Boolean and ortho fuzzy subset logics

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Abstract: Constructing a fuzzy subset logic L with Boolean properties is notoriously difficult because under a handful of "reasonable" conditions, we have the following three debilitating constraints: (1) Bellman and Giertz in 1973 showed that if L is *distributive*, then it must be *idempotent*. (2) Dubois and Padre in 1980 showed that if L has the *excluded middle* or the *non-contradiction* property or both, then it must be *non-idempotent*. (3) Bellman and Giertz also demonstrated in 1973 that even if L is *idempotent*, then the *only* choice available for the (\land, \lor) logic operator pair is the (min, max) operator pair. Thus it would seem impossible to construct a non-trivial fuzzy subset logic with *Boolean* properties. However, this paper examines these three results in detail, and shows that "hidden" in the hypotheses of the three is the assumption that the operator pair (\land, \lor) is *pointwise evaluated*. It is further demonstrated that removing this constraint yields the following results: (A) It is indeed possible to construct *fuzzy subset logics* that have all the *Boolean* properties, including that of *idempotency*, *non-contradiction*, *excluded middle*, and *distributivity*. (B) Even if *idempotency* holds, (min, max) is *not* the only choice for (\land, \lor) .

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Introduction

The problem. This paper addresses a well known conflict in *fuzzy subset logic* theory. *Fuzzy subset logic* is the foundation for *fuzzy set theory*, just as *classical logic* is the foundation for *classical set theory*. Just as *classical logic* and the *algebras of sets* constructed upon it are *Boolean algebras* (and hence have all the "nice" *Boolean* properties such as *idempotency*, *excluded middle*, *non-contradiction*, *distributivity*, etc.), one might very much prefer to have a *fuzzy subset logic* and resulting *fuzzy subset algebras* that are *Boolean* as well.¹ But it has been found that under what would seem to be very "reasonable" conditions, this is simply not possible. In particular, we have the following crippling constraints:

- Bellman and Giertz in 1973 demonstrated that under very "reasonable" conditions, if we want a *fuzzy subset logic* that is *distributive*, then it also must be *idempotent*.²
- Dubois and Padre in 1980 demonstrated that under very "reasonable" conditions, if we want a *fuzzy subset logic* that has the *non-contradiction* and *excluded middle* properties, then that logic is *not idempotent*...and therefore not only fails to be a *Boolean algebra*, but also is not even a *lattice*.³

²see *fuzzy operators idempotency theorem* (Theorem 1.25 page 12)

¹ excluded middle: $x \lor \neg x = 1$. non-contradiction: $x \land \neg x = 0$. idempotency: $x \lor x = x$ and $x \land x = x$. distributivity: $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ and $x \land (y \lor z) = (x \land y) \lor (x \land z)$. classic Boolean properties: Theorem A.42 page 30.

³Dubois-Padre 1980 result: see fuzzy negation idempotency theorem (Theorem 1.28 page 14) and Dubois-Padre 1980 theorem (Corollary 1.29 page 15). Every lattice is a Boolean algebra, but not conversely (Definition A.11 page 24, Definition A.41 page 30). A lattice is idempotent, commutative, associative, and absorptive (Theorem A.14 page 25). A Boolean algebra has all these properties but is moreover bounded, distributive, complemented, de Morgan, involutory, and has identity (Theorem A.42 page 30).

Moreover, even if we are willing to give up the *non-contradiction* and *excluded middle* properties and retain *idempotency*, Bellman and Giertz also demonstrated in 1973 that the *only* choice we have for the logic operator pair (Λ, \vee) is the (min, max) operator pair such that $(\Lambda, \vee) =$ $(\min, \max).^4$

A solution. Section 1 of this paper examines these results in detail, and demonstrates that "hidden" in the hypotheses of these results is the assumption that the operator pair (\land,\lor) is *pointwise evaluated*.⁵ Section 2 demonstrates that if this constraint is removed, then it is indeed possible to construct *fuzzy* subset logics that have all the Boolean properties, including that of idempotency, non-contradiction, excluded middle, and distributivity.

A solution yielding ortho fuzzy subset logics. In this paper, a logic $L' \triangleq (X, \lor, \land, \neg, 0, 1; \le, \rightarrow)$ is defined as a *lattice* $L \triangleq (X, \lor, \land; \leq)$ with a *negation function* \neg and *implication function* \rightarrow defined on this lattice. And in this sense, the logic L' is said to be "constructed on" the lattice L. This paper demonstrates that it is possible to construct *fuzzy subset logics* on *Boolean lattices* yielding *Boolean* fuzzy subset logics. However, more generally, it is also demonstrated that it is possible to construct fuzzy subset logics on orthocomplemented lattices yielding ortho fuzzy subset logics. The main difference between a Boolean lattice and an orthocomplemented lattice is that the latter does not in general support distributivity.⁶ On finite sets, there are significantly more choices of orthocomplemented lattices than there are Boolean lattices.⁷And so having the option of constructing ortho fuzzy subset logics is arguably not without advantage. The disadvantage is that we give up the guarantee of *distributivity*. But some authors⁸ have investigated structures without this property anyways. In fact, one could argue that the "crucial" properties that we would really like a logic to have, if possible, are the following:

- (1). *disjunctive idempotence*: $x \lor x$ = *x* and
- (2). *conjunctive idempotence:* $x \land x$ x and =
- (3). *excluded middle*: $x \lor \neg x = 1$ and
- (4). *non-contradiction*: $x \wedge \neg x = 0$

Not all *fuzzy logics* have all the these properties. Of course all *Boolean logics* have them. But more generally than *Boolean logics* and less generally than *fuzzy logics*, all *ortho logics* have them as well.⁹



⁴Bellman-Giertz 1973 result: see fuzzy min-max theorem (Theorem 1.26 page 13) and Bellman-Giertz 1973 theorem (Corollary 1.27 page 14). (\wedge, \vee) in an ordered set: Definition A.9 page 24 and Definition A.8 page 24. (\wedge, \vee) in a lattice: Definition A.11 page 24. (\land, \lor) in a *logic*: Definition C.5 page 50. (min, max): Definition 1.15 page 7.

⁵ pointwise evaluated: (Definition 1.12 page 7) ⁶ logic: Definition C.5 page 50. lattice: Definition A.11 page 24. negation function: Definition B.2 page 35. implication function: Definition C.1 page 45 Boolean lattice: Definition A.41 page 30. orthocomplemented lattice: Definition A.44 page 31. ortho negation: Definition B.3 page 35. ortho+distributivity=Boolean: Proposition A.50 page 33

⁷There are a total of 5 orthocomplemented lattices with 8 elements; of these 5, only 1 is Boolean. There are a total of 10 orthocomplemented lattices with 8 elements or less; of these 10, only 4 are Boolean. For further details, see Example A.46 page 31.

⁸ 📃 [Alsina et al.(1980)Alsina, Trillas, and Valverde], 📃 [Hamacher(1976)] (referenced by 📃 [Alsina et al.(1983)Alsina, Trillas, and Valverdel)

⁹ properties of *fuzzy negations* and hence also *fuzzy logics*: Theorem B.11 page 36. properties of *ortho negations* and hence also ortho logics: Theorem B.15 page 37. relationships between logics: Figure 13 page 50.

Negation functions. There are several types of *negation functions* and information about these functions is scattered about in the literature. APPENDIX B introduces several types of negation, describes some of their properties, and shows where fuzzy negation, ortho negation, and Boolean negation "fit" into the larger structure of *negations* in general.

Implication functions. Defining an *implication* function for a logic constructed on a *Boolean lattice* is straightforward because we can simply use the *classical implication* $x \stackrel{c}{\rightarrow} y \triangleq \neg x \lor y$. However, defining an *implication* function for a *non-Boolean* logic is more difficult. Appendix C addresses the problem of defining implication functions on *lattices*, including lattices that are *non-Boolean*.

Fuzzy subset operators 1

A fuzzy subset is often specified in terms of a membership function. A fuzzy subset logic is a lattice of membership functions together with a fuzzy negation function and an implication function. Although its definition is simple and straightforward, fuzzy subset logic has some notorious problems attempting to provide some very standard Boolean properties.¹⁰

1.1 Indicator functions

In *classical subset theory*, a subset A of a set X can be specified using an *indicator function* $\mathbb{1}_{A}(x)$ (next definition). An indicator function specifies concretely whether or not an element is a member of A. That is, it is a convenient "indicator" of whether or not a particular element is in a subset. A subset that can be defined using an indicator function is a *crisp subset* (next definition).

Definition 1.1 ¹¹ Let 2^X be the *power set* of a set X. Let Y^X be the *set of all functions* mapping from X to a set Y. The **indicator function** $\mathbb{1}_A \in \{0, 1\}^X$ is defined as $\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \forall_{x \in X, A \in 2^X}.$

The parameter A of $\mathbb{1}_A$ is a **crisp subset** of X if $\mathbb{1}_A(x)$ is an *indicator function* on X.

Every set X has at least one crisp subset (itself). A set of subsets, together with the relation \subseteq , form an ordered set, and in some cases also form a *lattice*. Common set structures include the power set 2^X , topologies, rings of sets and algebras of sets. A set structure may be represented in terms of subsets, or equivalently, in terms of set indicator functions.¹²

¹⁰ fuzzy subset: Definition 1.7 page 6, fuzzy subset logic: Definition 1.11 page 6, membership function: Definition 1.7 page 6, *lattice*: Definition A.11 page 24, *fuzzy negation*: Definition B.2 page 35, *implication*: Definition C.1 page 45 and Definition C.5 page 50; problems: Theorem 1.26 page 13 and Theorem 1.28 page 14.

¹¹ 🐃 [Feller(1971)], page 104 (1. Baire Functions), 🐃 [Aliprantis and Burkinshaw(1998)], page 126, 🐃 [Hausdorff(1937)], page 22, 🖱 [de la Vallée-Poussin(1915)] page 440

¹² ordered set: Definition A.1 page 22, lattice (Definition A.11 page 24), set indicator function: Definition 1.1 page 4, topologies: Example 1.3 page 5 and Example 1.4 page 5; examples of set structures: Example 1.3 page 5 and Example 1.5 page 5.

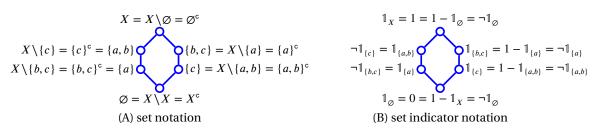


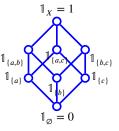
Figure 1: set structures on O_6 (see Example 1.5 page 5)

Remark 1.2 Often set structures are defined in terms of set operators like *intersection* \cap , *union* \cup , and *set complement* c. The set operators $(\cap, \cup, c, \Rightarrow, \emptyset, X)$ in turn can be defined in terms of arithmetic operators (min, max, $f(x) \triangleq 1 - x, g(x, y) \triangleq y - xy, 0, 1$) on the *set indicator* function¹³ or in terms of classic logic operators ($\wedge, \vee, \neg, \rightarrow, 0, 1$) like this:

 $\begin{array}{rcl} 0 & \triangleq \mathbb{1}_{\varnothing} & = & 0 \\ 1 & \triangleq \mathbb{1}_{X} & = & 1 \\ \mathbb{1}_{A} \lor \mathbb{1}_{B} & \triangleq \mathbb{1}_{A \cup B} & = & \max(\mathbb{1}_{A}, \mathbb{1}_{B}) \\ \mathbb{1}_{A} \land \mathbb{1}_{B} & \triangleq & \mathbb{1}_{A \cap B} & = & \min(\mathbb{1}_{A}, \mathbb{1}_{B}) \\ \neg \mathbb{1}_{A} & \triangleq & \mathbb{1}_{A^{c}} & = & 1 - \mathbb{1}_{A} \\ \mathbb{1}_{A} \to \mathbb{1}_{B} & \triangleq & \mathbb{1}_{A \Rightarrow B} & = & \max(1 - \mathbb{1}_{A}, \mathbb{1}_{B}) \end{array}$ where $A \Rightarrow B \triangleq A^{c} \cup B$ is the set implication from A to B.¹⁴

 $\begin{array}{c} X \\ \{a,b\} \\ \{a\} \\ \{a\} \\ \{b\} \\ \{b\} \\ \{c\} \\ \{c\}$

Example 1.3 The set structures illustrated to the left and right are the **power set** of the set $X \triangleq \{a, b, c\}$. A power set is a special case of an *algebra of sets* and also a *topology*. The lattice to the left uses set notation; the one to the right uses set indicators.



$\mathbf{Q}\{a,b\}$	Example 1.4 The set structures illustrated to the left
$\mathbf{Q}\left\{a\right\}$	and right are a topology on the set $X \triangleq \{a, b\}$. The
	lattice to the left uses set notation; the one to the right
o	uses set indicators.

Example 1.5 The set structures illustrated in Figure 1 (page 5) are not *topologies* (or *algebras of sets* or *power sets*), but are *set structures* none the less. The negation function in the structure is an *ortho negation* (Definition B.3 page 35). The lattice in (A) uses set notation; the one in (B) uses set indicators.

Definition 1.6 Let $\mathbb{1}^X$ be the *set of all indicator functions* on a set *X*. Let a *logic* be defined as in Definition C.5 (page 50). A **crisp subset logic** is a *logic*

 $\left(\mathbb{1}^X, \vee, \wedge, \neg, \mathbb{1}_{\varnothing}, \mathbb{1}_X; \mathbf{\vec{<}}, \rightarrow \right).$

¹³ S [Aliprantis and Burkinshaw(1998)], page 126, S [Hausdorff(1937)], pages 22–23

¹⁴ \square [Ellerman(2010)] (§1.7; $A \Rightarrow B = (A^c \cup B)^\circ$ where C° is the *interior* of a set C in a *topological space*)

1.2 Membership functions

In a crisp subset *A* of a crisp set *X* ($A \subseteq X$), an element $x \in X$ has only two possible "degrees of membership" in *A*: Either *x* is in *A* or *x* is *not* in *A*. Said another way, either *x* has "full membership" in *A*, or *x* has "absolute non-membership" in *A*. And this "degree of membership" is specified by an *indicator function* (Definition 1.1 page 4) $\mathbb{1}_A(x)$ which maps from *X* to the 2-valued set $\{0, 1\}$, where 0 represents "absolute non-membership" and 1 represents "full membership".

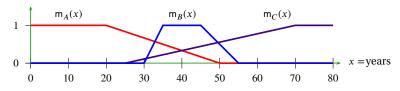
In a fuzzy subset *B* of a crisp set *X* ($B \subseteq X$), an element $x \in X$ has a range of possible degrees of membership in *B*. And this membership is specified by a *membership function* (next definition) $\mathbb{m}_B(x)$ which maps from *X* to the infinite set [0:1].

Definition 1.7 ¹⁵ Let [0:1] be the *closed interval* on \mathbb{R} such that $[0:1] \triangleq \{x \in \mathbb{R} \mid 0 \le x \le 1\}$. Let *X* be a set. A function $\mathbb{m}_A(x)$ is a **membership function** on *X* if $\mathbb{m}_A \in [0:1]^X$. The parameter *A* is called a **fuzzy subset** of *X*. For any value $x \in X$, $\mathbb{m}_A(x) \in [0:1]$ represents the "degree of membership" of *x* in *A*. The condition $\mathbb{m}_A(x) = 1$ indicates that *x* has "full membership" in *A*, and the condition $\mathbb{m}_A(x) = 0$ indicates that *x* has "absolute non-membership" in *A*.

