1}{b-1}\left(2 S_{b}(n)-(n-1) d_{b}(n)\right) \\
& \leq n \log _{b} n-\frac{n-1}{b-1} d_{b}(n) \leq n \log _{b}(n)-\frac{n-1}{b-1}
\end{aligned}
$$

as asserted.
For the case that $b=p$ is prime, we obtain a slight improvement on this upper bound in Theorem 6.7.
5.3. Summatory function of base $b$ digit sums: Delange's theorem. The detailed behavior of the running digit sum function $S_{b}(n)=\sum_{j=0}^{n-1} d_{b}(n)$ has complicated, interesting properties. In 1968 Trollope [53] obtained an exactly describable closed form for $S_{2}(n)$. In 1975 Delange [11] obtained the following definitive result applying to $S_{b}(n)$ for all bases $b \geq 2$.

Theorem 5.6. (Delange (1975)) Let $b \geq 2$ be an integer.
(1) For all integers $n \geq 1$,

$$
\begin{equation*}
S_{b}(n)=\left(\frac{b-1}{2}\right) n \log _{b} n+f_{b}\left(\log _{b} n\right) n \tag{5.12}
\end{equation*}
$$

in which $f_{b}(x)$ is a continuous real-valued function which is periodic of period 1 .
(2) The function $f_{b}(x)$ has a Fourier series expansion

$$
f_{b}(x)=\sum_{k \in \mathbb{Z}} c_{b}(k) e^{2 \pi i k x},
$$

whose Fourier coefficients are, for $k \neq 0$,

$$
\begin{equation*}
c_{b}(k)=-\frac{b-1}{2 k \pi i}\left(1+\frac{2 k \pi i}{\log b}\right)^{-1} \zeta\left(\frac{2 k \pi i}{\log b}\right) \tag{5.13}
\end{equation*}
$$

and, for the constant term $k=0$,

$$
\begin{equation*}
c_{b}(0)=\frac{b-1}{2 \log b}(\log (2 \pi)-1)-\left(\frac{b+1}{4}\right) . \tag{5.14}
\end{equation*}
$$

The function $f_{b}(x)$ is continuous but not differentiable.
Proof. (1) This statement is the main Theorem ${ }^{2}$ of Delange [11, p. 32].
(2) The explicit Fourier series expression is derived in Section 4 of 11]. The Fourier coefficients are complex-valued with $c_{-k}=\bar{c}_{k}$, as is required for a realvalued function $f_{b}(x)$. The Fourier series coefficients in (5.13) for $f_{b}(x)$ involve values of the Riemann zeta function evaluated at points on the line $\operatorname{Re}(s)=0$ which are, $b=p$ a prime the poles of the Euler product factor at $p$ in the Euler product for $\zeta(s)$, which is $\left(1-\frac{1}{p^{s}}\right)^{-1}$. The growth rate of the Riemann zeta function on the line $\operatorname{Re}(s)=0$ states that $|\zeta(i t)|=O\left((1+|t|)^{1 / 2+\epsilon}\right)$, which bounds the Fourier coefficients sufficiently to prove that $f_{b}(x)$ is a continuous function. Delange deduces the everywhere non-differentiable property of $f_{b}(x)$ from a self-similar functional relation that $f_{b}(x)$ satisfies.

Remark 5.7. Delange proved Theorem5.6 using methods from real analysis. Different approaches were introduced in 1983 by Mauclaire and Murata (40, [41), and in 1994 by Flajolet et al. [20, Theorem 3.1], using complex analysis and Mellin transform techniques. The latter methods obtain the Fourier expansion of $f_{b}(x)$ but do not establish the non-differentiability properties of the function $f_{b}(x)$. In another direction, in 1997 G. Tenebaum [52] extended the non-differentiability property of $f_{b}(x)$ to other periodic functions arising from summation formulas in a similar fashion. Generalizations of the Delange function connected to higher moments of digit sums were studied by Coquet [10] and Grabner and Hwang [24.

We next show that the function $f_{b}(n)$ is nonpositive, and give some estimates for its size, using results of Drazin and Griffiths [17.

Theorem 5.8. (1) For integer $b \geq 2$ and all real $x$,

$$
\begin{equation*}
f_{b}(x) \leq 0 \tag{5.15}
\end{equation*}
$$

and equality $f_{b}(x)=0$ holds only for $x=n, n \in \mathbb{Z}$.
(2) For integer $b \geq 3$ and all real $x$,

$$
\begin{equation*}
\frac{2}{b-1}\left|f_{b}(x)\right| \leq \frac{b-1}{b-2} \frac{\log (b-1)}{\log b} \tag{5.16}
\end{equation*}
$$

Proof. (1) Drazin and Griffiths [17] study a function $\Delta_{1}(b, n)$ for integer $(b, n)$ which is exactly $\Delta_{1}(b, n)=-\frac{2}{b-1} f_{b}\left(\log _{b} n\right)$ The nonpositivity for $x \in[0,1]$ follows from [17, Theorem 1], using the fact that $F_{b}(x)$ is periodic of period 1 and a continuous function, together with the fact that the fractional parts of $\log _{b} n$ are dense in $[0,1]$.
(2) This result follows from the bound on $\Delta_{1}(b, n)$ in [17, Theorem 2] in a similar fashion.

Figure 5.1 presents a picture of the function $f_{2}(x)$ computed by J. Arias de Reyna. It shows the non-positivity of $f_{2}(x)$ but can only hint at the property of $f_{2}(x)$ being non-differentiable at every point. In fact $f_{2}(x)$ is related to a famous

[^2]

Figure 5.1. The periodic function $f_{2}(x)$.
everywhere non-differentiable function, the Takagi function $\tau(x)$, introduced by Takagi 51] in 1903, which is given by

$$
\begin{equation*}
\tau(x):=\sum_{n=0}^{\infty} \frac{1}{2^{n}} \ll 2^{n} x \gg \tag{5.17}
\end{equation*}
$$

where $\ll x \gg$ is the distance from $x$ to the nearest integer. This connection can be deduced from work of Trollope [53], as explained in [34, Theorems 9.1 and 9.2]. One finds that

$$
\begin{equation*}
f_{2}(x)=-\frac{1}{2}\left(\frac{\tau\left(2^{x}-1\right)}{2^{x}}+\frac{2^{x}-1}{2}-\frac{2^{x}-1}{2^{x}}\right), \quad 0 \leq x \leq 1 . \tag{5.18}
\end{equation*}
$$

Although $\tau(x)$ is everywhere non differentiable, its oscillations on small scales are known to not be too large. There is a constant $C$ such that for all real $x$,

$$
|\tau(x+h)-\tau(x)| \leq C|h| \log \frac{1}{|h|}, \quad \text { for all } \quad|h| \leq \frac{1}{2}
$$

For our application to $\bar{G}_{n}, b=p$ is a prime, the formula 5.12 for $S_{b}(n)$ gives a smooth "main term" $\frac{p-1}{2} n \log _{p} n$ and a slowly oscillating "remainder term" $R_{p}(n):=f_{b}\left(\log _{b} n\right) n$ of order $O(n)$, with an explicit constant in the $O$-symbol, which encodes a logarithmic rescaling of the value of $n$.

## 6. Prime-power divisibility of binomial products $\bar{G}_{n}$ : Formulas using

 FRACTIONAL PARTS6.1. Prime-power divisibility of $\bar{G}_{n}$ : bilinear radix expansion form. We give a third formula for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$, which is also based on the base $p$ radix expansion of $n$, but which expresses it as a linear and bilinear expression in its base $p$ coefficients.

