# Boolean and ortho fuzzy subset logics 

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#### Abstract

Constructing a fuzzy subset logic $L$ with Boolean properties is notoriously difficult because under a handful of "reasonable" conditions, we have the following three debilitating constraints: (1) Bellman and Giertz in 1973 showed that if $L$ is distributive, then it must be idempotent. (2) Dubois and Padre in 1980 showed that if $L$ has the excluded middle or the non-contradiction property or both, then it must be non-idempotent. (3) Bellman and Giertz also demonstrated in 1973 that even if $L$ is idempotent, then the only choice available for the ( $\wedge, \vee$ ) logic operator pair is the (min, max) operator pair. Thus it would seem impossible to construct a non-trivial fuzzy subset logic with Boolean properties. However, this paper examines these three results in detail, and shows that "hidden" in the hypotheses of the three is the assumption that the operator pair $(\wedge, \vee)$ is pointwise evaluated. It is further demonstrated that removing this constraint yields the following results: (A) It is indeed possible to construct fuzzy subset logics that have all the Boolean properties, including that of idempotency, non-contradiction, excluded middle, and distributivity. (B) Even if idempotency holds, (min, max) is not the only choice for $(\wedge, \vee)$.


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## Introduction

The problem. This paper addresses a well known conflict in fuzzy subset logic theory. Fuzzy subset logic is the foundation for fuzzy set theory, just as classical logic is the foundation for classical set theory. Just as classical logic and the algebras of sets constructed upon it are Boolean algebras (and hence have all the "nice" Boolean properties such as idempotency, excluded middle, non-contradiction, distributivity, etc.), one might very much prefer to have a fuzzy subset logic and resulting fuzzy subset algebras that are Boolean as well. ${ }^{1}$ But it has been found that under what would seem to be very "reasonable" conditions, this is simply not possible. In particular, we have the following crippling constraints:

料 Bellman and Giertz in 1973 demonstrated that under very "reasonable" conditions, if we want a fuzzy subset logic that is distributive, then it also must be idempotent. ${ }^{2}$
4.4. Dubois and Padre in 1980 demonstrated that under very "reasonable" conditions, if we want a fuzzy subset logic that has the non-contradiction and excluded middle properties, then that logic is not idempotent... and therefore not only fails to be a Boolean algebra, but also is not even a lattice. ${ }^{3}$

[^0]料 Moreover，even if we are willing to give up the non－contradiction and excluded middle prop－ erties and retain idempotency，Bellman and Giertz also demonstrated in 1973 that the only choice we have for the logic operator pair $(\wedge, \vee)$ is the $(\min , \max )$ operator pair such that $(\wedge, \vee)=$ $(\min , \max ) .{ }^{4}$

A solution．Section 1 of this paper examines these results in detail，and demonstrates that＂hidden＂in the hypotheses of these results is the assumption that the operator pair $(\wedge, \vee)$ is pointwise evaluated．${ }^{5}$ Section 2 demonstrates that if this constraint is removed，then it is indeed possible to construct fuzzy subset logics that have all the Boolean properties，including that of idempotency，non－contradiction， excluded middle，and distributivity．

A solution yielding ortho fuzzy subset logics．In this paper，a logic $L^{\prime} \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is defined as a lattice $L \triangleq(X, \vee, \wedge ; \leq)$ with a negation function $\neg$ and implication function $\rightarrow$ defined on this lattice．And in this sense，the logic $L^{\prime}$ is said to be＂constructed on＂the lattice $L$ ．This paper demonstrates that it is possible to construct fuzzy subset logics on Boolean lattices yielding Boolean fuzzy subset logics．However，more generally，it is also demonstrated that it is possible to construct fuzzy subset logics on orthocomplemented lattices yielding ortho fuzzy subset logics．The main differ－ ence between a Boolean lattice and an orthocomplemented lattice is that the latter does not in general support distributivity．${ }^{6}$ On finite sets，there are significantly more choices of orthocomplemented lat－ tices than there are Boolean lattices．${ }^{7}$ And so having the option of constructing ortho fuzzy subset logics is arguably not without advantage．The disadvantage is that we give up the guarantee of distributivity． But some authors ${ }^{8}$ have investigated structures without this property anyways．In fact，one could argue that the＂crucial＂properties that we would really like a logic to have，if possible，are the following：
（1）．disjunctive idempotence：$x \vee x=x$ and
（2）．conjunctive idempotence：$x \wedge x=x$ and
（3）．excluded middle：$\quad x \vee \neg x=1$ and
（4）．non－contradiction：$\quad x \wedge \neg x=0$ ．
Not all fuzzy logics have all the these properties．Of course all Boolean logics have them．But more generally than Boolean logics and less generally than fuzzy logics，all ortho logics have them as well．${ }^{9}$

[^1]Boolean and ortho fuzzy subset logics
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Negation functions. There are several types of negation functions and information about these functions is scattered about in the literature. Appendix B introduces several types of negation, describes some of their properties, and shows where fuzzy negation, ortho negation, and Boolean negation "fit" into the larger structure of negations in general.

Implication functions. Defining an implication function for a logic constructed on a Boolean lattice is straightforward because we can simply use the classical implication $x \xrightarrow{c} y \triangleq \neg x \vee y$. However, defining an implication function for a non-Boolean logic is more difficult. Appendix C addresses the problem of defining implication functions on lattices, including lattices that are non-Boolean.

## 1 Fuzzy subset operators

A fuzzy subset is often specified in terms of a membership function. A fuzzy subset logic is a lattice of membership functions together with a fuzzy negation function and an implication function. Although its definition is simple and straightforward, fuzzy subset logic has some notorious problems attempting to provide some very standard Boolean properties. ${ }^{10}$

### 1.1 Indicator functions

In classical subset theory, a subset $A$ of a set $X$ can be specified using an indicator function $\mathbb{1}_{A}(x)$ (next definition). An indicator function specifies concretely whether or not an element is a member of $A$. That is, it is a convenient "indicator" of whether or not a particular element is in a subset. A subset that can be defined using an indicator function is a crisp subset (next definition).

Definition $1.1^{11}$ Let $2^{X}$ be the power set of a set $X$. Let $Y^{X}$ be the set of all functions mapping from $X$ to a set $Y$. The indicator function $\mathbb{1}_{A} \in\{0,1\}^{X}$ is defined as

$$
\mathbb{1}_{A}(x)=\left\{\begin{array}{ll}
1 & \text { if } x \in A \\
0 & \text { if } x \notin A
\end{array}\right\} \quad \forall x \in X, A \in 2^{X} .
$$

The parameter $A$ of $\mathbb{1}_{A}$ is a crisp subset of $X$ if $\mathbb{1}_{A}(x)$ is an indicator function on $X$.
Every set $X$ has at least one crisp subset (itself). A set of subsets, together with the relation $\subseteq$, form an ordered set, and in some cases also form a lattice. Common set structures include the power set $2^{X}$, topologies, rings of sets and algebras of sets. A set structure may be represented in terms of subsets, or equivalently, in terms of set indicator functions. ${ }^{12}$

[^2]
(A) set notation

(B) set indicator notation

Figure 1: set structures on $O_{6}$ (see Example 1.5 page 5)

Remark 1.2 Often set structures are defined in terms of set operators like intersection $\cap$, union $\cup$, and set complement c . The set operators ( $\cap, \cup, \mathrm{c}, \Rightarrow, \varnothing, X)$ in turn can be defined in terms of arithmetic operators (min, max, $\mathrm{f}(x) \triangleq 1-x, \mathrm{~g}(x, y) \triangleq y-x y, 0,1)$ on the set indicator function ${ }^{13}$ or in terms of classic logic operators $(\wedge, \vee, \neg, \rightarrow, 0,1)$ like this:

| 0 | $\triangleq \mathbb{1}_{\varnothing}=0$ |
| :--- | :--- |
| 1 | $\triangleq \mathbb{1}_{X}=1$ |
| $\mathbb{1}_{A} \vee \mathbb{1}_{B} \triangleq \mathbb{1}_{A \cup B}=\max \left(\mathbb{1}_{A}, \mathbb{1}_{B}\right)$ |  |
| $\mathbb{1}_{A} \wedge \mathbb{1}_{B} \triangleq \mathbb{1}_{A \cap B}=\min \left(\mathbb{1}_{A}, \mathbb{1}_{B}\right)$ |  |
| $\left.\mathfrak{T}_{A}\right)$ | $\triangleq \mathbb{1}_{A^{c}}=1-\mathbb{1}_{A}$ |
| $\mathbb{1}_{A} \rightarrow \mathbb{1}_{B} \triangleq \mathbb{1}_{A \Rightarrow B}=\max \left(1-\mathbb{1}_{A}, \mathbb{1}_{B}\right)$ |  |

where $A \Rightarrow B \triangleq A^{\mathrm{c}} \cup B$ is the set implication from $A$ to $B .^{14}$



Example 1.3 The set structures illustrated to the left and right are the power set of the set $X \triangleq\{a, b, c\}$. A power set is a special case of an algebra of sets and also a topology. The lattice to the left uses set notation; the one to the right uses set indicators.

Example 1.4 The set structures illustrated to the left and right are a topology on the set $X \triangleq\{a, b\}$. The lattice to the left uses set notation; the one to the right uses set indicators.


Example 1.5 The set structures illustrated in Figure 1 (page 5) are not topologies (or algebras of sets or power sets), but are set structures none the less. The negation function in the structure is an ortho negation (Definition B. 3 page 35). The lattice in (A) uses set notation; the one in (B) uses set indicators.

Definition 1.6 Let $\mathbb{1}^{X}$ be the set of all indicator functions on a set $X$. Let a logic be defined as in Definition C. 5 (page 50). A crisp subset logic is a logic $\left(\mathbb{1}^{X}, \vee, \wedge, \neg, \mathbb{1}_{\varnothing}, \mathbb{1}_{X} ;<, \rightarrow\right)$.

[^3]
## 1．2 Membership functions

In a crisp subset $A$ of a crisp set $X(A \subseteq X)$ ，an element $x \in X$ has only two possible＂degrees of mem－ bership＂in $A$ ：Either $x$ is in $A$ or $x$ is not in $A$ ．Said another way，either $x$ has＂full membership＂in $A$ ， or $x$ has＂absolute non－membership＂in $A$ ．And this＂degree of membership＂is specified by an indi－ cator function（Definition 1.1 page 4） $\mathbb{1}_{A}(x)$ which maps from $X$ to the 2 －valued set $\{0,1\}$ ，where 0 represents ＂absolute non－membership＂and 1 represents＂full membership＂．

In a fuzzy subset $B$ of a crisp set $X(B \subseteq X)$ ，an element $x \in X$ has a range of possible degrees of membership in $B$ ．And this membership is specified by a membership function（next definition） $\mathfrak{m}_{B}(x)$ which maps from $X$ to the infinte set $[0: 1]$ ．

Definition $1.7{ }^{15}$ Let $[0: 1]$ be the closed interval on $\mathbb{R}$ such that $[0: 1] \triangleq\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ ．Let $X$ be a set．A function $\mathfrak{m}_{A}(x)$ is a membership function on $X$ if $\mathfrak{m}_{A} \in[0: 1]^{X}$ ．The parameter $A$ is called a fuzzy subset of $X$ ．For any value $x \in X, \mathfrak{m}_{A}(x) \in[0: 1]$ represents the＂degree of membership＂of $x$ in $A$ ．The condition $\mathfrak{m}_{A}(x)=1$ indicates that $x$ has＂full membership＂in $A$ ，and the condition $m_{A}(x)=0$ indicates that $x$ has＂absolute non－membership＂in $A$ ．

Remark 1．8 ${ }^{16}$ What is typically called a＂fuzzy set＂arguably should more accurately be called a＂fuzzy subset＂because every element $x$ at any＂degree of membership＂in a fuzzy subset $A$ has absolute full membership in some universal crisp set $X$ ．And thus $A$ is a subset of the crisp set $X(A \subseteq X)$ ．

Remark 1．9 In a crisp set $X$ ，a fuzzy subset $A \subseteq X$ should not be confused with a random subset $B \subseteq X$ ．In the fuzzy subset $A$ ，an element $x \in X$ has a＂degree of membership＂in $A$ that specifies ＂to what extent＂$x$ can be considered a member of $A$ ．In the random subset $B$ ，the element $x \in X$ has a＂degree of likelihood＂that $x$ is in $B$ and that specifies the probability that $x$ is a member of $B$ ． Alternatively，a fuzzy subset is a result of＂inference under vagueness＂，while a random subset is a result of＂inference under randomness＂．${ }^{17}$

Example 1．10 Let $A$ be the set of all people who are＂young＂with membership function $\mathrm{m}_{A}(x)$ ．Let $B$ be the set of all people who are＂middle age＂with membership function $m_{B}(x)$ ．Let $C$ be the set of all people who are＂old＂with membership function $m_{C}(x)$ ．Of course all these are vague，or＂fuzzy＂， concepts；but the following figure illustrates what the membership functions（Definition 1.7 page 6）for these sets might look like．


Definition 1．11 Let $\mathbb{M}$ be a set of membership functions（Definition 1.7 page 6）． The structure $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ is a fuzzy subset logic if $L$ is a fuzzy logic（Definition C．5 page 50）．

[^4]
## 1．3 Operators on membership functions

The meet－join operator pair $(\wedge, \vee)$ on a set of indicator functions $\mathbb{1}^{X}$ induces an ordering relation on $\mathbb{1}^{X} .{ }^{18}$ ．So the operator pairs $(\wedge, \vee)$ can be defined on sets of membership functions to form lattices．But while lattices of set indicators effectively have just one choice for $(\wedge, \vee)$ ，membership function lattices have many choices．

In this paper，the operators $(\wedge, \vee)$ are called pointwise evaluated if at each single value $x$ ，the functions ［ $m \wedge \mathfrak{m}](x)$ and $[m \vee m](x)$ depend only on the values of $m(x)$（ $m$ evaluated at the single value $x$ ）and $m(x)$ （next definition）．

Definition $1.12{ }^{19}$ Let $L \triangleq(\mathbb{M}, \wedge, \vee)$ ，where $\mathbb{M}$ is a set of membership functions（Definition 1.7 page 6）with operators $(\wedge, \vee) . \quad L$ is pointwise evaluated，or said to have pointwise evaluation，if there exists $f, g \in$ $[0: 1]^{[0: 1]^{2}}$ such that

1．$[m \wedge \mathfrak{m}](x)=\mathrm{f}[m(x), \mathfrak{m}(x)] \quad \forall x \in \mathbb{R}$ ，and $\forall m, n \in \mathbb{M}$ and
2．$[\mathfrak{m} \vee \mathbb{\infty}](x)=\operatorname{g}[\mathfrak{m}(x), \mathfrak{n}(x)] \quad \forall x \in \mathbb{R}$ ，and $\forall m, \infty \in \mathbb{M}$

## Example 1.13

1．The function $\Delta$ defined as $[m \Delta m](x) \triangleq m(x)+m(x) \quad$ is pointwise evaluated ．
2．The function $\triangleq$ defined as $[m \triangleq m](x) \triangleq \underbrace{\int_{-\infty}^{x} m(u) m(x-u) \mathrm{d} u}_{\text {＂convolution＂}}$ is not pointwise evaluated．

Example 1．14 Examples of operators that are pointwise evaluated include the min－max operators （next definition），the product and probabilistic sum operators（Definition 1.17 page 8），and the Łukasiewicz $t$－norm and $t$－conorm（Definition 1.18 page 9 ）．

One of the most common fuzzy logic operator pairs is the min－max operator pair（next）．As will be demonstrated by the fuzzy min－max theorem（Theorem 1.26 page 13），under fairly＂reasonable＂conditions the min－max operators are the only choice available for a fuzzy subset logic．

Definition $1.15{ }^{20}$ Let $\mathbb{M}$ be a set of membership functions on a set $X$ ．Let $f(x)$ and $g(x)$ be functions both with domain $X$ ．Let $\min (\mathrm{f}(x), \mathrm{g}(x))$ and $\max (\mathrm{f}(x), \mathrm{g}(x))$ be the pointwise minimum and pointwise maximum，respectively，of $\mathrm{f}(x)$ and $\mathrm{g}(x)$ over $X$ ．The min－max operators $(\wedge, \vee)$ for $L$ are defined as

Proposition 1．16 Let $\mathbb{M}$ ，max，and min defined as in Definition 1．15．Let $L \triangleq(\mathbb{M}, \vee, \wedge ; \leq)$ be an alge－ braic structure with $x \leq y \Longleftrightarrow x \wedge y=x$ ．

$$
(\wedge, \vee)=(\min , \max ) \quad \Longrightarrow \quad L \text { is } a \text { LATTICE (Definition A. } 11 \text { page 24). }
$$

[^5]Q Proof: To be a lattice, L must be commutative, associative, and absorptive (Theorem A. 18 page 25).

$$
\begin{aligned}
& m \vee \mathfrak{m}=\max (m, m) \quad \text { by left hypothesis } \\
& =\max (m, m) \\
& =m \vee m \\
& \Longrightarrow \vee \text { is commutative } \\
& m \wedge m=\min (m, n) \\
& =\min (n, m) \\
& =\mathfrak{n} \wedge \mathrm{m} \\
& \Longrightarrow \wedge \text { is commutative } \\
& m \vee(m \vee p)=\max [m, \max (m, p)] \\
& =\max [\max (m, n), p] \\
& =(m \vee m) \vee p \\
& \Longrightarrow \quad \vee \text { is associative } \\
& \mathfrak{m} \wedge(\mathfrak{n} \wedge \mathfrak{p})=\min [m, \min (\mathfrak{m}, \mathfrak{p})] \\
& =\min [\min (m, n), p] \\
& =(m \wedge n) \wedge p \\
& \Longrightarrow \wedge \text { is associative } \\
& m \vee(m \wedge n)=\max [m, \min (m, m)] \\
& = \begin{cases}\max (m, m) & \text { if } m(x) \leq \mathfrak{n}(x) \quad \forall x \in X \\
\max (m, n) & \text { otherwise }\end{cases} \\
& = \begin{cases}m & \text { if } m(x) \leq m(x) \quad \forall x \in X \\
m & \text { otherwise }\end{cases} \\
& =m \\
& m \wedge(m \vee m)=\min [m, \max (m, n)] \quad \text { by left hypothesis } \\
& = \begin{cases}\min (m, m) & \text { if } \quad m(x) \leq m(x) \quad \forall x \in X \\
\min (m, m) & \text { otherwise }\end{cases} \\
& = \begin{cases}m & \text { if } m(x) \leq m(x) \quad \forall x \in X \\
m & \text { otherwise }\end{cases} \\
& =m \\
& \Longrightarrow(\wedge, \vee) \text { is absorptive } \\
& \text { by left hypothesis } \\
& \text { by definition of min } \\
& \text { by left hypothesis } \\
& \text { by left hypothesis } \\
& \text { by definition of min } \\
& \text { by left hypothesis } \\
& \text { by left hypothesis } \\
& \text { by definition of max } \\
& \text { by left hypothesis } \\
& \text { by left hypothesis } \\
& \text { by definition of min } \\
& \text { by left hypothesis } \\
& \text { by left hypothesis } \\
& \text { by left hypothesis }
\end{aligned}
$$

Definition $1.17{ }^{21}$ Let $\mathbb{M}$ be defined as in Definition 1.15. Then for all $m \in \mathbb{M}$, the probabilistic sum operator $\vee$ on $\mathbb{M}$ is defined as $\left[\mathfrak{m}_{A} \vee \mathfrak{m}_{B}\right](x) \triangleq m_{A}(x)+m_{B}(x)-m_{A}(x) \mathfrak{m}_{B}(x)$ and the product sum operator $\wedge$ on $\mathbb{M}$ is defined as $\left[\mathbb{m}_{A} \wedge \mathrm{~m}_{B}\right](x) \triangleq \mathrm{m}_{A}(x) \mathrm{m}_{B}(x)$

Note that the product and probabilistic sum operators (previous definition) do not in general form a lattice because, for example, they are not in general idempotent (a necessary condition for being a lattice-Theorem A. 14 page 25). Suppose for example $m(p)=1 / 2$ at some point $p$. Then at that point $p$


[^6]Definition $1.18{ }^{22}$ Let $L, \mathcal{D}$ ，min and max be defined as in Definition 1．15．Then for all $m \in \mathbb{M}$ ， the Lukasiewicz t－conorm $\vee$ is defined as $\quad\left[\mathfrak{m}_{A} \wedge \mathfrak{m}_{B}\right](x) \triangleq \max \left[0, \mathfrak{m}_{A}(x)+\mathfrak{m}_{B}(x)-1\right] \quad \forall m \in \mathbb{M}, x \in X$ and the Lukasiewicz t－norm $\wedge$ is defined as $\left[\mathfrak{m}_{A} \vee \mathfrak{m}_{B}\right](x) \triangleq \min \left[1, \mathfrak{m}_{A}(x)+\mathfrak{m}_{B}(x)\right] \quad \forall m \in \mathbb{M}, x \in X$ The Łukasiewicz $t$－conorm is also called the bold sum，and the Łukasiewicz $t$－norm is also called the bold intersection．

Note that the Łukasiewicz operators（previous definition）do not in general form a lattice because，for example，they are not in general idempotent．Suppose for example $m(p)=1 / 2$ at some point $p$ ．Then

$$
\mathfrak{m} \vee \mathfrak{m} \triangleq \min (1, m+m) \quad=\min (1,1 / 2+1 / 2) \quad=1 \neq \mathrm{m}
$$

$$
m \wedge m \triangleq \max (0, m+m-1)=\max (0,1 / 2+1 / 2-1)=0 \neq m
$$

There are several choices for negations in a fuzzy subset logic．Arguably the＂simplest＂is the discrete negation（Example B． 16 page 38）．Perhaps the most＂common＂is the standard negation（next definition）． More generally there is the $\lambda$－negation（Definition 1.20 page 9）which reduces to the standard negation at $\lambda=0$ and approaches the discrete negation as $\lambda \rightarrow \infty$ ．Alternatively there is also the Yager negation（Defini－ tion 1.21 page 9）which reduces to the standard negation at $p=1$ ．

Definition $1.19{ }^{23}$ The function $\neg m(x)$ is the standard negation（or Lukasiewicz negation）of $m$ if $\neg \operatorname{mon}(x) \triangleq 1-m(x) \quad \forall x \in \mathbb{R}$.