Remark 1.8 ¹⁶ What is typically called a "fuzzy set" arguably should more accurately be called a "fuzzy subset" because every element *x* at any "degree of membership" in a fuzzy subset *A* has absolute full membership in some universal crisp set *X*. And thus *A* is a subset of the crisp set *X* ($A \subseteq X$).

Remark 1.9 In a crisp set *X*, a *fuzzy subset* $A \subseteq X$ should not be confused with a *random subset* $B \subseteq X$. In the fuzzy subset *A*, an element $x \in X$ has a "degree of membership" in *A* that specifies "to what extent" *x* can be considered a member of *A*. In the random subset *B*, the element $x \in X$ has a "degree of likelihood" that *x* is in *B* and that specifies the probability that *x* is a member of *B*. Alternatively, a fuzzy subset is a result of "inference under vagueness", while a random subset is a result of "inference under random subset is a result of "inference under vagueness".

Example 1.10 Let *A* be the set of all people who are "young" with *membership function* $\mathbb{m}_A(x)$. Let *B* be the set of all people who are "middle age" with *membership function* $\mathbb{m}_B(x)$. Let *C* be the set of all people who are "old" with *membership function* $\mathbb{m}_C(x)$. Of course all these are vague, or "fuzzy", concepts; but the following figure illustrates what the *membership functions* (Definition 1.7 page 6) for these sets **might** look like.



Definition 1.11 Let \mathbb{M} be a set of *membership functions* (Definition 1.7 page 6). The structure $L \triangleq (\mathbb{M}, \vee, \wedge, \neg, 0, 1; \leq)$ is a **fuzzy subset logic** if L is a *fuzzy logic* (Definition C.5 page 50).

¹⁵ [Hájek(2011)], page 68 ("absolutely true", "absolutely false), \square [Dubois(1980)] page 10, \square [Dubois et al.(2000)Dubois, Ostasiewicz, and Padre], page 42 ("full membership", "absolute non-membership"), \square [Zadeh(1965)] page 339 ("grade of membership")

¹⁶ 🖻 [Dubois(1980)] page 10 (Remarks 1), 🛸 [Kaufmann(1975)]

¹⁷ [Hájek(2011)], page 67 (5.1 Introduction)

1.3 Operators on membership functions

The *meet-join* operator pair (\land, \lor) on a set of indicator functions $\mathbb{1}^X$ induces an ordering relation on $\mathbb{1}^X$.¹⁸. So the operator pairs (\land, \lor) can be defined on sets of membership functions to form lattices. But while lattices of set indicators effectively have just one choice for (\land, \lor) , membership function lattices have many choices.

In this paper, the operators (\land, \lor) are called *pointwise evaluated* if at each single value x, the functions $[m \land m](x)$ and $[m \lor m](x)$ depend only on the values of m(x) (m evaluated at the single value x) and m(x) (next definition).

Definition 1.12 ¹⁹ Let $L \triangleq (\mathbb{M}, \wedge, \vee)$, where \mathbb{M} is a set of *membership functions* (Definition 1.7 page 6) with operators (\wedge, \vee) . *L* is **pointwise evaluated**, or said to have **pointwise evaluation**, if there exists $f, g \in [0:1]^{[0:1]^2}$ such that

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1. [\mathfrak{m} \wedge \mathfrak{m}](x) = \mathbf{f}[\mathfrak{m}(x), \mathfrak{m}(x)] \quad \forall x \in \mathbb{R}, \text{ and } \forall \mathfrak{m}, \mathfrak{n} \in \mathbb{M} and
2. [\mathfrak{m} \vee \mathfrak{m}](x) = \mathbf{g}[\mathfrak{m}(x), \mathfrak{m}(x)] \quad \forall x \in \mathbb{R}, \text{ and } \forall \mathfrak{m}, \mathfrak{n} \in \mathbb{M}
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Example 1.13

1. The function \triangle defined as $[\mathbb{m} \triangle \mathbb{m}](x) \triangleq \mathbb{m}(x) + \mathbb{m}(x)$ is pointwise evaluated. 2. The function \triangle defined as $[\mathbb{m} \triangle \mathbb{m}](x) \triangleq \underbrace{\int_{-\infty}^{x} \mathbb{m}(u)\mathbb{m}(x-u) \, du}_{\text{"convolution"}}$ is not pointwise evaluated.

Example 1.14 Examples of operators that *are* pointwise evaluated include the *min-max operators* (next definition), the *product and probabilistic sum operators* (Definition 1.17 page 8), and the *Łukasiewicz t-norm and t-conorm* (Definition 1.18 page 9).

One of the most common fuzzy logic operator pairs is the *min-max operator pair* (next). As will be demonstrated by the *fuzzy min-max theorem* (Theorem 1.26 page 13), under fairly "reasonable" conditions the *min-max operators* are the *only* choice available for a *fuzzy subset logic*.

Definition 1.15 ²⁰ Let \mathbb{M} be a *set of membership functions* on a set X. Let f(x) and g(x) be functions both with *domain* X. Let min(f(x), g(x)) and max(f(x), g(x)) be the *pointwise minimum* and *pointwise maximum*, respectively, of f(x) and g(x) over X. The **min-max operators** (\wedge, \vee) for L are defined as

 $\begin{bmatrix} \mathsf{m}_A \lor \mathsf{m}_B \\ \mathsf{m}_A \land \mathsf{m}_B \end{bmatrix} (x) \triangleq \max \begin{bmatrix} \mathsf{m}_A(x), \mathsf{m}_B(x) \\ \mathsf{m}_A \land \mathsf{m}_B \end{bmatrix} (x) \triangleq \min \begin{bmatrix} \mathsf{m}_A(x), \mathsf{m}_B(x) \\ \forall \mathsf{m} \in \mathbb{M}, x \in X \end{bmatrix}$

Proposition 1.16 Let \mathbb{M} , max, and min defined as in Definition 1.15. Let $\mathbf{L} \triangleq (\mathbb{M}, \lor, \land; \le)$ be an algebraic structure with $x \le y \iff x \land y = x$.

 $(\land,\lor) = (\min, \max) \implies L \text{ is a LATTICE (Definition A.11 page 24).}$



¹⁸ see Remark 1.2 page 5, Proposition A.10 page 24, and Example 1.3 page 5–Example 1.5 page 5

¹⁹ \blacktriangleright [Dubois(1980)] page 11 (B.a(i))

²⁰ [Fodor and Yager(2000)], page 133, [] [Zadeh(1965)] pages 340–341 ((3),(5)); *pointwise ordering*: Definition A.7 page 23

 $m \vee n = max(m, n)$ by left hypothesis by definition of min $= \max(n, m)$ by left hypothesis $= n \vee m$ \implies \lor is commutative by left hypothesis $m \wedge n = \min(m, n)$ $= \min(n, m)$ by definition of min $= n \wedge m$ by left hypothesis \implies \land is commutative $m \lor (n \lor p) = max[m, max(n, p)]$ by left hypothesis by definition of max $= \max[\max(m, n), p]$ $= (m \lor n) \lor p$ by left hypothesis \implies \lor is associative $m \wedge (n \wedge p) = \min[m, \min(n, p)]$ by left hypothesis by definition of min $= \min[\min(m, n), p]$ $= (m \land n) \land p$ by left hypothesis \implies \land is associative $m \lor (m \land n) = max[m, min(m, n)]$ by left hypothesis $= \begin{cases} \max(\mathfrak{m},\mathfrak{m}) & \text{if } \mathfrak{m}(x) \leq \mathfrak{n}(x) \quad \forall x \in X \\ \max(\mathfrak{m},\mathfrak{n}) & \text{otherwise} \end{cases}$ $= \begin{cases} \mathfrak{m} & \text{if } \mathfrak{m}(x) \leq \mathfrak{n}(x) \quad \forall x \in X \\ \mathfrak{m} & \text{otherwise} \end{cases}$ = m $m \wedge (m \vee n) = \min[m, max(m, n)]$ by left hypothesis $= \begin{cases} \min(m, n) & \text{if } m(x) \le n(x) \quad \forall x \in X \\ \min(m, m) & \text{otherwise} \end{cases}$ $= \begin{cases} m & \text{if } m(x) \le n(x) \quad \forall x \in X \\ m & \text{otherwise} \end{cases}$ = m \implies (\land , \lor) is *absorptive*

SPROOF: To be a lattice, *L* must be *commutative*, *associative*, and *absorptive* (Theorem A.18 page 25).

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Definition 1.17 ²¹ Let M be defined as in Definition 1.15. Then for all $m \in M$, the **probabilistic sum operator** \lor on M is defined as and the **product sum operator** \land on M is defined as $\begin{bmatrix} m_A \lor m_B \\ m_A \land m_B \end{bmatrix} (x) \triangleq m_A(x) + m_B(x) - m_A(x)m_B(x) = m_A(x)m_B(x)$

Note that the *product and probabilistic sum operators* (previous definition) do *not* in general form a lattice because, for example, they are not in general *idempotent* (a necessary condition for being a lattice—Theorem A.14 page 25). Suppose for example $m(p) = \frac{1}{2}$ at some point p. Then at that point p $m \lor m \triangleq m + m - mm = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4} \neq m \implies \lor is non-idempotent$ $m \land m \triangleq mm = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq m \implies \land is non-idempotent$

²¹ [Fodor and Yager(2000)], page 133

Definition 1.18 ²² Let L, \mathcal{D} , min and max be defined as in Definition 1.15. Then for all $m \in M$, the **Łukasiewicz t-conorm** \lor is defined as $[m_A \land m_B](x) \triangleq \max [0, m_A(x) + m_B(x) - 1] \forall m \in M, x \in X$ and the **Łukasiewicz t-norm** \land is defined as $[m_A \lor m_B](x) \triangleq \min [1, m_A(x) + m_B(x)] \forall m \in M, x \in X$ The *Łukasiewicz t-conorm* is also called the **bold sum**, and the *Łukasiewicz t-norm* is also called the **bold sum**, and the *Łukasiewicz t-norm* is also called the **bold sum**.

Note that the *Łukasiewicz operators* (previous definition) do *not* in general form a lattice because, for example, they are not in general *idempotent*. Suppose for example $m(p) = \frac{1}{2}$ at some point *p*. Then $m \lor m \triangleq \min(1, m + m) = \min(1, \frac{1}{2} + \frac{1}{2}) = 1 \neq m$

 $m \wedge m \triangleq max(0, m + m - 1) = max(0, \frac{1}{2} + \frac{1}{2} - 1) = 0 \neq m$

There are several choices for *negations* in a *fuzzy subset logic*. Arguably the "simplest" is the *discrete negation* (Example B.16 page 38). Perhaps the most "common" is the *standard negation* (next definition). More generally there is the λ -*negation* (Definition 1.20 page 9) which reduces to the standard negation at $\lambda = 0$ and approaches the discrete negation as $\lambda \to \infty$. Alternatively there is also the *Yager negation* (Definition 1.21 page 9) which reduces to the standard negation (Definition 1.21 page 9) which reduces to the standard negation (Definition 1.21 page 9) which reduces to the standard negation (Definition 1.21 page 9) which reduces to the standard negation (Definition 1.21 page 9) which reduces to the standard negation at p = 1.

Definition 1.19 ²³ The function $\neg m(x)$ is the **standard negation** (or **Łukasiewicz negation**) of m if $\neg m(x) \triangleq 1 - m(x) \quad \forall x \in \mathbb{R}$.

Definition 1.20 ²⁴ The function $\neg m(x)$ is the λ -negation of a function m(x) if $\neg m(x) \triangleq \frac{1 - m(x)}{1 + \lambda m(x)}$ $\forall \lambda \in (-1 : \infty).$

Definition 1.21 ²⁵ The function $\neg m(x)$ is the **Yager negation** of a function m(x) if $\neg m(x) \triangleq (1 - m^p)^{1/p}$ $\forall p \in (0 : \infty)$.

If $\neg m$ is a λ -negation, then the function \neg in a fuzzy subset lattice L is a de Morgan negation (Definition B.3 page 35) and thus the de Morgan properties hold in L (Theorem B.14 page 37). The standard negation (Definition 1.19 page 9) is a λ -negation (at $\lambda = 0$) and so the standard negation is also de Morgan.

Theorem 1.22 Let $L \triangleq (\mathbb{M}, \vee, \wedge, \neg, 0, 1; \leq)$ be a LATTICE WITH NEGATION (Definition B.5 page 35). $\begin{cases} \neg \mathfrak{m}(x) \text{ is } a \lambda \text{-NEGATION } \forall \mathfrak{m} \in \mathbb{M} \\ \text{(Definition 1.7 page 6)} \end{cases} \implies \begin{cases} \neg \text{ is } a \text{ DE MORGAN NEGATION } on L \\ \text{(Definition B.3 page 35)} \end{cases}$





²² [Fodor and Yager(2000)], page 133

²³ [Zadeh(1965)] page 340, S [Jager(1995)] page 243 (Appendix A)

²⁴ ^[] [Fodor and Yager(2000)], page 129, ^[] [Hájek(2011)], page 68 ⟨Definition 5.1⟩, ^[] [Sugeno(1977)] page 95

⁽⁽²³⁾ " λ -complement", see also p.94(12), p.96(28) \rangle , \square [Jager(1995)] page 243 (Appendix A)

²⁵ \blacksquare [Yager(1980a)] (cf Jager(1995)), \blacksquare [Jager(1995)] page 243 (Appendix A)

by property of *real numbers* \mathbb{R}

by property of *real numbers* \mathbb{R}

by definition of λ -negation (Definition 1.20 page 9)

by definition of λ -negation (Definition 1.20 page 9)

by definition of λ -negation (Definition 1.20 page 9)

because $1 + \lambda m > 0$

SPROOF: To be a *de Morgan negation*, $\neg m_A(x)$ must be *antitone* and *involutory* (Definition B.3 page 35).

$$\begin{split} \mathsf{m}_{A}(x) &\leq \mathsf{m}_{B}(x) \Longrightarrow - \mathsf{m}_{B}(x) \leq -\mathsf{m}_{A}(x) \\ & \Longrightarrow 1 - \mathsf{m}_{B}(x) \leq 1 - \mathsf{m}_{A}(x) \\ & \Longrightarrow \frac{1 - \mathsf{m}_{B}(x)}{1 + \lambda \mathsf{m}_{B}(x)} \leq \frac{1 - \mathsf{m}_{A}(x)}{1 + \lambda \mathsf{m}_{A}(x)} \\ & \Longrightarrow \forall \mathsf{m}(x) \leq \forall \mathsf{m}(x) \\ & \Longrightarrow \forall \mathsf{m}(x) \leq \forall \mathsf{m}(x) \\ & \Rightarrow \mathsf{m}(x) \leq \forall \mathsf{m}(x) \\ & = \frac{1 - \frac{1 - \mathsf{m}_{A}(x)}{1 + \lambda \mathsf{m}_{A}(x)}}{1 + \lambda \frac{1 - \mathsf{m}_{A}(x)}{1 + \lambda \mathsf{m}_{A}(x)}} \\ & = \frac{(1 + \lambda \mathsf{m}_{A}(x)) - (1 - \mathsf{m}_{A}(x))}{(1 + \lambda \mathsf{m}_{A}(x)) + \lambda (1 - \mathsf{m}_{A}(x))} \\ & = \frac{(1 + \lambda)\mathsf{m}_{A}(x)}{1 + \lambda} \\ & = \mathsf{m}_{A}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \mathsf{m}(x) \\ & \Rightarrow \forall \mathsf{m}(x) \mathsf{m}(x)$$

 \square

Corollary 1.23 Let $L \triangleq (M, \lor, \land, \neg, 0, 1; \le)$ be a LATTICE WITH NEGATION (Definition B.5 page 35).

$$\left\{ \begin{array}{l} A. \neg m(x) \text{ is } a \lambda \text{-NEGATION } \forall m \in \mathbb{M} \quad and \\ B. \neg m_1 = m_0 \end{array} \right\} \implies \left\{ \begin{array}{l} 1. \neg \text{ is } a \text{ DE MORGAN NEGATION } on L \\ 2. \neg \text{ is } a \text{ FUZZY NEGATION } on L \end{array} \right\}$$

[®]Proof:

- (1) Proof for (1): by Theorem 1.22 (page 9)
- (2) Proof for (2): To be a *fuzzy negation*, $\neg m_A(x)$ must be *antitone*, have *weak double negation*, and have *bound-ary condition* $\neg m_1(x) = m_0(x)$ (Definition B.2 page 35).
 - (a) Proof that \neg is *antitone*: by Theorem 1.22 (page 9).
 - (b) Proof that ¬ has *weak double negation*: by Theorem 1.22 (page 9), ¬ is *involutory*, which implies ¬ has *weak double negation*.
 - (c) Proof that $\neg m_1(x) = m_0(x)$: by left hypothesis (B).