Theorem 6.1. Let $p$ be prime and write the base $p$ expansion of $n=\sum_{j=0}^{k} a_{j} p^{j}$, with $a_{k} \neq 0$. Then

$$
\begin{equation*}
\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=\sum_{j=1}^{k} j a_{j} p^{j}-\left(\sum_{j=1}^{k} a_{j}\left(\frac{p^{j}-1}{p-1}\right)+\sum_{j=0}^{k} \frac{1}{p^{j+1}}\left(\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} p^{v}\right)\right) \tag{6.1}
\end{equation*}
$$

The formula 6.1 has several interesting features.
(i) The formula expresses $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ as a difference of two positive terms,

$$
\mathrm{S}_{p, 3}^{+}(n):=\sum_{j=1}^{k} j a_{j} p^{j}
$$

and

$$
\mathrm{S}_{p, 3}^{-}(n):=\sum_{j=1}^{k} a_{j} \frac{p^{j}-1}{p-1}+\sum_{j=0}^{k} \frac{1}{p^{j}}\left(\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} p^{v}\right)
$$

An immediate consequence is the upper bound $\operatorname{ord}_{p}\left(\bar{G}_{n}\right) \leq S_{p, 3}^{+}(n)$ which is useful in obtaining upper bounds for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$, see Section 6.2 .
(ii) The first two sums on the right in 6.1) are linear functions of the base $p$ digits of $n$, while the third sum is bilinear in the base $p$ digits.
(iii) The first sum on the right in 6.1 makes sense as a $p$-adic function. That is, it extends continuously to a $p$-adically convergent series for $n \in \mathbb{Z}_{p}$, the $p$-adic integers. However the last two sums on the right, treated separately or together, do not have continuous extensions to $\mathbb{Z}_{p}$.
The right side of 6.1 makes sense for arbitrary bases $b \geq 2$, so we make the following definition for arbitrary $b$.

Definition 6.2. For each integer $b \geq 2$ and $n \geq 1$ set

$$
\nu_{b}^{*}\left(\bar{G}_{n}\right):=\sum_{j=1}^{k} j a_{j} b^{j}-\left(\sum_{j=1}^{k} a_{j}\left(\frac{b^{j}-1}{b-1}\right)+\sum_{j=0}^{k} \frac{1}{b^{j+1}}\left(\sum_{u=0}^{j} a_{u} b^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} b^{v}\right)\right)
$$

in which $n=\sum_{j=0}^{k} a_{j} b^{j}$ is its base $b$ radix expansion.
For $b=p$ a prime, we have $\nu_{p}^{*}\left(\bar{G}_{n}\right)=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ by Theorem 6.1. A priori this definition looks different from that of the generalized order $\nu_{b}(n)$ to base $b$ introduced in Section 5, but in Appendix B we show they coincide: For each $b \geq 2$ one has

$$
\nu_{b}^{*}\left(\bar{G}_{n}\right)=\nu_{b}\left(\bar{G}_{n}\right) \quad \text { for all } \quad n \geq 1
$$

We will deduce Theorem 6.1 starting from Kummer's formula for the maximal power of $p$ dividing a binomial coefficient (Kummer [33]).

Theorem 6.3. (Kummer (1852)) Given a prime p, the exact divisibility pe of $\binom{n}{t}$ by a power of $p$ is found by writing $t, n-t$ and $n$ in base $p$ arithmetic. Then $e$ is the number of carries that occur when adding $n-t$ to $t$ in base $p$ arithmetic, using digits $\{0,1,2, \ldots, p-1\}$, working from the least significant digit upward.

Proof. Kummer's theorem easily follows from Proposition 5.2 as Granville [25] observes. By inspection we see that each carry operation adding $t$ to $n-t$ in the $j$-th place reduces the sum of the digits in the sum $n$ by $p-1$, since it adds a 1 in the $(j+1)$-st place while removing a sum of $p$ in the $j$-th place. Thus the formula on the right in (5.7) counts the number of carries made in adding $t$ to $n-t$.

To establish Theorem6.1 we first reinterpret Kummer's formula as counting the number of borrowings involved in subtracting $j$ from $n$ in base $p$ arithmetic, now working from the least significant digit upwards. As an example, take base $p=3$ and consider

$$
n=13=(111)_{3}, \quad t=5=(12)_{3} \quad \text { and } \quad n-t=8=(22)_{3}
$$

In the following table we add $(n-t)$ to $t$ on the left and subtract $t$ from $n$ on the right. We list the carries $(+1)$ and the borrowings $(-1)$ on the top line of the table.

| carries : | 1 | 1 | 0 | borrows : | -1 | -1 | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 2 | 2 |  | 1 | 1 | 1 |
|  | + | 1 | 2 |  | - | 1 | 2 |
|  | 1 | 1 | 1 |  | 0 | 2 | 2 |

In this example there are 2 carries in the additive form versus 2 borrowings in the subtractive form.

We will derive the formula 6.1) of Theorem 6.1 by keeping track of the total number of borrowings in the addition made for the $j$-digit of $n$, but treating the contributions to each digit separately, specified by a function $c_{j}(n)$ defined below. For a given $n$ and $0 \leq t \leq n$ set the $j$-th carry digit $c_{j}(n, t)=1$ or 0 according as whether the addition of $t$ to $n-t$ in base $p$ expansion has a carry digit, or not, added to the $(j+1)$-place from the $j$-th place. Equivalently $c_{j}(n, t)$ specifies whether there is a borrowing from the $(j+1)$-st place in subtracting $t$ from $n$ in base $p$ arithmetic. Here $c_{j}(n, t)$ depends only on $n\left(\bmod p^{j+1}\right)$ and $t\left(\bmod p^{j+1}\right)$. The table above computes that $c_{2}(13,5)=0$.

Definition 6.4. Let a prime $p$ be fixed, and let a digit position $j \geq 0$ be given. The $j$-th position total carry function $c_{j}(n)$ is

$$
\begin{equation*}
c_{j}(n):=\sum_{t=0}^{n} c_{j}(n, t) \tag{6.2}
\end{equation*}
$$

Kummer's theorem applied to 5.1 yields

$$
\begin{equation*}
\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=\sum_{j=0}^{\infty} c_{j}(n) \tag{6.3}
\end{equation*}
$$

The sum on the right is always finite since $c_{j}(n)=0$ for all $j \geq k=\left\lfloor\log _{p} n\right\rfloor$.
Lemma 6.5. (1) For a fixed prime $p$,

$$
\begin{equation*}
c_{j}(n)=\left(\left(p^{j+1}-1\right)-\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{t=j+1}^{k} a_{t} p^{t-j-1}\right) \tag{6.4}
\end{equation*}
$$

(2) Alternatively we have

$$
\begin{equation*}
c_{j}(n)=\left(p^{j+1}-1-p^{j+1}\left\{\frac{n}{p^{j+1}}\right\}\right)\left(\frac{n}{p^{j+1}}-\left\{\frac{n}{p^{j+1}}\right\}\right) \tag{6.5}
\end{equation*}
$$

Where $j \geq 0,[x]$ is the floor function and $\{x\}$ is the fractional part of the rational number.
Proof. (1) Denote $t$ in base $p$ as $t=t_{k} p^{k}+t_{k-1} p^{k-1}+\cdots+t_{0}$ and note that since $t \leq n-1, t_{k} \leq a_{k}$. We prove the result by induction on $j \geq 0$. For the base case $j=0$ whenever $t_{0}>a_{0}$ then that value of $t$ contributes 1 towards $c_{0}(n)$. The number of values that $t_{0}$ can take while being greater than $a_{0}$ is $p-1-a_{0}$. The number of values $t$ can take with $t_{0}>a_{0}$ for a fixed $t_{0}$ is $\left(n-a_{0}\right) / p$ and so we have

$$
c_{0}(n)=\left(p-1-a_{0}\right) \frac{\left(n-a_{0}\right)}{p}=\left(p^{0+1}-1-a_{0} p^{0}\right)\left(\sum_{t=1}^{k} a_{t} p^{t-1}\right)
$$

For the induction step we observe that a value $t$ contributes to $c_{j}(n)$ only if its $j$ smallest base $p$ digits satisfy:

$$
\begin{equation*}
\sum_{v=0}^{j} t_{v} p^{v}>\sum_{v=0}^{j} a_{v} p^{v} \tag{6.6}
\end{equation*}
$$