Definition 1．20 ${ }^{24}$ The function $\neg m(x)$ is the $\lambda$－negation of a function $m(x)$ if $\neg m(x) \triangleq \frac{1-\operatorname{mon}(x)}{1+\lambda \operatorname{mon}(x)} \quad \forall \lambda \in(-1: \infty)$.

Definition 1．21 ${ }^{25}$ The function $\neg m(x)$ is the Yager negation of a function $m(x)$ if $\neg m(x) \triangleq\left(1-m^{p}\right)^{1 / p} \quad \forall p \in(0: \infty)$.

If $\neg m$ is a $\lambda$－negation，then the function $\neg$ in a fuzzy subset lattice $L$ is a de Morgan negation（Definition B． 3 page 35）and thus the de Morgan properties hold in $L$（Theorem B． 14 page 37）．The standard negation（Definition 1.19 page 9）is a $\lambda$－negation（at $\lambda=0$ ）and so the standard negation is also de Morgan．

Theorem 1.22 Let $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ be $a$ LATTICE WITH NEGATION（Definition B． 5 page 35）．
$\left\{\begin{array}{l}\neg m 0(x) \text { is } a \lambda \text {－NEGATION } \\ \text {（Definition } 1.7 \text { page 6）}\end{array}\right\} \Longrightarrow\left\{\begin{array}{l}\text { m } \\ \begin{array}{l}\neg \text { is } a \text { DE MORGAN NEGATION } \text { on } L \\ \text {（Definition B．3 page 35）}\end{array}\end{array}\right\}$

[^7]Q Proof: To be a de Morgan negation, $\neg \mathrm{m}_{A}(x)$ must be antitone and involutory (Definition B. 3 page 35).

$$
\begin{aligned}
& \mathfrak{m}_{A}(x) \leq \mathfrak{m}_{B}(x) \Longrightarrow-\mathfrak{m}_{B}(x) \leq-\mathfrak{m}_{A}(x) \\
& \Longrightarrow 1-\mathfrak{m}_{B}(x) \leq 1-\mathfrak{m}_{A}(x) \\
& \Longrightarrow \frac{1-\mathfrak{m}_{B}(x)}{1+\lambda \mathfrak{m}_{B}(x)} \leq \frac{1-\mathfrak{m}_{A}(x)}{1+\lambda \mathfrak{m}_{A}(x)} \\
& \Longrightarrow \neg \mathfrak{m}_{B}(x) \leq \neg \mathfrak{m}_{A}(x) \\
& \Longrightarrow \mathbb{m} \text { is antitone } \\
& \neg \neg \mathfrak{m}_{A}(x) \triangleq \neg\left(\frac{1-\mathfrak{m}_{A}(x)}{1+\lambda m_{A}(x)}\right) \\
& \triangleq \frac{1-\frac{1-m_{A}(x)}{1+\lambda m_{A}(x)}}{1+\lambda \frac{1-m_{A}(x)}{1+\lambda m_{A}(x)}} \\
& =\frac{\left(1+\lambda \mathfrak{m}_{A}(x)\right)-\left(1-\mathfrak{m}_{A}(x)\right)}{\left(1+\lambda \mathfrak{m}_{A}(x)\right)+\lambda\left(1-\mathfrak{m}_{A}(x)\right)} \\
& =\frac{(1+\lambda) m_{A}(x)}{1+\lambda} \\
& =\mathfrak{m}_{A}(x) \\
& \Longrightarrow \neg \mathrm{m} \text { is involutory } \\
& \text { by property of real numbers } \mathbb{R} \\
& \text { by property of real numbers } \mathbb{R} \\
& \text { because } 1+\lambda m>0 \\
& \text { by definition of } \lambda \text {-negation (Definition } 1.20 \text { page 9) } \\
& \text { by definition of } \lambda \text {-negation (Definition } 1.20 \text { page } 9 \text { ) } \\
& \text { by definition of } \lambda \text {-negation (Definition } 1.20 \text { page } 9 \text { ) }
\end{aligned}
$$

Corollary 1.23 Let $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ be $a$ LATTICE WITH NEGATION (Definition B.5 page 35).


## (2)PROF:

(1) Proof for (1): by Theorem 1.22 (page 9)
(2) Proof for (2): To be a fuzzy negation, $\neg \mathfrak{m}_{A}(x)$ must be antitone, have weak double negation, and have boundary condition $\neg \mathfrak{m}_{1}(x)=\mathfrak{m}_{0}(x)$ (Definition B. 2 page 35).
(a) Proof that $\neg$ is antitone: by Theorem 1.22 (page 9).
(b) Proof that $\neg$ has weak double negation: by Theorem 1.22 (page 9), $\neg$ is involutory, which implies $\neg$ has weak double negation.
(c) Proof that $\neg \mathfrak{m}_{1}(x)=\mathfrak{m}_{0}(x)$ : by left hypothesis (B).

We can now define fuzzy subset operators ( $(\Omega, \cup, c)$ in terms of the fuzzy logic operators $(\wedge, \vee, \neg)$ like this (cross reference Remark 1.2 page 5):

| ${ }^{m 0} \varnothing$ | $\triangleq$ | 0 | (Definition A. 19 page 25) |
| :---: | :---: | :---: | :---: |
| ${ }^{100}{ }_{X}$ | $\triangleq$ | 1 | (Definition A. 19 page 25) |
| ${ }^{m_{0}}{ }_{A \cup B}$ | $\triangleq$ | $\mathrm{mon}_{A} \vee \mathrm{~mm}_{B}$ | (Section 1.3 page 7) |
| ${ }^{m}{ }_{A \cap B}$ | $\triangleq$ | $\mathrm{mon}_{A} \wedge \mathrm{~mm}_{B}$ | (Section 1.3 page 7) |
| $\mathrm{mm}_{A^{\text {c }}}$ | $\triangleq$ | $\neg \mathrm{mm}_{A}$ | (Section 1.3 page 9) |

In the case of set indicator functions, defining $(\wedge, \vee)$ is straightforward. But again here in fuzzy subset logics, it is not.

## 1．4 Key theorems

This section contains the following key theorems which under very＂reasonable＂conditions say very roughly the following about the fuzzy subset logic operator pair $(\wedge, \vee)$ ：

```
4 fuzzy operators idempotency theorem (Theorem 1.25 page 12):
    distributive \(\quad \Longrightarrow\) idempotent and conversely
    non-idempotent \(\Longrightarrow\) non-distributive
幽 fuzzy negation idempotency theorem (Theorem 1.28 page 14)
    excluded middle or non-contradiction \(\Longrightarrow\) non-idempotent \(\quad\) and conversely
    idempotent \(\Longrightarrow\) excluded middle or non-contradiction or both fails
* fuzzy min-max theorem (Theorem 1.26 page 13):
    idempotent \(\quad \Longrightarrow \quad(\wedge, \vee)=(\min , \max )\) and conversely
    \((\wedge, \vee) \neq(\min , \max ) \Longrightarrow\) non-idempotent
```

The fuzzy min－max boundary theorem（next theorem）shows that under three pairs of arguably＂reason－ able＂conditions（including pointwise evaluation），the functions $\min (m, m)$ and $\max (m, m)$ act as bounds for any possible operators（ $\wedge, \vee$ ）．

Theorem 1.24 （fuzzy min－max boundary theorem）${ }^{26}$ Let $\mathbb{M}$ be a set of membership functions（Defi－ nition 1.7 page 6）．

$$
\begin{aligned}
& \Longrightarrow \quad\{m \wedge m \leq \min (m, m) \quad \text { and } \quad \max (m, m) \leq m \vee m \quad \forall m \in \mathbb{M}\}
\end{aligned}
$$

## （2）Proof：

$$
\begin{aligned}
\max (\mathfrak{m}, \mathfrak{m}) & =\max ([\mathfrak{m} \vee 0],[0 \vee \mathfrak{m}]) & & \text { by disjunctive identity property } \\
& \leq \max (m \vee \mathfrak{m}, 0 \vee \mathfrak{m}) & & \text { by disjunctive isotone property: } 0 \leq \mathfrak{m} \Longrightarrow \mathfrak{m} \vee 0 \leq \mathfrak{m} \vee \mathfrak{m} \\
& \leq \max (m \vee \mathfrak{m}, \mathfrak{m} \vee \mathfrak{m}) & & \text { by disjunctive isotone property: } 0 \leq \mathfrak{m} \Longrightarrow 0 \vee \mathfrak{m} \leq \mathfrak{m} \vee \mathfrak{m} \\
& =m \vee \mathfrak{m} & & \text { by definition of } \max (\cdot, \cdot \cdot) \\
m \mathfrak{m} \wedge \mathfrak{m} & =\min (m \wedge \mathfrak{m}, \mathfrak{m} \wedge \mathfrak{m}) & & \text { by definition of } \min (\cdot, \cdot) \\
& \leq \min ([\mathfrak{m} \wedge 1],[\mathfrak{m} \wedge \mathfrak{m}]) & & \text { by conjunctive isotone property: } \mathfrak{m} \leq 1 \Longrightarrow \mathfrak{m} \wedge \mathfrak{m} \leq \mathfrak{m} \wedge 1 \\
& \leq \min ([\mathfrak{m} \wedge 1],[1 \wedge \mathfrak{m}]) & & \text { by conjunctive isotone property: } \mathfrak{m} \leq 1 \Longrightarrow \mathfrak{m} \wedge \mathfrak{m} \leq 1 \wedge \mathfrak{m} \\
& =\min (m, \mathfrak{m}) \quad & & \text { by conjunctive identity property }
\end{aligned}
$$

How reasonable are the＂reasonable conditions＂of Theorem 1．24？Let＇s discuss them briefly：
遬 The strength of the pointwise evalution condition is perhaps more in its simplicity than in it＇s reasonableness．In mathematics in general，functions are often mapped to other functions in blatant disregard to this property or one like it．Often such a mapping is referred to as an ＂operator＂．

[^8]＊In fuzzy logic，the identity properties are＂reasonable＂because if either the＂degree of mem－ bership＂of $x$ is $\mathrm{man}_{(x)}$ or $x$ has＂full membership＂，then arguably the＂degree of membership＂ of $x$ is $\mathfrak{m}(x)$ ．Likewise，if both the＂degree of membership＂of $x$ is $m(x)$ and $x$ has＂absolute non－membership＂，then arguably the＂degree of membership＂of $x$ is $m(x)$ ．In order theory， $x \vee 0=x$ and $x \wedge 1=x$ are true of any bounded lattice（Proposition A． 21 page 26）．Their commuted counterparts follow from a weakened form of the commutative property．Note that all lattices are commutative（Theorem A． 14 page 25）．
幽 The isotone properties are a natural requirement of fuzzy logic－if the＂degree of member－ ship＂$m(x)$ increases，then we might expect that the＂degrees of membership＂$[m \vee \sim](x)$ ，$[m \vee$ $m \mathrm{~m}](x),[m \wedge m](x)$ ，and $[m \wedge m](x)$ to also increase．In order theory，the isotone properties hold for all lattices（Proposition A． 15 page 25）．

The fuzzy operators idempotency theorem（next theorem）shows that under a handful of additional ar－ guably＂somewhat reasonable＂conditions（including the rather＂strong＂distributivity property），the functions $\wedge$ and $\vee$ are both idempotent．

Theorem 1.25 （fuzzy operators idempotency theorem）${ }^{27}$ Let $\mathbb{M}$ be a set of MEMBERSHIP functions （Definition 1.7 page 6）．


$$
\Longrightarrow\left\{\begin{array}{lllll}
1 . & \mathbb{m}=\mathbb{m} \vee \mathbb{m} & \forall m \in \mathbb{M} & \text { (DISJUNCTIVE IDEMPOTENT) } & \text { and } \\
2 . & \mathbb{W}=\mathbb{W} \wedge \mathbb{W} & \forall m \in \mathbb{M} & \text { (CONJUNCTIVE IDEMPOTENT) }
\end{array}\right\}
$$

（Proof：

$$
\begin{aligned}
\mathfrak{m} & =m \wedge 1 \\
& =m \wedge(1 \vee 1) \\
& =(m \wedge 1) \vee(m \wedge 1) \\
& =m \vee m \\
m & =m \vee 0 \\
& =m \vee(0 \wedge 0) \\
& =(m \vee 0) \wedge(m \vee 0) \\
& =m \wedge m
\end{aligned}
$$

by conjunctive identity property
by boundary condition
by conjunctive distributive property
by conjunctive identity property
by disjunctive identity property
by boundary condition
by disjunctive distributive property
by disjunctive identity property

How reasonable are the＂reasonable conditions＂of Theorem 1．25？Let＇s discuss them briefly：
料 In fuzzy logic，the boundary conditions are＂reasonable＂because if $x$ has both＂absolute non－membership＂and＂absolute non－membership＂，then arguably $x$ has＂absolute non－membership＂．

[^9]Likewise，if $x$ has either＂full membership＂or＂full membership＂，then arguably $x$ has＂full membership＂．In order theory，the boundary conditions are simply a weakened form of the idempotent property，which holds for all lattices（Theorem A． 14 page 25）．
幽 The distributive properties hold in classical logic（2－valued logic）and more generally in any Boolean logic，but not necessarily in any other form of logic（Definition C． 5 page 50）．In order theory， a comparatively small but important class of lattices are distributive．But note that in any lattice，the distributive inequalities always hold（Theorem A． 16 page 25）；and if one of the distributive properties hold，then they both hold（Theorem A． 28 page 27）．

The fuzzy min－max theorem（next theorem）shows that under the identity and isotone conditions（The－ orem 1.24 page 11）and the additional condition of weak idempotency，the only functions for $(\wedge, \vee)$ are $(\wedge, \vee)=(\min , \max ) \ldots$.

Theorem 1.26 （fuzzy min－max theorem）${ }^{28}$ Let $\mathbb{M}$ be a set of MEMBERSHIP FUNCTION $S$（Definition 1.7 page 6 ）．

（Proof：

$$
\begin{aligned}
\max (m, m) & \leq m \vee m \\
& \leq \max (m, n) \vee m \\
& \leq \max (m, n) \vee \max (m, m) \\
& \leq \max (m, n) \\
\min (m, n) & \leq \min (m, n) \wedge \min (m, n) \\
& \leq m \wedge \min (m, m) \\
& \leq m \wedge m \\
& \leq \min (m, m)
\end{aligned}
$$

by fuzzy min－max boundary theorem（Theorem 1.24 page 11）
by disjunctive isotone property：$m \leq \max (m, m)$
by disjunctive isotone property：$n \leq \max (m, m)$
by weak idempotent property
by weak idempotent property
by isotone property of $\wedge: \min (m, m) \leq m$
by isotone property of $\wedge$ ： $\min (m, m) \leq m$
by fuzzy min－max boundary theorem（Theorem 1.24 page 11）

How reasonable are the＂reasonable conditions＂of Theorem 1．26？Let＇s discuss them briefly：
＊One way to get the weak idempotent property or even the stronger idempotent property is to force（min，max）to have the boundary and distributive properties（Theorem 1.25 page 12）．However， this is arguably a kind of sledge hammer approach and is not really necessary．

[^10]Boolean and ortho fuzzy subset logics
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制 In fuzzy logic，even the stronger idempotent property is arguably＂reasonable＂because if an element $x$ both has a＂degree of membership＂$m(x)$ and a＂degree of membership＂$m(x)$ ，then arguably $x$ has a＂degree of membership＂$m(x)$ ．Likewise，if $x$ either has a＂degree of member－ ship＂$m^{m(x)}$ or a＂degree of membership＂$m(x)$ ，then arguably $x$ has a＂degree of membership＂ $m(x)$ ．In order theory，all lattices are idempotent（Theorem A． 14 page 25）．But，again，here we only require weak idempotency，not idempotency．

Corollary 1.27 （Bellman－Giertz 1973 theorem）${ }^{29}$ Let $\mathbb{M}$ be a set of MEMBERSHIP FUNCTIONS（Definition 1.7 page 6）．


QProof：This follows directly from Theorem 1.26 （page 13）．$\Leftrightarrow$
One big difficulty in fuzzy subset logic（Definition 1.11 page 6）is that under＂reasonable＂conditions，if the fuzzy subset logic is required to have either the excluded middle property or the non－contradiction property （Boolean algebras have both），then the fuzzy subset logic cannot be idempotent（next theorem）．Fur－ thermore，if a structure is not idempotent，then it is not a lattice（Theorem A． 14 page 25）．

Theorem 1.28 （fuzzy negation idempotency theorem）Let $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ be $a$ FUZZY sub－ SET LOGIC（Definition 1.11 page 6）．Let（ $\wedge, \vee$ ）be POINTWISE EVALUATED（Definition 1.12 page 7 ）．


©PROOF：

$$
\begin{aligned}
1 & =m(p) \vee \neg m(p) & & \text { by excluded middle hypothesis (A) } \\
& =m(p) \vee m(p) & & \text { by fixed point hypothesis } \\
& =m(p) & & \text { if } \vee \text { is } \text { idempotent } \\
& \Longrightarrow \neg m(p)=0 & & \text { because } \neg m(p)=\neg 1=0 \\
& \Longrightarrow \quad \operatorname{mon}(p)=0 & & \text { by fixed point hypothesis } \\
& \text { contradiction } & & \text { because } m(p)=1 \neq 0=m(p) \text { is a contradiction }
\end{aligned}
$$

$0=m(p) \wedge \neg m(p) \quad$ by non－contradiction hypothesis（B）

[^11]\[

$$
\begin{aligned}
& =\mathfrak{m}(p) \wedge \mathfrak{m}(p) \\
& =\mathfrak{m}(p) \\
& \Longrightarrow \quad \neg m(p)=1 \\
& \Longrightarrow \quad m(p)=0 \\
& \Longrightarrow \quad \text { contradiction } \\
& \Longrightarrow \quad \wedge \text { is non-idempotent }
\end{aligned}
$$
\]

by fixed point hypothesis
if $\wedge$ is idempotent
because $\neg m(p)=\neg 0=1$
by fixed point hypothesis
because $m(p)=0 \neq 1=m(p)$ is a contradiction

How reasonable are the＂reasonable conditions＂of Theorem 1．28？Let＇s discuss them briefly：
One of these＂reasonable conditions＂is that at some point $p, \neg m(p)=m(p) \in$ （ $0: 1$ ）．Because fuzzy negations are antitone，in some cases this is arguably a ＂reasonable＂assumption，especially if $m(x)$ is continuous and strictly antitone． However，be warned that it is not always the case that there is such a point $p$ in a fuzzy subset logic $(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ ．For example，under standard negation， and if the universal set is finite，then it is certainly possible that $p$ does not exist，

| $x$ | $\mathfrak{m}(x)$ | $\neg \mathfrak{m}(x)$ |
| :---: | :---: | :---: |
| $d$ | 1 | 0 |
| $c$ | $3 / 4$ | $1 / 4$ |
| $b$ | $1 / 4$ | $3 / 4$ |
| $a$ | 0 | 1 | as in the example illustrated to the right with $X \triangleq\{a, b, c, d\}$ ：

Corollary 1.29 （Dubois－Padre 1980 theorem）${ }^{30}$ Let $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ be a FUZZY SUBSET LOGIC （Definition 1.11 page 6）．Let $(\wedge, \vee)$ be POINTWISE EVALUATED（Definition 1.12 page 7 ）． If $\neg(x)$ is CONTINUOUS and STRICTLY ANTITONE then

$$
\begin{aligned}
& \text { (A). } \mathbb{M D} \vee \neg \mathbb{M}=1 \quad \forall m \in \mathbb{M} \text { (EXCLUDED MIDDLE) } \quad \Longrightarrow \quad \mathbb{O} \vee \mathbb{O} \neq \mathbb{M} \quad \text { (NON-IDEMPOTENT) } \\
& \text { (B). } \mathbb{M O} \wedge \neg \mathbb{M}=0 \quad \forall m \in \mathbb{M} \text { (NON-CONTRADICTION) } \Longrightarrow \quad \mathbb{O} \wedge \mathbb{M} \neq \mathbb{M} \text { (NON-IDEMPOTENT) }
\end{aligned}
$$

（2roof：This follows directly from Theorem 1.28 （page 14）．

## 1．5 Examples of non－ortho and non－Boolean fuzzy subset

This section presents some examples of fuzzy subset logics．They all have＂problems＂．The problem of the first example is just that is a kind of trivial fuzzy subset logic in that it is 2 －valued and equivalent to the classical subset logic．In all the other examples，the＂problem＂involves not having one or more of the following four properties：
（1）．disjunctive idempotence：$x \vee x=x$ and
（2）．conjunctive idempotence：$x \wedge x=x$ and
（3）．excluded middle：$\quad x \vee \neg x=1$ and
（4）．non－contradiction：$\quad x \wedge \neg x=0$
Actually，this is a problem only as far as not having an ortho or Boolean logic is a problem—because all ortho logics and all Boolean logics have these properties．And so if even one is missing，the logic is neither an ortho logic nor a Boolean logic．Also note that if a logic does not have both（1）and（2），then it cannot even be constructed on a lattice at all．．．and as defined in this paper，is not even a logic．

Example 1．30 Consider the structure $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ in Figure 2 page 16 （A）．
1．L is a Boolean lattice（Definition A． 41 page 30）．

[^12]Boolean and ortho fuzzy subset logics
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（A）fuzzy classical logic （see Example 1.30 page 15）
料 $\neg$ is standard negation $(\lambda=0)$
（ $\mathrm{A}, \mathrm{\vee})=(\mathrm{min}, \max )$
excluded middle holds
4．4．non－contradiction holds
＊distributive and idempotent
（trivial）Boolean fuzzy subset logic

（B）fuzzy Kleene 3－valued logic （see Example 1.31 page 16）
选 $\neg$ is standard negation $(\lambda=0)$
整 $(\wedge, \vee)=(\min , \max )$
构 excluded middle does not hold
䠳 non－contradiction does not hold
幽 distributive and idempotent
租 non－ortho fuzzy subset logic