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We can now define fuzzy subset operators (\cap, \cup, c) in terms of the fuzzy logic operators (\wedge, \lor, \neg) like this (cross reference Remark 1.2 page 5):

In the case of *set indicator functions*, defining (\land, \lor) is straightforward. But again here in *fuzzy subset logics*, it is not.

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page 11

1.4 Key theorems

This section contains the following key theorems which under very "reasonable" conditions say very roughly the following about the *fuzzy subset logic* operator pair (\land, \lor) :

fuzzy operators idempotency theorem (Theorem 1.25 page 12): *distributive* ⇒ *idempotent* and conversely *non-idempotent* ⇒ *non-distributive fuzzy negation idempotency theorem* (Theorem 1.28 page 14) *excluded middle* or *non-contradiction* ⇒ *non-idempotent* and conversely *idempotent* ⇒ *excluded middle* or *non-contradiction* or both fails *fuzzy min-max theorem* (Theorem 1.26 page 13): *idempotent* ⇒ (∧, ∨) = (min, max) and conversely (∧, ∨) ≠ (min, max) ⇒ *non-idempotent*

The *fuzzy min-max boundary theorem* (next theorem) shows that under three pairs of arguably "reasonable" conditions (including *pointwise evaluation*), the functions $\min(\mathfrak{m}, \mathfrak{n})$ and $\max(\mathfrak{m}, \mathfrak{n})$ act as bounds for any possible operators (\land, \lor) .

Theorem 1.24 (fuzzy min-max boundary theorem) ²⁶ Let M be a set of MEMBERSHIP FUNCTIONS (Definition 1.7 page 6).

1. $\exists f \in [0:1]^{[0:1]^2}$ such that $[m \land n](x) = f[m(x), n(x)] \quad \forall m, n \in \mathbb{M}$ and (POINTWISE EVALUATED) 2. $\exists g \in [0:1]^{[0:1]^2}$ such that $[m \lor n](x) = g[m(x), n(x)] \forall m, n \in M$ (POINTWISE EVALUATED) and 3. $m \vee 0 = m$ $0 \vee m =$ m ∀m∈M (DISJUNCTIVE IDENTITY) and 4. $m \wedge 1 = m$ $1 \wedge m =$ m ∀m∈M (CONJUNCTIVE IDENTITY) and5. $n \leq p \implies m \lor n \leq m \lor p$ and $n \lor m \leq$ p V m ∀m,n,p∈M (DISJUNCTIVE ISOTONE) and $n \leq p \implies m \land n \leq m \land p$ and $n \land m \leq$ 6. p∧m ∀m,n,p∈M (CONJUNCTIVE ISOTONE) $\{m \land n \le \min(m, n)\}$ and $\max(m, n) \le m \lor n$ ∀m∈M }

Proof:

$\max(\mathfrak{m},\mathfrak{n}) = \max([\mathfrak{m} \lor 0], [0 \lor \mathfrak{n}])$	by disjunctive identity property
$\leq \max(m \lor n, 0 \lor n)$	by <i>disjunctive isotone</i> property: $0 \le m \implies m \lor 0 \le m \lor m$
$\leq \max(m \lor n, m \lor n)$	by <i>disjunctive isotone</i> property: $0 \le m \implies 0 \lor n \le m \lor n$
	by definition of $max(\cdot, \cdot)$
$\boxed{\texttt{m} \land \texttt{n}} = \min(\texttt{m} \land \texttt{n}, \texttt{m} \land \texttt{n})$	by definition of $\min(\cdot, \cdot)$
$\leq \min([m \land 1], [m \land n])$	by <i>conjunctive isotone</i> property: $m \le 1 \implies m \land m \le m \land 1$
$\leq \min([m \land 1], [1 \land m])$	by <i>conjunctive isotone</i> property: $m \le 1 \implies m \land m \le 1 \land m$
= min(m, m)	by conjunctive identity property

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How reasonable are the "reasonable conditions" of Theorem 1.24? Let's discuss them briefly:

5 The strength of the **pointwise evalution** condition is perhaps more in its simplicity than in it's reasonableness. In mathematics in general, functions are often mapped to other functions in blatant disregard to this property or one like it. Often such a mapping is referred to as an "operator".



²⁶ [] [Alsina et al.(1983)Alsina, Trillas, and Valverde] page 16 (§1)

- In fuzzy logic, the **identity** properties are "reasonable" because if either the "degree of membership" of *x* is m(x) or *x* has "full membership", then arguably the "degree of membership" of *x* is m(x). Likewise, if both the "degree of membership" of *x* is m(x) and *x* has "absolute non-membership", then arguably the "degree of membership" of *x* is m(x). In order theory, $x \lor 0 = x$ and $x \land 1 = x$ are true of any *bounded lattice* (Proposition A.21 page 26). Their commuted counterparts follow from a weakened form of the *commutative* property. Note that all lattices are *commutative* (Theorem A.14 page 25).
- Solution A.15 page 25). The **isotone** properties are a natural requirement of fuzzy logic—if the "degree of membership" $[m \lor n](x)$, $[n \lor m](x)$, $[m \land m$

The *fuzzy operators idempotency theorem* (next theorem) shows that under a handful of additional arguably "somewhat reasonable" conditions (including the rather "strong" *distributivity* property), the functions \land and \lor are both *idempotent*.

Theorem 1.25 (fuzzy operators idempotency theorem) 27 Let \mathbb{M} be a set of MEMBERSHIP FUNCTIONS (Definition 1.7 page 6).

ſ			such that $[m \land n](x) = f[m(x), n(x)]$		(POINTWISE EVALUATED)	and
	2.	$\exists g \in [0:1]^{[0:1]^2}$	such that $[m \lor n](x) = g[m(x), n(x)]$	∀m,n∈M	(POINTWISE EVALUATED)	and
	З.	$0 \wedge 0 =$	$0 1 \lor 1 = 1$		(BOUNDARY CONDITION)	and
ł	4.	$m \lor 0 =$	$m \mid 0 \lor m = m$	∀m∈M	(DISJUNCTIVE IDENTITY)	and }
	5.	$m \wedge 1 =$	$m \mid 1 \land m = m$	∀m∈M	(CONJUNCTIVE IDENTITY)	and
	6.	$m \land (n \lor p) =$	$(m \land n) \lor (m \land p)$	∀m,n,p∈M	(DISJUNCTIVE DISTRIBUTIVE)	and
l	7.		$(m \lor n) \land (m \lor p)$		(CONJUNCTIVE DISTRIBUTIVE)	J
		∫ 1. m	$= \mathfrak{m} \vee \mathfrak{m} \forall \mathfrak{m} \in \mathbb{M} (\text{disjunctive idemposities})$ $= \mathfrak{m} \wedge \mathfrak{m} \forall \mathfrak{m} \in \mathbb{M} (\text{conjunctive idemposities})$	TENT) and	}	
		→ { 2. m	$=$ $\mathbb{M} \land \mathbb{M} \forall m \in \mathbb{M}$ (conjunctive idempo) () () () () () () () () () () () () ()	5	

[®]Proof:

$m = m \wedge 1$	by conjunctive identity property
$= m \wedge (1 \vee 1)$	by boundary condition
$= (\mathfrak{m} \land 1) \lor (\mathfrak{m} \land 1)$	by conjunctive distributive property
$= m \lor m$	by conjunctive identity property
$m = m \lor 0$	by disjunctive identity property
$= m \lor (0 \land 0)$	by boundary condition
$= (m \lor 0) \land (m \lor 0)$	by disjunctive distributive property
$= m \wedge m$	by disjunctive identity property

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How reasonable are the "reasonable conditions" of Theorem 1.25? Let's discuss them briefly:

In fuzzy logic, the **boundary conditions** are "reasonable" because if x has both "absolute non-membership" **and** "absolute non-membership", then arguably x has "absolute non-membership".

²⁷ [] [Bellman and Giertz(1973)] page 154 $\langle a \lor a = a \land a = a...(10) \rangle$, [] [Alsina et al.(1983)Alsina, Trillas, and Valverde] page 15 $\langle x = G(x, x) \rangle$

page 13

Likewise, if x has either "full membership" **or** "full membership", then arguably x has "full membership". In order theory, the boundary conditions are simply a weakened form of the *idempotent* property, which holds for all lattices (Theorem A.14 page 25).

⁶⁵ The distributive properties hold in *classical logic* (2-valued logic) and more generally in any *Boolean logic*, but not necessarily in any other form of *logic* (Definition C.5 page 50). In order theory, a comparatively small but important class of lattices are *distributive*. But note that in any lattice, the *distributive inequalities* always hold (Theorem A.16 page 25); and if *one* of the distributive properties hold, then they *both* hold (Theorem A.28 page 27).

The *fuzzy min-max theorem* (next theorem) shows that under the *identity* and *isotone* conditions (Theorem 1.24 page 11) and the additional condition of *weak idempotency*, the **only** functions for (\land, \lor) are $(\land, \lor) = (\min, \max) \dots$

Theorem 1.26 (fuzzy min-max theorem) 28 Let \mathbb{M} be a set of MEMBERSHIP FUNCTIONS (Definition 1.7 page 6).

		$\exists f \in [0:1]^{[0:1]^2}$	such that	[m ^	[n](x) =	f[m	(x), n(x)]	∀m,n∈M	(POINTWISE EVALUATED)	and
		$\exists g \in [0:1]^{[0:1]^2}$	such that	[m \	[n](x) =	g [m	n(x), n(x)]	∀m,n∈M	(POINTWISE EVALUATED)	and
		$m \lor 0 = m$			$0 \vee m$	=	m	∀m∈M	(DISJUNCTIVE IDENTITY)	and
ł	4.	$m \wedge 1 = m$			$1 \land m$	=	m	∀m∈M	(CONJUNCTIVE IDENTITY)	and
	5.	$m \land m \ge m$			$m \lor m$	\leq	m	∀m∈M	(WEAK IDEMPOTENT)	and
	6.	$n \leq p \implies m \lor n$	\leq m \vee p	and	$n \lor m$	\leq	p∨m	∀m,n,p∈M	(DISJUNCTIVE ISOTONE)	and
l		$n \leq p \implies m \wedge n$						∀m,n,p∈M	(CONJUNCTIVE ISOTONE)	J
		$\implies \begin{cases} 1. & m \lor \\ 2. & m \land \end{cases}$	m = max m = min	x(m, 1(m,	n) ∀m,n¢ n) ∀m,n¢	∈M ∈M	and $\left.\right\}$			

[∞]Proof:

$\max(m,n) \leq m \lor n$	by fuzzy min-max boundary theorem (Theorem 1.24 page 11)
$\leq \max(m, n) \lor n$	by <i>disjunctive isotone</i> property: $m \le max(m, m)$
$\leq \max(m, n) \vee \max(m, n)$	by <i>disjunctive isotone</i> property: $m \le max(m, m)$
\leq max(m, n)	by weak idempotent property
$\boxed{\min(\mathfrak{m},\mathfrak{n})} \leq \min(\mathfrak{m},\mathfrak{n}) \wedge \min(\mathfrak{m},\mathfrak{n})$	by weak idempotent property
$\leq m \wedge \min(m, n)$	by <i>isotone</i> property of \land : min(m, m) \leq m
\leq m \wedge n	by <i>isotone</i> property of \land : min(m, m) \leq m
\leq min(m, m)	by <i>fuzzy min-max boundary theorem</i> (Theorem 1.24 page 11)

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How reasonable are the "reasonable conditions" of Theorem 1.26? Let's discuss them briefly:

One way to get the *weak idempotent* property or even the stronger *idempotent* property is to force (min, max) to have the *boundary* and *distributive* properties (Theorem 1.25 page 12). However, this is arguably a kind of sledge hammer approach and is not really necessary.



²⁸ This result is very similar to the celebrated result of Bellman and Giertz (1973): **E** [Bellman and Giertz (1973)] pages 153–154 (\$4)

In fuzzy logic, even the stronger *idempotent* property is arguably "reasonable" because if an element x both has a "degree of membership" m(x) and a "degree of membership" m(x), then arguably x has a "degree of membership" m(x). Likewise, if x either has a "degree of membership" m(x). Likewise, if x either has a "degree of membership" m(x). In order theory, all lattices are *idempotent* (Theorem A.14 page 25). But, again, here we only require *weak idempotency*, not *idempotency*.

Corollary 1.27 (Bellman-Giertz 1973 theorem) ²⁹ Let M be a set of MEMBERSHIP FUNCTIONS (Definition 1.7 page 6).

ſ		$\exists f \in [0:1]^{[0:1]^2}$	such that [m	$(\land m](x) =$	f[m	n(x), n(x)]	∀m,n∈M	(POINTWISE EVALUATED)	and
	2.	$\exists g \in [0:1]^{[0:1]^2}$	such that [m	$v \vee n](x) =$: g [n	n(x), n(x)]	∀m,n∈M	(POINTWISE EVALUATED)	and
	З.	$m \lor 0 = m$		$0 \lor m$	=	m	∀m∈M	(DISJUNCTIVE IDENTITY)	and
ſ	4.	$m \wedge 1 = m$		$1 \land m$	=	m	∀m∈M	(CONJUNCTIVE IDENTITY)	and A
	5.	$m \wedge m = m$		m v m	=	m	∀m∈M	(IDEMPOTENT)	and
	6.	$n \leq p \implies m \lor n$	$1 \leq m \vee p$ and	d n∨m	\leq	p∨m	∀m,n,p∈M	(DISJUNCTIVE ISOTONE)	and
l	7.	$n \le p \implies m \land n$	$1 \leq m \wedge p$ and	d n∧m	\leq	p∧m	∀m,n,p∈M	(CONJUNCTIVE ISOTONE)	J
		$\implies \begin{cases} 1. m \lor \\ 2. m \checkmark$	$m = \max(m)$	n,n) ∀m,r	n∈M	and 🔪			
		2. m∧	<pre>\m = min(m</pre>	n,n) ∀m,r	n∈M	5			

PROOF: This follows directly from Theorem 1.26 (page 13).