The number of last $j$ digits that would contribute to $c_{j}(n)$ is consequently

$$
\begin{equation*}
p^{j+1}-1-\sum_{v=0}^{j} a_{v} p^{v} \tag{6.7}
\end{equation*}
$$

The number of $t$ that would satisfy 6.7 is therefore:

$$
\begin{equation*}
\frac{n-\sum_{v=0}^{j} a_{v} p^{v}}{p^{j+1}}=\sum_{t=j+1}^{k} a_{t} p^{t-j-1} \tag{6.8}
\end{equation*}
$$

And so we obtain the value of $c_{j}(n)$ by taking the product of 6.7 and 6.8

$$
c_{j}(n)=\left(\left(p^{j+1}-1\right)-\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{t=j+1}^{k} a_{t} p^{t-j-1}\right)
$$

(2) The formula (6.5) follows by rewriting the sums in (6.4), observing that

$$
\sum_{t=j+1}^{k} a_{t} p^{t-j-1}=\left[\frac{n}{p^{j}}\right]=\frac{n}{p^{j}}-\left\{\frac{n}{p^{j}}\right\}
$$

and

$$
p^{j}-1-\sum_{v=0}^{j} a_{v} p^{v}=p^{j}-1-p^{j}\left\{\frac{n}{p^{j}}\right\},
$$

as required.
We apply Lemma 6.5 to prove our first formula for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$.
Proof of Theorem 6.1. Substituting in (6.3) the formula of Lemma 6.5 yields

$$
\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=\sum_{j=0}^{k-1} c_{j}(n)=\sum_{j=0}^{k-1}\left(\left(\left(p^{j+1}-1\right)-\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{t=j+1}^{k} a_{t} p^{t-j-1}\right)\right)
$$

Expanding the latter sum yields

$$
\begin{equation*}
\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=\sum_{j=0}^{k-1} \sum_{t=j+1}^{k} a_{t} p^{t}-\sum_{j=0}^{k-1} \sum_{t=j+1}^{k} a_{t} p^{t-j-1}-\sum_{j=0}^{k-1} \sum_{u=0}^{j} \sum_{t=j+1}^{k} a_{u} a_{t} p^{u+t-j-1} \tag{6.9}
\end{equation*}
$$

Now we simplify the three sums in $\sqrt{6.9}$ individually. The first of these sums is

$$
\begin{equation*}
\sum_{j=0}^{k-1} \sum_{t=j+1}^{k} a_{t} p^{t}=\sum_{j=1}^{k} j a_{j} p^{j} \tag{6.10}
\end{equation*}
$$

This sum extends to a $p$-adically convergent series, which for $\alpha:=\sum_{j=0}^{\infty} a_{j} p^{j}$ is $f(\alpha)=\sum_{j=0}^{\infty} j a_{j} p^{j}$. In fact $f: \mathbb{Z}_{p} \rightarrow \mathbb{Z}_{p}$ is not only a continuous function, but is a $p$-adic analytic function on $\mathbb{Z}_{p}$.

The second sum of $(\sqrt{6.9})$ can be re-written as:

$$
\begin{equation*}
\sum_{j=0}^{k-1} \sum_{t=j+1}^{k} a_{t} p^{t-j-1}=\sum_{j=1}^{k} a_{j}\left(p^{j-1}+p^{j-2}+\cdots+1\right)=\sum_{j=1}^{k} a_{j}\left(\frac{p^{j}-1}{p-1}\right) \tag{6.11}
\end{equation*}
$$

The third sum is a bilinear sum, which satisfies the identity

$$
\begin{equation*}
\sum_{j=0}^{k-1} \sum_{b=0}^{j} \sum_{t=j+1}^{k} a_{b} a_{t} p^{b+t-j-1}=\sum_{j=0}^{k} \frac{1}{p^{j+1}}\left(\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} p^{v}\right) \tag{6.12}
\end{equation*}
$$

By substituting (6.12), (6.10) and (6.11) into 6.9 we obtain the desired result.
Remark 6.6. The total carry functions $c_{j}(n)$ seem of interest in their own right. Bergelson and Leibman [6] study the class of all bounded functions obtainable as finite iterated combinations of the fractional part function $\{\cdot\}$, calling them generalized polynomials. They relate generalized polynomials to piecewise polynomial maps on nilmanifolds and use this relation to derive recurrence and distribution properties of values of such functions. The formula in Lemma 6.5(2) involves such functions. It can be written as $c_{j}(n)=n g_{1, j}(n)+g_{2, j}(n)$ where $g_{i, j}(n)$ are the bounded generalized polynomials

$$
g_{1, j}(n):=1-\frac{1}{p^{j+1}}-\left\{\frac{n}{p^{j+1}}\right\}
$$

and

$$
g_{2, j}(n):=-g_{1}(n)\left(p^{j+1}\left\{\frac{n}{p^{j+1}}\right\}\right) .
$$

each of which is a periodic function of $n$ with period $p^{j+1}$. However the function $c_{j}(n)$ itself is unbounded, so is not a generalized polynomial.
6.2. Prime-power divisibility of binomial products $\bar{G}_{n}$ : extreme values. Theorem 6.1 permits an exact determination of the extreme behaviors of $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$. We obtain a useful upper bound on $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ by retaining only those terms in Theorem 6.1 that are linear in the $a_{i}$, and this upper bound turns out to be sharp.
Theorem 6.7. Let the prime $p$ be fixed. Then we have for all $n>0$ that

$$
\begin{equation*}
0 \leq \operatorname{ord}_{p}\left(\bar{G}_{n}\right) \leq M_{p}(n):=\sum_{j=0}^{k} j a_{j} p^{j}-\sum_{j=1}^{k} a_{j}\left(\frac{p^{j}-1}{p-1}\right) \tag{6.13}
\end{equation*}
$$

The equality $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=0$ holds if and only if $n=a p^{k}-1$, with $1 \leq a \leq p-1$. The equality $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=M_{p}(n)$ holds if and only if $n=a p^{k}$ and in that case

$$
\operatorname{ord}_{p}\left(\bar{G}_{a p^{k}}\right)=a\left(k p^{k}-\frac{p^{k}-1}{p-1}\right) .
$$

Proof. The lower bound in (6.7) is immediate since $\bar{G}_{n}$ is an integer. The case of equality can be deduced directly from Kummer's theorem. To have $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=0$ using Kummer's theorem, all binomial coefficients $\binom{n}{j}$ must be prime to $p$, so there can be no value $0 \leq j \leq n$ such that subtracting $j$ from $n$ in base $p$ arithmetic results in borrowing a digit. This fact requires that the base $p$ digits $a_{j}$ of $n$ except the top digit be $p-1$, i.e. $a_{j}=p-1$ for $0 \leq j \leq k-1$. There is no constraint on the top digit $a_{k}$ other than $a_{k} \neq 0$, since no borrowing can occur in this digit.

The upper bound inequality $\operatorname{ord}_{p}\left(\bar{G}_{n}\right) \leq M_{p}(n)$ follows immediately from Theorem 6.1. The equality case will hold only if the bilinear term in that theorem vanishes; it is

$$
\sum_{j=1}^{k} \frac{1}{p^{j}}\left(\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} p^{v}\right) .
$$

This term will be positive whenever two nonzero coefficients appear in the base $p$ expansion of $n$, since one can find a nonzero cross term in this expression. Therefore equality can hold only for those $n$ having one nonzero base $p$ digit, i.e. $n=a p^{k}$ with $a \neq 0$. Direct substitution of $a_{j}=0$ for $0 \leq j \leq p-1$ yields the explicit formula for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ above.