（C）fuzzy Heyting 3－valued logic （see Example 1.32 page 17）
（4）$\neg$ is discrete negation $(\lambda \rightarrow \infty)$
为 $(\wedge, \vee)=(\min , \max )$
＊excluded middle does not hold
（44．4．non－contradiction holds
私 distributive and idempotent
（4n－non－ortho fuzzy subset logic

Figure 2：fuzzy logics（Definition C． 5 page 50）on linear lattices（Definition A． 11 page 24）$L_{2}$ and $L_{3}$

2．The function $\neg$ is an ortho negation（Definition B．3 page 35）（and hence also is a fuzzy negation Defini－ tion B． 2 page 35 ，Figure 9 page 34 ）．
3．The negation $\neg m$ of each membership function $m$（Definition 1.7 page 6）is the standard negation （Definition 1.19 page 9）．
4．$L$ together with the classical implication（Example C．4 page 46）is the classical logic（Example C． 6 page 50 ） and is also a fuzzy logic（Definition C． 5 page 50）．
5．Because the membership functions $m(x)$ equal 0 or 1 only，the fuzzy subsets are equivalent to crisp sets．
6．$L$ is linear（Definition A． 11 page 24）and therefore distributive（Theorem A． 30 page 27）；and therefore（ $\wedge, \vee$ ） are idempotent（Theorem 1.25 page 12）．
7．The excluded middle and non－contradiction properties hold in $L$ ，but $L$ is also idempotent． This does not contradict Theorem 1.28 （page 14），because $\neg$ does not satisfy the fixed point condition（there is no point $p$ such that $\neg m(p)=m(p) \in(0: 1)$ ）．

Example 1．31 Consider the structure $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ in Figure 2 page 16 （B）．
1．The function $\neg$ is a Kleene negation（Definition B． 3 page 35）（and hence a de Morgan negation），and is also a fuzzy negation（Example B． 25 page 41）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation because for exam－ ple $m_{A}(x) \triangleq 1 / 2=1-1 / 2=1-m_{A}(x) \triangleq \neg m_{A}(x)$ ．
3．$L$ is linear（Definition A． 11 page 24）and therefore distributive（Definition A． 27 page 27，Theorem A． 30 page 27）；and therefore $(\wedge, \vee$ ）are idempotent（Theorem 1.25 page 12）．
4．$L$ does not have the excluded middle property because

$$
\mathfrak{m}_{A} \vee \neg \mathfrak{m}_{A}=\mathfrak{m}_{A} \vee \mathfrak{m}_{A}=\mathfrak{m}_{A} \triangleq 1 / 2 \neq 1 .
$$

5．$L$ does not have the non－contradiction property because

$$
\mathfrak{m}_{A} \wedge \neg \mathfrak{m}_{A} \mathfrak{m}_{A} \wedge \mathfrak{m}_{A}=\mathfrak{m}_{A} \triangleq 1 / 2 \neq 0 .
$$

6．$(\wedge, \vee)=(\min , \max )($ Definition 1.15 page 7$)$ ，which together with the idempotence property agrees with Theorem 1.26 （page 13）．

（A）min－max operators（see Example 1.33 page 17）
$\checkmark \operatorname{mon}(x)$ is standard negation $(\lambda=0)$

速 distributive and idempotent
段 non－ortho fuzzy subset logic

（B）Łukasiewicz operators（see Example 1.34 page 18）

excluded middle and non－contradiction hold
幽，non－idempotent
＊not a logic

Figure 3：fuzzy logic on $\boldsymbol{M}_{2}$ lattice

7．$L$ together with the classical implication（Example C． 4 page 46）is a Kleene 3－valued logic（Example C． 7 page 51）and also a fuzzy logic（Definition C． 5 page 50）．

Example 1．32 Consider the structure $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ in Figure 2 page 16 （C）．
1．The function $\neg$ is an intuitionistic negation（Definition B．3 page 35）（and hence also a fuzzy negation Example B． 26 page 41）．
2．The negation $\neg m$ of each membership function $m$ is the discrete negation（Example B． 16 page 38 ）．
3．$L$ does not have the excluded middle property because $m_{A} \vee \neg m_{A} \neq 1$
4．$L$ does have the non－contradiction property．
5．$L$ is linear（Definition A． 11 page 24）and therefore distributive（Definition A． 27 page 27，Theorem A． 30 page 27）；and therefore（ $\wedge, \vee$ ）are idempotent（Theorem 1.25 page 12）．
6．Note that having both non－contradiction and idempotency does not conflict with Theorem 1.28 （page 14）because it does not satisfy the fixed point condition．
7．$(\wedge, \vee)=(\min , \max )($ Definition 1.15 page 7$)$ ，which together with the idempotence property agrees with （Theorem 1.26 page 13）．
8．$L$ together with the classical implication（Example C． 4 page 46）is a Heyting 3－valued logic（Example C． 10 page 52）and also a fuzzy logic（Definition C． 5 page 50）．

Example 1．33 Consider the structure $L$ illustrated in Figure 3 page 17 （A）．
1．The function $\neg$ is a Kleene negation（Definition B． 3 page 35）and also a fuzzy negation（Definition B． 2 page 35）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation（Definition 1.19 page 9 ）．
3．The $\wedge$ and $\vee$ operators are the min－max operators（Definition 1.15 page 7 ）．
4．Because $(\wedge, \vee)=(\min , \max ), L$ is a lattice（Proposition 1.16 page 7 ）．
5．Because $L$ is a lattice，$L$ is idempotent（Theorem A． 14 page 25）．Conversely，idempotence and（min，max） are in agreement with Theorem 1.26 （page 13）．
6．$L$ does not have the excluded middle property because $m_{A} \vee \neg m_{A}=m_{1} \neq 1$ ．
7．$L$ does not have the non－contradiction property because $\mathfrak{m}_{A} \wedge \neg m_{A}=m_{0} \neq 0$ ．
8．The idempotence property is not in disagreement with Theorem 1.28 （page 14）because $L$ does not have the excluded middle or non－contradiction properties．


Figure 4：fuzzy logic on $O_{6}$ lattice

9．$L$ together with any of the six implication functions listed in Example C． 4 （page 46）is a fuzzy subset logic（Definition 1.11 page 6）．

Example 1．34 Consider the structure $L$ illustrated in Figure 3 page 17 （B）．
1．The function $\neg$ is an ortho negation（Definition B．3 page 35）（and thus also a fuzzy negation）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation（Definition 1.19 page 9 ）．
3．The $\wedge$ and $\vee$ operators are the Łukasiewicz operators（Definition 1.18 page 9 ）．Under these operators， $L$ has the non－contradiction and excluded middle properties，but $L$ is not idempotent（e．g． $m_{A} \vee m_{A} \neq m_{A}$ ），and so $L$ is not a lattice（Theorem 1.28 page 14，Theorem A． 14 page 25）．

Example 1．35 Consider the structure $L$ illustrated in Figure 4 page 18 （A）．
1．The function $\neg$ is an ortho negation（Definition B．3 page 35）（and thus also a fuzzy negation）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation（Definition 1.19 page 9）．
3．The $\wedge$ and $\vee$ operators are the min－max operators（Definition 1.15 page 7 ）．
4．Because $(\wedge, \vee)=(\min , \max ), L$ is a lattice（Proposition 1.16 page 7 ）．
5．Because $L$ is a lattice，$L$ is idempotent（Theorem A． 14 page 25）．Conversely，idempotence and（min，max） are in agreement with Theorem 1.26 （page 13）．
6．L does not have the excluded middle property because $m_{A} \vee \neg m_{A}=m_{1} \neq 1$ ．
7．$L$ does not have the non－contradiction property because $\mathfrak{m}_{A} \wedge \neg m_{A}=m_{0} \neq 0$ ．
8．$L$ together with any of the six implication functions listed in Example C． 4 （page 46）is a fuzzy subset logic（Definition 1.11 page 6 ）．

Example 1．36 Consider the structure $L$ illustrated in Figure 4 page 18 （B）．
1．The function $\neg$ is an ortho negation（Definition B .3 page 35）（and thus also a fuzzy negation）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation（Definition 1.19 page 9 ）．

(440 min-max operators (see Example 1.37 page 19)
納 $\neg \mathrm{mo}(x)$ is standard negation ( $\lambda=0)$

* exluded middle and non-contradiction do not hold
* distributive and idempotent
( ${ }^{(4)}$ non-ortho fuzzy subset logic
Figure 5: fuzzy logic on lattice with $\boldsymbol{L}_{2}^{3}$ sublattice

3. The $\wedge$ and $\vee$ operators are the Łukasiewicz operators (Definition 1.18 page 9). Under these operators, $L$ has the non-contradiction and excluded middle properties, but $L$ is not idempotent, and so $L$ is not a lattice (Theorem 1.28 page 14).

Example 1.37 Consider the structure $L$ illustrated in Figure 5 (page 19).

1. The function $\neg$ is an ortho negation (Definition B .3 page 35) (and thus also a fuzzy negation).
2. The negation $\neg m$ of each membership function $m$ is the standard negation (Definition 1.19 page 9).
3. The $\wedge$ and $\vee$ operators are the min-max operators (Definition 1.15 page 7 ).
4. Because $(\wedge, \vee)=(\min , \max ), L$ is a lattice (Proposition 1.16 page 7 ).
5. Because $L$ is a lattice, $L$ is idempotent (Theorem A. 14 page 25). Conversely, idempotence and (min, max) are in agreement with Theorem 1.26 (page 13).
6. $L$ does not have the excluded middle property because for example

$$
\mathfrak{m}_{A} \vee \neg \mathfrak{m}_{A}=m_{A} \vee m_{R}=m_{K} \neq 1 .
$$

7. $L$ does not have the non-contradiction property because for example

$$
\mathfrak{m}_{A} \wedge \neg \mathfrak{m}_{A}=\mathfrak{m}_{A} \wedge \mathfrak{m}_{R}=\mathfrak{m}_{G} \neq 0 .
$$

8. $L$ does not contain $M_{3}$ or $N_{5}$ and so is distibutive (Theorem A. 30 page 27). (also cross reference Theorem 1.25 page 12 and Theorem 1.28 page 14).
9. $L$ is non-Boolean, but has an $L_{2}^{3}$ Boolean sublattice ( shaded in Figure 5).

## 2 Boolean and ortho fuzzy subset logics

The Introduction described the problem of constructing Boolean fuzzy subet logics and more generally ortho fuzzy subet logics. It also briefly described a "solution". This section presents this solution in more detail.

Simply put, a solution is available if we are willing to give up the pointwise evaluation condition (Definition 1.12 page 7 ). In particular, we can proceed as follows:
(1) We give up the pointwise evaluation condition.
(2) We define the ordering relation (Definition A.1 page 22) $\leq$ in the fuzzy subset logic $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ to be the pointwise ordering relation (Definition A.7 page 23).
(3) In a lattice (Definition A. 11 page 24), the definitions of the ordering relation $\leq$ and operators $(\wedge, \vee)$ are not independent-the ordering relation defines the operators (Definition A. 9 page 24, Definition A.8 page 24) and the operators define the ordering relation (Proposition A. 10 page 24).
(4) Traditionally in fuzzy logic literature, we first define a pointwise evaluated (Definition 1.12 page 7) pair of operators $(\wedge, \vee)$, and then define the ordering relation $\leq$ in terms of $(\wedge, \vee)$. For example, if $(\wedge, \vee)=(\min , \max )$, then

$$
\begin{array}{lll}
x \leq y & \stackrel{\text { def }}{\Longleftrightarrow} & \max (x, y)=y \\
x \leq y & \stackrel{\text { def }}{\Longleftrightarrow} & \min (x, y)=x
\end{array}
$$

(5) However, here we take a kind of converse approach: We first define a pointwise ordering relation $\leq$ (Definition A. 7 page 23), and then define the operators $(\wedge, \vee)$ in terms of $\leq$. In doing so, $(\wedge, \vee)$ may possibly no longer satisfy the pointwise evaluation condition.
(6) By carefully constructing a set of membershipfunctions (Definition 1.7 page 6) $\mathbb{M}$, we can construct fuzzy subset logics (Definition 1.11 page 6) on Boolean and other types of lattice structures.
(7) A fuzzy subset logic then inherits the properties of the lattice it is constructed on. So, for example, if a fuzzy subset logic is constructed on a Boolean lattice, then that fuzzy subset logic is also Boolean with all the properties of a Boolean algebra (Theorem A. 42 page 30) including the noncontradiction, excluded middle, idempotent, and distributive properties.
(8) Despite Theorem 1.26 page 13 and Theorem 1.28, this is all possible because $(\wedge, \vee)$ is no longer pointwise evaluated (Definition 1.12 page 7). The result of, say, [ $m \vee \mathrm{~m}](x)$ at the point $x$ is no longer necessarily the result of the two values $m(x)$ and $\mathfrak{m}(x)$ alone, but instead $[m \vee m](x)$ at the point $x$ may be the result of entire membership functions in the structure or even the position of $m$ and $m$ in the structure.
(9) Examples follow.

Example 2.1 Consider the structure $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ with $M \triangleq\left\{\mathfrak{m}_{0}, \mathfrak{m}_{A}, \mathfrak{m}_{B}, \mathfrak{m}_{1}\right\}$ illustrated in Figure 6 page 21 (A).


Figure 6：fuzzy logic on $\boldsymbol{M}_{\mathbf{2}}$ lattice

1．The function $\neg$ is an ortho negation（Definition B．3 page 35）（and thus also a fuzzy negation）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation（Definition 1.19 page 9 ）．
3．$L$ is very similar to the structure in Example 1.34 （page 18），which fails to even be a logic．
4．However the structure of this example has a valid ordering relation $\leq$（pointwise ordering re－ lation），has valid operators（ $\wedge, \vee$ ）defined in terms of $\leq$（Definition A．9 page 24，Definition A． 8 page 24）， and is a Boolean lattice with all the accompanying Boolean properties including the non－ contradiction，excluded middle，idempotency，and distributivity．
5．In this example，the operators are no longer Łukasiewicz operators（as in Example 1．34），but some other operators（not explicitly given in terms of a function of the form given in Theo－ rem 1.26 （page 13））．
6．This Boolean lattice together with the classical implication（Example C． 4 page 46）is an ortho logic （and thus also a fuzzy subset logic—Definition 1.11 page 6）．

Example 2．2 Consider the structure $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ with $M \triangleq\left\{m_{0}, m_{A}, m_{B}, m_{P}, m_{Q}, m_{1}\right\}$ illustrated in Figure 6 page 21 （B）．
1．The function $\neg$ is an ortho negation（Definition B．3 page 35）（and thus also a fuzzy negation）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation（Definition 1.19 page 9 ）．
3．$L$ is very similar to the structure in Example 1.36 （page 18），which fails to be a logic．
4．However the structure of this example has a valid ordering relation $\leq$ ，has valid operators $(\wedge, \vee)$ defined in terms of $\leq$ ，and is an orthocomplemented lattice（Definition A． 44 page 31）with all the accompanying properties of an orthocomplemented lattice including the non－contradiction， excluded middle and idempotency properties（Theorem A． 14 page 25，Definition A． 44 page 31，Theorem A． 47 page 32）．
5．In this example，the operators are no longer Łukasiewicz operators（as in Example 1．36，but some other operators．
6．This orthocomplemented lattice together with any one of the implications given in Exam－ ple C． 4 （page 46）is an ortho logic（and thus also a fuzzy subset logic－Definition 1.11 page 6）．

Boolean and ortho fuzzy subset logics


[^13]Figure 7：fuzzy logic on Boolean lattice $L_{2}^{3}$

Example 2．3 Consider the structure $L \triangleq(\mathbb{M}, \vee, \wedge, \neg, 0,1 ; \leq)$ illustrated in Figure 7 （page 22）．
1．The function $\neg$ is an ortho negation（Definition B．3 page 35）（and thus also a fuzzy negation）．
2．The negation $\neg m$ of each membership function $m$ is the standard negation（Definition 1.19 page 9）．
3．$L$ is somewhat similar to the fuzzy subset logic of Example 1.37 （page 19），which fails to be Boolean．
4．However the structure of this example has a valid ordering relation $\leq$ ，has valid operators（ $\wedge, \vee$ ） defined in terms of $\leq$ ，and is a Boolean lattice with the accompanying Boolean properties in－ cluding the non－contradiction，excluded middle，idempotent，and distributivity properties（The－ orem A． 42 page 30）．
5．In this example，the operators are no longer min－max operators（as in Example 1．37），but some other operators．
6．This Boolean lattice together with the classical implication（Example C． 4 page 46）is an ortho logic （and thus also a fuzzy logic）．

## Appendix A Background：Order

## A． 1 Ordered sets

Definition A． $1{ }^{31}$ Let $2^{X X}$ be the set of all relations on a set $X$ ．
A relation $\leq$ is an order relation in $2^{X X}$ if

[^14]$\left.\begin{array}{llll}\text { 1．} x \leq x & \forall x \in X & \text {（reflexive）} & \text { and } \\ \text { 2．} x \leq y \text { and } y \leq z \Longrightarrow x \leq z & \forall x, y, z \in X & \text {（transitive）} & \text { and } \\ \text { 3．} x \leq y \text { and } y \leq x \Longrightarrow x=y & \forall x, y \in X & \text {（anti－symmetric）}\end{array}\right]$ preorder

The pair $(X, \leq)$ is an ordered set if $\leq$ is an order relation on a set $X$ ．If $x \leq y$ or $y \leq x$ ，then elements $x$ and $y$ are said to be comparable，denoted $x \sim y$ ．Otherwise they are incomparable，denoted $x \| y$ ．

Definition A． $2{ }^{32}$ Let $(X, \leq)$ be an ordered set．Let $2^{X X}$ be the set of all relations on $X$ ．The relations $\geq,<,>\in 2^{X X}$ are defined as follows：

$$
\begin{array}{llll}
x \geq y & \stackrel{\text { def }}{\Longleftrightarrow} y \leq x & & \forall x, y \in X \\
x<y & \stackrel{\text { def }}{\Longleftrightarrow} x \leq y \text { and } x \neq y & \forall x, y \in X \\
x>y & \stackrel{\text { def }}{\Longleftrightarrow} x \geq y \text { and } x \neq y & \forall x, y \in X
\end{array}
$$

Definition A． $3{ }^{33}$ An ordered set $(X, \leq)$（Definition A． 1 page 22）is linear，or is a linearly ordered set，if $x \leq y \quad$ or $y \leq x \quad \forall x, y \in X \quad$（comparable）．
A linearly ordered set is also called a totally ordered set，a fully ordered set，and a chain．
Definition A． $4{ }^{34} y$ covers $x$ ，denoted $x<y$ ，in the ordered set $(X, \leq)$ if
1．$x \leq y \quad(y$ is greater than $x) \quad$ and
2．$(x \leq z \leq y) \Longrightarrow \quad(z=x$ or $z=y) \quad$（there is no element between $x$ and $y)$ ．

An ordered set can be represented graphically by a Hasse diagram（next definition）．

Definition A． 5 Let $(X, \leq)$ be an ordered pair．A diagram is a Hasse diagram of $(X, \leq)$ if
1．Each element in $X$ is represented by a dot or small circle and
2．for each $x, y \in X$ ，if $x<y$ ，then $y$ appears at a higher position than $x$ and a line connects $x$ and $y$ ．

Example A． 6 Here are three ways of representing the ordered set $\left(2^{\{x, y\}}, \subseteq\right)$ ；
（1）Hasse diagram：

（2）Sets of ordered pairs specifying order relations：

$$
\subseteq=\left\{\begin{array}{ccc}
(\varnothing, \varnothing), & (\{x\},\{x\}), & (\{y\},\{y\}), \\
(\varnothing,\{x\}), & (\varnothing,\{y\}), & (\varnothing,\{x, y\}),
\end{array}(\{x\},\{x, y\}),(\{y\},\{x, y\}), \quad\left\{\begin{array}{c}
(x, y\})
\end{array}\right\}\right.
$$

（3）Sets of ordered pairs specifying covering relations：

$$
\prec=\{(\varnothing,\{x\}), \quad(\varnothing,\{y\}), \quad(\{x\},\{x, y\}),(\{y\},\{x, y\})\}
$$

Definition A． 7 Let $Y^{X}$ be the set of all functions that map from a set $X$ to a set $Y$ ．Let $(Y,<)$ be an ordered set．The relation $\leq$ is a pointwise ordering relation on $Y^{X}$ with respect to $<$ if for all $\mathrm{f}, \mathrm{g} \in Y^{X}$ $\mathrm{f} \leq \mathrm{g} \quad \Longrightarrow \quad\{\mathrm{f}(x)<\mathrm{g}(x) \quad \forall x \in X\}$

[^15]Definition A． 8 Let $(X, \leq)$ be an ordered set and $2^{X}$ the power set of $X$ ．
For any set $A \in 2^{X}, c$ is an upper bound of $A$ in $(X, \leq)$ if
1．$x \in A \Longrightarrow x \leq c$ ．
An element $b$ is the least upper bound，or l．u．b．，of $A$ in $(X, \leq)$ if
2．$b$ and $c$ are upper bounds of $A \Longrightarrow b \leq c$ ．
The least upper bound of the set $A$ is denoted $\bigvee A$ ．It is also called the supremum of $A$ ，which is denoted $\sup A$ ．The join $x \vee y$ of $x$ and $y$ is defined as $x \vee y \triangleq \bigvee\{x, y\}$ ．

Definition A． 9 Let $(X, \leq)$ be an ordered set and $2^{X}$ the power set of $X$ ．For any set $A \in 2^{X}, p$ is a lower bound of $A$ in $(X, \leq)$ if

1．$p \leq x \quad \forall x \in A$ ．
An element $a$ is the greatest lower bound，or $\mathbf{g l b}$ ，of $A$ in $(X, \leq)$ if
2．$a$ and $p$ are lower bounds of $A \Longrightarrow p \leq a$ ．
The greatest lower bound of the set $A$ is denoted $\Lambda A$ ．It is also called the infimum of $A$ ，which is denoted inf $A$ ．The meet $x \wedge y$ of $x$ and $y$ is defined as $x \wedge y \triangleq \bigwedge\{x, y\}$ ．