One big difficulty in *fuzzy subset logic* (Definition 1.11 page 6) is that under "reasonable" conditions, if the fuzzy subset logic is required to have either the *excluded middle* property *or* the *non-contradiction* property (Boolean algebras have both), then the fuzzy subset logic cannot be *idempotent* (next theorem). Furthermore, if a structure is not idempotent, then it is *not* a *lattice* (Theorem A.14 page 25).

Theorem 1.28 (fuzzy negation idempotency theorem) Let $L \triangleq (\mathbb{M}, \lor, \land, \neg, 0, 1; \le)$ be a FUZZY SUB-SET LOGIC (Definition 1.11 page 6). Let (\land, \lor) be POINTWISE EVALUATED (Definition 1.12 page 7). If there exists p such that $\neg m(p) = m(p) \in (0: 1)$ then

(FIXED POINT CONDITION)										
(A).	$m \lor \neg m$	=	1	∀m∈M	(EXCLUDED MIDDLE)	\Rightarrow	$m \lor m$	\neq	m	(NON-IDEMPOTENT)
<i>(B)</i> .	$m \land \neg m$	=	0	∀m∈M	(NON-CONTRADICTION)	\Rightarrow	$m \land m$	\neq	m	(NON-IDEMPOTENT)

[®]Proof:

$1 = m(p) \lor \neg m(p)$	by excluded middle hypothesis (A)
$= m(p) \vee m(p)$	by fixed point hypothesis
= m(p)	if \lor is <i>idempotent</i>
$\implies \neg m(p) = 0$	because $\neg m(p) = \neg 1 = 0$
\implies m(p) = 0	by fixed point hypothesis
\implies contradiction	because $m(p) = 1 \neq 0 = m(p)$ is a contradiction
\implies \lor is non-idempotent	

 $0 = \mathsf{m}(p) \land \neg \mathsf{m}(p)$

by non-contradiction hypothesis (B)

²⁹ [] [Bellman and Giertz(1973)] pages 153–154 (§4)

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= m(p)	$) \wedge m(p)$	by fixed point hypothesis
= m(p))	if \land is <i>idempotent</i>
\Rightarrow	$\neg m(p) = 1$	because $\neg m(p) = \neg 0 = 1$
\Rightarrow	$\mathbf{m}(p) = 0$	by fixed point hypothesis
\Rightarrow	contradiction	because $m(p) = 0 \neq 1 = m(p)$ is a contradiction
\implies	∧ is <i>non-idempotent</i>	

How reasonable are the "reasonable conditions" of Theorem 1.28? Let's discuss them briefly:

One of these "reasonable conditions" is that at some point $p, \neg m(p) = m(p) \in (0:1)$. Because fuzzy negations are *antitone*, in some cases this is arguably a "reasonable" assumption, especially if m(x) is *continuous* and *strictly antitone*. However, be warned that it is not always the case that there is such a point p in a fuzzy subset logic $(M, \lor, \land, \neg, 0, 1; \le)$. For example, under standard negation, and if the universal set is finite, then it is certainly possible that p does not exist, as in the example illustrated to the right with $X \triangleq \{a, b, c, d\}$:

 $\begin{array}{c|ccc} x & m(x) & \neg m(x) \\ \hline d & 1 & 0 \\ c & \frac{3}{4} & \frac{1}{4} \\ b & \frac{1}{4} & \frac{3}{4} \\ a & 0 & 1 \\ \hline \end{array}$

Corollary 1.29 (Dubois-Padre 1980 theorem) ³⁰ Let $L \triangleq (M, \lor, \land, \neg, 0, 1; \le)$ be a FUZZY SUBSET LOGIC (Definition 1.11 page 6). Let (\land, \lor) be POINTWISE EVALUATED (Definition 1.12 page 7).

If $\neg(x)$ is continuous and strictly antitone then

(A).	$m \lor \neg m$	=	1	∀m∈M	(EXCLUDED MIDDLE)	\implies	$m \lor m$	\neq	m	(NON-IDEMPOTENT)
<i>(B)</i> .	$m \land \neg m$	=	0	∀m∈M	(NON-CONTRADICTION)	\Rightarrow	$m \land m$	¥	m	(NON-IDEMPOTENT)

PROOF: This follows directly from Theorem 1.28 (page 14).

1.5 Examples of non-ortho and non-Boolean fuzzy subset

This section presents some examples of *fuzzy subset logics*. They all have "problems". The problem of the first example is just that is a kind of trivial fuzzy subset logic in that it is 2-valued and equivalent to the classical subset logic. In all the other examples, the "problem" involves not having one or more of the following four properties:

- (1). *disjunctive idempotence*: $x \lor x = x$ and (2). *conjunctive idempotence*: $x \land x = x$ and
- (3). excluded middle: $x \lor \neg x = 1$ and
- (4). *non-contradiction*: $x \land \neg x = 0$

Actually, this is a problem only as far as not having an *ortho* or *Boolean* logic is a problem—because all *ortho logics* and all *Boolean logics* have these properties. And so if even one is missing, the logic is neither an *ortho logic* nor a *Boolean logic*. Also note that if a logic does not have both (1) and (2), then it cannot even be constructed on a *lattice* at all...and as defined in this paper, is not even a *logic*.

Example 1.30 Consider the structure $L \triangleq (M, \lor, \land, \neg, 0, 1; \le)$ in Figure 2 page 16 (A).

1. *L* is a *Boolean lattice* (Definition A.41 page 30).





³⁰ \square [Dubois and Padre(1980)], page 62 (P1, requires $\neg(x)$ be to *continuous* and *strictly antitone*) \square [Fodor and Yager(2000)], pages 130–131 (Theorem 2, reference to previous without proof)

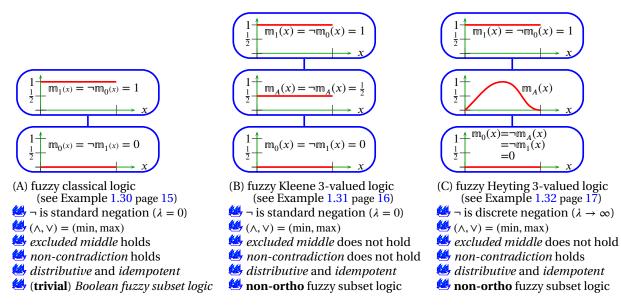


Figure 2: fuzzy logics (Definition C.5 page 50) on linear lattices (Definition A.11 page 24) L_2 and L_3

- 2. The function ¬ is an *ortho negation* (Definition B.3 page 35) (and hence also is a *fuzzy negation* Definition B.2 page 35, Figure 9 page 34).
- 3. The negation ¬m of each *membership function* m (Definition 1.7 page 6) is the *standard negation* (Definition 1.19 page 9).
- 4. *L* together with the *classical implication* (Example C.4 page 46) is the *classical logic* (Example C.6 page 50) and is also a *fuzzy logic* (Definition C.5 page 50).
- 5. Because the membership functions m(x) equal 0 or 1 only, the fuzzy subsets are equivalent to crisp sets.
- 6. *L* is *linear* (Definition A.11 page 24) and therefore *distributive* (Theorem A.30 page 27); and therefore (\land, \lor) are *idempotent* (Theorem 1.25 page 12).
- 7. The *excluded middle* and *non-contradiction* properties hold in L, but L is also *idempotent*. This does not contradict Theorem 1.28 (page 14), because \neg does not satisfy the *fixed point condition* (there is no point *p* such that $\neg m(p) = m(p) \in (0 : 1)$).

Example 1.31 Consider the structure $L \triangleq (M, \lor, \land, \neg, 0, 1; \leq)$ in Figure 2 page 16 (B).

- 1. The function \neg is a *Kleene negation* (Definition B.3 page 35) (and hence a *de Morgan negation*), and is also a *fuzzy negation* (Example B.25 page 41).
- 2. The negation $\neg m$ of each *membership function* m is the *standard negation* because for example $m_A(x) \triangleq \frac{1}{2} = 1 \frac{1}{2} = \frac{1}{2} =$
- 3. *L* is *linear* (Definition A.11 page 24) and therefore *distributive* (Definition A.27 page 27, Theorem A.30 page 27); and therefore (∧, ∨) are *idempotent* (Theorem 1.25 page 12).
- 4. *L* does not have the *excluded middle* property because $m_A \lor \neg m_A = m_A \lor m_A = m_A \triangleq \frac{1}{2} \neq 1.$
- 5. *L* does not have the *non-contradiction* property because

 $\mathbf{m}_A\wedge \neg \mathbf{m}_A\mathbf{m}_A\wedge \mathbf{m}_A=\mathbf{m}_A\triangleq \mathbf{1}_{\!\!\!\!/}\neq 0.$

6. $(\land, \lor) = (\min, \max)$ (Definition 1.15 page 7), which together with the *idempotence* property agrees with Theorem 1.26 (page 13).

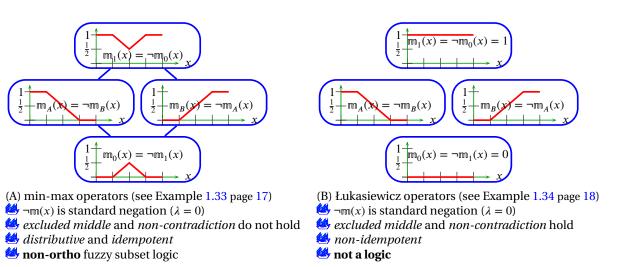


Figure 3: *fuzzy logic* on *M*₂ *lattice*

7. L together with the *classical implication* (Example C.4 page 46) is a *Kleene 3-valued logic* (Example C.7 page 51) and also a *fuzzy logic* (Definition C.5 page 50).

Example 1.32 Consider the structure $L \triangleq (M, \lor, \land, \neg, 0, 1; \leq)$ in Figure 2 page 16 (C).

- 1. The function ¬ is an *intuitionistic negation* (Definition B.3 page 35) (and hence also a *fuzzy negation* Example B.26 page 41).
- 2. The negation ¬m of each membership function m is the discrete negation (Example B.16 page 38).
- 3. *L* does **not** have the *excluded middle* property because $m_A \lor \neg m_A \neq 1$
- 4. *L* does have the *non-contradiction* property.
- 5. *L* is *linear* (Definition A.11 page 24) and therefore *distributive* (Definition A.27 page 27, Theorem A.30 page 27); and therefore (∧, ∨) are *idempotent* (Theorem 1.25 page 12).
- 6. Note that having both *non-contradiction* and *idempotency* does not conflict with Theorem 1.28 (page 14) because it does not satisfy the *fixed point condition*.
- 7. $(\land, \lor) = (\min, \max)$ (Definition 1.15 page 7), which together with the *idempotence* property agrees with (Theorem 1.26 page 13).
- 8. L together with the *classical implication* (Example C.4 page 46) is a *Heyting 3-valued logic* (Example C.10 page 52) and also a *fuzzy logic* (Definition C.5 page 50).

Example 1.33 Consider the structure *L* illustrated in Figure 3 page 17 (A).

- 1. The function ¬ is a *Kleene negation* (Definition B.3 page 35) and also a *fuzzy negation* (Definition B.2 page 35).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).
- 3. The \land and \lor operators are the *min-max operators* (Definition 1.15 page 7).
- 4. Because $(\land, \lor) = (\min, \max)$, *L* is a lattice (Proposition 1.16 page 7).
- 5. Because *L* is a lattice, *L* is *idempotent* (Theorem A.14 page 25). Conversely, *idempotence* and (min, max) are in agreement with Theorem 1.26 (page 13).
- 6. *L* does *not* have the *excluded middle* property because $m_A \lor \neg m_A = m_1 \neq 1$.
- 7. *L* does *not* have the *non-contradiction* property because $m_A \wedge \neg m_A = m_0 \neq 0$.
- 8. The *idempotence* property is *not* in disagreement with Theorem 1.28 (page 14) because *L* does not have the *excluded middle* or *non-contradiction* properties.





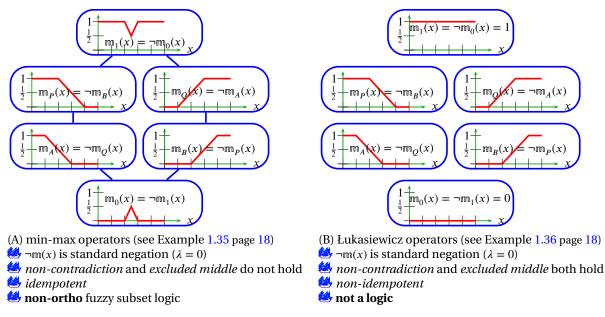


Figure 4: *fuzzy logic* on O₆ *lattice*

9. *L* together with any of the six *implication* functions listed in Example C.4 (page 46) is a *fuzzy subset logic* (Definition 1.11 page 6).

Example 1.34 Consider the structure *L* illustrated in Figure 3 page 17 (B).

- 1. The function \neg is an *ortho negation* (Definition B.3 page 35) (and thus also a *fuzzy negation*).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).
- 3. The \land and \lor operators are the *Łukasiewicz operators* (Definition 1.18 page 9). Under these operators, *L* has the *non-contradiction* and *excluded middle* properties, but *L* is *not idempotent* (e.g. $\mathbb{m}_A \lor \mathbb{m}_A \neq \mathbb{m}_A$), and so *L* is not a lattice (Theorem 1.28 page 14, Theorem A.14 page 25).

Example 1.35 Consider the structure *L* illustrated in Figure 4 page 18 (A).

- 1. The function \neg is an *ortho negation* (Definition B.3 page 35) (and thus also a *fuzzy negation*).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).
- 3. The \land and \lor operators are the *min-max operators* (Definition 1.15 page 7).
- 4. Because $(\land, \lor) = (\min, \max)$, *L* is a lattice (Proposition 1.16 page 7).
- 5. Because *L* is a lattice, *L* is *idempotent* (Theorem A.14 page 25). Conversely, *idempotence* and (min, max) are in agreement with Theorem 1.26 (page 13).
- 6. *L* does *not* have the *excluded middle* property because $m_A \vee \neg m_A = m_1 \neq 1$.
- 7. *L* does *not* have the *non-contradiction* property because $m_A \wedge \neg m_A = m_0 \neq 0$.
- 8. *L* together with any of the six *implication* functions listed in Example C.4 (page 46) is a *fuzzy subset logic* (Definition 1.11 page 6).

Example 1.36 Consider the structure *L* illustrated in Figure 4 page 18 (B).

- 1. The function \neg is an *ortho negation* (Definition B.3 page 35) (and thus also a *fuzzy negation*).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).