The previous result implies the following bounds.
Theorem 6.8. For each prime $p$, there holds for all $n \geq 1$,

$$
\begin{equation*}
0 \leq \operatorname{ord}_{p}\left(\bar{G}_{n}\right)<n \log _{p} n \tag{6.14}
\end{equation*}
$$

The value $n=p^{k}$ has $\operatorname{ord}_{p}\left(\bar{G}_{n}\right) \geq n\left(\log _{p} n-1\right)$.
Proof. Only the upper bound in (6.14) needs to be verified. By Theorem 6.1 we have

$$
\operatorname{ord}_{p}\left(\bar{G}_{n}\right)<\sum_{j=1}^{k} j a_{j} p^{j}=k n-\sum_{u=0}^{k-1}(k-u) a_{u} p^{u} \leq k n=n\left\lfloor\log _{p} n\right\rfloor \leq n \log _{p} n .
$$

For $n=p^{k}$ we have $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)=k n-\left(1+p+\cdots+p^{k-1}\right) \geq k n-n=n\left(\log _{p} n-1\right)$.

## 7. Formulas for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ : Comparison and Implications

We presented three formulas for $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ in Sections 4,5 and 6 , respectively. Now we compare the formulas and discuss what they imply about features of the graph of $\nu_{2}\left(\bar{G}_{n}\right)=\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$ given in Figure 1.1 and in the rescaled Figure 7.1 following.

Each of the three formulas express $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ as a difference of positive terms $\mathrm{S}_{p, j}^{+}(n)$ and $\mathrm{S}_{p, j}^{-}(n)$, for $j=1,2,3$. The term $\mathrm{S}_{p, j}^{+}(n)$ is nondecreasing in $n$ in each formula, while the term $\mathrm{S}_{p, j}^{-}(n)$ is smooth for $j=1$ but is oscillatory in the other two formulas.

| $n$ | $\operatorname{ord}_{2}\left(D_{n}^{*}\right)$ | $\operatorname{ord}_{2}\left(N_{n}^{*}\right)$ | $\mathrm{S}_{2,2}^{+}(n)$ | $\mathrm{S}_{2,2}^{-}(n)$ | $\mathrm{S}_{2,3}^{+}(n)$ | $\mathrm{S}_{2,3}^{-}(n)$ | $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$ |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | :---: |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 2 | 1 | 2 | 1 | 2 | 1 | 1 |
| 3 | 2 | 2 | 4 | 4 | 2 | 2 | 0 |
| 4 | 10 | 5 | 8 | 3 | 8 | 3 | 5 |
| 5 | 10 | 8 | 10 | 8 | 8 | 6 | 2 |
| 6 | 16 | 12 | 14 | 10 | 10 | 6 | 4 |
| 7 | 16 | 16 | 18 | 18 | 10 | 10 | 0 |
| 8 | 40 | 23 | 24 | 7 | 24 | 7 | 17 |
| 9 | 40 | 30 | 26 | 16 | 24 | 14 | 10 |
| 10 | 50 | 38 | 30 | 18 | 26 | 14 | 12 |
| 11 | 50 | 46 | 34 | 30 | 26 | 22 | 4 |
| 12 | 74 | 56 | 40 | 22 | 32 | 14 | 18 |
| 13 | 74 | 66 | 44 | 36 | 32 | 24 | 8 |
| 14 | 88 | 77 | 50 | 39 | 34 | 23 | 11 |
| 15 | 88 | 88 | 56 | 56 | 34 | 34 | 0 |
| 16 | 152 | 103 | 64 | 15 | 64 | 15 | 49 |

TABLE 7.1. Comparison of Theorems 4.1, 5.1 and 6.1 for $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$, $1 \leq n \leq 16$, divided in blocks $2^{k} \leq n<2^{k+1}$.

Table 7.1 presents numerical data on the three formulas for $p=2$ and small $n$, which illustrate their differences. The lead term $\mathrm{S}_{p, 1}^{+}(n)=\operatorname{ord}_{p}\left(D_{n}^{*}\right)$ in the first formula grows much more rapidly than the corresponding terms $\mathrm{S}_{p, j}^{+}(n)$ for $j=2,3$, evidenced by the asymptotic formula in Theorem 4.2. The second and third formulas differ qualitatively in their second terms, in that $\mathrm{S}_{p, 2}^{+}(n):=\frac{2}{p-1} S_{p}(n)$ grows smoothly, being of size $n \log _{p} n+O(n)$, while $S_{p, 3}^{+}(n)$ grows less smoothly, having occasional jumps proportional to $n \log n$ as the base $p$ digits $a_{i}$ vary. It appears that the third formula gives the best upper bound of the three formulas, in the sense of minimal growth of the positive term $S_{2,3}^{+}(n)$ among the $S_{2, j}^{+}(n)$. In particular $\mathrm{S}_{2,2}^{+}(n) \geq S_{2,3}^{+}(n)$ holds in the table entries, with equality holding for $n=2^{k}$.

We have shown that the function $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ is of average size about $\frac{1}{2} n \log _{p} n$ and of size at $\operatorname{most} n \log _{p} n$. It is natural to make a companion plot to Figure 1.1 that rescales the values of $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$ by a factor $\frac{1}{2} n \log _{2} n$, which we present in Figure 7.1 below. In the rescaled plot all values fall in the interval [ 0,2 ], and have average size around 1, according to the discussion of $d_{p}(n)$ in feature (iii) after Theorem 5.1 .

There are several patterns visible in Figure 1.1 and in Figure 7.1 above.
(i) Between $n=2^{k}-1$ to $n=2^{k}$ there is a large jump visible in the value of $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$. The large jump between $n=2^{k-1}$ and $n=2^{k}$ is quantified more generally in Theorem 6.7 for $\operatorname{ord}_{p}\left(G_{n}\right)$ between $n=p^{k}-1$ and $n=p^{k}$.
(ii) In Figure 1.1 the values of $\operatorname{ord}_{2}\left(\bar{G}_{n}\right)$ between $n=2^{k}$ to $n=2^{k+1}$ show a pattern of diagonal lines or "stripes". These diagonal lines are sloping upwards, have different lengths, and are roughly parallel to each other. The lengths of the "stripes" vary in a predictable manner: from top to bottom, they first increase in length, starting from the left, until they extend nearly


Figure 7.1. Values of $\operatorname{ord}_{2}\left(\bar{G}_{n}\right) /\left(0.5 n \log _{2} n\right), 1 \leq n \leq 1023=2^{10}-1$.
to the next level $2^{k+1}$, remain stable at this width for a while, and thereafter decrease in length while continuing to end near the next level $2^{k+1}$. In the rescaled Figure 7.1 these lines flatten out to give parallel "stripes".

Theorem 5.1 accounts for the "stripes" visible in Figure 7.1 via its term $-\frac{n-1}{p-1} d_{p}(n)$, which shows that each stripe is occupied by integers having a fixed value $d_{p}(n)=j$, in which $d_{p}(n)=1$ labels the highest stripe, and each stripe downward increases $j$ by one. The odometer behavior of the function $d_{p}(n)$ also accounts for the horizontal width of the "stripes" and the motion of their behavior over the interval $\left[p^{k}, p^{k+1}-1\right.$ ], and determines when they start near the left endpoint $p^{k}$ (small values of $d_{p}(n)$ ) or end near the right endpoint $p^{k+1}-1$ (large values of $d_{p}(n)$ ).
(iii) On comparing "stripes" in the interval $n=2^{k}$ and $n=2^{k+1}-1$, with those at the next interval between $n=2^{k+1}$ and $n=2^{k+2}-1$, the number of "stripes" increases by 1. This increase in the number of stripes from the interval from $\left[p^{k}, p^{k+1}-1\right]$ and the interval $\left[p^{k+1}, p^{k+2}-1\right]$ is accounted for by the allowed values of $d_{p}(n)$ labeling the given interval. For $p=2$ there are exactly $k+1$ such stripes on the region $2^{k} \leq n \leq 2^{k+1}-1$, there is an increase of exactly one stripe in the new interval, and the spacing between the stripes is of width approximately $\frac{1}{k} \approx \frac{1}{\log _{p} n}$. For a general prime $p$, the increase in the number of stripes between the interval $\left[p^{k}, p^{k+1-1}\right]$ and $\left[p^{k+1}, p^{k+2}-1\right]$ is exactly $p-1$.
(iv) The values between successive powers $2^{k}$ and $2^{k+1}$ have an apparent envelope of largest growth. The envelope appears to be of size proportional to $k 2^{k}$, a value approximately equal to $n \log _{2} n$. Under the rescaling by a factor proportional to $n \log n$, as is done in Figure 7.1, the "stripes" become approximately flat, and they fall in the range $0 \leq \operatorname{ord}_{2}\left(\bar{G}_{n}\right) /\left(0.5 n \log _{2} n\right) \leq 2$.