## Proposition A． 10

$$
x \leq y \Longleftrightarrow\left\{\begin{array}{l}
1 . \\
x \wedge y=x \quad \text { and } \\
2 . \\
x \vee y=y
\end{array}\right\} \quad \forall x, y \in X
$$

## A． 2 Lattices

## A．2．1 General lattices

Definition A． 1135 An algebraic structure $L \triangleq(X, \vee, \wedge ; \leq)$ is a lattice if
1．$(X, \leq)$ is an ordered set $\quad((X, \leq)$ is a partially or totally ordered set）and
2．$x, y \in X \quad \Longrightarrow \quad \exists(x \vee y) \in X \quad$（every pair of elements in $X$ has a least upper bound in $X$ ）and
3．$x, y \in X \quad \Longrightarrow \exists(x \wedge y) \in X \quad$（every pair of elements in $X$ has a greatest lower bound in $X$ ）．
The lattice $L$ is linear if $(X, \leq)$ is a linearly ordered set（Definition A． 3 page 23）．

Example A． $12{ }^{36}$ The ordered set $(X, \leq)$ illustrated by the Hasse diagram to the right is not a lattice because，$a$ and $b$ have no lower bound in $X$ ．


Example A． $13{ }^{37}$ The ordered set illustrated by the Hasse diagram to the right is not a lattice because，for example，while $a$ and $b$ have upper bounds $c, d$ ，and 1，still $a$ and $b$ have no least upper bound．The element 1 is not the least upper bound because $c \leq 1$ and $d \leq 1$ ．And neither $c$ nor $d$ is a least upper bound because $c \not \leq d$ and $d \not \leq c$ ；rather， $c$ and $d$ are incomparable $(a \| b)$ ．Note that if we remove either or both of the two lines crossing the center，the ordered set becomes a lattice．


[^16]Theorem A． $14^{38}(X, \vee, \wedge$ ；$\leq)$ is a LATTICE $\quad \Longleftrightarrow$

Proposition A． 15 （Monotony laws）${ }^{39} \operatorname{Let}(X, \vee, \wedge$ ；$\leq$ ）be a LATTICE．

$$
\left\{\begin{array}{l}
a \leq b \text { and } \\
x \leq y
\end{array}\right\} \quad \Longrightarrow \quad\left\{\begin{array}{l}
a \wedge x \leq b \wedge y \text { and } \\
a \vee x \leq b \vee y
\end{array}\right\} \quad \forall a, b, x, y \in X
$$

Theorem A． 16 （distributive inequalities）${ }^{40}(X, \vee, \wedge ; \leq)$ is $a$ LATTICE $\Longrightarrow$

$$
\left\{\begin{array}{rllll}
x \wedge(y \vee z) & \geq(x \wedge y) \vee(x \wedge z) & \forall x, y, z \in X & \text { (JoIN SUPER-DISTRIBUTIVE) } & \text { and } \\
x \vee(y \wedge z) & \leq(x \vee y) \wedge(x \vee z) & \forall x, y, z \in X & \text { (MEET SUB-DISTRIBUTIVE) } & \text { and } \\
(x \wedge y) \vee(x \wedge z) \vee(y \wedge z) & \leq(x \vee y) \wedge(x \vee z) \wedge(y \vee z) & \forall x, y, z \in X & \text { (MEDIAN INEQUALITY). }
\end{array}\right\}
$$

Theorem A． 17 （Modular inequality）${ }^{41}$ Let $(X, \vee, \wedge ; \leq)$ be $a$ Lattice．

$$
x \leq y \quad \Longrightarrow \quad x \vee(y \wedge z) \leq y \wedge(x \vee z)
$$

Theorem A． 14 （page 25）gives 4 necessary and sufficient pairs of properties for a structure（ $X, \vee, \wedge$ ；$\leq$ ） to be a lattice．However，these 4 pairs are actually overly sufficient（they are not independent），as demonstrated next．

Theorem A． $18{ }^{42}$

## A．2．2 Bounded lattices

Definition A． 19 Let $L \triangleq(X, \vee, \wedge$ ；$\leq$ ）be a lattice．Let $\bigvee X$ be the least upper bound of $(X, \leq)$ and let $\bigwedge X$ be the greatest lower bound of $(X, \leq) . \quad L$ is upper bounded if $(\bigvee X) \in X . L$ is lower bounded if $(\bigwedge X) \in X . L$ is bounded if $L$ is both upper and lower bounded．A bounded lattice is optionally denoted（ $X, \vee, \wedge, 0,1 ; \leq)$ ，where $0 \triangleq \wedge X$ and $1 \triangleq \bigvee X$ ．

Proposition A． 20 Let $L \triangleq(X, \vee, \wedge ; \leq)$ be $a$ Lattice．
$\{L$ is FINITE $\} \quad \Longrightarrow \quad\{L$ is bOUNDED $\}$

[^17]

Figure 8：relationships between selected lattice types
Proposition A． 21 Let $L \triangleq(X, \vee, \wedge ; \leq)$ be $a$ Lattice with $\bigvee X \triangleq 1$ and $\wedge X \triangleq 0$ ．

$$
\{\boldsymbol{L} \text { is } \text { BOUNDED }\} \quad\left\{\begin{array}{llllll}
x \vee 1 & = & \forall x \in X & \text { (upper bounded) } & \text { and } \\
x \wedge 0 & =0 & \forall x \in X & \text { (lower bounded) } & \text { and } \\
x \vee 0 & = & x & \forall x \in X & \text { (join-identity) } & \text { and } \\
x \wedge 1= & x & \forall x \in X & \text { (meet-identity) }
\end{array}\right\}
$$

## A．2．3 Modular lattices

Definition A． $22{ }^{43}$ Let $(X, \vee, \wedge ; \leq)$ be a lattice．The modularity relation $\Perp \in \mathbb{2}^{X X}$ is defined as $x @ y \stackrel{\text { def }}{\Longleftrightarrow}\left\{(x, y) \in X^{2} \mid a \leq y \quad \Longrightarrow \quad y \wedge(x \vee a)=(y \wedge x) \vee a \quad \forall a \in X\right\}$.

Modular lattices are a generalization of distributive lattices in that all distributive lattices are modular， but not all modular lattices are distributive（Example A． 33 page 28，Example A． 34 page 28）．
Definition A． $23{ }^{44}$ A lattice $(X, \vee, \wedge$ ；$\leq$ ）is modular if $\quad x @ y \quad \forall x, y \in X$ ．
Definition A． 24 （N5 lattice／pentagon）${ }^{45}$ The N5 lattice is the ordered set （ $\{0, a, b, p, 1\}, \leq$ ）with cover relation $<=\{(0, a),(a, b),(b, 1),(p, 1),(0, p)\}$.
The N5 lattice is also called the pentagon．The N5 lattice is illustrated by the Hasse
 diagram to the right．

[^18]
## Theorem A． $25{ }^{46}$ Let $L$ be $a$ LATtice（Definition A． 11 page 24）． <br> $L$ is modular $\quad \Longleftrightarrow \quad L$ does not contain the N5 lattice

Examples of modular lattices are provided in Example A． 33 （page 28）and Example A． 34 （page 28）．

## A．2．4 Distributive lattices

Definition A． $26{ }^{47}$ Let（ $X, \vee, \wedge$ ；$\leq$ ）be a lattice（Definition A．11 page 24）．
The distributivity relation（1）$\in 3^{X X X}$ and the dual distributivity relation $\mathbb{D}^{*} \in 2^{X X X}$ are defined as
（®）$\triangleq\left\{(x, y, z) \in X^{3} \mid x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z)\right\} \quad$（each $(x, y, z)$ is disjunctive distributive）．
（D）${ }^{*} \triangleq\left\{(x, y, z) \in X^{3} \mid x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)\right\} \quad$（each $(x, y, z)$ is conjunctive distributive）．
A triple $(x, y, z)$ is distributive if $(x, y, z) \in(®)$ and such a triple is alternatively denoted as $(x, y, z)(\mathbb{D}$ ．
Definition A． $27{ }^{48}$ A lattice $(X, \vee, \wedge ; \leq)$ is distributive if $(x, y, z) \in ® \quad \forall x, y, z \in X$
Not all lattices are distributive．But if a lattice $L$ does happen to be distributive－that is all triples in $L$ satisfy the distributive property－then all triples in $L$ also satisfy the dual distributive property，as well as another property called the median property．The converses also hold（next theorem）．

Theorem A． $28{ }^{49}$ Let $L \triangleq(X, \vee, \wedge$ ；$\leq$ ）be $a$ LATtice．The following statements are all equivalent：

|  | （1）．Lis DISTRIBUTIVE |  | （Definition A．27 page 27） |
| :--- | :--- | :--- | :--- |
| $\Longleftrightarrow$ | （2）．$x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z)$ | $\forall x, y, z \in X$ | （DISJUNCTIVE DISTRIBUTIVE） |
| $\Longleftrightarrow$ | （3）．$x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)$ | $\forall x, y, z \in X$ | （CONJUNCTIVE DISTRIBUTIVE） |
| $\Longleftrightarrow$ | （4）．$(x \vee y) \wedge(x \vee z) \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z) \vee(y \wedge z)$ | $\forall x, y, z \in X$ | （MEDIAN PROPERTY） |

Definition A． 29 （M3 lattice／diamond）${ }^{50}$ The M3 lattice is the ordered set （ $\{0, p, q, r, 1\}, \leq$ ）with covering relation
$<=\{(p, 1),(q, 1),(r, 1),(0, p),(0, q),(0, r)\}$ ．
The M3 lattice is also called the diamond，and is illustrated by the Hasse diagram to the right．


Theorem A． 30 （Birkhoff distributivity criterion）${ }^{51}$ Let $L \triangleq(X, \vee, \wedge$ ；$\leq$ ）be $a$ LATTICE．
$L$ is DISTRIbUTIVE $\Longleftrightarrow\left\{\begin{array}{l}L \text { does not contain } N 5 \text { as a sublattice } \\ L \text { does not contain } M 3 \text { as a sublattice }\end{array}\right.$

[^19]Boolean and ortho fuzzy subset logics

Distributive lattices are a special case of modular lattices．That is，all distributive lattices are modular， but not all modular lattices are distributive（next theorem）．An example is the M3 lattice－it is modular， but yet it is not distributive．

Theorem A． $31{ }^{52} \operatorname{Let}(X, \vee, \wedge ; \leq)$ be a lattice．

$$
(X, \vee, \wedge ; \leq) \text { is DISTRIBUTIVE } \quad \Longrightarrow \quad(X, \vee, \wedge ; \leq) \text { is MODULAR. }
$$

Proposition A． $32{ }^{53}$ Let $X_{n}$ be a finite set with order $n=\left|X_{n}\right|$ ．Let $l_{n}$ be the number of unlabeled lattices on $X_{n}, m_{n}$ the number of unlabeled modular lattices on $X_{n}$ ．and $d_{n}$ the number of unlabeled distributive lattices on $X_{n}$ ．

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $l_{n}$ | 1 | 1 | 1 | 1 | 2 | 5 | 15 | 53 | 222 | 1078 | 5994 | 37622 | 262776 | 2018305 |
| $m_{n}$ | 1 | 1 | 1 | 1 | 2 | 4 | 8 | 16 | 34 | 72 | 157 | 343 | 766 | 1718 |
| $d_{n}$ | 1 | 1 | 1 | 1 | 2 | 3 | 5 | 8 | 15 | 26 | 47 | 82 | 151 | 269 |

Example A． $33{ }^{54}$ There are a total of 5 unlabeled lattices on a five element set．Of these， 3 are dis－ tributive（Proposition A． 32 page 28，and thus also modular），one is modular but non－distributive，and one is non－distributive（and non－modular）．
distributive（and modular）$\quad$ modular non－distributive

Example A． $34{ }^{55}$ There are a total of 15 unlabeled lattices on a six element set；and of these 15，five are distributive（Proposition A． 32 page 28）．The following illustrates the 5 distributive lattices．Note that none of these lattices are complemented（none are Boolean Definition A． 41 page 30）．
distributive lattices on 6 element sets


## A． 3 Complemented lattices

## A．3．1 Definitions

Definition A． $35{ }^{56}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition A．19 page 25）．An element $x^{\prime} \in X$ is a complement of an element $x$ in $L$ if

[^20]1．$x \wedge x^{\prime}=0$（non－contradiction）and
2．$x \vee x^{\prime}=1 \quad$（excluded middle）．
An element $x^{\prime}$ in $L$ is the unique complement of $x$ in $L$ if $x^{\prime}$ is a complement of $x$ and $y^{\prime}$ is a complement of $x \Longrightarrow x^{\prime}=y^{\prime}$ ．$L$ is complemented if every element in $X$ has a complement in $X$ ．$L$ is uniquely complemented if every element in $X$ has a unique complement in $X$ ．A complemented lattice that is not uniquely complemented is multiply complemented．

Example A． 36 Here are some examples：


Example A． 37 Of the 53 unlabeled lattices on a 7 element set， 0 are uniquely complemented， 17 are multiply complemented，and 36 are non－complemented．

Theorem A． 38 （next）is a landmark theorem in mathematics．

Theorem A． $38{ }^{57}$ For every lattice $\mathbf{L}$ ，there exists a lattice $\boldsymbol{U}$ such that
1．$L \subseteq \boldsymbol{U}(\boldsymbol{L}$ is a sublattice of $\boldsymbol{U})$ and
2． $\boldsymbol{U}$ is UNIQUELY COMPLEMENTED．

Corollary A． $39{ }^{58}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice．

$$
\left\{\begin{array}{lll}
\text { 1. } L \text { L is DISTRIBUTIVE } \\
2 . & L \text { is COMPLEMENTED }
\end{array}\right\} \quad \Longleftrightarrow \quad\{L \text { is UNIQUELY COMPLEMENTED }\}
$$

Theorem A． 40 （Huntington properties）${ }^{59}$ Let $L$ be a lattice．


[^21]
## A．3．2 Boolean lattices

Definition A．41 ${ }^{60}$ A lattice（Definition A． 11 page 24） $\boldsymbol{L}$ is Boolean if
1．$L$ is bounded（Definition A． 19 page 25）and
2．L is distributive（Definition A． 27 page 27）and
3．$L$ is complemented（Definition A． 35 page 28）．
In this case，$L$ is a Boolean algebra or a Boolean lattice．In this paper，a Boolean lattice is denoted $(X, \vee, \wedge, 0,1 ; \leq)$ ，and a Boolean lattice with $2^{N}$ elements is sometimes denoted $L_{2}^{N}$ ．

Theorem A． 42 （classic 10 Boolean properties）${ }^{61}$ Let $\boldsymbol{A} \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be an algebraic structure． In the event that $\boldsymbol{A}$ is $a$ BOUNDED LATTICE（Definition A． 19 page 25），let $x^{\prime}$ represent $a$ COMPLEMENT（Definition A． 35 page 28）of an element $x$ in $\boldsymbol{A}$ ．
$\boldsymbol{A}$ is a Boolean algebra $\Leftrightarrow \quad \forall x, y, z \in X$


## Lemma A． 43

$\left.\begin{array}{l}(X, \vee, \wedge, 0,1 ; \leq) \\ \text { is } a \text { Boolean algebra }\end{array}\right\} \Longrightarrow\left\{\begin{array}{lllll}1 . & x^{\prime} \vee(x \wedge y)= & x^{\prime} \vee y \quad \forall x, y \in X \quad \text {（SAsAкі ноок）} & \text { and } \\ 2 . & x \vee\left(x^{\prime} \wedge y\right)=x \vee y \quad \forall x, y \in X & \end{array}\right.$
（Proof：

$$
\begin{array}{rlrl}
x^{\prime} \vee(x \wedge y) & =\underbrace{x^{\prime} \vee\left(x^{\prime} \wedge y\right)}_{x^{\prime}} \vee(x \wedge y) & & \text { by absorption property (Theorem A.42 page 30) } \\
& =x^{\prime} \vee\left[\left(x^{\prime} \vee x\right) \wedge y\right] & & \text { by associative and distributive properties (Theorem A.42 page 30) } \\
& =x^{\prime} \vee[1 \wedge y] & & \text { by excluded middle property (Theorem A.42 page 30) } \\
& =x^{\prime} \vee y & & \text { by definition of 1 (Definition A.8 page 24) } \\
x \vee\left(x^{\prime} \wedge y\right) & =\underbrace{x \vee(x \wedge y) \vee(x \wedge y)}_{x} & & \text { by absorption property (Theorem A.42 page 30) } \\
& =x \vee\left[\left(x \vee x^{\prime}\right) \wedge y\right] & & \\
& =x \vee[1 \wedge y] & & \text { by associative and distributive properties (Theorem A.42 page 30) } \\
& =x \vee y & & \text { by excluded middle property (Theorem A.42 page 30) } \\
\text { by definition of 1 (Definition A.8 page 24) }
\end{array}
$$

[^22]
## A．3．3 Orthocomplemented Lattices

Definition A． $44{ }^{62}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition A． 19 page 25）．An element $x^{\perp} \in X$ is an orthocomplement of an element $x \in X$ if

| 1． | $x^{\perp \perp}$ | $=x$ | $\forall x \in X$ | （involutory） | and |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 2． | $x \wedge x^{\perp}$ | $=0$ | $\forall x \in X$ | （non－contradiction） | and |
| 3． | $x \leq y$ | $\Longrightarrow y^{\perp} \leq x^{\perp}$ | $\forall x, y \in X$ | （antitone）． |  |

The lattice $L$ is orthocomplemented（ $L$ is an orthocomplemented lattice）if every element $x$ in $X$ has an orthocomplement．

Definition A． $45{ }^{63}$ The $\mathbf{O}_{6}$ lattice is the ordered set $\left(\left\{0, p, q, p^{\perp}, q^{\perp}, 1\right\}, \leq\right)$ with cover relation

$$
\ll\left\{(0, p),(0, q),\left(p, q^{\perp}\right),\left(q, p^{\perp}\right),\left(p^{\perp}, 1\right),\left(q^{\perp}, 1\right)\right\} .
$$

The $O_{6}$ lattice is illustrated by the Hasse diagram to the right．


Example A．46 ${ }^{64}$ There are a total of 10 orthocomplemented lattices with 8 elements or less．These along with some other orthocomplemented lattices are illustrated next：${ }^{65}$

Lattices that are orthocomplemented but non－orthomodular and hence also non－modular－orthocomplemented and non－Boolean：


1．$O_{6}$ lattice

5.

3.

6.

7.
4.


Lattices that are orthocomplemented and orthomodular but non－modular－orthocomplemented and hence also non－Boolean：

[^23]
8.

9.
Lattices that are orthocomplemented，orthomodular，and modular－orthocomplemented but non－Boolean：


10．$M_{4}$ lattice


11．$M_{6}$ lattice
Lattices that are orthocomplemented，orthomodular，modular－orthocomplemented and Boolean：


16．$\quad L_{2}^{4}$ lattice


14．$\quad L_{2}^{2}$ lattice


15．$\quad L_{2}^{3}$ lattice
12．$L_{1}$ lattice


17．$\quad L_{2}^{5}$ lattice

Theorem A． $47{ }^{66}$ Let $x^{\perp}$ be the orthocomplement of an element $x$ in $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ．

＊Proof：Let $x^{\perp} \triangleq \neg x$ ，where $\neg$ is an ortho negation function（Definition B．3 page 35）．Then this theorem follows

[^24]directly from Theorem B. 15 (page 37).

Corollary A. 48 Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a LATTICE (Definition A. 11 page 24). $\left\{\begin{array}{l}L \text { is orthocomplemented } \\ (\text { Definition A.44 page 31) }\end{array}\right\} \Longrightarrow\left\{\begin{array}{l}L \text { is complemented } \\ (\text { Definition A.35 page 28) }\end{array}\right\}$

Proof: This follows directly from the definition of orthocomplemented lattices (Definition A. 44 page 31) and complemented lattices (Definition A. 35 page 28).


Example A. 49 The $O_{6}$ lattice (Definition A. 45 page 31) illustrated to the left is both orthocomplemented (Definition A. 44 page 31) and multiply complemented (Definition A. 35 page 28). The lattice illustrated to the right is multiply complemented, but is non-orthocomplemented.

(1) Proof that $O_{6}$ lattice is multiply complemented: $b$ and $q$ are both complements of $p$.
(2) Proof that the right side lattice is multiply complemented: $a, p$, and $q$ are all complements of $r$.

Proposition A. $50{ }^{67}$ Let $L=(X, \vee, \wedge, 0,1 ; \leq)$ be $a$ BOUNDED LATTICE (Definition A. 19 page 25). $\left\{\begin{array}{lll}\text { 1. } & \text { L is orthocomplemented } & \text { (Definition A. } 44 \text { page 31) } \\ \text { 2. } & \text { Lis distributive } & \text { (Definition A. } 27 \text { page 27) }\end{array}\right\} \Longrightarrow\left\{\begin{array}{l}L \text { is Boolean } \\ \text { (Definition A.41 page 30) }\end{array}\right\}$


Example A. 51 The $O_{6}$ lattice (Definition A. 45 page 31) illustrated to the left is orthocomplemented (Definition A. 44 page 31) but non-join-distributive (Definition A. 27 page 27),and hence non-Boolean. The lattice illustrated to the right is orthocomplemented and distributive and hence also Boolean (Proposition A. 50 page 33).