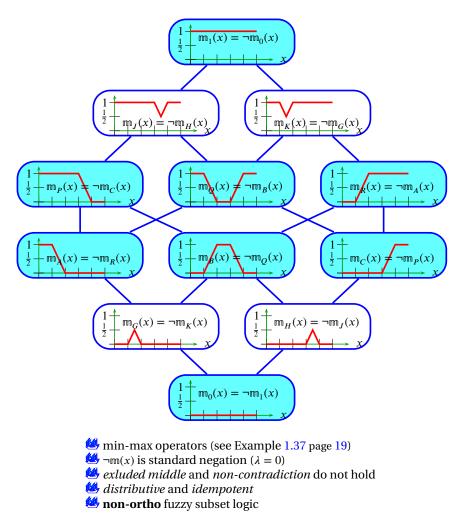


Figure 5: *fuzzy logic* on lattice with L_2^3 sublattice

3. The ∧ and ∨ operators are the *Łukasiewicz operators* (Definition 1.18 page 9). Under these operators,
 L has the *non-contradiction* and *excluded middle* properties, but *L* is *not idempotent*, and so
 L is not a lattice (Theorem 1.28 page 14).

Example 1.37 Consider the structure *L* illustrated in Figure 5 (page 19).

- 1. The function \neg is an *ortho negation* (Definition B.3 page 35) (and thus also a *fuzzy negation*).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).
- 3. The \land and \lor operators are the *min-max operators* (Definition 1.15 page 7).
- 4. Because $(\land, \lor) = (\min, \max)$, *L* is a lattice (Proposition 1.16 page 7).
- 5. Because *L* is a lattice, *L* is *idempotent* (Theorem A.14 page 25). Conversely, *idempotence* and (min, max) are in agreement with Theorem 1.26 (page 13).
- 6. *L* does *not* have the *excluded middle* property because for example

$$\mathbf{m}_A \vee \neg \mathbf{m}_A = \mathbf{m}_A \vee \mathbf{m}_R = \mathbf{m}_K \neq 1.$$

7. *L* does *not* have the *non-contradiction* property because for example $m_A \wedge \neg m_A = m_A \wedge m_B = m_G \neq 0.$





- 8. *L* does not contain M_3 or N_5 and so is *distibutive* (Theorem A.30 page 27). (also cross reference Theorem 1.25 page 12 and Theorem 1.28 page 14).
- 9. *L* is *non-Boolean*, but has an L_2^3 Boolean sublattice (shaded in Figure 5).

2 Boolean and ortho fuzzy subset logics

The Introduction described the problem of constructing *Boolean fuzzy subet logics* and more generally *ortho fuzzy subet logics*. It also briefly described a "solution". This section presents this solution in more detail.

Simply put, a solution is available if we are willing to give up the *pointwise evaluation* condition (Definition 1.12 page 7). In particular, we can proceed as follows:

- (1) We give up the *pointwise evaluation* condition.
- (2) We define the *ordering relation* (Definition A.1 page 22) \leq in the *fuzzy subset logic* $L \triangleq (\mathbb{M}, \vee, \wedge, \neg, 0, 1; \leq)$ to be the *pointwise ordering relation* (Definition A.7 page 23).
- (3) In a *lattice* (Definition A.11 page 24), the definitions of the ordering relation \leq and operators (\land , \lor) are not independent—the ordering relation defines the operators (Definition A.9 page 24, Definition A.8 page 24) and the operators define the ordering relation (Proposition A.10 page 24).
- (4) Traditionally in fuzzy logic literature, we first define a *pointwise evaluated* (Definition 1.12 page 7) pair of operators (\land, \lor) , and then define the ordering relation \leq in terms of (\land, \lor) . For example, if $(\land, \lor) = (\min, \max)$, then

$$\begin{array}{ccc} x \leq y & \stackrel{\text{def}}{\iff} & \max(x,y) = y \\ x \leq y & \stackrel{\text{def}}{\iff} & \min(x,y) = x \end{array}$$

- (5) However, here we take a kind of converse approach: We first define a *pointwise* ordering relation \leq (Definition A.7 page 23), and then define the operators (\land, \lor) in terms of \leq . In doing so, (\land, \lor) may possibly no longer satisfy the *pointwise evaluation* condition.
- (6) By carefully constructing a set of *membership functions* (Definition 1.7 page 6) M, we can construct *fuzzy subset logics* (Definition 1.11 page 6) on Boolean and other types of lattice structures.
- (7) A fuzzy subset logic then inherits the properties of the lattice it is constructed on. So, for example, if a fuzzy subset logic is constructed on a Boolean lattice, then that fuzzy subset logic is also *Boolean* with all the properties of a Boolean algebra (Theorem A.42 page 30) including the *non-contradiction, excluded middle, idempotent,* and *distributive* properties.
- (8) Despite Theorem 1.26 page 13 and Theorem 1.28, this is all possible because (∧, ∨) is no longer *pointwise evaluated* (Definition 1.12 page 7). The result of, say, [m ∨ m](x) at the point x is no longer necessarily the result of the two values m(x) and m(x) alone, but instead [m ∨ m](x) at the point x may be the result of entire membership functions in the structure or even the position of m and m in the structure.
- (9) Examples follow.

Example 2.1 Consider the structure $L \triangleq (\mathbb{M}, \vee, \wedge, \neg, 0, 1; \leq)$ with $M \triangleq \{\mathbb{m}_0, \mathbb{m}_A, \mathbb{m}_B, \mathbb{m}_1\}$ illustrated in Figure 6 page 21 (A).

page 21

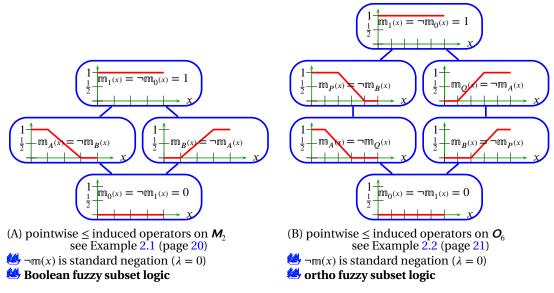


Figure 6: *fuzzy logic* on M_2 *lattice*

- 1. The function \neg is an *ortho negation* (Definition B.3 page 35) (and thus also a *fuzzy negation*).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).
- 3. *L* is very similar to the structure in Example 1.34 (page 18), which fails to even be a logic.
- 4. However the structure of this example has a valid ordering relation \leq (*pointwise ordering relation*), has valid operators (\land , \lor) defined in terms of \leq (Definition A.9 page 24, Definition A.8 page 24), and is a *Boolean lattice* with all the accompanying Boolean properties including the *non-contradiction, excluded middle, idempotency*, and *distributivity*.
- 5. In this example, the operators are no longer *Łukasiewicz operators* (as in Example 1.34), but some other operators (not explicitly given in terms of a function of the form given in Theorem 1.26 (page 13)).
- 6. This Boolean lattice together with the *classical implication* (Example C.4 page 46) is an *ortho logic* (and thus also a *fuzzy subset logic*—Definition 1.11 page 6).

Example 2.2 Consider the structure $L \triangleq (\mathbb{M}, \lor, \land, \neg, 0, 1; \le)$ with $M \triangleq \{\mathbb{m}_0, \mathbb{m}_A, \mathbb{m}_B, \mathbb{m}_P, \mathbb{m}_Q, \mathbb{m}_1\}$ illustrated in Figure 6 page 21 (B).

- 1. The function \neg is an *ortho negation* (Definition B.3 page 35) (and thus also a *fuzzy negation*).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).
- 3. *L* is very similar to the structure in Example 1.36 (page 18), which fails to be a logic.
- 4. However the structure of this example has a valid ordering relation \leq , has valid operators (\land, \lor) defined in terms of \leq , and is an *orthocomplemented lattice* (Definition A.44 page 31) with all the accompanying properties of an orthocomplemented lattice including the *non-contradiction*, *excluded middle* and *idempotency* properties (Theorem A.14 page 25, Definition A.44 page 31, Theorem A.47 page 32).
- 5. In this example, the operators are no longer *Łukasiewicz operators* (as in Example 1.36, but some other operators.
- 6. This orthocomplemented lattice together with any one of the *implications* given in Example C.4 (page 46) is an *ortho logic* (and thus also a *fuzzy subset logic*—Definition 1.11 page 6).





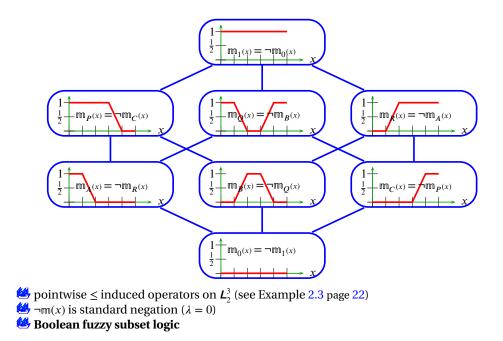


Figure 7: *fuzzy logic* on *Boolean lattice* L_2^3

Example 2.3 Consider the structure $L \triangleq (M, \lor, \land, \neg, 0, 1; \le)$ illustrated in Figure 7 (page 22).

- 1. The function \neg is an *ortho negation* (Definition B.3 page 35) (and thus also a *fuzzy negation*).
- 2. The negation ¬m of each membership function m is the *standard negation* (Definition 1.19 page 9).
- 3. *L* is somewhat similar to the fuzzy subset logic of Example 1.37 (page 19), which fails to be *Boolean*.
- 4. However the structure of this example has a valid ordering relation ≤, has valid operators (∧, ∨) defined in terms of ≤, and is a *Boolean lattice* with the accompanying Boolean properties including the *non-contradiction, excluded middle, idempotent*, and *distributivity* properties (Theorem A.42 page 30).
- 5. In this example, the operators are no longer *min-max operators* (as in Example 1.37), but some other operators.
- 6. This Boolean lattice together with the *classical implication* (Example C.4 page 46) is an *ortho logic* (and thus also a *fuzzy logic*).

Appendix A Background: Order

A.1 Ordered sets

Definition A.1 ³¹ Let 2^{XX} be the set of all *relation*s on a set *X*. A relation \leq is an **order relation** in 2^{XX} if

1.	$x \leq x$	$\forall x \in X$	(reflexive)	and	preorder
2.	$x \le y$ and $y \le z \implies x \le z$	$\forall x, y, z \in X$	(transitive)	and	
3.	$x \le y$ and $y \le x \implies x = y$	$\forall x, y \in X$	(anti-symmetric)	_	

The pair (X, \leq) is an **ordered set** if \leq is an *order relation* on a set X. If $x \leq y$ or $y \leq x$, then elements x and y are said to be **comparable**, denoted $x \sim y$. Otherwise they are **incomparable**, denoted x||y.

Definition A.2 ³² Let (X, \leq) be an *ordered set*. Let 2^{XX} be the set of all relations on X. The relations $\geq, <, > \in 2^{XX}$ are defined as follows:

 $x \ge y \quad \stackrel{\text{def}}{\iff} \quad y \le x$ $x < y \quad \stackrel{\text{def}}{\longleftrightarrow} \quad x \le y \quad \text{and} \quad x \ne y \quad \forall x, y \in X$ $x > y \quad \stackrel{\text{def}}{\iff} \quad x \ge y \quad \text{and} \quad x \ne y \quad \forall x, y \in X$

Definition A.3 ³³ An ordered set (X, \leq) (Definition A.1 page 22) is **linear**, or is a **linearly ordered set**, if $x \le y$ or $y \le x$ $\forall x, y \in X$ (comparable).

A *linearly ordered set* is also called a **totally ordered set**, a **fully ordered set**, and a **chain**.

Definition A.4 ³⁴ *y* covers *x*, denoted $x \prec y$, in the ordered set (X, \leq) if

1. $x \leq y$			$(y ext{ is greater than } x)$	and
2. $(x \le z \le y)$	\implies	(z = x or z = y)	(there is no element between x and y).	

An ordered set can be represented graphically by a *Hasse diagram* (next definition).

Definition A.5 Let (X, \leq) be an ordered pair. A diagram is a **Hasse diagram** of (X, \leq) if

- 1. Each element in *X* is represented by a dot or small circle and
- 2. for each x, $y \in X$, if $x \prec y$, then y appears at a higher position than x and a line connects x and v.

Example A.6 Here are three ways of representing the ordered set $(2^{\{x,y\}}, \subseteq)$;

(1) Hasse diagram: $\{x\} \bigcirc \{x, y\}$

(2) Sets of ordered pairs specifying *order relations*: $\subseteq = \begin{cases} (\emptyset, \emptyset), & (\{x\}, \{x\}), & (\{y\}, \{y\}), & (\{x, y\}, \{x, y\}), \\ (\emptyset, \{x\}), & (\emptyset, \{y\}), & (\emptyset, \{x, y\}), & (\{x\}, \{x, y\}), & (\{y\}, \{x, y\}) \end{cases}$

(3) Sets of ordered pairs specifying *covering relations*: $\prec = \left\{ (\emptyset, \{x\}), (\emptyset, \{y\}), (\{x\}, \{x, y\}), (\{y\}, \{x, y\}) \right\}$

Definition A.7 Let Y^X be the set of all functions that map from a set X to a set Y. Let (Y, \leq) be an ordered set. The relation \leq is a pointwise ordering relation on Y^X with respect to \leq if for all f, g $\in Y^X$ $f \leq g$ $\{f(x) \ge g(x) \quad \forall x \in X\}$ \implies

³² [[Peirce(1880)] page 2

³³ S [MacLane and Birkhoff(1999)] page 470, []. [Ore(1935)] page 410

³⁴ [] [Birkhoff(1933)] page 445

Definition A.8 Let (X, \leq) be an ordered set and 2^X the power set of *X*.

For any set $A \in 2^X$, *c* is an **upper bound** of *A* in (X, \leq) if

1. $x \in A \implies x \leq c$.

An element *b* is the **least upper bound**, or **l.u.b.**, of *A* in (X, \leq) if

2. *b* and *c* are *upper bounds* of $A \implies b \le c$.

The *least upper bound* of the set *A* is denoted $\bigvee A$. It is also called the **supremum** of *A*, which is denoted sup *A*. The **join** $x \lor y$ of *x* and *y* is defined as $x \lor y \triangleq \bigvee \{x, y\}$.

Definition A.9 Let (X, \leq) be an ordered set and 2^X the power set of X. For any set $A \in 2^X$, p is a **lower bound** of A in (X, \leq) if

1.
$$p \le x \quad \forall x \in A$$
.

An element *a* is the **greatest lower bound**, or **glb**, of *A* in (X, \leq) if

2. *a* and *p* are *lower bounds* of $A \implies p \le a$.

The *greatest lower bound* of the set *A* is denoted $\bigwedge A$. It is also called the **infimum** of *A*, which is denoted inf *A*. The **meet** $x \land y$ of *x* and *y* is defined as $x \land y \triangleq \bigwedge \{x, y\}$.

Proposition A.10

 $x \leq y \iff \left\{\begin{array}{rrrr} 1. & x \wedge y &= x & and \\ 2. & x \vee y &= y \end{array}\right\} \quad \forall x, y \in X$

A.2 Lattices

A.2.1 General lattices

Definition A.11 ³⁵ An algebraic structure $L \triangleq (X, \lor, \land; \leq)$ is a **lattice** if

1. (X, \leq) is an ordered set $((X, \leq)$ is a partially or totally ordered set)and2. $x, y \in X \implies \exists (x \lor y) \in X$ (every pair of elements in X has a *least upper bound* in X)and3. $x, y \in X \implies \exists (x \land y) \in X$ (every pair of elements in X has a *greatest lower bound* in X).andThe *lattice* L is *linear* if (X, \leq) is a *linearly ordered set* (Definition A.3 page 23).