The envelope of largest growth for $d_{p}(n)$ is $n \log _{p}(n)-n$, as quantified in Theorem6.8. On rescaling by a factor $1 / n \log _{p} n$, as done in Figure 7.1 the "stripe" of values with $d_{p}(n)=j$ over the interval $\left[p^{n}, p^{n+1}-1\right]$ becomes
approximately flat. Note that the smooth main term $\frac{2}{p-1} S_{p}(n)$ in Theorem 5.1. changes by at most $n$ over this interval, which becomes after rescaling of size $O\left(\frac{1}{\log n}\right)$ which is asymptotically negligible.
(v) Inside the envelope of largest growth are irregular, correlated fluctuations in size consisting of a structured set of negative "jumps" of various sizes. The "stripes" are a structure that indirectly emerges from correlations within the pattern of "jumps". We observe that the odometer behavior of $d_{p}(n)$ completely accounts for the pattern of downward "jumps" in the values of $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$, The odometer behavior also produces the self-similar structure in the locations of values $n$ with $d_{p}(n)=j$ which produce the "stripes".
A remaining mystery of the functions $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ concerns the interpretation of their generalizations $\nu_{b}\left(\bar{G}_{n}\right)$ introduced for all integers $b \geq 2$ in Section 5.2 using base $b$ radix expansions. One can view $\bar{G}_{n}$ as a universal object that encodes all the data $\nu_{p}(n)=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ for prime $p$. If so, what might be the universal object encoding the data for all $\nu_{b}\left(\bar{G}_{n}\right), b \geq 1$ ?

## 8. Binomial Products and Prime Counting Estimates

Binomial coefficients are well known to encode information about the distribution of prime numbers. This holds more generally for integer factorial ratios, which are one-parameter families of ratios of products of factorials that are integers for all parameter values. Let $\pi(x)$ count the number of primes $p \leq x$. Chebyshev [9] used the integer factorial ratios $A_{n}:=\frac{(30 n)!n!}{(15 n)!(10 n)!(6 n)!}$ to obtain his bounds

$$
0.92 \frac{x}{\log x} \leq \pi(x) \leq 1.11 \frac{x}{\log x}
$$

It is known that the in principle the ensemble of integer factorial ratios contain enough information to give a proof of the prime number theorem, see Diamond and Erdős [15]. However their method to show the existence of a suitable sequence of such ratios used the prime number theorem as an input, so did not yield an elementary proof of the prime number theorem, see the discussion in Diamond 14, Sect. 9].

We now consider the restricted set of all products of binomial coefficients. The asymptotic formula for $\log \left(\bar{G}_{n}\right)$ in Theorem 9.2 gives $\log \left(\bar{G}_{n}\right)=\frac{1}{2} n^{2}+O(n \log n)$. The prime number theorem says that $\pi(n) \sim \frac{n}{\log n}$ and is therefore equivalent to the assertion

$$
\begin{equation*}
\log \left(\bar{G}_{n}\right)=\frac{1}{2} \pi(n) n \log n+o\left(n^{2}\right) \tag{8.1}
\end{equation*}
$$

The individual binomial products $\bar{G}_{n}$ imply Chebyshev-type bounds for $\pi(x)$. The product formula expressing unique factorization ([2]) yields on taking a logarithm

$$
\begin{equation*}
\log \left(\bar{G}_{n}\right)=\sum_{p \leq n} \operatorname{ord}_{p}\left(\bar{G}_{n}\right) \log p \tag{8.2}
\end{equation*}
$$

The left side is estimated by the asymptotic formula for $\log \left(G_{n}\right)$ in Theorem 9.2.

$$
\log \left(\bar{G}_{n}\right)=\frac{1}{2} n^{2}-\frac{1}{2} n \log n+\left(1-\frac{1}{2} \log (2 \pi)\right) n+O(\log n)
$$

Since $1-\frac{1}{2} \log (2 \pi)>0$, one has for sufficiently large $n$, and in fact for all $n \geq 3$,

$$
\begin{equation*}
\log \left(\bar{G}_{n}\right) \geq \frac{1}{2} n^{2}-\frac{1}{2} n \log n \tag{8.3}
\end{equation*}
$$

Theorem 6.8 states $\operatorname{ord}_{p}\left(\bar{G}_{n}\right) \leq n \log _{p}(n)$, which upper bounds the right side of (8.1) by

$$
\begin{equation*}
\log \left(\bar{G}_{n}\right)=\sum_{p \leq n} \operatorname{ord}_{p}\left(\bar{G}_{n}\right) \log p \leq \sum_{p \leq n}\left(n \log _{p} n\right) \log p=\pi(n) n \log n \tag{8.4}
\end{equation*}
$$

Combining these two inequalities yields the Chebyshev-type lower bound,

$$
\pi(n) \geq \frac{1}{2}\left(\frac{n}{\log n}\right)-\frac{1}{2}
$$

valid for all $n \geq 2$. This bound loses a constant factor of 2 compared to the prime number theorem, so is much worse than Chebyshev's lower bound. But it has a redeeming feature: in an average sense most $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ are of size near $\frac{1}{2} n \log _{p} n$. This observation follows from Theorem 5.1, combined with the fact that $d_{p}(n)$ has mean $\frac{p-1}{2} \log _{p} n$, provided that one averages over $n$. This (heuristic) observation would save back exactly the factor of 2 lost in the argument above on the right side of (8.4), which therefore suggests the possibility of an approach to proving the prime number theorem ${ }^{3}$ via radix expansions.

There remain serious obstacles to obtaining a proof of the prime number theorem along these lines. If one holds $p$ fixed and varies $n$, then one can rigorously show that $\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ is usually of size near $\frac{1}{2} n \log _{p} n$. However the sum $\sum_{p \leq n} \operatorname{ord}_{p}\left(\bar{G}_{n}\right) \log p$ appearing in (8.4) makes a different averaging: holding $n$ fixed and letting $p$ vary, restricting to $p \leq n$. The analysis of this new averaging leads to new kinds of arithmetical sums involving radix expansions and we leave their investigation to future work.

Another relation of binomial products $\bar{G}_{n}$ to the distribution of prime numbers arises via their connection to products of Farey fractions. This connection via Möbius inversion creates other products of binomial coefficients which may be directly related to the Riemann hypothesis, for which see 36. Moreover, there are general relations known between families of integer factorial ratios and the Riemann hypothesis. For some recent work on their structure, see Bell and Bober [7] and Bober [8].

## 9. Appendix A: Asymptotic expansions for $\log \left(D_{n}^{*}\right), \log \left(N_{n}^{*}\right)$ and $\log \left(\bar{G}_{n}\right)$

In this appendix we derive full asymptotic expansions for the logarithms of the superfactorial function $N_{n}^{*}=\prod_{k=1}^{n} k$ !, the hyperfactorial function $D_{n}^{*}=\prod_{k=1}^{n} k^{k}$ and the binomial products $\bar{G}_{n}=D_{n}^{*} / N_{n}^{*}$.