## A.3.4 Orthomodular lattices

Definition A. $52{ }^{68}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition A. 19 page 25). $L$ is orthomodular if

1. $L$ is orthocomplemented
and
2. $x \leq y \quad \Longrightarrow \quad x \vee\left(x^{\perp} \wedge y\right)=y \quad \forall x, y \in X \quad$ (orthomodular identity)

Theorem A. $53{ }^{69}$ Let $L=(X, \vee, \wedge, 0,1 ; \leq)$ be an algebraic structure.

$$
\left\{\begin{array}{ll}
\begin{array}{ll}
L \text { is an orthomodular lattice } & \text { and } \\
\underbrace{\left(x \wedge y^{\perp}\right)^{\perp}=y \vee\left(x^{\perp} \wedge y^{\perp}\right)}_{\text {ELKAN'S LAW }}
\end{array} & \forall x, y \in x
\end{array}\right\} \quad \Longrightarrow \quad\left\{\begin{array}{l}
L \text { is a } \\
\text { Boolean algebra } \\
\text { (Definition A.41 page 30) }
\end{array}\right\}
$$

[^25]Definition A. 54 Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition A.19 page 25). $L$ is a modular orthocomplemeted lattice if

1. $L$ is orthocomplemented (Definition A. 44 page 31) and
2. $L$ is modular (Definition A. 23 page 26)

Theorem A. $55{ }^{70}$ Let $L$ be a lattice.
$\{L$ is Boolean $\} \quad \Longrightarrow \quad\{L$ is MODULAR ORTHOCOMPLEMENTED (Definition A. 54 page 34)\}
$\begin{array}{ll}\Longrightarrow & \text { \{L is ORTHOMODULAR }\end{array} \quad$ (Definition A.52 page 33) $\}$

## Appendix B Background: Negation



Figure 9: lattice of negations

## B. 1 Definitions

Definition B. $1{ }^{71}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition A. 19 page 25). A function $\neg \in X^{X}$ is a subminimal negation on $L$ if ${ }^{72}$

$$
x \leq y \quad \Longrightarrow \quad \neg y \leq \neg x \quad \forall x, y \in X \quad \text { (antitone). }
$$

[^26]Definition B． $2{ }^{73}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition A． 19 page 25）． A function $\neg \in X^{X}$ is a negation，or minimal negation，on $L$ if

1．$x \leq y \quad \Longrightarrow \quad \neg y \leq \neg x \quad \forall x, y \in X \quad$（antitone）and
2．$x \quad \leq \quad \neg \neg x \quad \forall x \in X \quad$（weak double negation）．
A minimal negation $\neg$ is an intuitionistic negation on $L$ if
3．$x \wedge \neg x=0 \quad \forall x, y \in X \quad$（non－contradiction）． A minimal negation $\neg$ is a fuzzy negation on $L$ if
4．$\neg 1=0$
（boundary condition）．

Definition B． $3{ }^{74}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition A． 19 page 25）． A minimal negation $\neg$ is a de Morgan negation on $L$ if

$$
\text { 5. } x \quad=\quad \neg \neg x \quad \forall x \in X \quad \text { (involutory). }
$$

A de Morgan negation $\neg$ is a Kleene negation on $L$ if
6．$x \wedge \neg x \leq y \vee \neg y \quad \forall x, y \in X \quad$（Kleene condition）．
A de Morgan negation $\neg$ is an ortho negation on $L$ if
7．$x \wedge \neg x=0 \quad \forall x, y \in X \quad$（non－contradiction）．
A de Morgan negation $\neg$ is an orthomodular negation on $L$ if
$\begin{array}{lllll}\text { 8．} x \wedge \neg x & =0 & \forall x, y \in X & \text {（non－contradiction）} & \text { and } \\ \text { 9．} x \leq y & \Longrightarrow x \vee\left(x^{\perp} \wedge y\right)=y & \forall x, y \in X & \text {（orthomodular）．}\end{array}$
Remark B． $4{ }^{75}$ The Kleene condition is a weakened form of the non－contradiction and excluded middle properties in the sense

$$
\underbrace{x \wedge \neg x=0}_{\text {non-contradiction }} \leq \underbrace{1=y \vee \neg y}_{\text {excluded middle }} .
$$

Definition B． 5 Let $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be a bounded lattice（Definition A． 19 page 25）with a function $\neg \in X^{X}$ ．If $\neg$ is a negation（Definition B .2 page 35），then $L$ is a lattice with negation．

## B． 2 Properties of negations

Lemma B． $6{ }^{76}$ Let $\neg \in X^{X}$ be a function on a Lattice $L \triangleq(X, \vee, \wedge ; \leq)$（Definition A．11 page 24）． $\left.\begin{array}{l}x \leq y \Longrightarrow \\ \neg y \leq \neg x\end{array}\right\} \Longrightarrow\left\{\begin{array}{llll}\neg x \vee \neg y & \leq & \neg(x \wedge y) & \forall x, y \in X\end{array}\right.$
$\neg(x \vee y) \leq$

（CONJUNCTIVE DE MORGAN INEQUALITY） and

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Lemma B． $7{ }^{77}$ Let $\neg \in X^{X}$ be a function on $a \operatorname{LATtICE} L \triangleq(X, \vee, \wedge ; \leq)$（Definition A． 11 page 24）． If $x=(\neg \neg x)$ for all $x \in X$（INVOLUTORY），then

$$
\underbrace{x \leq y \Longrightarrow \neg y \leq \neg x\}}_{\text {ANTITONE }} \Longleftrightarrow \underbrace{\left\{\begin{array}{lllll}
\neg(x \vee y) & =\neg x \wedge \neg y & \forall x, y \in X & \text { (DISUUNCTIVE DE Morgan) } \\
\neg(x \wedge y) & =\neg x \vee \neg y & \forall x, y \in X & \text { (Conjunctive de Morgan) }
\end{array}\right.}_{\text {DE MORGAN }}
$$

Lemma B． 8 Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ．
$\left\{\begin{array}{llll}1 . & x \leq \neg \neg x & \forall x \in X & \text {（weak double negation）} \\ \text { 2．} & \neg 1=0 & \text {（boundary condition）}\end{array}\right\} \Longrightarrow\{\neg 0=1 \quad$（boundary condition）$\}$
QProof：

$$
\begin{aligned}
\neg 0 & =\neg \neg 1 & & \text { by boundary condition hypothesis (2) } \\
& \geq 1 & & \text { by weak double negation hypothesis (1) } \\
& \Longrightarrow \neg 0=1 & & \text { by upper bound property (Definition A. } 19 \text { page 25) }
\end{aligned}
$$

Lemma B． 9 Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ． $\{x \wedge \neg x=0 \quad \forall x \in X$（non－Contradiction）$\} \Longrightarrow\{\neg 1=0 \quad$（boundary condition）$\}$

## ＊Proof：

$$
\begin{aligned}
0 & =1 \wedge \neg 1 & & \text { by non-contradiction hypothesis } \\
& =\neg 1 & & \text { by definition of g.u.b. } 1 \text { and } \wedge
\end{aligned}
$$

Lemma B． $10{ }^{78}$ Let $\neg \in X^{X}$ be a function on a bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ．

Theorem B． 11 Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ． $\left\{\begin{array}{l}\neg \text { is } a \\ \text { FUZZY NEGATION }\end{array}\right\} \Longrightarrow\{\neg 0=1 \quad$（boundary Condition）$\}$
＊Proof：This follows directly from Definition B． 2 （page 35）and Lemma B． 8 （page 36）．
Theorem B． 12 Let $\neg \in X^{X}$ be a function on a bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ．

$$
\left\{\begin{array}{l}
\neg \text { is an } \\
\text { INTUITIONISTIC NEGATION }
\end{array}\right\} \Longrightarrow\left\{\begin{array}{lllll}
\text { (a) } \neg 1=0 & \text { (Boundary condition) } & \text { and } \\
\text { (b) } & \neg 0 & =1 & \text { (Boundary condition) } & \text { and } \\
\text { (c) } & \neg \text { is } a & \text { FUZZY NEGATION }
\end{array}\right\}
$$

[^28](2roof:
\[

$$
\begin{aligned}
\neg \text { is an intuitionistic negation } & \Longrightarrow x \wedge \neg x=0 \\
& \Longrightarrow \neg 1=0 \\
& \Longrightarrow \neg \text { is a fuzzy negation } \\
& \Longrightarrow \neg 0=1
\end{aligned}
$$
\]

by Definition B. 2 page 35
by Lemma B. 9 page 36
by Definition B. 2 page 35
by Theorem B. 11 page 36

Theorem B. 13 Let $\neg \in X^{X}$ be a function on a BOUNDED LATTICE $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$.
$\left\{\begin{array}{l}\neg \text { is a } \\ \text { minimal } \\ \text { negation }\end{array}\right\} \Longrightarrow\left\{\begin{array}{lllll}\neg x \vee \neg y & \leq & \neg(x \wedge y) & \forall x, y \in X & \text { (conjunctive de Morgan inequality) and } \\ \neg(x \vee y) & \leq & \text { and }\end{array}\right\}$

QProof: This follows directly from Definition B. 5 (page 35) and Lemma B. 6 (page 35).
Theorem B. 14 Let $\neg \in X^{X}$ be a function on a BOUNDED LATTICE $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$. $\left.\begin{array}{l}\neg \text { is } a \\ \text { de Morgan negation }\end{array}\right\} \Longrightarrow\left\{\begin{array}{llll}\neg(x \vee y) & =\neg x \wedge \neg y & \forall x, y \in X & \text { (disjunctive de Morgan) and } \\ \neg(x \wedge y) & =\neg x \vee \neg y & \forall x, y \in X & \text { (conjunctive de Morgan) }\end{array}\right.$

QProof: This follows directly from Definition B. 5 (page 35) and Lemma B. 7 (page 36).
Theorem B. 15 Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$.

QProof:
(1) Proof for $0=\neg 1$ boundary condition: by Lemma B. 9 (page 36)
(2) Proof for boundary conditions:

$$
\begin{aligned}
1 & =\neg \neg 1 & & \text { by involutory property } \\
& =\neg 0 & & \text { by previous result }
\end{aligned}
$$

(3) Proof for de Morgan properties:
(a) By Definition B. 5 (page 35), ortho negation is involutory and antitone.
(b) Therefore by Lemma B. 7 (page 36), de Morgan properties hold.
(4) Proof for excluded middle property:

$$
\begin{aligned}
x \vee \neg x & =\neg \neg(x \vee \neg x) \\
& =\neg(x \neg \wedge[\neg \neg x]) \\
& =\neg(\neg x \wedge x) \\
& =\neg(x \wedge \neg x) \\
& =\neg 0 \\
& =1
\end{aligned}
$$

by involutory property of ortho negation (Definition B. 5 page 35)
by disjunctive de Morgan property
by involutory property of ortho negation (Definition B. 5 page 35)
by commutative property of lattices (Definition A. 11 page 24)
by non-contradiction property of ortho negation (Definition B. 5 page 35) by boundary condition (item (2) page 37) of minimal negation

Proof for Kleene condition：

$$
\begin{align*}
x \wedge \neg x & =0  \tag{5}\\
& \leq 1 \\
& =y \vee \neg y
\end{align*}
$$

by non－contradiction property（Definition B． 5 page 35）
by definition of 0 and 1
by excluded middle property（item（4）page 37）

## B． 3 Examples

Example B． 16 （discrete negation）${ }^{79}$ Let $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be a bounded lattice（Definition A． 19 page 25）with a function $\neg \in X^{X}$ ．The function $\neg x$ defined as

$$
\neg x \triangleq \begin{cases}1 & \text { for } x=0 \\ 0 & \text { otherwise }\end{cases}
$$

is an intuitionistic negation（Definition B .2 page 35，and a fuzzy negation）．
QProof：To be an intuitionistic negation，$\neg x$ must be antitone，have weak double negation，and have the non－contradiction property（Definition B． 2 page 35）．

$$
\begin{aligned}
& \left\{\begin{array}{l}
\neg \neg x=\neg 1=0 \geq 0=x \text { for } x=0 \\
\neg \neg x=\neg 0=1 \geq x=x \text { for } x \neq 0
\end{array}\right\} \Longrightarrow \neg x \text { has weak double negation } \\
& \left\{\begin{array}{l}
x \wedge \neg x=x \wedge 1=0 \wedge 0=0 \\
\text { for } x=0 \\
x \wedge \neg x=x \wedge 0=x \wedge 0=0 \quad \text { for } x \neq 0
\end{array}\right\} \Longrightarrow \neg x \text { has non-contradiction property }
\end{aligned}
$$

Example B． 17 （dual discrete negation）${ }^{80}$ Let $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be a bounded lattice（Defini－ tion A． 19 page 25）with a function $\neg \in X^{X}$ ．The function $\neg x$ defined as

$$
\neg x \triangleq \begin{cases}0 & \text { for } x=1 \\ 1 & \text { otherwise }\end{cases}
$$

is a subminimal negation（Definition B． 1 page 34）but it is not a minimal negation（Definition B． 2 page 35）（and not any other negation defined here）．

QProof：To be a subminimal negation，$\neg x$ must be antitone（Definition B． 1 page 34）．To be a minimal negation，$\neg x$ must be antitone and have weak double negation（Definition B． 2 page 35）．

$$
\begin{aligned}
& \left\{\begin{array}{rll}
\neg y \leq \neg x & \Longleftrightarrow 0 \leq 0 & \text { for } x=y=1 \\
\neg y \leq \neg x & \Longleftrightarrow & 0 \leq 1
\end{array} \text { for } x \leq y=1\right. \\
& \left\{\begin{array}{l}
\neg \neg x=\neg 0=1 \geq x \text { for } x=1 \\
\neg \neg x=\neg 1=0 \leq x \text { for } x \neq 1
\end{array}\right\} \Longrightarrow \neg x \text { does not have weak double negation }
\end{aligned}
$$

[^29]
## Example B. 18

The function $\neg$ illustrated to the right is an ortho negation (Definition B. 3 page 35).

$$
\begin{aligned}
& \text { O1 }=\neg 0 \\
& \mathbf{O}_{0}=\neg 1
\end{aligned}
$$

(2roof:
(1) Proof that $\neg$ is antitone:

$$
0 \leq 1 \Longrightarrow \neg 1=0 \leq x=\neg 0 \Longrightarrow \neg \text { is antitone over }(0,1)
$$

(2) Proof that $\neg$ is involutory: $1=\neg 0=\neg \neg 1$
(3) Proof that $\neg$ has the non-contradiction property: $1 \wedge \neg 1=1 \wedge 0=0$

$$
0 \wedge \neg 0=0 \wedge 1=0
$$

## Example B. 19

The functions $\neg$ illustrated to the right are not any negation defined here. In particular, none of them is antitone.

$$
\begin{array}{ccc}
\mathrm{O}_{1}=\neg 1 & \text { O} 1=\neg 0 & \text { O } 1=\neg a \\
\mathrm{O}_{2}=\neg a & \text { O} a=\neg 1 & \text { O } a=\neg 0 \\
\mathrm{O}_{0}=\neg 0 & \text { O}=\neg a & \text { O } 0=\neg 1 \\
\text { (a) } & \text { (b) } & \text { (c) }
\end{array}
$$

2Proof:

1. Proof that (a) is not antitone: $a \leq 1 \Longrightarrow \neg 1=1 \not \leq a=\neg a$
2. Proof that (b) is not antitone: $a \leq 1 \Longrightarrow \neg 1=a \not \leq 0=\neg a$
3. Proof that (c) is not antitone: $0 \leq a \Longrightarrow \neg a=1 \not \leq a=\neg 0$

Example B. 20 The function $\neg$ as illustrated to the right is not a subminimal negation (it is not antitone) and so is not any negation defined here. Note however that the problem is not the $O_{6}$ lattice-it is possible to define a negation on an $O_{6}$ lattice (Example B. 31 page 43).

$\otimes$ Proof: Proof that $\neg$ is not antitone: $a \leq c \Longrightarrow \neg c=d \not \leq b=\neg a$
$\Leftrightarrow$
Remark B. 21 The concept of a complement (Definition A. 35 page 28) and the concept of a negation are fundamentally different. A complement is a relation on a lattice $L$ and a negation is a function. In Example B. 20 (page 39), $b$ and $d$ are both complements of $a$ (and so the lattice is multiply complemented), but yet $\neg$ is not a negation. In the right side lattice of Example B. 31 (page 43), both $b$ and $d$ are complements of $a$, but yet only $d$ is equal to the negation of $a(d=\neg a)$. It can also be said that complementation is a property of a lattice, whereas negation is a function defined on a lattice.

Remark B. 22 If a lattice is complemented, then by definition each element $x$ in the lattice has a complement $x^{\prime}$ such that $x \wedge x^{\prime}=0$ (non-contradiction property) and $x \vee x^{\prime}=1$ (excluded middle property). If a lattice $L$ is both distributive and complemented, then $L$ is uniquely complemented (Definition A. 41 page 30 , Theorem A. 42 page 30). If $L$ is uniquely complemented and satisfies any one of Huntington's properties ( $L$ is modular, atomic, ortho-complemented, has finite width, or de Morgan), then $L$ is distributive (Theorem A. 40 page 29).

Example B. 23 Each of the functions $\neg$ illustrated in Figure 10 (page 40) is a subminimal negation (Definition B. 1 page 34); none of them is a minimal negation (each fails to have weak double negation).

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(A)

(B)

(C)

Figure 10: subminimal negations on $L_{3}$ (Example B. 23 page 39)
$\mathrm{O} 1=\neg a=\neg 0$
O $a=\neg 1$
O 0
(A) minimal (but not fuzzy) see Example B. 24 page 40
$\mathrm{O} 1=\neg 0$
$\mathrm{O} a=\neg a$
$\mathrm{O}=\mathrm{a}=\neg 1$
(B) Kleene and fuzzy see Example B. 25 page 41

(C) intuitionistic (and fuzzy) see Example B. 26 page 41

Figure 11: negations on $L_{3}$
(Proof:
(1) Proof that $(\mathrm{A}) \neg$ is antitone:
$a \leq 1 \Longrightarrow \neg 1=0 \leq 0=\neg a \Longrightarrow \neg$ is antitone over $(a, 1)$
$0 \leq 1 \Longrightarrow \neg 1=0 \leq a=\neg 0 \Longrightarrow$ is antitone over $(0,1)$
$0 \leq a \Longrightarrow \neg a$ is antitone over $(0, a)$
(2) Proof that $(\mathrm{A}) \neg$ fails to have weak double negation:
$1 \not \leq a=\neg 0=\neg \neg 1$
(3) Proof that (B) $\neg$ is antitone:
$a \leq 1 \Longrightarrow \neg 1=a \leq a=\neg a \Longrightarrow \neg$ is antitone over $(a, 1)$
$0 \leq 1 \Longrightarrow \neg 1=a \leq a=\neg 0 \Longrightarrow \neg$ is antitone over $(0,1)$
$0 \leq a \Longrightarrow \neg a=a \leq a=\neg 0 \Longrightarrow$ is antitone over $(0, a)$
(4) Proof that (B) $\neg$ fails to have weak double negation: $1 \not \leq a=\neg a=\neg \neg 1$
(5) (C) is a special case of the dual discrete negation (Example B. 17 page 38).

Example B. 24 Consider the function $\neg$ on $L_{3}$ illustrated in Figure 11 page 40 (A):

1. $\neg$ is a minimal negation (Definition B. 2 page 35);
2. $\neg$ is not an intuitionistic negation and it is not a de Morgan negation.
*Proof:
(1) Proof that $\neg$ is antitone:
$a \leq 1 \Longrightarrow \neg 1=a \leq 1=\neg a \Longrightarrow \neg$ is antitone over $(a, 1)$
$0 \leq 1 \Longrightarrow \neg 1=a \leq 1=\neg 0 \Rightarrow \neg$ is antitone over $(0,1)$
$0 \leq a \Longrightarrow \neg a=1 \leq 1=\neg 0 \Rightarrow$ is antitone over $(0, a)$
(2) Proof that $\neg$ is a weak double negation (and so is a minimal negation, but is not a de Morgan negation):

$$
\begin{aligned}
& 1=1=\neg a=\neg \neg 1 \Longrightarrow \neg \text { is involutory at } 1 \\
& a=a=\neg 1=\neg \neg a \Longrightarrow \neg \text { is involutory at } a \\
& 0 \leq a=\neg 1=\neg \neg 0 \Longrightarrow \neg \text { is a weak double negation at } 0
\end{aligned}
$$

(3) Proof that $\neg$ does not have the non-contradiction property (and so is not an intuitionistic negation):

$$
1 \wedge \neg 1=1 \wedge a=a \neq 0
$$

(4) Proof that $\neg$ is not a fuzzy negation: $\neg 1=a \neq 0$

Example B． $25{ }^{81}$ Consider the function $\neg$ on $L_{3}$ illustrated in Figure 11 page 40 （B）．
1．$\neg$ is a Kleene negation（Definition B． 3 page 35）and is also a fuzzy negation（Definition B． 2 page 35，Example 1.31 page 16）．
2．$\neg$ is not an ortho negation and is not an intuitionistic negation（it does not have the non－ contradiction property）．
3．This negation on $L_{3}$ is used with an implication function to construct the Kleene 3－valued logic in Example C． 7 （page 51），with another implication to construct the Łukasiewicz 3－valued logic in Example C． 8 （page 52），and with yet another implication to construct the $R M_{3}$ logic in Example C． 9 （page 52）．

PRoof：
（1）Proof that $\neg$ is antitone：
$a \leq 1 \Longrightarrow \neg 1=0 \leq a=\neg a \Longrightarrow \neg$ is antitone over $(a, 1)$
$0 \leq 1 \Longrightarrow \neg 1=0 \leq 1=\neg 0 \Longrightarrow$ is antitone over $(0,1)$
$0 \leq a \Longrightarrow \neg a=a \leq 1=\neg 0 \Longrightarrow$ is antitone over $(0, a)$
（2）Proof that $\neg$ is involutory（and so is a de Morgan negation）：

$$
\begin{aligned}
& 1=\neg 0=\neg \neg 1 \Rightarrow \neg \text { is involutory at } 1 \\
& a=\neg a=\neg \neg a l \\
& 0=\neg 0=\neg \neg 0 \Longrightarrow \text { is involutory at } a \\
& 0=\neg \text { is involutory at } 0
\end{aligned}
$$

（3）Proof that $\neg$ does not have the non－contradiction property（and so is not an ortho negation）：

$$
x \wedge \neg x=x \wedge x=x \neq 0
$$

（4）Proof that $\neg$ satisfies the Kleene condition（and so is a Kleene negation）：


Example B． 26 （Heyting 3－valued logic／Jaśkowski＇s first matrix）${ }^{82}$ Consider the the function $\neg$ on $L_{3}$ illustrated in Figure 11 page 40 （C）：

1．$\neg$ is an intuitionistic negation（Definition B .2 page ${ }^{35}$ ）（and thus is also a fuzzy negation）．
2．$\neg$ is not a de Morgan negation．
3．This negation on $L_{3}$ is used with an implication function to construct the Heyting 3－valued logic in Example C． 10 （page 52）．

QProof：This is simply a special case of the discrete negation（Example B． 16 page 38）．