Example A.12 ³⁶The *ordered set* (X, \leq) illustrated by the *Hasse diagram* to the right is **not** a *lattice* because, *a* and *b* have no *lower bound* in *X*.

Example A.13 ³⁷ The *ordered set* illustrated by the *Hasse diagram* to the right is **not** a *lattice* because, for example, while *a* and *b* have *upper bounds c*, *d*, and 1, still *a* and *b* have no *least upper bound*. The element 1 is not the *least upper bound* because $c \le 1$ and $d \le 1$. And neither *c* nor *d* is a *least upper bound* because $c \nleq d$ and $d \nleq c$; rather, *c* and *d* are *incomparable* (*a*||*b*). Note that if we remove either or both of the two lines crossing the center, the *ordered set* becomes a *lattice*.

 $a \circ \circ b$

³⁵ S [MacLane and Birkhoff(1999)] page 473, S [Birkhoff(1948)] page 16, [[Ore(1935)], [] [Birkhoff(1933)] page 442, [Maeda and Maeda(1970)], page 1

³⁶ **(Dominich**(2008)] page 50 (Fig. 3.5)

³⁷ S [Birkhoff(1967)] pages 15–16, S [Oxley(2006)] page 54, S [Dominich(2008)] page 50 (Figure 3.6), [[Farley(1997)] page 3, [[Farley(1996)] page 5

Theorem A.14 ³⁸ $(X, \lor, \land; \leq)$ *is a* LATTICE \iff

$x \lor x$	=	x	$x \wedge x$	=	x	$\forall x \in X$	(IDEMPOTENT)	and
$x \lor y$	=	$y \lor x$	$x \wedge y$	=	$y \wedge x$	$\forall x, y \in X$	(COMMUTATIVE)	and
$(x \lor y) \lor z$	=	$x \lor (y \lor z)$	$(x \land y) \land z$	=	$x \wedge (y \wedge z)$	$\forall x, y, z \in X$	(ASSOCIATIVE)	and
$\int x \vee (x \wedge y)$	=	x	$x \land (x \lor y)$	=	x	$\forall x, y \in X$	(ABSORPTIVE).	J

Proposition A.15 (Monotony laws) ³⁹ Let $(X, \lor, \land; \le)$ be a LATTICE.

ſ	а	\leq	b	and	$a \wedge x$			and		$\forall a, b, x, y \in X$
J	x	\leq	у		$a \lor x$	\leq	$b \lor y$		ſ	$\forall a, b, x, y \in A$

Theorem A.16 (distributive inequalities) ⁴⁰ $(X, \lor, \land; \leq)$ is a LATTICE \implies

 $\begin{cases} x \land (y \lor z) \ge (x \land y) \lor (x \land z) & \forall x, y, z \in X \text{ (Join super-distributive) and} \\ x \lor (y \land z) \le (x \lor y) \land (x \lor z) & \forall x, y, z \in X \text{ (Meet sub-distributive) and} \\ (x \land y) \lor (x \land z) \lor (y \land z) \le (x \lor y) \land (x \lor z) \land (y \lor z) & \forall x, y, z \in X \text{ (Median inequality).} \end{cases}$

Theorem A.17 (Modular inequality) ⁴¹ Let $(X, \lor, \land; \le)$ be a LATTICE. $x \le y \implies x \lor (y \land z) \le y \land (x \lor z)$

Theorem A.14 (page 25) gives 4 necessary and sufficient pairs of properties for a structure $(X, \lor, \land; \leq)$ to be a *lattice*. However, these 4 pairs are actually *overly* sufficient (they are not *independent*), as demonstrated next.

Theorem A.18 42

A.2.2 Bounded *lattices*

Definition A.19 Let $L \triangleq (X, \lor, \land; \le)$ be a *lattice*. Let $\bigvee X$ be the least upper bound of (X, \le) and let $\bigwedge X$ be the greatest lower bound of (X, \le) . *L* is **upper bounded** if $(\bigvee X) \in X$. *L* is **lower bounded** if $(\bigwedge X) \in X$. *L* is **bounded** if *L* is both upper and lower bounded. A *bounded lattice* is optionally denoted $(X, \lor, \land, 0, 1; \le)$, where $0 \triangleq \bigwedge X$ and $1 \triangleq \bigvee X$.

Proposition A.20 Let $L \triangleq (X, \lor, \land; \le)$ be a LATTICE.

 $\{L \text{ is finite}\} \implies \{L \text{ is bounded}\}$





³⁸ **►** [MacLane and Birkhoff(1999)] pages 473–475 〈LEMMA 1, THEOREM 4〉, **►** [Burris and Sankappanavar(1981)] pages 4–7, **►** [Birkhoff(1938)], pages 795–796, **■** [Ore(1935)] page 409 〈(*α*)〉, **■** [Birkhoff(1933)] page 442, **■** [Dedekind(1900)] pages 371–372 〈(1)–(4)〉

³⁹ 🗢 [Givant and Halmos(2009)] page 39, 📃 [Doner and Tarski(1969)] pages 97–99

⁴⁰ \square [Davey and Priestley(2002)] page 85, \square [Grätzer(2003)] page 38, \blacksquare [Birkhoff(1933)] page 444, \blacksquare [Korselt(1894)] page 157, \square [Müller-Olm(1997)] page 13 (terminology)

⁴¹ (Birkhoff(1948)] page 19, (Burris and Sankappanavar(1981)] page 11, (Dedekind(1900)] page 374

⁴² 🖻 [Padmanabhan and Rudeanu(2008)] pages 7–8, 🛸 [Beran(1985)] page 5, 📃 [McKenzie(1970)] page 24

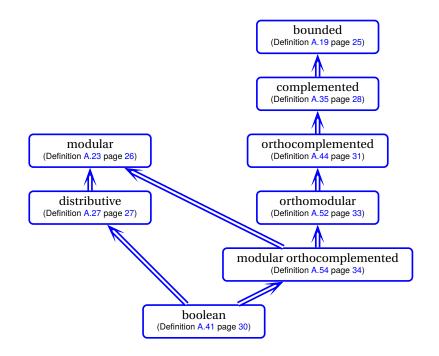


Figure 8: relationships between selected lattice types

Proposition A.21 Let $L \triangleq (X, \lor, \land; \leq)$ be a LATTICE with $\bigvee X \triangleq 1$ and $\bigwedge X \triangleq 0$.

		$\begin{cases} x \lor 1 = 1 \forall x \in X (upper bounded) and \end{cases}$
{ <i>L is</i> bounded }	\implies	$\left\{\begin{array}{ll} x \land 0 &= 0 \forall x \in X (lower bounded) and \\ x \lor 0 &= x \forall x \in X (join-identity) and \end{array}\right\}$
{ L IS BOUNDED }	\rightarrow	$x \lor 0 = x \forall x \in X$ (join-identity) and
		$x \wedge 1 = x \forall x \in X (meet-identity)$

A.2.3 Modular lattices

Definition A.22 ⁴³ Let
$$(X, \lor, \land; \le)$$
 be a lattice. The **modularity** relation $\circledast \in 2^{XX}$ is defined as $x \circledast y \iff \{(x, y) \in X^2 | a \le y \implies y \land (x \lor a) = (y \land x) \lor a \forall a \in X\}.$

Modular lattices are a generalization of *distributive lattices* in that all distributive lattices are modular, but not all modular lattices are distributive (Example A.33 page 28, Example A.34 page 28).

Definition A.23 ⁴⁴ A lattice $(X, \lor, \land; \leq)$ is **modular** if $x \otimes y \quad \forall x, y \in X$.

Definition A.24 (N5 lattice/pentagon) ⁴⁵ The **N5 lattice** is the ordered set $(\{0, a, b, p, 1\}, \leq)$ with cover relation

 $\prec = \{(0, a), (a, b), (b, 1), (p, 1), (0, p)\}.$

The N5 lattice is also called the **pentagon**. The N5 lattice is illustrated by the Hasse diagram to the right.

⁴⁴ ► [Birkhoff(1967)] page 82, ► [Maeda and Maeda(1970)], page 3 (Definition (1.7))

⁴³ \cong [Stern(1999)] page 11, \cong [Maeda and Maeda(1970)], page 1 (Definition (1.1)), \cong [Maeda(1966)] page 248

⁴⁵ [Beran(1985)] pages 12–13, [] [Dedekind(1900)] pages 391–392 ((44) and (45))

Theorem A.25 ⁴⁶ Let *L* be a LATTICE (Definition A.11 page 24). \iff *L does* Not *contain the* N5 LATTICE *L is* modular

Examples of *modular lattices* are provided in Example A.33 (page 28) and Example A.34 (page 28).

A.2.4 Distributive lattices

Definition A.26 ⁴⁷ Let $(X, \lor, \land; \le)$ be a *lattice* (Definition A.11 page 24). The **distributivity relation** $@ \in 2^{XXX}$ and the **dual distributivity relation** $@^* \in 2^{XXX}$ are defined as $\mathbb{D}^* \triangleq \left\{ (x, y, z) \in X^3 \mid x \lor (y \land z) = (x \lor y) \land (x \lor z) \right\} \quad (\text{each} (x, y, z) \text{ is conjunctive distributive}).$ A triple (x, y, z) is **distributive** if $(x, y, z) \in \mathbb{O}$ and such a triple is alternatively denoted as $(x, y, z) \otimes \mathbb{O}$.

Definition A.27 ⁴⁸ A lattice $(X, \lor, \land; \leq)$ is **distributive** if $(x, y, z) \in \bigcirc \forall x, y, z \in X$

Not all lattices are *distributive*. But if a lattice *L* does happen to be distributive—that is all triples in *L* satisfy the *distributive* property—then all triples in *L* also satisfy the *dual distributive* property, as well as another property called the *median property*. The converses also hold (next theorem).

Theorem A.28 ⁴⁹ Let $L \triangleq (X, \lor, \land; \le)$ be a LATTICE. The following statements are all equivalent:

	(1).	<i>L is</i> distributive		(Definition A.27 page 27)
\Leftrightarrow	(2).	$x \land (y \lor z) = (x \land y) \lor (x \land z)$	$\forall x, y, z \in X$	(DISJUNCTIVE DISTRIBUTIVE)
\Leftrightarrow	(3).	$x \lor (y \land z) = (x \lor y) \land (x \lor z)$	$\forall x, y, z \in X$	(CONJUNCTIVE DISTRIBUTIVE)
\Leftrightarrow	(4).	$(x \lor y) \land (x \lor z) \land (y \lor z) = (x \land y) \lor (x \land z) \lor (y \land z)$	$\forall x, y, z \in X$	(MEDIAN PROPERTY)

Definition A.29 (M3 lattice/diamond) ⁵⁰ The M3 lattice is the ordered set $(\{0, p, q, r, 1\}, \leq)$ with covering relation

 $\prec = \{ (p, 1), (q, 1), (r, 1), (0, p), (0, q), (0, r) \}.$

The M3 lattice is also called the **diamond**, and is illustrated by the Hasse diagram to the right.



Theorem A.30 (Birkhoff distributivity criterion) ⁵¹ Let $L \triangleq (X, \lor, \land; \leq)$ be a LATTICE.

 $\iff \begin{cases} L \text{ does } not \text{ contain N5 as a sublattice} \\ L \text{ does } not \text{ contain M3 as a sublattice} \end{cases}$ and *L is* DISTRIBUTIVE

⁴⁶ S [Burris and Sankappanavar(1981)] page 11, S [Grätzer(1971)] page 70, [[Dedekind(1900)] (cf Stern 1999 page 10





^{47 🛎 [}Maeda and Maeda(1970)], page 15 (Definition 4.1), 📃 [Foulis(1962)] page 67, 🛸 [von Neumann(1960)], page 32 (Definition 5.1), [[[Davis(1955)] page 314 (*disjunctive distributive and conjunctive distributive functions*)

⁸ 🛸 [Burris and Sankappanavar(1981)] page 10, 🛸 [Birkhoff(1948)] page 133, 📃 [Ore(1935)] page 414 (*arithmetic axiom*), [] [Birkhoff(1933)] page 453, 🛸 [Balbes and Dwinger(1975)] page 48 (Definition II.5.1)

⁴⁹ [[Dilworth(1984)] page 237,
[Burris and Sankappanavar(1981)] page 10, [[Ore(1935)] page 416 ((7), (8), Theorem 3), [] [Ore(1940)] (cf Gratzer 2003 page 159), (Schröder(1890)] page 286 (cf Birkhoff(1948)p.133), [Korselt(1894)] (cf Birkhoff(1948)p.133)

⁵⁰ [Beran(1985)] pages 12–13, [Korselt(1894)] page 157 $\langle p_1 \equiv x, p_2 \equiv y, p_3 \equiv z, g \equiv 1, 0 \equiv 0 \rangle$

⁵¹ 🖻 [Burris and Sankappanavar(1981)] page 12, 🛸 [Birkhoff(1948)] page 134, 📃 [Birkhoff and Hall(1934)]

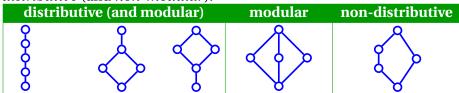
Distributive lattices are a special case of modular lattices. That is, all distributive lattices are modular, but not all modular lattices are distributive (next theorem). An example is the M3 lattice—it is modular, but yet it is not distributive.

Theorem A.31 ⁵² Let $(X, \lor, \land; \leq)$ be a lattice. $(X, \lor, \land; \leq)$ *is* modular. \Rightarrow $(X, \lor, \land; \leq)$ *is* distributive

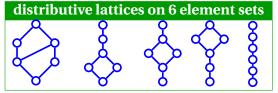
Proposition A.32 ⁵³ Let X_n be a finite set with order $n = |X_n|$. Let l_n be the number of unlabeled lattices on X_n , m_n the number of unlabeled modular lattices on X_n . and d_n the number of unlabeled distributive lattices on X_n .

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
l_n	1	1	1	1	2	5	15	53	222	1078	5994	37622	262776	2018305	16873364
m_n	1	1	1	1	2	4	8	16	34	72	157	343	766	1718	3899
d_n	1	1	1	1	2	3	5	8	15	26	47	82	151	269	494

Example A.33 ⁵⁴ There are a total of 5 unlabeled lattices on a five element set. Of these, 3 are *dis*tributive (Proposition A.32 page 28, and thus also modular), one is modular but non-distributive, and one is non-distributive (and non-modular).



Example A.34 ⁵⁵ There are a total of 15 unlabeled lattices on a six element set; and of these 15, five are distributive (Proposition A.32 page 28). The following illustrates the 5 distributive lattices. Note that none of these lattices are *complemented* (none are *Boolean* Definition A.41 page 30).