We start from the formulas

$$
\begin{align*}
N_{n}^{*} & =\Gamma(n+1) G(n+1)  \tag{9.1}\\
D_{n}^{*} & =\frac{\Gamma(n+1)^{n}}{G(n+1)} \tag{9.2}
\end{align*}
$$

[^3]in which $\Gamma(n)$ denotes the Gamma function and $G(n)$ denotes the Barnes $G$ function, both discussed below. The formulas 9.1 and 9.2 follow from the standard identities $\Gamma(n+1)=n$ ! and $G(n+1)=1!2!\cdots(n-1)$ !, respectively. These two formulas yield
\[

$$
\begin{equation*}
\bar{G}_{n}=\frac{D_{n}^{*}}{N_{n}^{*}}=\frac{\Gamma(n+1)^{n-1}}{G(n+1)^{2}} \tag{9.3}
\end{equation*}
$$

\]

The Gamma function $\Gamma(z)$ was originally defined to interpolate the factorial function, and was studied at length by Euler (see [35, Sect. 2.3], and Artin [?]). It satisfies a functional equation $\Gamma(z+1)=z \Gamma(z)$, and has $\Gamma(1)=1$, which yields $\Gamma(n+1)=n$ !. Its reciprocal is an entire function of order 1 (and maximal type) defined by the everywhere convergent Hadamard product

$$
\frac{1}{\Gamma(z)}=e^{\gamma z} z \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right) e^{-\frac{z}{k}}
$$

in which $\gamma \approx 0.57721$ denotes Euler's constant. The asymptotic expansion of the logarithm of the Gamma function is related to Stirling's formula. It was determined by Stieltjes, who gave a precise notion of asymptotic expansion (see [35] Sect. 3.1, and (3.1.10)], [45, Chapter $5,(5.11 .1)]$ ). It states $\square^{4}$ for any fixed $N \geq 1$ that

$$
\begin{align*}
& \log \Gamma(z+1)=z \log z-z+\frac{1}{2} \log z+\frac{1}{2} \log (2 \pi) \\
&+\sum_{k=1}^{N} \frac{B_{2 k}}{2 k(2 k-1)} \frac{1}{z^{2 k-1}}+O\left(\frac{1}{z^{2 N+1}}\right) \tag{9.4}
\end{align*}
$$

where $B_{k}$ denote the Bernoulli numbers, as determined by the generating function $\frac{t}{e^{t}-1}=\sum_{k=0}^{\infty} B_{k} \frac{t^{k}}{k!}$, in particular $B_{1}=-\frac{1}{2}$. This formula is known to be valid in any sector $-\pi+\epsilon<\operatorname{Arg}(z) \leq \pi-\epsilon$ of the complex plane, with the implied $O$-constant depending on both $N$ and $\epsilon$.

The Barnes $G$-function was introduced by Barnes [4] in 1900. It is a less well known than the Gamma function, and is closely related to a generalization of the gamma function, the double gamma function, also introduced by Barnes ([3], [5]), see also [45, Sect. 5.17]. It satisfies the functional equation

$$
G(z+1)=\Gamma(z) G(z)
$$

and has $G(1)=1$, which yields $G(n+1)=(n-1)!(n-2)!\cdots 1$ !, and also $G(n+2)=$ $N_{n}^{*}$. Recently the Barnes $G$-function has assumed prominence from its appearance in formulas relating the Riemann zeta function to random matrix theory. These formulas appear in random matrix theory for the Circular Unitary Ensemble, and in conjectured formulas for moments of the Riemann zeta function on the critical line $\operatorname{Re}(s)=\frac{1}{2}$, see Keating and Snaith [30] and Hughes [28].

The Barnes $G$-function is an entire function of order 2 defined by the everywhere convergent Weierstrass product

$$
G(z):=(2 \pi)^{\frac{z}{2}} \exp \left(-\frac{1}{2}\left(z+z^{2}(1+\gamma)\right)\right) \prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{k} \exp \left(\frac{z^{2}}{2 k}-z\right)
$$

[^4]where again $\gamma$ is Euler's constant. The asymptotic expansion for the Barnes $G$ function ${ }^{5}$ (4, p. 285]) has the form, for any fixed $N \geq 1$,
\[

$$
\begin{align*}
\log G(z+1)= & \frac{1}{2} z^{2} \log z-\frac{3}{4} z^{2}+\left(\frac{1}{2} \log (2 \pi)\right) z-\frac{1}{12} \log z \\
& +\left(\frac{1}{12}-\log A\right)+\sum_{k=1}^{N} \frac{B_{2 k+2}}{2 k(2 k+2)} \frac{1}{z^{2 k}}+O\left(\frac{1}{z^{2 N+2}}\right) \tag{9.5}
\end{align*}
$$
\]

where $A=\exp \left(\frac{1}{12}-\zeta^{\prime}(-1)\right)$ is the Glasher-Kinkelin constant (Kinkelin 31], Glaisher [22], [23]), which has a numerical value of $A \approx 1.2824271291 \ldots$ and where $B_{k}$ denote the Bernoulli numbers. This asymptotic expansion is valid in any sector $-\pi+\epsilon<\operatorname{Arg}(z) \leq \pi-\epsilon$ of the complex plane, with the implied $O$-constant depending on both $N$ and $\epsilon$. The original derivation of Barnes did not control the error term but Ferreira and Löpez [19] later obtained an asymptotic expansion ${ }^{6}$ with error term as above, see also Ferreira [18] and Nemes [44].

The asymptotic expansions for both $\Gamma(z+1)$ and $G(z+1)$ when extended to all orders are divergent series. That is, the associated power series in $w=\frac{1}{z}$, taking $N=\infty$, has radius of convergence zero, a fact which follows from the super exponential growth of the even Bernoulli numbers $\left|B_{2 n}\right| \sim 4 \sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2 n}$ as $n \rightarrow \infty$. The derived asymptotic expansions for $\log \left(D_{n}^{*}\right), \log \left(N_{n}^{*}\right)$ and and $\log \left(\bar{G}_{n}\right)$ below also involve Bernoulli numbers and are also divergent series when extended to all orders.

We now state asymptotic expansions for $\log \left(D_{n}^{*}\right)$ and $\log \left(N_{n}^{*}\right)$.
Theorem 9.1. (1) The superfactorial $N_{n}^{*}=\prod_{k=1}^{n} k$ ! has an asymptotic expansion for $\log \left(N_{n}^{*}\right)$ valid to any given order $N \geq 1$, valid uniformly for all $n \geq 2$, of the form

$$
\begin{gathered}
\left.\log \left(N_{n}^{*}\right)=\frac{1}{2} n^{2} \log n-\frac{3}{4} n^{2}+n \log n+\left(\frac{1}{2} \log (2 \pi)-1\right)\right) n+\frac{5}{12} \log n+ \\
+c_{0}+\sum_{j=1}^{N} c_{j}\left(\frac{1}{n^{j}}\right)+O\left(\frac{1}{n^{N+1}}\right)
\end{gathered}
$$

The constant $c_{0}=\frac{1}{2} \log (2 \pi)+\frac{1}{12}-\log A$ where $A=\exp \left(\frac{1}{12}-\zeta^{\prime}(-1)\right)$ is the Glaisher-Kinkelin constant, for $j \geq 1$ the coefficients $c_{j}$ are explicitly computable rational numbers, and the implied $O$-constant depends on $N$.
(2) The hyperfactorial $D_{n}^{*}=\prod_{k=1}^{n} k^{k}$ has an asymptotic expansion for $\log \left(D_{n}^{*}\right)$ up to any given order $N \geq 1$, valid uniformly for all $n \geq 2$, of the form

$$
\begin{array}{r}
\log \left(D_{n}^{*}\right)=\frac{1}{2} n^{2} \log n-\frac{1}{4} n^{2}+\frac{1}{2} n \log n+\frac{1}{12} \log n+ \\
+d_{0}+\sum_{j=1}^{N} d_{j}\left(\frac{1}{n^{j}}\right)+O\left(\frac{1}{n^{N+1}}\right)
\end{array}
$$

The constant $d_{0}=\log A$, where $A$ is the Glaisher-Kinkelin constant, for $j \geq 1$ the coefficients $d_{j}$ are explicitly computable rational numbers, and the implied $O$ constant depends on $N$.