[^30]
(A) fuzzy negation
(Example B. 27 page 42)

(B) ortho negation
(Example B. 28 page 42)

(C) de Morgan negation
(Example B. 29 page 43)

Figure 12: negations on $\boldsymbol{M}_{2}$

Example B. 27 The function $\neg$ illustrated in Figure 12 page 42 (A) is a fuzzy negation (Definition B. 2 page 35 ). It is not an intuitionistic negation (it does not have the non-contradiction property) and it is not a de Morgan negation (it is not involutory).
© Proof: Note that

(Example B. 27 page 42)

fuzzy and intuitionistic
(Example B. 26 page 41)

Kleene negation
(Example B. 25 page 41)
(1) Proof that $\neg$ is antitone:

(2) Proof that $\neg$ has weak double negation property (and so is a minimal negation, but not a de Morgan negation):

| $1=\neg 0=\neg \neg 1$ |  | $\Longrightarrow \neg$ is involutory at 1 |
| :--- | :--- | :--- |
| $a \leq 1=\neg 0$ | $=\neg \neg a$ | $\Longrightarrow \neg$ has weak double negation at $a$ |
| $0=\neg 1=\neg \neg 0$ |  |  |
| $b=\neg b=\neg \neg b=$ | $\Longrightarrow \neg$ is involutory at 0 |  |
| $b$ | $\Longrightarrow \neg$ is involutory at $b$ |  |

(3) Proof that $\neg$ does not have the non-contradiction property (and so is not an intuitionistic negation): $b \wedge \neg b=b \wedge b=b \neq 0$
(4) Proof that $\neg$ is has boundary conditions (and so is a fuzzy negation): $\neg 1=0, \quad \neg 0=1$

Example B. $28{ }^{83}$ The function $\neg$ illustrated in Figure 12 page $42(\mathrm{~B})$ is an ortho negation (Definition B. 3 page 35).
*Proof:
(1) Proof that $\neg$ is antitone: a


[^31]（2）Proof that $\neg$ is involutory（and so is a de Morgan negation）： $1=\neg 0=\neg \neg 1$
$a=\neg a=\neg \neg a$
$$
b=\neg b=\neg \neg b
$$
$$
0=\neg 0=\neg \neg 0
$$
（3）Proof that $\neg$ is has the non－contradiction property（and so is an ortho negation）：
$1 \wedge \neg \neg=1 \wedge \wedge=0$
$a \wedge \neg a=a \wedge b=0$
$b \wedge \neg b=b \wedge a=0$
$0 \wedge \neg 0=0 \wedge 1=0$

Example B． $29\left(\mathrm{BN}_{4}\right)^{84}$ The function $\neg$ illustrated in Figure 12 page $42(\mathrm{C})$ is a de Morgan negation （Definition B .3 page 35），but it is not a Kleene negation and not an ortho negation（it does not satisfy the Kleene condition）．
＊Proof：
（1）Proof that $\neg$ is antitone：

（2）Proof that $\neg$ is involutory（and so is a de Morgan negation）： $1=\neg 0=\neg \neg 1$
$a=\neg a=\neg \neg a$
$b=\neg b=\neg \neg b$
$0=\neg 0=\neg \neg 0$
（3）Proof that $\neg$ does not have the non－contradiction property（and so is not an ortho negation）：

$$
\begin{aligned}
& a \wedge \neg a=a \wedge a=a \neq 0 \\
& b \wedge \neg b=b \wedge b=b \neq 0
\end{aligned}
$$

（4）Proof that $\neg$ does not satisfy the Kleene condition（and so is a de Morgan negation）：

$$
a \wedge \neg a=a \wedge a=a \not \leq b \wedge \neg b=b
$$

## Example B． 30



The function $\neg$ illustrated to the left is a de Morgan negation，but it is not a Kleene negation and not an or－ tho negation．The negation illustrated to the right is a Kleene negation，but it is not an ortho negation．


## Example B． 31



The function $\neg$ illustrated to the left is a de Morgan nega－
tion（Definition B． 3 page 35）；it is not a Kleene negation（it does not satisfy the Kleene condition）．The negation illustrated to the right is an ortho negation（Definition B． 3 page 35）．


[^32]
## Example B. 32



The function $\neg$ illustrated to the left is not antitone and therefore is not a negation (Definition B. 2 page 35). The function $\neg$ illustrated to the right is a Kleene negation (Definition B. 3 page 35); it is not an ortho negation (it does not have the non-

*Proof:
(1) Proof that left $\neg$ is not antitone: $a \leq c$ but $\neg c \not \subset \neg a$.
(2) Proof that right $\neg$ satisfies the Kleene condition:

$$
\begin{aligned}
& x \wedge \neg x=\left\{\begin{array}{ll}
b & \text { for } x=b \\
0 & \text { otherwise }
\end{array} \quad \forall x \in X\right. \\
& x \wedge \neg x \leq y \vee \neg y
\end{aligned} \quad \forall x, y \in X \quad \text { and } \quad y \wedge \neg y=\left\{\begin{array}{ll}
c & \text { for } y=c \\
0 & \text { otherwise }
\end{array} \quad \forall y \in X\right.
$$

(3) Proof that right $\neg$ does not have the non-contradiction property: $b \wedge \neg b=b \wedge c=b \neq 0$

## Example B. 33



The lattices illustrated to the left and right are Boolean (Definition A. 41 page 30). The function $\neg$ illustrated to the left is a Kleene negation (Definition B .3 page 35), but it is not an ortho negation (it does not have the noncontradiction property). The negation illustrated to the right is an ortho negation (Definition B. 3 page 35).


2Proof:
(1) Proof that left side negation does not have non-contradiction property (and so is not an ortho negation): $a \wedge \neg a=a \wedge d=a \neq 0$
(2) Proof that left side negation does not satisfy Kleene condition (and so is not a Kleene negation): $a \wedge \neg a=a \wedge d=a \not \leq f=c \vee f=c \vee \neg c$

## Appendix C New implication functions for non-Boolean logics

## C. 1 Implication functions

This paper deals with how to construct a fuzzy subset logic not only on a Boolean lattice, but more generally on other types of lattices as well. However, any logic (fuzzy or otherwise) is arguably not complete without the inclusion of an implication function $\rightarrow$. If we were only concerned with logics on Boolean lattices, then arguably the classical implication $x \rightarrow y \triangleq \neg x \vee y$ would suffice. However, for some non-Boolean lattices, we may do well to have other options. Two common properties of classical implication are entailment and modus ponens. However, these properties do not always support well known logic systems that are constructed on non-orthocomplented (and hence also non-Boolean) lattices. For example,
＊4 the $R M_{3}$ logic does not support the strong entailment property，
慗 the Łukasiewicz 3－valued logic does not support the strong modus ponens property，and
＊the Kleene 3－valued logic and $B N_{4}$ logic do not support either property．
This section introduces a new definition for an implication function with weakened forms of entail－ ment and modus ponens（herein called weak entailment and weak modus ponens），and that supports logics constructed on a large class of lattices including non－orthocomplemented（and non－Boolean） ones．

Definition C． 1 Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition A． 19 page 25）．The function $\rightarrow$ in $X^{X}$ is an implication on $L$ if

1．$\{x \leq y\} \Longrightarrow x \rightarrow y \geq x \vee y \quad \forall x, y \in X \quad$（weakentailment）$\quad$ and
2．$\quad x \wedge(x \rightarrow y) \leq \neg x \vee y \quad \forall x, y \in X \quad$（weak modus ponens）
Proposition C． 2 Let $\rightarrow$ be an IMPLICATION（Definition C． 1 page 45）on $a$ BOUNDED Lattice
$L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$（Definition A． 19 page 25）．
$\{x \leq y\} \quad \Longleftrightarrow \quad\{x \rightarrow y \geq x \vee y\} \quad \forall x, y \in X$
QProof：
（1）Proof for $\Longrightarrow$ case：by weak entailment property of implications（Definition C． 1 page 45）．
（2）Proof for $\Longleftarrow$ case：

$$
\begin{aligned}
y & \geq x \wedge(x \rightarrow y) & & \text { by right hypothesis } \\
& \geq x \wedge(x \vee y) & & \text { by modus ponens property of } \rightarrow \text { (Definition C. } 1 \text { page 45) } \\
& =x & & \text { by absorptive property of lattices (Definition A.11 page 24) }
\end{aligned}
$$

Remark C． 3 Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition A． 19 page 25）．In the context of ortho－ complemented lattices，a more common（and stronger）definition of implication $\rightarrow$ might be ${ }^{85}$

1．$x \leq y \Longrightarrow x \rightarrow y=1 \quad \forall x, y \in X \quad$（entailment／strong entailment）and
2．$\quad x \wedge(x \rightarrow y) \leq y \quad \forall x, y \in X \quad$（modus ponens／strong modus ponens）
This definition yields a result stronger than that of Proposition C． 2 （page 45）：
$\{x \leq y\} \quad \Longleftrightarrow \quad\{x \rightarrow y=1\} \quad \forall x, y \in X$
The Heyting 3－valued logic（Example C． 10 page 52）and Boolean 4－valued logic（Example C． 12 page 53）have both strong entailment and strong modus ponens．However，for non－Boolean logics in general，these two properties seem inappropriate to serve as a definition for implication．For example，the Kleene 3－valued logic（Example C． 7 page 51），$R M_{3}$ logic（Example C． 9 page 52），and $B N_{4}$ logic（Example C． 13 page 54）do not have the strong entailment property；and the Kleene 3－valued logic，Lukasiewicz 3－valued logic（Example C． 8 page 52），and $B N_{4}$ logic do not have the strong modus ponens property．

PProof：
（1）Proof for $\Longrightarrow$ case：by entailment property of implications（Definition C． 1 page 45）．

[^33]Boolean and ortho fuzzy subset logics
（4
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Proof for $\Longleftarrow$ case：

$$
\begin{align*}
x \rightarrow y=1 & \Longrightarrow x \wedge 1 \leq y & & \text { by modus ponens } \text { property (Definition C. } 1 \text { page 45) }  \tag{2}\\
& \Longrightarrow x \leq y & & \text { by definition of } 1 \text { (least upper bound) (Definition A.8 page 24) }
\end{align*}
$$

Example C． $4{ }^{86}$ Let $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be a lattice with negation（Definition B． 5 page 35）．If $L$ is an orthocomplemented lattice，then under Definition C．1，functions（1）－（5）below are valid implication functions with strong entailment and weak modus ponens．The relevance implication（6）in this lat－ tice is not a valid implication：It does have weak modus ponens，but it does not have weak or strong entailment．However，if $L$ is an orthomodular lattice（Definition A． 23 page 26，a special case of an orthocom－ plemented lattice），then（6）is also a valid implication function with strong entailment．

1．$x \xrightarrow{c} y \triangleq \neg x \vee y \quad \forall x, y \in X \quad$（classical implication／material implication／horseshoe）
2．$x \xrightarrow{s} y \triangleq \neg x \vee(x \wedge y) \quad \forall x, y \in X \quad$（Sasaki hook／quantum implication）
3．$x \xrightarrow{d} y \triangleq y \vee(\neg x \wedge \neg y) \quad \forall x, y \in X \quad$（Dishkantimplication）
4．$x \xrightarrow{k} y \triangleq(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee(x \wedge(\neg x \vee y)) \quad \forall x, y \in X \quad$（Kalmbach implication）
5．$x \xrightarrow{n} y \triangleq(\neg x \wedge y) \vee(x \wedge y) \vee((\neg x \vee y) \wedge \neg y) \quad \forall x, y \in X \quad$（non－tollens implication）
6．$x \xrightarrow{r} y \triangleq(\neg x \wedge y) \vee(x \wedge y) \vee(\neg x \wedge \neg y) \quad \forall x, y \in X \quad$（relevance implication）
Moreover，if $L$ is a Boolean lattice，then all of these implications are equivalent to $\xrightarrow{c}$ ，and all of them have strong entailment and strong modus ponens．
Note that $\forall x, y \in X, \quad x \xrightarrow{d} y=\neg y \xrightarrow{s} \neg x \quad$ and $\quad x \xrightarrow{h} y=\neg y \xrightarrow{k} \neg x$ ．The values for the six implications on an orthocomplemented $O_{6}$ lattice（Definition A． 45 page 31）are listed in Example C． 14 （page 54）．
（2）Poof：
（1）Proofs for the classical implication $\xrightarrow{c}$ ：
（a）Proof that on an orthocomplemented lattice，$\xrightarrow{c}$ is an implication：

$$
\begin{aligned}
x \leq y \Longrightarrow x \xrightarrow{c} y & \triangleq \neg x \vee y & & \text { by definition of } c \\
& \geq \neg y \vee y & & \text { by } x \leq y \text { and antitone property of } \neg \text { (Definition B.3 page 35) } \\
& =1 & & \text { by excluded middle property of } \neg \text { (Theorem B.15 page 37) } \\
& \Longrightarrow \text { strong entailment } & & \text { by definition of strong entailment } \\
x \wedge(\neg x \vee y) & \leq \neg x \vee y & & \text { by definition of } \wedge \text { (Definition A.9 page 24) } \\
& \Longrightarrow \text { weak modus ponens } & & \text { by definition of weak modus ponens }
\end{aligned}
$$

Note that in general for an orthocomplemented lattice，the bound cannot be tightened to strong modus ponens because，for example in the $O_{6}$ lattice（Defini－ tion A． 45 page 31）illustrated to the right
$x \wedge(\neg x \vee y)=x \wedge 1=x \not \leq y \Longrightarrow$ not strong modus ponens

（b）Proof that on a Boolean lattice，$\xrightarrow{c}$ is an implication：

$$
\begin{aligned}
x \wedge(\neg x \vee y) & =(x \wedge \neg x) \vee(x \wedge y) & & \text { by distributive property (Definition A.41 page 30) } \\
& =1 \vee(x \wedge y) & & \text { by excluded middle property of Boolean lattices } \\
& =x \wedge y & & \text { by definition of } 1 \\
& \leq y & & \text { by definition of } \wedge \text { (Definition A.9 page 24) } \\
& \Longrightarrow \text { strong modus ponens } & & \text { by definition of strong modus ponens }
\end{aligned}
$$

[^34](2) Proofs for Sasaki implication $\xrightarrow{s}$ :
(a) Proof that on an orthocomplemented lattice, $\stackrel{s}{\rightarrow}$ is an implication:
\[

$$
\begin{aligned}
x \leq y & \Longrightarrow x \stackrel{s}{s} y \\
& \triangleq \neg x \vee(x \wedge y) \\
& =\neg x \vee x \\
& =1 \\
& \Longrightarrow \text { strong entailment } \\
x \wedge(x \stackrel{s}{s} y) & \triangleq x \wedge[\neg x \vee(x \wedge y)] \\
& \leq[\neg x \vee(x \wedge y)] \\
& \leq \neg x \vee y \\
& \Longrightarrow \text { weak modus ponens }
\end{aligned}
$$
\]

by definition of $\stackrel{k}{\rightarrow}$
by $x \leq y$ hypothesis
by excluded middle prop. (Theorem B.15 page 37)
by definition of strong entailment
by definition of $s$
by definition of $\wedge$ (Definition A.9 page 24)
by definition of $\wedge$ (Definition A.9 page 24)
(b) Proof that on a Boolean lattice, $\xrightarrow[\rightarrow]{\stackrel{c}{c} \text { : }}$

$$
\begin{aligned}
x \stackrel{s}{\rightarrow} y & \triangleq \neg x \vee(x \wedge y) \\
& =\neg x \vee y \\
& =x \stackrel{\leftrightarrow}{\hookrightarrow} y
\end{aligned}
$$

by definition of $\stackrel{s}{\rightarrow}$
by Lemma A. 43 (page 30)
by definition of $\stackrel{c}{\rightarrow}$
(3) Proofs for Dishkant implication $\xrightarrow{d}$ :
(a) Proof that $x \xrightarrow{d} y \equiv \neg y \xrightarrow{s} \neg x$ :

$$
\begin{aligned}
x \xrightarrow{d} y & \triangleq y \vee(\neg x \wedge \neg y) & & \text { by definition of } \xrightarrow{d} \\
& =y \vee(\neg y \wedge \neg x) & & \text { by commutative property of lattices (Theorem A.14 page 25) } \\
& =\neg \neg y \vee(\neg y \wedge \neg x) & & \text { by involutory property of ortho negations (Definition B.3 page 35) } \\
& \triangleq \neg y \stackrel{ }{\leftrightarrows} \neg x & & \text { by definition of } \xrightarrow{s}
\end{aligned}
$$

(b) Proof that on an orthocomplemented lattice, $\xrightarrow[\rightarrow]{d}$ is an implication:

$$
\begin{aligned}
x \leq y & \Longrightarrow x \xrightarrow{d} y \\
& \triangleq y \vee(\neg x \wedge \neg y) \\
& =y \vee \neg y \\
& =1 \\
& \Longrightarrow \text { strong entailment } \\
x \xrightarrow[\rightarrow]{d}) & \triangleq y \vee(\neg x \wedge \neg y) \\
& =y \vee \neg x \\
& \Longrightarrow \text { weak modus ponens }
\end{aligned}
$$

$$
\triangleq y \vee(\neg x \wedge \neg y) \quad \text { by definition of } \xrightarrow{d}
$$

$$
=y \vee \neg y \quad \text { by } x \leq y \text { hypothesis and antitone property }
$$

$$
=1 \quad \text { by excluded middle property of ortho negation }
$$

$$
\Longrightarrow \text { strong entailment } \quad \text { by definition of strong entailment }
$$

$$
x \wedge(x \xrightarrow{d} y) \triangleq y \vee(\neg x \wedge \neg y) \quad \text { by definition of } \xrightarrow{d}
$$

$$
\text { by definition of } \wedge \text { (Definition A.9 page 24) }
$$

(c) Proof that on a Boolean lattice, $\xrightarrow[\rightarrow]{d}=\stackrel{c}{\rightarrow}$ :

$$
\begin{aligned}
x \xrightarrow{d} y & \triangleq y \vee(\neg x \wedge \neg y) & & \text { by definition of } \xrightarrow{d} \\
& =\neg x \vee y & & \text { by Lemma A.43 (page 30) } \\
& =x \xrightarrow{c} y & & \text { by definition of } \stackrel{c}{\rightarrow}
\end{aligned}
$$

(4) Proofs for the Kalmbach implication $\xrightarrow{k}$ :
(a) Proof that on an orthocomplemented lattice, $\xrightarrow{k}$ is an implication:

$$
\begin{aligned}
x \leq y & \Longrightarrow x \stackrel{k}{\rightarrow} y & & \\
& \triangleq(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee[x \wedge(\neg x \vee y)] & & \text { by definition of } \xrightarrow{k} \\
& =(\neg x \wedge y) \vee(\neg y) \vee[x \wedge(\neg x \vee y)] & & \text { by antitone proper } \\
& =(\neg x \wedge y) \vee \neg y \vee[x \wedge(1)] & & \\
& =(\neg x \wedge y) \vee(x \vee \neg y) & & \text { by definition of } 1(\mathrm{D} \\
& =\neg \neg(\neg x \wedge y) \vee(x \vee \neg y) & & \text { by involutory prop } \\
& =\neg(\neg \neg x \vee \neg y) \vee(x \vee \neg y) & & \text { by de Morgan prop } \\
& =\neg(x \vee \neg y) \vee(x \vee \neg y) & & \text { by involutory prop } \\
& =1 & & \text { by excluded middle } \\
& \Longrightarrow \text { strong entailment } & &
\end{aligned}
$$

$$
\begin{aligned}
& x \wedge(x \xrightarrow{k} y) \\
& \triangleq x \wedge[(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee[x \wedge(\neg x \vee y)]] \\
& \leq(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee[x \wedge(\neg x \vee y)] \\
& \leq(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee(\neg x \vee y) \\
& \leq y \vee(\neg x \wedge \neg y) \vee \neg x \vee y \\
& =y \vee \neg x \vee(\neg x \wedge \neg y) \\
& \leq y \vee \neg x \vee \neg x \\
& =\neg x \vee y \\
& \Longrightarrow \text { weak modus ponens }
\end{aligned}
$$

(b) Proof that on a Boolean lattice, $\xrightarrow[\rightarrow]{\boldsymbol{k}}=\stackrel{c}{\rightarrow}$ :

$$
\begin{aligned}
& x \stackrel{k}{\rightarrow} y \\
& \triangleq(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee[x \wedge(\neg x \vee y)] \\
& =(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee[(x \wedge \neg x) \vee(x \wedge y)] \\
& =(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee[(0) \vee(x \wedge y)] \\
& =(\neg x \wedge y) \vee(\neg x \wedge \neg y) \vee(x \wedge y) \\
& =\neg x \wedge(y \vee \neg y) \vee(x \wedge y) \\
& =\neg x \wedge 1 \vee(x \wedge y) \\
& =\neg x \vee(x \wedge y) \\
& =\neg x \vee y \\
& \triangleq x \neg y
\end{aligned}
$$

by definition of $\xrightarrow{\boldsymbol{k}}$
by definition of $\wedge$ (Definition A. 9 page 24)
by definition of $\wedge$ (Definition A. 9 page 24)
by definition of $\wedge$ (Definition A.9 page 24)
by idempotent p. (Theorem A. 14 page 25) by definition of $\wedge$ (Definition A.9 page 24) by idempotent p. (Theorem A. 14 page 25)
by definition of $\xrightarrow{k}$
by distributive property (Definition A. 41 page 30) by non-contradiction property
by bounded property (Definition A. 19 page 25)
by distributive property (Definition A. 41 page 30)
by excluded middle property
by definition of 1 (Definition A. 8 page 24)
by Lemma A. 43 (page 30)
by definition of $\xrightarrow{c}$
(5) Proofs for the non-tollens implication $\xrightarrow{n}$ :
(a) Proof that $x \xrightarrow{n} y \equiv \neg y \xrightarrow{k} \neg x$ :

$$
\begin{aligned}
x \xrightarrow[\rightarrow]{n} y & \triangleq(\neg x \wedge y) \vee(x \wedge y) \vee[(\neg x \vee y) \wedge \neg y] & & \text { by definition of } \xrightarrow{n} \\
& =(y \wedge \neg x) \vee(y \wedge x) \vee[\neg y \wedge(y \vee \neg x)] & & \\
& =(\neg \neg y \wedge \neg x) \vee(\neg \neg y \wedge \neg \neg x) \vee[\neg y \wedge(\neg \neg y \vee \neg x)] & & \text { by definition of } \xrightarrow{k}
\end{aligned}
$$