A.3 Complemented lattices

A.3.1 Definitions

Definition A.35 ⁵⁶ Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25). An element $x' \in X$ is a **complement** of an element x in L if

⁵² 🖱 [Birkhoff(1948)] page 134, 🖱 [Burris and Sankappanavar(1981)] page 11

⁵³ 🖵 [oei(2014)] (http://oeis.org/A006966), 🖵 [oei(2014)] (http://oeis.org/A006982), 🖵 [oei(2014)] $\langle \text{http://oeis.org/A006981} \rangle$, [Heitzig and Reinhold(2002)] $\langle l_n \rangle$, [Erné et al.(2002)Erné, Heitzig, and Reinhold] page 17 $\langle d_n \rangle$, [] [Thakare et al.(2002)Thakare, Pawar, and Waphare]

⁵⁴ 📃 [Erné et al.(2002)Erné, Heitzig, and Reinhold] pages 4–5

 ⁵⁵ [] [Erné et al.(2002)Erné, Heitzig, and Reinhold] pages 4–5
 ⁵⁶ ∞ [Stern(1999)] page 9, ∞ [Birkhoff(1948)] page 23

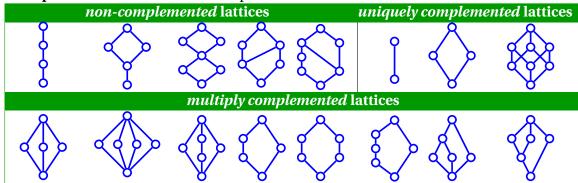
page 29

1. $x \wedge x' = 0$ (non-contradiction) and

2. $x \lor x' = 1$ (excluded middle).

An element x' in L is the *unique complement* of x in L if x' is a *complement* of x and y' is a *complement* of $x \implies x' = y'$. L is **complemented** if every element in X has a complement in X. L is **uniquely complemented** if every element in X has a unique complement in X. A complemented lattice that is *not* uniquely complemented is **multiply complemented**.

Example A.36 Here are some examples:



Example A.37 Of the 53 unlabeled lattices on a 7 element set, 0 are *uniquely complemented*, 17 are *multiply complemented*, and 36 are *non-complemented*.

Theorem A.38 (next) is a landmark theorem in mathematics.

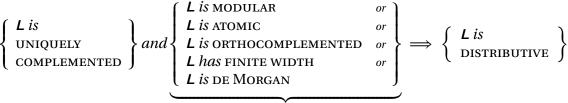
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Theorem A.38 <sup>57</sup> For every lattice L, there exists a lattice U such that
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- 1. $L \subseteq U$ (*L* is a sublattice of U) and
- 2. *U is* Uniquely complemented.

Corollary A.39 ⁵⁸ Let $L \triangleq (X, \lor, \land; \le)$ be a lattice.

 $\left\{\begin{array}{c} 1. \quad L \text{ is DISTRIBUTIVE} \\ 2. \quad L \text{ is COMPLEMENTED}\end{array}\right\} \stackrel{\text{def}}{\rightleftharpoons} \left\{\begin{array}{c} L \text{ is UNIQUELY COMPLEMENTED}\right\}$

Theorem A.40 (Huntington properties) ⁵⁹ Let *L* be a lattice.



HUNTINGTON PROPERTIES





⁵⁷ 📃 [Dilworth(1945)] page 123, 🐃 [Saliĭ(1988)] page 51, 🐃 [Grätzer(2003)] page 378 (Corollary 3.8)

⁵⁸ 🖻 [MacLane and Birkhoff(1999)] page 488, 🛸 [Saliĭ(1988)] page 30 (Theorem 10)

⁵⁹ S [Roman(2008)] page 103, S [Adams(1990)] page 79, S [Salii(1988)] page 40, [[Dilworth(1945)] page 123, S [Grätzer(2007)], page 698

A.3.2 Boolean lattices

Definition A.41 ⁶⁰ A *lattice* (Definition A.11 page 24) *L* is **Boolean** if

- 1. *L* is bounded (Definition A.19 page 25) and
- 2. *L* is *distributive* (Definition A.27 page 27) and
- 3. L is complemented (Definition A.35 page 28) .

In this case, *L* is a **Boolean algebra** or a **Boolean lattice**. In this paper, a *Boolean lattice* is denoted $(X, \vee, \wedge, 0, 1; \leq)$, and a *Boolean lattice* with 2^N elements is sometimes denoted L_2^N .

Theorem A.42 (classic 10 Boolean properties) ⁶¹ Let $\mathbf{A} \triangleq (X, \vee, \wedge, 0, 1; \leq)$ be an algebraic structure. In the event that **A** is a BOUNDED LATTICE (Definition A. 19 page 25), let x' represent a COMPLEMENT (Definition A.35) page 28) of an element x in A. _ _

A is a Boole	an i	algebra 👄	$\forall x, y, z \in X$	(
$x \lor x$	=	x	$x \wedge x$	=	x	(IDEMPOTENT)	and
$x \lor y$	=	$y \lor x$	$x \wedge y$	=	$y \wedge x$	(COMMUTATIVE)	and
$x \lor (y \lor z)$	=	$(x \lor y) \lor z$	$x \wedge (y \wedge z)$	=	$(x \land y) \land z$	(ASSOCIATIVE)	and
$x \lor (x \land y)$	=	x	$x \land (x \lor y)$	=	x	(ABSORPTIVE)	and
$x \lor 1$	=	1	$x \wedge 0$	=	0	(BOUNDED)	and
$x \lor 0$	=	x	$x \wedge 1$	=	x	(IDENTITY)	and
$x \lor (y \land z)$	=	$(x \lor y) \land (x \lor z)$	$x \land (y \lor z)$	=	$(x \land y) \lor (x \land z)$	(DISTRIBUTIVE)	and
$x \lor x'$	=	1	$x \wedge x'$	=	0	(COMPLEMENTED)	and
$(x \lor y)'$	=	$x' \wedge y'$	$(x \wedge y)'$	=	$x' \lor y'$	(de Morgan)	and
		(x')	y' = x			(INVOLUTORY)	
disju	nctiı	ve properties	conjunctive properties			property name	?

Lemma A.43

 $\begin{cases} (X, \lor, \land, 0, 1; \le) \\ is a \text{ BOOLEAN ALGEBRA} \end{cases} \implies \begin{cases} 1. \quad x' \lor (x \land y) = x' \lor y \quad \forall x, y \in X \quad (\text{SASAKI HOOK}) \quad and \\ 2. \quad x \lor (x' \land y) = x \lor y \quad \forall x, y \in X \end{cases}$

SPROOF:

$$x' \lor (x \land y) = \underbrace{x' \lor (x' \land y)}_{x'} \lor (x \land y)$$
by absorption property (Theorem A.42 page 30) $= x' \lor [(x' \lor x) \land y]$ by associative and distributive properties (Theorem A.42 page 30) $= x' \lor [1 \land y]$ by excluded middle property (Theorem A.42 page 30) $= x' \lor y$ by definition of 1 (Definition A.8 page 24) $x \lor (x' \land y) = \underbrace{x \lor (x \land y)}_{x} \lor (x \land y)$ by associative and distributive properties (Theorem A.42 page 30) $= x \lor [(x \lor x') \land y]$ by associative and distributive properties (Theorem A.42 page 30) $= x \lor [1 \land y]$ by excluded middle property (Theorem A.42 page 30) $= x \lor y$ by definition of 1 (Definition A.8 page 24)

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⁶⁰ 🖱 [MacLane and Birkhoff(1999)] page 488, 🖱 [Jevons(1864)]

⁶¹ [Huntington(1904)] pages 292–293 ("1st set"), [[Huntington(1933)] page 280 ("4th set"), 🖱 [MacLane and Birkhoff(1999)] page 488, 🐃 [Givant and Halmos(2009)] page 10, 🛸 [Müller(1909)], pages 20–21, 🐃 [Schröder(1890)], Solution [Whitehead(1898)] pages 35–37

A.3.3 Orthocomplemented Lattices

Definition A.44 ⁶² Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25). An element $x^{\perp} \in X$ is an **orthocomplement** of an element $x \in X$ if

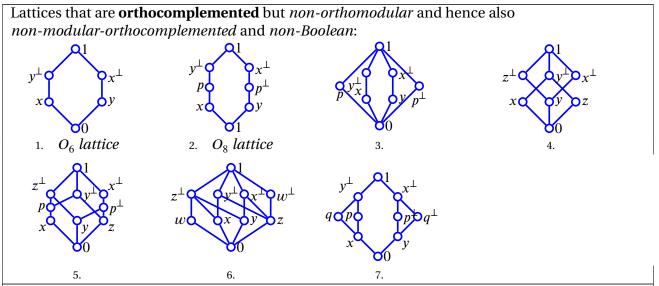
1. $x^{\perp \perp} = x \quad \forall x \in X \quad (involutory)$ and 2. $x \wedge x^{\perp} = 0 \quad \forall x \in X \quad (non-contradiction)$ and 3. $x \leq y \implies y^{\perp} \leq x^{\perp} \quad \forall x, y \in X \quad (antitone).$

The lattice *L* is **orthocomplemented** (*L* is an **orthocomplemented lattice**) if every element *x* in *X* has an *orthocomplement*.

Definition A.45 ⁶³ The **O**₆ **lattice** is the ordered set $(\{0, p, q, p^{\perp}, q^{\perp}, 1\}, \leq)$ with cover relation

 $<= \left\{ (0,p), (0,q), (p,q^{\perp}), (q,p^{\perp}), (p^{\perp},1), (q^{\perp},1) \right\}.$ The O_6 lattice is illustrated by the Hasse diagram to the right.

Example A.46 ⁶⁴ There are a total of 10 **orthocomplemented lattices** with 8 elements or less. These along with some other orthocomplemented lattices are illustrated next:⁶⁵



Lattices that are **orthocomplemented** and **orthomodular** but *non-modular-orthocomplemented* and hence also *non-Boolean*:





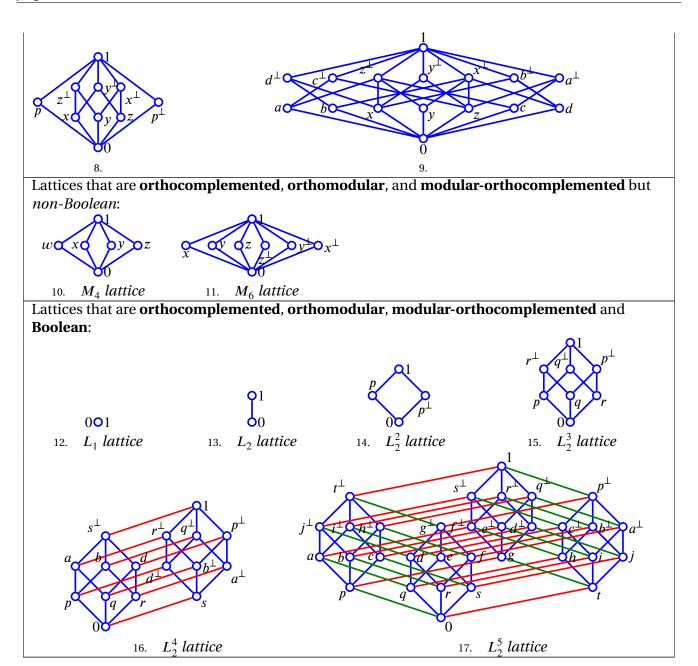
⁶² [Stern(1999)] page 11, [Beran(1985)] page 28, [Kalmbach(1983)] page 16, [Gudder(1988)] page 76, [Loomis(1955)] page 3, [Birkhoff and Neumann(1936)] page 830 ⟨L71–L73⟩

⁶³ [Kalmbach(1983)] page 22, [] [Holland(1970)], page 50, [Beran(1985)] page 33, [Stern(1999)] page 12. The O_6 lattice is also called the **hexagon** or **Benzene ring**.

⁶⁴ S [Beran(1985)] pages 33–42, [[Maeda(1966)] page 250, S [Kalmbach(1983)] page 24 (Figure 3.2), S [Stern(1999)] page 12, ↑ [Holland(1970)], page 50

⁶⁵ As can be seen in this example, the number of *orthocomplemented lattices* with (1, 2, 3, ...) elements is (1, 1, 0, 1, 0, 2, 0, 5, 0, ...). It is interesting to note that at least the first 9 terms (and possibly more?) of this sequence are the same as the "expansion of $\frac{1+2x}{1+\sqrt{1-4x^2}} \supseteq$ [oei(2014)] (http://oeis.org/A097331) and the "Catalan numbers ...interpolated with 0's" \supseteq [oei(2014)] (http://oeis.org/A126120)





Theorem A.47 ⁶⁶ Let x^{\perp} be the ORTHOCOMPLEMENT of an element x in a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$.

$$\left\{\begin{array}{l} L \text{ is}\\ \text{ORTHOCOMPLEMENTED}\end{array}\right\} \implies \left\{\begin{array}{ll} (1). & 0^{\perp} = 1 & (\text{BOUNDARY CONDITION}) & and\\ (2). & 1^{\perp} = 0 & (\text{BOUNDARY CONDITION}) & and\\ (3). & (x \lor y)^{\perp} = x^{\perp} \land y^{\perp} & \forall x, y \in X & (\text{DISJUNCTIVE DE MORGAN}) & and\\ (4). & (x \land y)^{\perp} = x^{\perp} \lor y^{\perp} & \forall x, y \in X & (\text{CONJUNCTIVE DE MORGAN}) & and\\ (5). & x \lor x^{\perp} = 1 & \forall x \in X & (\text{EXCLUDED MIDDLE}).\end{array}\right\}$$

SPROOF: Let $x^{\perp} \triangleq \neg x$, where \neg is an *ortho negation* function (Definition B.3 page 35). Then this theorem follows $\frac{^{66} \boxtimes [Beran(1985)] \text{ pages 30-31,}}{[Birkhoff and Neumann(1936)] \text{ page 830 } \langle L74 \rangle, \cong [Cohen(1989)] \text{ page 37}} \langle 3B.13. \text{ Theorem} \rangle$

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directly from Theorem B.15 (page 37).

Corollary A.48 Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a LATTICE (Definition A.11 page 24). $\begin{array}{c} \textbf{L} \text{ is orthocomplemented} \\ \text{(Definition A.44 page 31)} \end{array} \end{array} \right\} \implies \left\{ \begin{array}{c} \textbf{L} \text{ is complemented} \\ \text{(Definition A.35 page 28)} \end{array} \right\}$

This follows directly from the definition of orthocomplemented lattices (Definition A.44 page 31) and com-[©]Proof: plemented lattices (Definition A.35 page 28).

PROOF:

Example A.49 The O_6 *lattice* (Definition A.45 page 31) illustrated to the left is both orthocomplemented (Definition A.44 page 31) and multiply complemented (Definition A.35 page 28). The lattice illustrated to the right is multiply complemented, but is non-orthocomplemented.

- (1) Proof that O_6 lattice is multiply complemented: b and q are both complements of p.
- (2) Proof that the right side lattice is multiply complemented: *a*, *p*, and *q* are all *complements* of *r*.

Proposition A.50 ⁶⁷ Let $L = (X, \lor, \land, 0, 1; \le)$ be a BOUNDED LATTICE (Definition A.19 page 25). 1. L is orthocomplemented
2. L is distributive(Definition A.44 page 31)
(Definition A.27 page 27)and
 $\Rightarrow \begin{cases} L \text{ is Boolean} \\ (Definition A.41 page 30) \end{cases}$



Example A.51 The O_6 *lattice* (Definition A.45 page 31) illustrated to the left is orthocomplemented (Definition A.44 page 31) but non-join-distributive (Definition A.27 page 27), and hence non-Boolean. The lattice illustrated to the right is **orthocomplemented** and **distributive** and hence also Boolean (Proposition A.50 page 33).