[^5]Proof of Theorem 4.1. (1) We have

$$
\log \left(N_{n}^{*}\right)=\log \Gamma(n+1)+\log G(n+1)
$$

Substituting the asymptotic series for $\Gamma(n+1)$ and $G(n+1)$ each term by term yields 9.6). Here for $k \geq 1$ we have $c_{2 k}=\frac{B_{2 k+2}}{2 k(2 k+2)}$ while $c_{2 k-1}=\frac{B_{2 k}}{(2 k)(2 k-1)}$.
(2) We have

$$
\log \left(D_{n}^{*}\right)=n \log \Gamma(n+1)-\log G(n+1)
$$

Substituting the asymptotic series $\Gamma(n+1)$ and $G(n+1)$ term by term on the right side yields (9.6). Multiplying by $n$ in the first term on the right shifts the coefficient indices of the asymptotic expansion of $\log \Gamma(n+1)$ down by 1 . For $k \geq 1$ we have

$$
d_{2 k}=c_{2 k+1}-c_{2 k}=\frac{B_{2 k+2}}{(2 k+2)(2 k+1)}-\frac{B_{2 k+2}}{2 k(2 k+2)}=\frac{B_{2 k+2}}{2 k(2 k+1)(2 k+2)}
$$

while $d_{2 k-1}=0$.

Theorem4.1 immediately yields asymptotic expansion for $\log \left(\bar{G}_{n}\right)$. The resulting asymptotic behavior of $\log \left(\bar{G}_{n}\right)$ is of smaller order of magnitude, since the leading term in the asymptotic series of $\log \left(N_{n}^{*}\right)$ and $\log \left(D_{n}^{*}\right)$ on the right side of (9.6) cancel.

Theorem 9.2. The complete binomial products $\bar{G}_{n}=\prod_{j=1}^{n}\binom{n}{j}$ have an asymptotic expansion for $\log \left(\bar{G}_{n}\right)$ to any given order $N \geq 1$, valid uniformly for all $n \geq 2$, of the form
$\left.\log \left(\bar{G}_{n}\right)=\frac{1}{2} n^{2}-\frac{1}{2} n \log n+\left(1-\frac{1}{2} \log (2 \pi)\right)\right) n-\frac{1}{3} \log n+g_{0}+\sum_{j=1}^{N} g_{j}\left(\frac{1}{n^{j}}\right)+O\left(\frac{1}{n^{N+1}}\right)$.
Here $g_{0}=-\frac{1}{2} \log (2 \pi)-\frac{1}{12}+2 \log A$ where $A$ is the Glaisher-Kinkelin constant, for $j \geq 1$ the coefficients $g_{j}$ are explicitly computable rational numbers, and the implied $O$-constant depends on $N$.

Proof. We have

$$
\begin{equation*}
\log \left(\bar{G}_{n}\right)=\log \left(D_{n}^{*}\right)-\log \left(N_{n}^{*}\right) \tag{9.6}
\end{equation*}
$$

Direct substation from Theorem 4.1 then gives the result. Here the terms $g_{j}$ for $j \geq 1$ are given by $g_{j}=d_{j}-c_{j}$, so are

$$
g_{2 k}=d_{2 k}-c_{2 k}=-\frac{B_{2 k+2}}{2 k(2 k+1)(2 k+2)}-\frac{B_{2 k+2}}{2 k(2 k+2)}=-\frac{B_{2 k+2}}{2 k(2 k+1)}
$$

while $g_{2 k-1}=-c_{2 k-1}=-\frac{B_{2 k}}{2 k(2 k-1)}$.
Table A. 1 below gives coefficients of the first few terms $\frac{1}{n^{k}}$ in the asymptotic expansions above in Section 3. The small size of the coefficients in the table is quite misleading; later coefficients become very large, since the Bernoulli numbers satisfy $\left|B_{2 n}\right| \sim 4 \sqrt{\pi n}\left(\frac{n}{\pi e}\right)^{2 n}$ as $n \rightarrow \infty$.

| Coefficient | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | $k=6$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\log \Gamma(z+1)$ | $\frac{1}{12}$ | 0 | $-\frac{1}{360}$ | 0 | $\frac{1}{1260}$ | 0 |
| $\log G(z+1)$ | 0 | $-\frac{1}{240}$ | 0 | $\frac{1}{1008}$ | 0 | $-\frac{1}{1440}$ |
| $c_{k}$ | $\frac{1}{12}$ | $-\frac{1}{240}$ | $-\frac{1}{360}$ | $\frac{1}{1008}$ | $\frac{1}{1260}$ | $-\frac{1}{1440}$ |
| $d_{k}$ | 0 | $\frac{1}{720}$ | 0 | $-\frac{1}{5040}$ | 0 | $\frac{1}{10080}$ |
| $g_{k}$ | $-\frac{1}{12}$ | $\frac{1}{180}$ | $\frac{1}{360}$ | $-\frac{1}{860}$ | $-\frac{1}{1260}$ | $\frac{1}{1260}$ |

TABLE A.1. Asymptotic expansion coefficients $c_{k}, d_{k}, g_{k}$.

Remark 9.3. (1) We may extend $\bar{G}_{n}$ to an analytic function $\bar{G}^{a n}(z)$ of a complex variable $z$ on the complex plane cut along the nonpositive real axis, using the right side of 9.3 as a definition:

$$
\bar{G}^{a n}(z):=\frac{\Gamma(z+1)^{z-1}}{G(z+1)^{2}} .
$$

Since the Gamma function has no zeros, the function $\log \Gamma(z+1)$ is well-defined on the cut plane, and we may set $\Gamma(z+1)^{z-1}:=\exp ((z-1) \log \Gamma(z+1))$, choosing that branch of the logarithm that is real on the positive real axis. The function $\bar{G}^{a n}(z)$ is not a meromorphic function; instead, it analytically continues to a multi-valued function on a suitable Riemann surface which covers the complex plane punctured at the negative integers. It is an example of an "endlessly continuable" function, as discussed in Sternin and Shalatov [49] or Sauzin 47].
(2) For number-theoretic applications (as in 36]) one extends the values $\bar{G}_{n}$ to positive real $x$ another way, making it a step function setting $\bar{G}_{x}:=\bar{G}_{\lfloor x\rfloor}$. For the step function definition the function $\log \left(\bar{G}_{x}\right)$, viewed as a function of a real variable $x$, has jumps of size $\gg n$ at integer values of $x$. These jumps are of much larger size than most terms in the asymptotic expansion of Theorem 9.2 . In this case the asymptotic expansion in Theorem 9.2 is valid to all orders $\frac{1}{n^{k}}$ exactly at integer points $x=n$.
10. Appendix B: Equality of $\nu_{b}\left(\bar{G}_{n}\right)$ and $\nu_{b}^{*}\left(\bar{G}_{n}\right)$

In this Appendix we shows that the functions $\nu_{b}\left(\bar{G}_{n}\right)$ and $\nu_{b}^{*}\left(\bar{G}_{n}\right)$ introduced in Section 5.2 and 6.1 are equal. Recall from Section 5.2 that

$$
\nu_{b}\left(\bar{G}_{n}\right):=\frac{1}{b-1}\left(2 S_{b}(n)-(n-1) d_{b}(n)\right)
$$

Recall from Section 6.1 that

$$
\nu_{b}^{*}\left(\bar{G}_{n}\right):=\sum_{j=1}^{k} j a_{j} p^{j}-\left(\sum_{j=1}^{k} a_{j}\left(\frac{p^{j}-1}{p-1}\right)+\sum_{j=0}^{k} \frac{1}{p^{j+1}}\left(\sum_{u=0}^{j} a_{\ell} p^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} p^{v}\right)\right)
$$

Both these functions are expressed in terms of the base $b$ expansion $n=\sum_{j=0}^{k} a_{j} b^{j}$. These functions were defined so that they satisfy $\nu_{p}\left(\bar{G}_{n}\right)=\nu_{p}^{*}\left(\bar{G}_{n}\right)=\operatorname{ord}_{p}\left(\bar{G}_{n}\right)$ for $p$ a prime.