(b) Proof that on an orthocomplemented lattice, $\xrightarrow{n}$ is an implication:

$$
\begin{aligned}
x \leq y & \Longrightarrow x \xrightarrow{n} y & & \\
& \equiv \neg y \xrightarrow{k} \neg x & & \text { by item (5a) page } 48 \\
& =1 & & \text { by item (4a) page } 48 \\
& \Longrightarrow \text { strong entailment } & & \\
x \wedge(x \rightarrow y) & =x \wedge(\neg y \xrightarrow{n} \neg x) & & \text { by item (5a) page } 48 \\
& \leq \neg \neg y \vee \neg x & & \text { by item (4a) page } 48 \\
& =y \vee \neg x & & \text { by involutory property of } \neg \text { (Definition B.3 page 35) } \\
& =\neg x \vee y & & \text { by commutative property of lattices }
\end{aligned}
$$

(c) Proof that on a Boolean lattice, $\xrightarrow{n}=\stackrel{c}{\rightarrow}$ :

$$
\begin{aligned}
x \xrightarrow{n} y & =\neg y \stackrel{k}{\rightarrow} \neg x & & \text { by item (5a) page 48 } \\
& =\neg \neg y \vee \neg x & & \text { by item (4b) page } 48 \\
& =y \vee \neg x & & \text { by involutory property of } \neg \text { (Definition B.3 page 35) } \\
& =\neg x \vee y & & \text { by commutative property of lattices (Definition A.11 page 24) } \\
& \triangleq x \xrightarrow[\rightarrow]{c} y & & \text { by definition of } \stackrel{c}{\rightarrow}
\end{aligned}
$$

(6) Proofs for the relevance implication $\xrightarrow{\rightarrow}$ :
(a) Proof that on an orthocomplemented lattice, $\xrightarrow{r}$ does not have weak entailment: In the orthocomplemented lattice to the right...

$$
\begin{aligned}
x \leq y & \Longrightarrow x \stackrel{r}{\rightarrow} y \\
& \triangleq(\neg x \wedge y) \vee(x \wedge y) \vee(\neg x \wedge \neg y) \quad \text { by definition of } \xrightarrow{r} \\
& =0 \vee x \vee \neg y \\
& =x \vee \neg y \\
& \neq x \vee y
\end{aligned}
$$

(b) Proof that on an orthomodular lattice, $\xrightarrow{r}$ does have strong entailment:

$$
\begin{aligned}
x \leq y & \Longrightarrow x \xrightarrow{r} y & & \\
& \triangleq(\neg x \wedge y) \vee(x \wedge y) \vee(\neg x \wedge \neg y) & & \text { by definition of } \xrightarrow{r} \\
& =(\neg x \wedge y) \vee x \vee(\neg x \wedge \neg y) & & \text { by } x \leq y \text { hypothesis } \\
& =(\neg x \wedge y) \vee x \vee \neg y & & \text { by } x \leq y \text { and } \text { antitone property (Definition B.3 page 35) } \\
& =y \vee \neg y & & \text { by orthomodular identity (Definition B.3 page 35) } \\
& =1 & & \text { by excluded middle property of } \neg \text { (Theorem B. } 15 \text { page 37) }
\end{aligned}
$$

(c) Proof that on an orthocomplemented lattice, $\xrightarrow{\rightarrow}$ does have weak modus ponens:

$$
\begin{aligned}
x \wedge(x \xrightarrow{r} y) & \triangleq x \wedge[(\neg x \wedge y) \vee(x \wedge y) \vee(\neg x \wedge \neg y)] & & \text { by definition of } \xrightarrow{r} \\
& \leq[(\neg x \wedge y) \vee(x \wedge y) \vee(\neg x \wedge \neg y)] & & \text { by definition of } \wedge \text { (Definition A.9 page 24) } \\
& \leq \neg x \vee(x \wedge y) \vee(\neg x \wedge \neg y) & & \text { by definition of } \wedge \text { (Definition A.9 page 24) } \\
& \leq \neg x \vee y \vee(\neg x \wedge \neg y) & & \text { by definition of } \wedge \text { (Definition A.9 page 24) } \\
& \leq \neg x \vee y & & \text { by absorption property (Theorem A.14 page 25) } \\
& \Longrightarrow \text { weak modus ponens } & &
\end{aligned}
$$

（d）Proof that on a Boolean lattice，$\xrightarrow[\rightarrow]{\stackrel{c}{c} \text { ：}}$

$$
\begin{array}{rlrl}
x \xrightarrow{r} y & \triangleq & \triangleq(\neg x \wedge y) \vee(x \wedge y) \vee(\neg x \wedge \neg y) & \\
\text { by definition of } \xrightarrow{r} \\
& =[\neg x \wedge(y \vee \neg y)] \vee(x \wedge y) & & \text { by distributive property (Definition A.41 page 30) } \\
& =[\neg x \wedge 1] \vee(x \wedge y) & & \text { by excluded middle property of } \neg \text { (Theorem B.15 page 37) } \\
& =\neg x \vee(x \wedge y) & & \text { by definition of } 1 \text { and } \wedge \text { (Definition A.9 page 24) } \\
& =\neg x \vee y & & \text { by property of Boolean lattices (Lemma A.43 page 30) } \\
& \triangleq x \stackrel{\rightarrow}{\rightarrow} y & & \text { by definition of } \xrightarrow[\rightarrow]{c}
\end{array}
$$

## C． 2 Logics



Figure 13：lattice of logics

Definition C． $5{ }^{87}$ Let $\rightarrow$ be an implication（Definition C． 1 page 45）defined on a lattice with negation $L \triangleq$ $(X, \vee, \wedge, \neg, 0,1 ; \leq)$（Definition B． 5 page 35）．
$(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a logic if $\neg$ is a minimal negation．
$(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a fuzzy logic if $\neg$ is a fuzzy negation．
$(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is an intuitionalistic logic if $\neg$ is an intuitionalistic negation．
$(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a de Morgan logic $\quad$ if $\neg$ is a de Morgan negation．
$(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a Kleene logic if $\neg$ is a Kleene negation．
$(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is an ortho logic if $\neg$ is an ortho negation．
$(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a Boolean logic if $\neg$ is an ortho negation and $L$ is Boolean．
Example C． 6 （Aristotelian logic／classicallogic）${ }^{88}$ The classical bi－variate logic is defined below．It is a 2 element Boolean logic（Definition C． 5 page 50 ）．with $L \triangleq(\{1,0\}, \wedge, \neg, 0,1, \leq ; \vee)$ and a classical implication $\rightarrow$ with strong entailment and strong modus ponens．The value 1 represents＂true＂and 0 represents

[^35]＂false＂．
\[

\mathbf{O}_{0=\neg 1}^{1=\neg 0} \quad x \rightarrow y \triangleq\left\{$$
\begin{array}{ll}
1 & \forall x \leq y \\
y & \text { otherwise }
\end{array}
$$\right\}=\left\{$$
\begin{array}{c|cc}
\rightarrow & 1 & 0 \\
\hline 1 & 1 & 0 \\
0 & 1 & 1
\end{array}
$$ \quad \forall x, y \in X\right\}=\neg x \vee y
\]

2Proof：
（1）Proof that $\neg$ is an ortho negation：by Definition B． 3 （page 35）
（2）Proof that $\rightarrow$ is an implication with strong entailment and strong modus ponens：
（a）$L$ is Boolean and therefore is orthocomplemented．
（b）$\rightarrow$ is equivalent to the classical implication $\stackrel{c}{\rightarrow}$（Example C． 4 page 46）．
（c）By Example C． 4 （page 46），$\rightarrow$ has strong entailment and strong modus ponens．

[^36]The classical logic（previous example）can be generalized in several ways．Arguably one of the simplest of these is the 3 －valued logic due to Kleene（next example）．

Example C． $7{ }^{89}$ The Kleene 3－valued logic（ $X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow$ ）is defined below．The function $\neg$ is a Kleene negation（Definition B． 3 page 35）and is presented in Example B． 25 （page 41）．The function $\rightarrow$ is the classic implication $x \rightarrow y \triangleq \neg x \vee y$ ．The values 1 represents＂true＂， 0 represents＂false＂，and $n$ represents ＂neutral＂or＂undecided＂．

$$
\left\{\begin{array}{l}
n=\neg n \\
0=\neg 1
\end{array} \quad x \rightarrow y \triangleq\{\neg x \vee y \quad \forall x \in X \quad\}=\left\{\begin{array}{c|ccc}
\rightarrow & 1 & n & 0 \\
\hline 1 & 1 & n & 0 \\
n & 1 & n & n \\
0 & 1 & 1 & 1
\end{array} \quad \forall x, y \in X\right\}\right.
$$

QProof：
（1）Proof that $\neg$ is a Kleene negation：see Example B． 25 （page 41）
（2）Proof that $\rightarrow$ is an implication：This follows directly from the definition of $\rightarrow$ and the definition of an implication（Definition C． 1 page 45）．
（3）Proof that $\rightarrow$ does not have strong entailment：$n \rightarrow n=n=n \vee n \neq 1$ ．
（4）Proof that $\rightarrow$ does not have strong modus ponens：$n \rightarrow 0=n=\neg n \vee 0 \not \leq 0$ ．

A lattice and negation alone do not uniquely define a logic．Łukasiewicz also introduced a 3－valued logic with identical lattice structure to Kleene，but with a different implication relation（next example）． Historically，Łukasiewicz＇s logic was introduced before Kleene＇s．

[^37]
## Example C． $8{ }^{90}$

The Lukasiewicz 3－valued logic（ $X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow$ ）is defined to the right and be－ low．The function $\neg$ is a Kleene negation（Definition B． 3 page 35）and is presented in Exam－ ple B． 25 （page 41）．The implication has strong entailment but weak modus ponens．In the implication table below，values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded．


$$
x \rightarrow y \triangleq\left\{\begin{array}{ll}
1 & \forall x \leq y \\
\neg x \vee y & \text { otherwise }
\end{array}\right\}=\left\{\begin{array}{l|lll}
\rightarrow & 1 & n & 0 \\
\hline 1 & 1 & n & 0 \\
n & 1 & 1 & n \\
0 & 1 & 1 & 1
\end{array} \quad \forall x, y \in X \quad\right\}=\left\{\begin{array}{ll}
1 & \text { for } x=y=n \\
\neg x \vee y & \text { otherwise }
\end{array}\right\}
$$

＊Proof：
（1）Proof that $\neg$ is a Kleene negation：see Example B． 25 （page 41）
（2）Proof that $\rightarrow$ is an implication：This follows directly from the definition of $\rightarrow$ and the definition of an implication（Definition C． 1 page 45）．
（3）Proof that $\rightarrow$ does not have strong modus ponens：$n \rightarrow 0=n=\neg n \vee 0 \nless 0$ ．

Example C． $9{ }^{91}$ The $\mathbf{R M}_{3} \operatorname{logic}(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is defined below．The function $\neg$ is a Kleene negation（Definition B． 3 page 35）and is presented in Example B． 25 （page 41）．The implication function has weak entailment but strong modus ponens．In the implication table below，values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded ．

$$
\left\{\begin{array}{l}
1=\neg 0 \\
n=\neg n \\
0=\neg 1
\end{array}\right.
$$

$$
x \rightarrow y \triangleq\left\{\begin{array}{cc}
1 & \forall x<y \\
n & \forall x=y \\
0 & \forall x>y
\end{array}\right\}=\left\{\begin{array}{c|ccc}
\rightarrow & 1 & n & 0 \\
\hline 1 & 1 & 0 & 0 \\
n & 1 & n & 0 \\
0 & 1 & 1 & 1
\end{array} \quad \forall x, y \in X\right\}
$$

＊Proof：
（1）Proof that $\neg$ is a Kleene negation：see Example B． 25 （page 41）
（2）Proof that $\rightarrow$ is an implication：This follows directly from the definition of $\rightarrow$ and the definition of an implication（Definition C． 1 page 45）．
（3）Proof that $\rightarrow$ does not have strong entailment：$n \rightarrow n=n=n \vee n \neq 1$ ．

In a 3－valued logic，the negation does not necessarily have to be as in the previous three examples．The next example offers a different negation．

Example C． 10 （Heyting 3－valued logic／Jaśkowski＇s first matrix）${ }^{92}$
The Heyting 3－valued logic $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is defined below．The negation $\neg$ is both intuition－ istic and fuzzy（Definition B． 2 page 35），and is defined on a 3 element linearly ordered lattice（Definition A． 3 page 23）．

[^38]The implication function has both strong entailment and strong modus ponens．In the implication ta－ ble below，values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded ．

$$
\begin{aligned}
& \text { O1 } 1=\neg 0 \\
& 00=\neg n=\neg 1
\end{aligned} \quad x \rightarrow y \triangleq\left\{\begin{array}{ll}
1 & \forall x \leq y \\
y & \text { otherwise }
\end{array}\right\}=\left\{\begin{array}{c|ccc}
\rightarrow & 1 & n & 0 \\
\hline 1 & 1 & n & 0 \\
n & 1 & 1 & 0 \\
0 & 1 & 1 & 1
\end{array} \quad \forall x, y \in X\right\}
$$

PRoof：
（1）Proof that $\neg$ is a Kleene negation：see Example B． 26 （page 41）
（2）Proof that $\rightarrow$ is an implication：by definition of implication（Definition C． 1 page 45）

## $\Leftrightarrow$

Of course it is possible to generalize to more than 3 values（next example）．
Example C． $11{ }^{93}$ The Lukasiewicz 5－valued logic（ $X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow$ ）is defined below．The impli－ cation function has strong entailment but weak modus ponens．In the implication table below，values that differ from the classical $x \rightarrow y \triangleq \neg x \vee y$ are shaded ．

$$
\begin{aligned}
& \text { O} 1=\neg 0 \\
& \mathbf{O}^{p}=\neg m \\
& \text { O } \\
& n=\neg n \\
& \text { O } \\
& m=\neg p \\
& \text { O } 0=\neg 1
\end{aligned} \quad x \rightarrow y \triangleq\left\{\begin{array}{c|ccccc}
\rightarrow & 1 & p & n & m & 0 \\
\hline 1 & 1 & p & n & m & 0 \\
p & 1 & 1 & n & m & m \\
n & 1 & 1 & 1 & m & n \\
m & 1 & 1 & 1 & 1 & p \\
0 & 1 & 1 & 1 & 1 & 1
\end{array} \quad \forall x, y \in X\right.
$$

QProof：

All the previous examples in this section are linearly ordered．The following examples employ logics that are not．

Example C． $12{ }^{94}$ The Boolean 4－valued logic is defined below．The negation function $\neg$ is an ortho negation（Example B． 28 page 42）defined on an $M_{2}$ lattice．The value 1 represents＂true＂， 0 represents＂false＂， and $m$ and $n$ represent some intermediate values．


[^39]All the previous examples in this section are distributive；the previous example was Boolean．The next example is non－distributive，and de Morgan（but non－Boolean）．Note for a given order structure，the method of negation may not be unique；in the previous and following examples both have identical lattices，but are negated differently．

Example C． $13{ }^{95}$ The $\mathbf{B N}_{4}$ logic is defined below．The function $\neg$ is a de Morgan negation（Example B． 29 page 43）defined on a 4 element $M_{2}$ lattice．The value 1 represents＂true＂， 0 represents＂false＂，$b$ represents ＂both＂（both true and false），and $n$ represents＂neither＂．In the implication table below，the values that differ from those of the classical implication $\stackrel{c}{ }$ are shaded．


$$
x \rightarrow y \triangleq\left\{\begin{array}{c|cccc}
\rightarrow & 1 & n & b & 0 \\
\hline 1 & 1 & n & 0 & 0 \\
n & 1 & 1 & n & n \\
b & 1 & n & b & 0 \\
0 & 1 & 1 & 1 & 1
\end{array} \quad \forall x, y \in X\right.
$$

## Example C． 14

The tables that follow are the 6 implications defined in Example C． 4 （page 46） on the $O_{6}$ lattice with ortho negation（Definition B． 3 page 35），or the $O_{6}$ orthocom－ plemented lattice（Definition A． 45 page 31），illustrated to the right．In the tables，the values that differ from those of the classical implication $\stackrel{c}{\rightarrow}$ are shaded．

| $c$ | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| $d$ | 1 | 1 | $c$ | 1 | $a$ | $a$ |
| $c$ | 1 | $d$ | 1 | $b$ | 1 | $b$ |
| $b$ | 1 | 1 | $c$ | 1 | $c$ | $c$ |
| $a$ | 1 | $d$ | 1 | $d$ | 1 | $d$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
|  | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| 1 | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| $d$ | 1 | 1 | $a$ | 1 | $a$ | $a$ |
| $c$ | 1 | $b$ | 1 | $b$ | 1 | $b$ |
| $b$ | 1 | 1 | $c$ | 1 | $a$ | $c$ |
| $a$ | 1 | $d$ | 1 | $b$ | 1 | $d$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |


| $\xrightarrow{s}$ | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| $d$ | 1 | 1 | $a$ | 1 | $a$ | $a$ |
| $c$ | 1 | $b$ | 1 | $b$ | 1 | $b$ |
| $b$ | 1 | 1 | $c$ | 1 | $c$ | $c$ |
| $a$ | 1 | 1 | 1 | $d$ | 1 | $d$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\xrightarrow{n}$ | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| 1 | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| $d$ | 1 | 1 | $a$ | 1 | $a$ | $a$ |
| $c$ | 1 | $b$ | 1 | $b$ | 1 | $b$ |
| $b$ | 1 | 1 | $c$ | 1 | $a$ | $c$ |
| $a$ | 1 | $d$ | 1 | $b$ | 1 | $d$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |


| $\underset{\rightarrow}{d}$ | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| $d$ | 1 | 1 | $c$ | 1 | $a$ | $a$ |
| $c$ | 1 | $d$ | 1 | $b$ | 1 | $b$ |
| $b$ | 1 | 1 | $c$ | 1 | $a$ | $c$ |
| $a$ | 1 | $d$ | 1 | $b$ | 1 | $d$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\xrightarrow{r}$ | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| 1 | 1 | $d$ | $c$ | $b$ | $a$ | 0 |
| $d$ | 1 | 1 | $a$ | 1 | $a$ | $a$ |
| $c$ | 1 | $b$ | 1 | $b$ | 1 | $b$ |
| $b$ | 1 | 1 | $c$ | 1 | $a$ | $c$ |
| $a$ | 1 | $d$ | 1 | $b$ | 1 | $d$ |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |

Example C．15 ${ }^{96}$ A 6 element logic is defined below．The function $\neg$ is a Kleene negation（Example B． 32 page 44）．The implication has strong entailment but weak modus ponens．In the implication table below， the values that differ from those of the classical implication $\xrightarrow{c}$ are shaded ．

[^40]
\[

x \rightarrow y \triangleq\left\{$$
\begin{array}{c|cccccc}
\rightarrow & 1 & p & q & m & n & 0 \\
\hline 1 & 1 & p & q & m & n & 0 \\
p & 1 & 1 & q & p & q & n \\
q & 1 & p & 1 & m & p & m \\
m & 1 & 1 & q & 1 & q & q \\
n & 1 & 1 & 1 & p & 1 & p \\
0 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$ \quad \forall x, y \in X\right.
\]

PRoof:
(1) Proof that $\neg$ is a Kleene negation: see Example B. 32 (page 44)
(2) Proof that $\rightarrow$ is an implication: This follows directly from the definition of $\rightarrow$ and the definition of an implication (Definition C. 1 page 45).
(3) Proof that $\rightarrow$ does not have strong modus ponens:

$$
\begin{aligned}
& \neg p \wedge(p \rightarrow m)=n \wedge p=n \leq p=\neg p \vee m \\
& \neq m \\
& \neg n \wedge(n \rightarrow m)=n \wedge p=n \leq p=\neg p \vee m \\
& \neq m \\
& \neg p \wedge(p \rightarrow 0)=n \wedge n=n \leq n=\neg p \vee 0 \\
& \not \leq 0 \\
& \neg n \wedge(n \rightarrow 0)=p \wedge n=n \leq p=\neg n \vee 0 \\
& \not \leq=
\end{aligned}
$$

For an example of an 8 -valued logic, see 圊 [Kamide(2013)]. For examples of 16 -valued logics, see 圈 [Shramko and Wansing(2005)].

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[^0]:    ${ }^{1}$ excluded middle: $x \vee \neg x=1$. non-contradiction: $x \wedge \neg x=0$. idempotency: $x \vee x=x$ and $x \wedge x=x$. distributivity: $x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)$ and $x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z)$. classic Boolean properties: Theorem A. 42 page 30.
    ${ }^{2}$ see fuzzy operators idempotency theorem (Theorem 1.25 page 12)
    ${ }^{3}$ Dubois-Padre 1980 result: see fuzzy negation idempotency theorem (Theorem 1.28 page 14) and Dubois-Padre 1980 theorem (Corollary 1.29 page 15). Every lattice is a Boolean algebra, but not conversely (Definition A. 11 page 24, Definition A. 41 page 30). A lattice is idempotent, commutative, associative, and absorptive (Theorem A. 14 page 25). A Boolean algebra has all these properties but is moreover bounded, distributive, complemented, de Morgan, involutory, and has identity (Theorem A. 42 page 30).