Theorem 10.1. For each integer $b \geq 2$ there holds

$$
\nu_{b}\left(\bar{G}_{n}\right)=\nu_{b}^{*}\left(\bar{G}_{n}\right) \quad \text { for all } \quad n \geq 1
$$

Proof. We re-express both functions using the floor function. We first have

$$
\begin{aligned}
\nu_{b}\left(\bar{G}_{n}\right) & =\frac{1}{b-1}\left(2 \sum_{m=1}^{n-1}\left(m-(b-1) \sum_{j \geq 1}\left\lfloor\frac{m}{b^{j}}\right\rfloor\right)-(n-1) \sum_{j \geq 0}\left(\left\lfloor\frac{n}{b^{j}}\right\rfloor-b\left\lfloor\frac{n}{p^{j+1}}\right\rfloor\right)\right) \\
& =(n-1) \sum_{j \geq 1}\left\lfloor\frac{n}{b^{j}}\right\rfloor-2 \sum_{m=1}^{n-1}\left(\sum_{j \geq 1}\left\lfloor\frac{m}{b^{j}}\right\rfloor\right) .
\end{aligned}
$$

We also have

$$
\begin{aligned}
\nu_{b}^{*}\left(\bar{G}_{n}\right) & =\sum_{j=1}^{k} j b^{j}\left(\left\lfloor\frac{n}{b^{j}}\right\rfloor-b\left\lfloor\frac{n}{b^{j+1}}\right\rfloor\right)-\left(\sum_{j=1}^{k} a_{j}\left(\frac{b^{j}-1}{b-1}\right)+\sum_{j=0}^{k} \frac{1}{b^{j+1}}\left(\sum_{u=0}^{j} a_{u} p^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} b^{v}\right)\right) \\
& =\sum_{j \geq 1} b^{j}\left\lfloor\frac{n}{b^{j}}\right\rfloor-\left(\sum_{j=1}^{k}\left(\left\lfloor\frac{n}{b^{j}}\right\rfloor-b\left\lfloor\frac{n}{b^{j+1}}\right\rfloor\right)\left(\frac{b^{j}-1}{b-1}\right)+\sum_{j=0}^{k} \frac{1}{b^{j+1}}\left(\sum_{u=0}^{j} a_{u} b^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} b^{v}\right)\right) \\
& =\sum_{j \geq 1} b^{j}\left\lfloor\frac{n}{b^{j}}\right\rfloor-\left(\sum_{j \geq 1}\left\lfloor\frac{n}{b^{j}}\right\rfloor+\sum_{j=0}^{k} \frac{1}{b^{j+1}}\left(\sum_{u=0}^{j} a_{u} b^{u}\right)\left(\sum_{v=j+1}^{k} a_{v} b^{v}\right)\right) \\
& =\sum_{j \geq 1} b^{j}\left\lfloor\frac{n}{b^{j}}\right\rfloor-\left(\sum_{j \geq 1}\left\lfloor\frac{n}{b^{j}}\right\rfloor+\sum_{j \geq 1}\left(\frac{n}{b^{j}}-\left\lfloor\frac{n}{b^{j}}\right\rfloor\right)\left(b^{j}\left\lfloor\frac{n}{b^{j}}\right\rfloor\right)\right) .
\end{aligned}
$$

Combining these two formulas yields

$$
\begin{aligned}
\nu_{b}\left(\bar{G}_{n}\right)-\nu_{b}^{*}\left(\bar{G}_{n}\right) & =n \sum_{j \geq 1}\left\lfloor\frac{n}{b^{j}}\right\rfloor-\sum_{m=1}^{n-1}\left(\sum_{j \geq 1}\left\lfloor\frac{m}{b^{j}}\right\rfloor\right)-\sum_{j \geq 1} b^{j}\left(\sum_{k=1}^{\left\lfloor n / b^{j}\right\rfloor} k\right) \\
& =\sum_{j \geq 1}\left(n\left\lfloor\frac{n}{b^{j}}\right\rfloor-\sum_{m=1}^{n-1}\left\lfloor\frac{m}{b^{j}}\right\rfloor-b^{j}\left(\sum_{k=1}^{\left\lfloor n / b^{j}\right\rfloor} k\right)\right) .
\end{aligned}
$$

We assert that each inner sum (for fixed $j$ ) on the right side of this sum is 0 . To see this, we have

$$
\begin{aligned}
n\left\lfloor\frac{n}{b^{j}}\right\rfloor-\sum_{m=1}^{n-1}\left\lfloor\frac{m}{b^{j}}\right\rfloor-b^{j}\left(\sum_{k=1}^{\left\lfloor n / b^{j}\right\rfloor} k\right) & =\sum_{k=1}^{\left\lfloor n / b^{j}\right\rfloor}\left(n-b^{j} k\right)-\sum_{m=1}^{n-1}\left\lfloor\frac{m}{b^{j}}\right\rfloor \\
& =\sum_{k=1}^{\left\lfloor n / b^{j}\right\rfloor}\left(n-b^{j} k\right)-\sum_{m=1}^{n-1}\left\lfloor\frac{n-m}{b^{j}}\right\rfloor=0 .
\end{aligned}
$$

The equality to zero on the last line follows by a counting argument. We evaluate the last sum on the right in blocks of length $b^{j}$, taking $1+(k-1) b^{j} \leq m \leq$ $k b^{j}$ for $1 \leq k \leq\left\lfloor n / b^{j}\right\rfloor$. If $n=b_{j}\left\lfloor n / b^{j}\right\rfloor+a$ then the first block contributes
$b^{j}\left\lfloor n / b_{j}\right\rfloor-\left(b^{j}-a\right)=n-b^{j}$. The $k$-th block contributes $n-b^{j} k$ similarly, and a possible final "short" block contributes 0 .

Acknowledgments. We thank J. Arias de Reyna for supplying the plot in Fig. 2 and for several corrections. We thank the reviewer for helpful comments. The first author is indebted to Harm Derksen for raising the topic of products of Farey fractions, see [12, [13]. The work of the second author began as part of an REU project at the University of Michigan in 2013 with the first author as mentor. Work of the first author was partially supported by NSF grants DMS-1101373 and DMS-1401224.

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[^0]:    2010 Mathematics Subject Classification. Primary: 11B65, Secondary: 05A10, 11B57, 11N05, 11N64.

    Work of the first author was supported by NSF Grants DMS-1101373 and DMS-1401224.

[^1]:    ${ }^{1}$ The hyperfactorials occur as the discriminants of the Hermite polynomials up to factor of a power of 2 in the usual normalization $H_{n}(x)=(-1)^{n} e^{x^{2} / 2} \frac{d^{n}}{d x^{n}}\left(e^{-x^{2} / 2}\right)$, see Szego [50, (6.71.7)]. The Hermite polynomials are a fundamental family of orthogonal polynomials, and this connection hints at a deep importance of the hyperfactorial function.

[^2]:    ${ }^{2}$ Delange uses a different notation for digit sums. He writes $S_{p}(n)$ for the function that we call $d_{p}(n)$.

[^3]:    ${ }^{3}$ The lower bound $\pi(x) \geq \frac{x}{\log x}+o\left(\frac{x}{\log x}\right)$ is known to be equivalent to the full prime number theorem

[^4]:    ${ }^{4}$ The asymptotic expansion of $\log \Gamma(z)$ is extremely similar to that of $\log \Gamma(z+1)$, changing only one term $\frac{1}{2} \log z$ to $-\frac{1}{2} \log z$, via the identity $\log \Gamma(z+1)=\log \Gamma(z)+\log z$. The expansion of $\log \Gamma(z)$ given in Whittaker and Watson 54 Sect. 12.33] uses an older notation for Bernoulii numbers: their notation $B_{k}$ corresponds to $\left|B_{2 k}\right|$ in our notation.

[^5]:    ${ }^{5}$ Barnes 4 follows a different convention for Bernoulli numbers: his $B_{k}$ corresponds to $\left|B_{2 k}\right|$ in the notation used here. We have altered his formula accordingly.

    6 Their expansion contains a term $z \log \Gamma(z+1)$ so the asymptotic expansion of $\log \Gamma(z+1)$ must be substituted in their formula.