[^1]:    ${ }^{4}$ Bellman－Giertz 1973 result：see fuzzy min－max theorem（Theorem 1.26 page 13）and Bellman－Giertz 1973 theorem （Corollary 1.27 page 14）．（ $\wedge, \vee$ ）in an ordered set：Definition A． 9 page 24 and Definition A． 8 page $24 .(\wedge, \vee)$ in a lattice： Definition A． 11 page $24 .(\wedge, \vee$ ）in a logic：Definition C． 5 page 50．（min，max）：Definition 1.15 page 7.
    ${ }^{5}$ pointwise evaluated：（Definition 1.12 page 7 ）
    ${ }^{6}$ logic：Definition C． 5 page 50．lattice：Definition A． 11 page 24．negation function：Definition B． 2 page 35．im－ plication function：Definition C． 1 page 45 Boolean lattice：Definition A． 41 page 30．orthocomplemented lattice： Definition A． 44 page 31．ortho negation：Definition B． 3 page 35．ortho＋distributivity＝Boolean：Proposition A． 50 page 33
    ${ }^{7}$ There are a total of 5 orthocomplemented lattices with 8 elements；of these 5 ，only 1 is Boolean．There are a total of 10 orthocomplemented lattices with 8 elements or less；of these 10，only 4 are Boolean．For further details， see Example A． 46 page 31.
    ${ }^{8}$ 暑［Alsina et al．（1980）Alsina，Trillas，and Valverde］，圈［Hamacher（1976）］〈referenced by 圊［Alsina et al．（1983）Alsina，Trillas，and Valverde］〉
    ${ }^{9}$ properties of fuzzy negations and hence also fuzzy logics：Theorem B． 11 page 36．properties of ortho negations and hence also ortho logics：Theorem B． 15 page 37．relationships between logics：Figure 13 page 50.

[^2]:    ${ }^{10}$ fuzzy subset: Definition 1.7 page 6, fuzzy subset logic: Definition 1.11 page 6, membership function: Definition 1.7 page 6, lattice: Definition A. 11 page 24, fuzzy negation: Definition B. 2 page 35, implication: Definition C. 1 page 45 and Definition C. 5 page 50; problems: Theorem 1.26 page 13 and Theorem 1.28 page 14.
    ${ }^{11} \Rightarrow$ [Feller(1971)], page 104 〈1. Baire Functions〉, $\Delta$ [Aliprantis and Burkinshaw(1998)], page 126, $\Delta$ [Hausdorff(1937)], page 22, $\Delta$ [de la Vallée-Poussin(1915)] page 440
    ${ }^{12}$ ordered set: Definition A. 1 page 22, lattice (Definition A. 11 page 24), set indicator function: Definition 1.1 page 4, topologies: Example 1.3 page 5 and Example 1.4 page 5; examples of set structures: Example 1.3 page 5 and Example 1.5 page 5.

[^3]:    ${ }^{13} \boxminus$ [Aliprantis and Burkinshaw(1998)], page 126, $\otimes$ [Hausdorff(1937)], pages 22-23
    ${ }^{14}$ 圈 [Ellerman(2010)] $\left\langle\S 1.7 ; A \Rightarrow B=\left(A^{\mathrm{C}} \cup B\right)^{\circ}\right.$ where $C^{\circ}$ is the interior of a set $C$ in a topological space $\rangle$

[^4]:    ${ }^{15}$ 日［Hájek（2011）］，page 68 〈＂absolutely true＂，＂absolutely false〉，$\Delta$［Dubois（1980）］page 10，日［Dubois et al．（2000）Dubois，Ostasiewicz，and Padre］，page 42 〈＂full membership＂，＂absolute non－membership＂〉，圈［Zadeh（1965）］ page 339 〈＂grade of membership＂〉
    ${ }^{16} \boxminus$［Dubois（1980）］page $10\langle$ Remarks 1$\rangle, \boxminus$［Kaufmann（1975）］
    ${ }^{17}$ ■［Hájek（2011）］，page 67 〈5．1 Introduction〉

[^5]:    ${ }^{18}$ see Remark 1.2 page 5，Proposition A． 10 page 24，and Example 1.3 page 5－Example 1.5 page 5
    ${ }^{19} \otimes$［Dubois（1980）］page $11\langle$ B．a（i）$\rangle$
    ${ }^{20}$（ ［Fodor and Yager（2000）］，page 133，嘖［Zadeh（1965）］pages 340－341〈（3），（5）$\rangle$ ；pointwise ordering：Defini－ tion A． 7 page 23

[^6]:    ${ }^{21}$ [ [Fodor and Yager(2000)], page 133

[^7]:    ${ }^{22}$ 自［Fodor and Yager（2000）］，page 133
    ${ }^{23}$ 䀝［Zadeh（1965）］page 340，$\theta$［Jager（1995）］page 243 〈Appendix A〉
    ${ }^{24}$－［Fodor and Yager（2000）］，page 129，白［Hájek（2011）］，page $68\langle$ Definition 5．1〉，$\otimes$［Sugeno（1977）］page 95 $\langle(23)$＂$\lambda$－complement＂，see also p．94（12），p．96（28）$\rangle$ ，$\boxminus$ JJager（1995）］page 243 〈Appendix A〉
    ${ }^{25}$ 圈［Yager（1980a）］〈cf Jager（1995）$\rangle, \square$［Jager（1995）］page 243 〈Appendix A〉

[^8]:    ${ }^{26}$ 圈［Alsina et al．（1983）Alsina，Trillas，and Valverde］page $16\langle \$ 1\rangle$

[^9]:    ${ }^{27}$ 圏［Bellman and Giertz（1973）］page $154\langle a \vee a=a \wedge a=a \ldots$（10）$\rangle$ ，誊［Alsina et al．（1983）Alsina，Trillas，and Valverde］page $15\langle x=G(x, x)\rangle$

[^10]:    ${ }^{28}$ This result is very similar to the celebrated result of Bellman and Giertz（1973）：圈［Bellman and Giertz（1973）］ pages $153-154\langle \$ 4\rangle$

[^11]:    ${ }^{29}$ 圊［Bellman and Giertz（1973）］pages 153－154〈\＄4〉

[^12]:    ${ }^{30} \square^{-1}$［Dubois and Padre（1980）］，page $62\langle\mathrm{P} 1$ ，requires $\neg(x)$ be to continuous and strictly antitone $\rangle$
    ［Fodor and Yager（2000）］，pages 130－131〈Theorem 2，reference to previous without proof〉

[^13]:    速 pointwise $\leq$ induced operators on $L_{2}^{3}$（see Example 2.3 page 22）
    然 $\neg \mathrm{mm}(x)$ is standard negation $(\lambda=0)$
    料 Boolean fuzzy subset logic

[^14]:    ${ }^{31} \boxminus$［MacLane and Birkhoff（1999）］page 470，$\Delta$［Beran（1985）］page 1，圈［Korselt（1894）］page $156\langle\mathrm{I}, \mathrm{II},(1)\rangle$ ，围［Dedekind（1900）］page 373 〈I－III〉

[^15]:    ${ }^{32}$ 圈［Peirce（1880）］page 2
    ${ }^{33} \boxminus$［MacLane and Birkhoff（1999）］page 470，圈［Ore（1935）］page 410
    ${ }^{34}$ 圈［Birkhoff（1933）］page 445

[^16]:    ${ }^{35} \Rightarrow$［MacLane and Birkhoff（1999）］page 473，$\Delta$［Birkhoff（1948）］page 16，圈［Ore（1935）］，圏［Birkhoff（1933）］ page $442, \square$［Maeda and Maeda（1970）］，page 1
    ${ }^{36} \triangleq$［Dominich（2008）］page $50\langle$ Fig． 3.5$\rangle$
    ${ }^{37} \boxminus$［Birkhoff（1967）］pages 15－16，$\ominus[O x l e y(2006)]$ page 54，$\ominus$［Dominich（2008）］page $50\langle$ Figure 3．6〉，圈 ［Farley（1997）］page 3，圈［Farley（1996）］page 5

[^17]:    $38 \boxminus$［MacLane and Birkhoff（1999）］pages 473－475 〈Lemma 1，Theorem 4〉，$\boxminus$［Burris and Sankap－ panavar（1981）］pages 4－7，$\boxminus$［Birkhoff（1938）］，pages 795－796，圈［Ore（1935）］page $409\langle(\alpha)\rangle$ ，圈［Birkhoff（1933）］ page 442，圈［Dedekind（1900）］pages 371－372〈（1）－（4）〉
    ${ }^{39} \boxminus$［Givant and Halmos（2009）］page 39，圈［Doner and Tarski（1969）］pages 97－99
    ${ }^{40} \nabla$［Davey and Priestley（2002）］page 85，$\square$［Grätzer（2003）］page 38，圁［Birkhoff（1933）］page 444，圊［Ko－ rselt（1894）］page 157，$\Delta$［Müller－Olm（1997）］page 13 〈terminology〉
    ${ }^{41} \otimes$［Birkhoff（1948）］page 19，$\Delta$［Burris and Sankappanavar（1981）］page 11，圈［Dedekind（1900）］page 374
    ${ }^{42} \triangleq$［Padmanabhan and Rudeanu（2008）］pages 7－8，$\Delta$［Beran（1985）］page 5，圈［McKenzie（1970）］page 24

[^18]:    ${ }^{43} \square$［Stern（1999）］page 11，$\theta$［Maeda and Maeda（1970）］，page 1 〈Definition（1．1）$\rangle, \square$［Maeda（1966）］ page 248
    ${ }^{44} \triangleq$［Birkhoff（1967）］page 82，$\nabla$［Maeda and Maeda（1970）］，page 3 〈Definition（1．7）〉
    ${ }^{45} \boxminus$［Beran（1985）］pages 12－13，圊［Dedekind（1900）］pages 391－392〈（44）and（45）〉

[^19]:    ${ }^{46} \otimes$［Burris and Sankappanavar（1981）］page 11，$\otimes$［Grätzer（1971）］page 70，圈［Dedekind（1900）］〈cf Stern 1999 page 10
    ${ }^{47}$［Maeda and Maeda（1970）］，page $15\langle$ Definition 4．1〉，嘖［Foulis（1962）］page 67，$\triangle$［von Neumann（1960）］， page 32 〈Definition 5．1〉，圈［Davis（1955）］page 314 〈disjunctive distributive and conjunctive distributive functions〉
    ${ }^{48} \otimes$［Burris and Sankappanavar（1981）］page 10，$\theta$［Birkhoff（1948）］page 133，圊［Ore（1935）］page 414 $\langle$ arithmetic axiom $\rangle$ ，圈［Birkhoff（1933）］page 453，$\boxminus$［Balbes and Dwinger（1975）］page 48 〈Definition II．5．1〉
    ${ }^{49}$ 圈［Dilworth（1984）］page 237，$\otimes$［Burris and Sankappanavar（1981）］page 10，䁪［Ore（1935）］page 416 〈（7），（8）， Theorem 3 $\rangle$ ，圈［Ore（1940）］〈cf Gratzer 2003 page 159〉，$\otimes$［Schröder（1890）］page $286\langle$ cf Birkhoff（1948）p．133〉，圈［Korselt（1894）］〈cf Birkhoff（1948）p．133〉
    ${ }^{50} \boxminus[B e r a n(1985)]$ pages $12-13, \boxminus[K o r s e l t(1894)]$ page $157\left\langle p_{1} \equiv x, p_{2} \equiv y, p_{3} \equiv z, g \equiv 1,0 \equiv 0\right\rangle$
    ${ }^{51} \boxminus$［Burris and Sankappanavar（1981）］page 12，$\theta$［Birkhoff（1948）］page 134，圈［Birkhoff and Hall（1934）］

[^20]:    ${ }^{52} \boxminus$［Birkhoff（1948）］page 134，$\Delta$［Burris and Sankappanavar（1981）］page 11
    ${ }^{53}$ ㅁ．ᅩ．［oei（2014）］〈http：／／oeis．org／A006966〉，모［oei（2014）］〈http：／／oeis．org／A006982〉，모․［oei（2014）］ $\langle\mathrm{http}: / /$ oeis．org／A006981 $\rangle$ ，圈［Heitzig and Reinhold（2002）］$\left\langle l_{n}\right\rangle$ ，圈［Erné et al．（2002）Erné，Heitzig，and Rein－ hold］page $17\left\langle d_{n}\right\rangle$ ，㾇［Thakare et al．（2002）Thakare，Pawar，and Waphare］

    54 圈［Erné et al．（2002）Erné，Heitzig，and Reinhold］pages 4－5
    ${ }^{55}$ 圈［Erné et al．（2002）Erné，Heitzig，and Reinhold］pages 4－5
    ${ }^{56}$－［Stern（1999）］page 9，$\boxminus$［Birkhoff（1948）］page 23

[^21]:    ${ }^{57}$ 圊［Dilworth（1945）］page 123，\＆［Salií（1988）］page 51，\＆［Grätzer（2003）］page 378 〈Corollary 3．8〉
    ${ }^{58}$［MacLane and Birkhoff（1999）］page 488，［Saliǐ（1988）］page 30 〈Theorem 10〉
    ${ }^{59}$［Roman（2008）］page 103，［Adams（1990）］page 79，［Saliir（1988）］page 40，圈［Dilworth（1945）］ page 123，［Grätzer（2007）］，page 698

[^22]:    ${ }^{60} \square$［MacLane and Birkhoff（1999）］page 488，$\Delta$［Jevons（1864）］
    ${ }^{61}$ 圈［Huntington（1904）］pages 292－293 〈＂1st set＂〉，圈［Huntington（1933）］page 280 〈＂4th set＂〉，$\boxminus$［MacLane and Birkhoff（1999）］page 488，$\boxminus$［Givant and Halmos（2009）］page 10，$\boxminus$［Müller（1909）］，pages 20－21，$\boxminus$ ［Schröder（1890）］，日［Whitehead（1898）］pages 35－37

[^23]:    ${ }^{62} \otimes[\operatorname{Stern}(1999)]$ page 11，$\otimes$［Beran（1985）］page 28，$\Delta$［Kalmbach（1983）］page 16，$\Delta$［Gudder（1988）］ page 76，$\rightarrow$［Loomis（1955）］page 3，圈［Birkhoff and Neumann（1936）］page 830 〈L71－L73〉
    ${ }^{63} \square$［Kalmbach（1983）］page 22，$\quad$［Holland（1970）］，page 50，$\square$［Beran（1985）］page 33，$\nabla$［Stern（1999）］ page 12．The $O_{6}$ lattice is also called the hexagon or Benzene ring．
    ${ }^{64} \nabla$［Beran（1985）］pages 33－42，圈［Maeda（1966）］page 250，$\theta$［Kalmbach（1983）］page 24 〈Figure 3．2〉，$\theta$ ［Stern（1999）］page 12，回［Holland（1970）］，page 50
    ${ }^{65}$ As can be seen in this example，the number of orthocomplemented lattices with $(1,2,3, \ldots)$ elements is $(1,1,0,1,0,2,0,5,0, \ldots)$ ．It is interesting to note that at least the first 9 terms（and possibly more？）of this se－ quence are the same as the＂expansion of $\frac{1+2 x}{1+\sqrt{1-4 x^{2}}}$ ㅁ．［oei（2014）］〈http：／／oeis．org／A097331〉 and the＂Catalan numbers ．．．interpolated with 0＇s＂모．［oei（2014）］〈http：／／oeis．org／A126120〉

[^24]:    ${ }^{66} \theta$［Beran（1985）］pages 30－31，萬［Birkhoff and Neumann（1936）］page $830\langle\mathrm{~L} 74\rangle$ ，$\theta$［Cohen（1989）］page 37〈3B．13．Theorem〉

[^25]:    ${ }^{67} \Rightarrow$ [Kalmbach(1983)] page 22
    ${ }^{68} \otimes$ [Kalmbach(1983)] page 22, $\otimes$ [Lidl and Pilz(1998)] page 90, 圈 [Husimi(1937)]
    ${ }^{69}$ 圏 [Renedo et al.(2003)Renedo, Trillas, and Alsina] page 72

[^26]:    ${ }^{70} \triangleq$ [Kalmbach(1983)] page $32\langle 20$.$\rangle , 回 [Iturrioz(1985)], page 57$
    ${ }^{71} \boxminus[$ Dunn(1996)] pages 4-6, $\exists$ [Dunn(1999)] pages 24-26 〈2 The Kite of Negations $\rangle$

[^27]:    ${ }^{72}$ In the context of natural language，D．Devidi has argued that，subminimal negation（Definition B． 1 page 34）is ＂difficult to take seriously as＂a negation．For further details see $⿴$［Devidi（2010）］，page 511，回［Devidi（2006）］， page 568
    ${ }^{73} \boxminus[D u n n(1996)]$ pages $4-6, \boxtimes[D u n n(1999)]$ pages $24-26\langle 2$ The Kite of Negations $\rangle, ~[T r o e l-$ stra and van Dalen（1988）］page 4 〈 1.6 Intuitionism．（b）$\rangle$ ，圈［De Vries（2007）］page 11 〈Definition 16〉， $\otimes$［Gottwald（1999）］page 21 〈Definition 3.3$\rangle$ ，$\boxminus$［Novák et al．（1999）Novák，Perfilieva，and Močkoř］ page 50 〈Definition 2．26〉，$\boxminus$［Nguyen and Walker（2006）］pages 98－99 〈5．4 Negations〉，圈［Bellman and Giertz（1973）］PAGES $155-156\langle(N 1) ~ \neg 0=1$ AND $\neg 1=0$ ，（N3）$\neg \neg x=x\rangle$
    ${ }^{74} \nabla$［Dunn（1999）］pages $24-26\langle 2$ The Kite of Negations $\rangle, \Delta$［Jenei（2003）］page 283，$\Delta$［Kalmbach（1983）］ page 22，$\theta$［Lidl and Pilz（1998）］Page 90，圈［Husimi（1937）］
    ${ }^{75} \otimes$［Cattaneo and Ciucci（2009）］page 78
    ${ }^{76}$［Beran（1985）］page 31 〈Theorem 1．2 Proof〉，圊［Fáy（1967）］page 268 〈Lemma 1 Proof $\rangle$ ，圈［deVries（2007）］ page $12\langle$ Theorem 18〉

[^28]:    $77 \otimes$［Beran（1985）］pages 30－31 〈Theorem 1．2〉，圈［Fáy（1967）］page 268〈Lemma 1〉，圈［Nakano and Romberger（1971）］〈cf Beran 1985〉
    ${ }^{78} \otimes$［Varadarajan（1985）］page 42

[^29]:    ${ }^{79}$－［Fodor and Yager（2000）］page 128，圏［Yager（1980b）］pages 256－257，圈［Yager（1979）］〈cf Fodor $\rangle$
    ${ }^{80} \boxminus$［Fodor and Yager（2000）］page 128，誊［Ovchinnikov（1983）］page 235 〈Example 4〉

[^30]:     pages 332－339 〈§64．The 3－valued logic〉，圈［Sobociński（1952）］
    ${ }^{82} \boxminus[K a r p e n k o(2006)]$ page 45，$\theta$［Johnstone（1982）］page $9\langle \$ 1.12\rangle$ ，圊［Heyting（1930a）］，圈［Heyt－ ing（1930b）］，圏［Heyting（1930c）］，圏［Heyting（1930d）］，圈［Jaskowski（1936）］，$\triangleright$［Mancosu（1998）］

[^31]:    〈Definition 2, classical implication〉

[^32]:    ${ }^{84}$ 圊［Cignoli（1975）］page 270，$\forall$［Restall（2000）］page 171 〈Example 8．39〉，盽［de Vries（2007）］pages 15－16〈Example 26〉，圈［Dunn（1976）］，圈［Belnap（1977）］

[^33]:    ${ }^{85}$ 誊［Hardegree（1979）］page 59 〈（E），（MP），（E＊） ，圈［Kalmbach（1973）］page 498，$\triangle$［Kalmbach（1983）］ pages 238－239＜Chapter $4 \S 15\rangle$ ，圈［Pavičić and Megill（2009）］page 24，$\quad$［Xu et al．（2003）Xu，Ruan，Qin，and Liu］page 27 〈Definition 2．1．1〉，圊［Xu（1999）］page 25，圈［Jun et al．（1998）Jun，Xu，and Qin］page 54

[^34]:    ${ }^{86}$ 圈［Kalmbach（1973）］page 499，園［Kalmbach（1974）］，童［Mittelstaedt（1970）］〈Sasaki hook〉，圈［Finch（1970）］ page 102 〈Sasaki hook（1．1）〉，$\boxminus$［Kalmbach（1983）］page 239 〈Chapter 4 §15，3．Theorem〉

[^35]:    ${ }^{87}$ 圈［Straßburger（2005）］page 136 〈Definition 2.1$\rangle$ ，圈［de Vries（2007）］page $11\langle$ Definition 16〉
    ${ }^{88}$［Novák et al．（1999）Novák，Perfilieva，and Močkoř］pages 17－18 〈Example 2．1〉

[^36]:    $\Leftrightarrow$

[^37]:    ${ }^{89}$ 圈［Kleene（1938）］page 153，$\ominus$［Kleene（1952）］，pages 332－339〈§64．The 3－valued logic〉，圈［Avron（1991）］ page 277

[^38]:    ${ }^{90}$ 圊［Łukasiewicz（1920）］page 17 〈II．The principles of consequence〉，誊［Avron（1991）］page 277〈Łukasiewicz．〉
    ${ }^{91}$ 圈［Avron（1991）］pages 277－278，圏［Sobociński（1952）］
    $92 \boxminus[K a r p e n k o(2006)]$ page 45，$\Delta[J o h n s t o n e(1982)]$ page $9\langle\S 1.12\rangle$ ，圈［Heyting（1930a）］，圊［Heyt－ ing（1930b）］，圈［Heyting（1930c）］，圈［Heyting（1930d）］，圈［Jaskowski（1936）］，$\otimes$［Mancosu（1998）］

[^39]:    ${ }^{93} \boxminus$［Xu et al．（2003）Xu，Ruan，Qin，and Liu］page 29 〈Example 2．1．3〉，穵［Jun et al．（1998）Jun，Xu，and Qin］ page 54 〈Example 2．2〉
    ${ }^{94}$ 圈［Belnap（1977）］page 13，$\ominus$［Restall（2000）］page 177 〈Example 8．44〉，圈［Pavičić and Megill（2009）］page 28〈Definition 2，classical implication〉，圈［Mittelstaedt（1970）］，圈［Finch（1970）］page $102\langle(1.1)\rangle$ ，圈［Smets（2006）］ page 270

[^40]:    ${ }^{95}$［Restall（2000）］page 171 〈Example 8．39〉
    ${ }^{96} \theta$［Xu et al．（2003）Xu，Ruan，Qin，and Liu］pages 29－30 〈Example 2．1．4〉