A.3.4 Orthomodular lattices

Definition A.52 ⁶⁸ Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25). L is orthomodular if

1. *L* is orthocomplemented and 2. $x \le y \implies x \lor (x^{\perp} \land y) = y \forall x, y \in X$ (orthomodular identity)

Theorem A.53 ⁶⁹ Let $L = (X, \lor, \land, 0, 1; \le)$ be an algebraic structure.

 $\underbrace{L \text{ is an orthomodular lattice}}_{(x \land y^{\perp})^{\perp} = y \lor (x^{\perp} \land y^{\perp})} \qquad \forall x, y \in X \\ \underbrace{Frank = (x \land y^{\perp})^{\perp} = y \lor (x^{\perp} \land y^{\perp})}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})} \\ \underbrace{L \text{ is a}}_{(Definition A.41 \text{ page 30})}$

⁶⁷ 🖱 [Kalmbach(1983)] page 22





^{68 🖻 [}Kalmbach(1983)] page 22, 🛸 [Lidl and Pilz(1998)] page 90, 📃 [Husimi(1937)]

⁶⁹ []. [Renedo et al.(2003)Renedo, Trillas, and Alsina] page 72

Definition A.54 Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25).

L is a modular orthocomplemeted lattice if

- 1. L is orthocomplemented (Definition A.44 page 31) and
- 2. *L* is modular (Definition A.23 page 26)

Theorem A.55 ⁷⁰ Let	L be a	lattice.	
{ L is Boolean}	\implies	$\{L is modular orthocomplemented\}$	(Definition A.54 page 34)}
	\implies	{ L <i>is</i> orthomodular	(Definition A.52 page 33)}
	\Rightarrow	{L is orthocomplemented	(Definition A.44 page 31)}

Appendix B Background: Negation

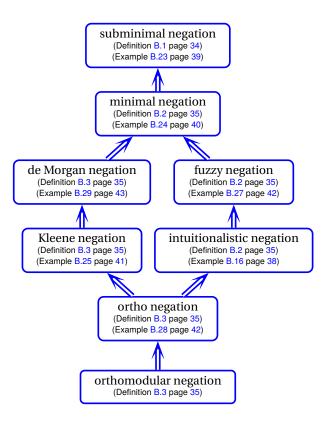


Figure 9: lattice of negations

B.1 Definitions

Definition B.1 ⁷¹ Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25). A *function* $\neg \in X^X$ is a **subminimal negation** on L if ⁷²

 $x \leq y \implies \neg y \leq \neg x \quad \forall x, y \in X \quad (antitone).$

⁷⁰ 🖻 [Kalmbach(1983)] page 32 (20.), 📋 [Iturrioz(1985)], page 57

⁷¹ (Dunn(1996)] pages 4–6, (Dunn(1999)] pages 24–26 (2 The Kite of Negations)

page 35

Definition B.2 ⁷³ Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25).

A function $\neg \in X^X$ is a **negation**, or **minimal negation**, on L if 1. $x \le y \implies \neg y \le \neg x \quad \forall x, y \in X$ (antitone) and 2. $x \le \neg \neg x \quad \forall x \in X$ (weak double negation). A minimal negation \neg is an **intuitionistic negation** on L if 3. $x \land \neg x = 0 \quad \forall x, y \in X$ (non-contradiction). A minimal negation \neg is a **fuzzy negation** on L if 4. $\neg 1 = 0$ (boundary condition).

Definition B.3 ⁷⁴ Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25). A *minimal negation* \neg is a **de Morgan negation** on L if 5. $x = \neg \neg x \qquad \forall x \in X \quad (involutory).$ A *de Morgan negation* \neg is a **Kleene negation** on L if

6. $x \wedge \neg x \leq y \vee \neg y \quad \forall x, y \in X$ (*Kleene condition*). A *de Morgan negation* \neg is an **ortho negation** on *L* if 7. $x \wedge \neg x = 0 \quad \forall x, y \in X$ (*non-contradiction*). A *de Morgan negation* \neg is an **orthomodular negation** on *L* if 8. $x \wedge \neg x = 0 \quad \forall x, y \in X$ (*non-contradiction*) and 9. $x \leq y \implies x \vee (x^{\perp} \wedge y) = y \quad \forall x, y \in X$ (*orthomodular*).

Remark B.4 ⁷⁵ The *Kleene condition* is a weakened form of the *non-contradiction* and *excluded middle* properties in the sense $x \land \neg x = 0 \leq 1 = y \lor \neg y$.

Definition B.5 Let $L \triangleq (X, \lor, \land, \neg, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25) with a function $\neg \in X^X$. If \neg is a *negation* (Definition B.2 page 35), then L is a **lattice with negation**.

B.2 Properties of negations

Lemma B.6 ⁷⁶ Let $\neg \in X^X$ be a function on a LATTICE $L \triangleq (X, \lor, \land; \leq)$ (Definition A.11 page 24). $x \leq y \Longrightarrow$ $\neg y \leq \neg x$ $\downarrow y \leq \neg x$ $\downarrow x \lor y = \langle \neg x \lor \neg y \rangle \leq \neg (x \land y) \forall x, y \in X$ (CONJUNCTIVE DE MORGAN INEQUALITY) and $\neg (x \lor y) \leq \neg x \land \neg y \forall x, y \in X$ (DISJUNCTIVE DE MORGAN INEQUALITY)



⁷² In the context of natural language, D. Devidi has argued that, *subminimal negation* (Definition B.1 page 34) is "difficult to take seriously as" a negation. For further details see [] [Devidi(2010)], page 511, [] [Devidi(2006)], page 568

⁷³ ■ [Dunn(1996)] pages 4–6, ■ [Dunn(1999)] pages 24–26 (2 The Kite of Negations), ■ [Troelstra and van Dalen(1988)] page 4 (1.6 Intuitionism. (b)), [] [De Vries(2007)] page 11 (Definition 16), ■ [Gottwald(1999)] page 21 (Definition 3.3), ■ [Novák et al.(1999)Novák, Perfilieva, and Močkoř] page 50 (Definition 2.26), ■ [Nguyen and Walker(2006)] pages 98–99 (5.4 Negations), [] [Bellman and Giertz(1973)] pages 155–156 ((N1) ¬0 = 1 and ¬1 = 0, (N3) ¬¬x = x)

⁷⁴ [Dunn(1999)] pages 24–26 (2 The Kite of Negations), [Jenei(2003)] page 283, [Kalmbach(1983)] page 22, [Lidl and Pilz(1998)] page 90, [[Husimi(1937)]]

⁷⁵ 🔁 [Cattaneo and Ciucci(2009)] page 78

⁷⁶ \blacksquare [Beran(1985)] page 31 (Theorem 1.2 Proof), [] [Fáy(1967)] page 268 (Lemma 1 Proof), [] [de Vries(2007)] page 12 (Theorem 18)

Lemma B.7 ⁷⁷ Let $\neg \in X^X$ be a function on a LATTICE $L \triangleq (X, \lor, \land; \leq)$ (Definition A.11 page 24). If $x = (\neg \neg x)$ for all $x \in X$ (INVOLUTORY), then

$$\underbrace{x \leq y \implies \neg y \leq \neg x}_{\text{ANTITONE}} \iff \underbrace{\begin{cases} \neg (x \lor y) = \neg x \land \neg y \quad \forall x, y \in X \text{ (DISJUNCTIVE DE MORGAN)} & and \\ \neg (x \land y) = \neg x \lor \neg y \quad \forall x, y \in X \text{ (CONJUNCTIVE DE MORGAN)} \\ \text{DE MORGAN} \\ \text{DE MORGAN} \end{cases}$$

Lemma B.8 Let $\neg \in X^X$ be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$.

 $\left\{\begin{array}{ll} 1. & x \leq \neg \neg x & \forall x \in X & (\text{weak double negation}) & and \\ 2. & \neg 1 = 0 & (\text{boundary condition}) \end{array}\right\} \implies \left\{\neg 0 = 1 & (\text{boundary condition})\right\}$

[®]Proof:

by boundary condition hypothesis (2)

$$\geq 1$$
 by weak double negation hypothesis (1)
 $\Rightarrow \neg 0 = 1$ by upper bound property (Definition A.19 page 25)

Lemma B.9 Let $\neg \in X^X$ be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$. $\begin{cases} x \land \neg x = 0 \quad \forall x \in X \quad (\text{NON-CONTRADICTION}) \end{cases} \implies \begin{cases} \neg 1 = 0 \quad (\text{BOUNDARY CONDITION}) \end{cases}$

[®]Proof:

$$0 = 1 \land \neg 1$$
by non-contradiction hypothesis $= \neg 1$ by definition of g.u.b. 1 and \land

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Lemma B.10 ⁷⁸ Let $\neg \in X^X$ be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$. $\begin{cases}
(A). \neg is BIJECTIVE & and \\
(B). x \le y \implies \neg y \le \neg x \quad \forall x, y \in X \quad (ANTITONE)
\end{cases} \implies \underbrace{\begin{cases}
(1). \neg 0 = 1 & and \\
(2). \neg 1 = 0
\end{cases}}_{BOUNDARY CONDITIONS}
\end{cases}$

Theorem B.11 Let $\neg \in X^X$ be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$. $\begin{cases} \neg is a \\ FUZZY NEGATION \end{cases} \implies \{ \neg 0 = 1 \text{ (BOUNDARY CONDITION)} \}$

PROOF: This follows directly from Definition B.2 (page 35) and Lemma B.8 (page 36).

Theorem B.12 Let $\neg \in X^X$ be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \leq)$.

 $\left\{ \begin{array}{l} \neg is \ an \\ \text{INTUITIONISTIC NEGATION} \end{array} \right\} \implies \left\{ \begin{array}{l} (a) \ \neg 1 \ = \ 0 \ (\text{BOUNDARY CONDITION}) \ and \\ (b) \ \neg 0 \ = \ 1 \ (\text{BOUNDARY CONDITION}) \ and \\ (c) \ \neg \ is \ a \ \text{FUZZY NEGATION} \end{array} \right\}$

⁷⁷ \blacksquare [Beran(1985)] pages 30–31 (Theorem 1.2), [[Fáy(1967)] page 268 (Lemma 1), [[[Nakano and Romberger(1971)] (cf Beran 1985)

⁷⁸ 🔁 [Varadarajan(1985)] page 42

[∞]Proof:

$$\neg$$
 is an intuitionistic negation $\Rightarrow x \land \neg x = 0$ by Definition B.2 page 35 $\Rightarrow \neg 1 = 0$ by Lemma B.9 page 36 $\Rightarrow \neg$ is a fuzzy negationby Definition B.2 page 35 $\Rightarrow \neg 0 = 1$ by Theorem B.11 page 36

 \square

page 37

Theorem B.13 Let $\neg \in X^X$ be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$. $\begin{cases} \neg is a \\ minimal \\ negation \end{cases} \implies \begin{cases} \neg x \lor \neg y \le \neg (x \land y) \forall x, y \in X \text{ (conjunctive de Morgan inequality) and} \\ \neg (x \lor y) \le \neg x \land \neg y \forall x, y \in X \text{ (disjunctive de Morgan inequality)} \end{cases}$

PROOF: This follows directly from Definition B.5 (page 35) and Lemma B.6 (page 35).

Theorem B.14 Let $\neg \in X^X$ be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$. \neg is a de Morgan negation $\rbrace \implies \begin{cases} \neg(x \lor y) = \neg x \land \neg y \quad \forall x, y \in X \text{ (DISJUNCTIVE DE MORGAN)} and \\ \neg(x \land y) = \neg x \lor \neg y \quad \forall x, y \in X \text{ (CONJUNCTIVE DE MORGAN)} \end{cases}$

PROOF: This follows directly from Definition B.5 (page 35) and Lemma B.7 (page 36).

Theorem B.15 Let
$$\neg \in X^X$$
 be a function on a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \le)$.

$$\left\{ \begin{array}{c} \neg \text{ is an} \\ \text{ortho} \\ \text{negation} \end{array} \right\} \implies \begin{cases} 1. & \neg 0 = 1 & (\text{BOUNDARY CONDITION}) & \text{and} \\ 2. & \neg 1 = 0 & (\text{BOUNDARY CONDITION}) & \text{and} \\ 3. & \neg (x \lor y) = \neg x \land \neg y & \forall x, y \in X & (\text{DISJUNCTIVE DE MORGAN}) & \text{and} \\ 4. & \neg (x \land y) = \neg x \lor \neg y & \forall x, y \in X & (\text{CONJUNCTIVE DE MORGAN}) & \text{and} \\ 5. & x \lor \neg x = 1 & \forall x \in X & (\text{EXCLUDED MIDDLE}) & \text{and} \\ 6. & x \land \neg x & \leq y \lor \neg y & \forall x, y \in X & (\text{KLEENE CONDITION}). \end{cases}$$

[∞]Proof:

- (1) Proof for $0 = \neg 1$ boundary condition: by Lemma B.9 (page 36)
- (2) Proof for boundary conditions:

1 = -1	by involutory property
$= \neg 0$	by previous result

- (3) Proof for *de Morgan* properties:
 - (a) By Definition B.5 (page 35), *ortho negation* is *involutory* and *antitone*.
 - (b) Therefore by Lemma B.7 (page 36), *de Morgan* properties hold.
- (4) Proof for *excluded middle* property:

$x \lor \neg x = \neg \neg (x \lor \neg x)$	by <i>involutory</i> property of <i>ortho negation</i> (Definition B.5 page 35)
$= \neg (x \neg \land [\neg \neg x])$	by <i>disjunctive de Morgan</i> property
$= \neg(\neg x \land x)$	by <i>involutory</i> property of <i>ortho negation</i> (Definition B.5 page 35)
$= \neg (x \land \neg x)$	by <i>commutative</i> property of <i>lattices</i> (Definition A.11 page 24)
$= \neg 0$	by non-contradiction property of ortho negation (Definition B.5 page 35)
= 1	by <i>boundary condition</i> (item (2) page 37) of <i>minimal negation</i>



(5) Proof for *Kleene condition*:

$x \wedge \neg x = 0$	by <i>non-contradiction</i> property (Definition B.5 page 35)
≤ 1	by definition of 0 and 1
$= y \vee \neg y$	by <i>excluded middle</i> property (item (4) page 37)

B.3 Examples

Example B.16 (discrete negation) ⁷⁹ Let $L \triangleq (X, \lor, \land, \neg, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25) with a function $\neg \in X^X$. The function $\neg x$ defined as

 $\neg x \triangleq \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{otherwise} \end{cases}$ is an **intuitionistic negation** (Definition B.2 page 35, and a *fuzzy negation*).

[©]Proof: To be an *intuitionistic negation*, $\neg x$ must be *antitone*, have *weak double negation*, and have the non-contradiction property (Definition B.2 page 35).

 $\begin{cases} \neg y \leq \neg x \iff 1 \leq 1 \quad \text{for } 0 = x = y \\ \neg y \leq \neg x \iff 0 \leq 1 \quad \text{for } 0 = x \leq y \\ \neg y \leq \neg x \iff 0 \leq 0 \quad \text{for } 0 = x \leq y \end{cases} \implies \neg x \text{ is antitone}$ $\begin{cases} (\neg x) = \neg 1 = 0 \geq 0 = x \quad \text{for } x = 0 \\ \neg x = \neg 0 = 1 \geq x = x \quad \text{for } x \neq 0 \end{cases} \implies \neg x \text{ has weak double negation}$ $\begin{cases} (x \land \neg x) = x \land 1 = 0 \land 0 = 0 \quad \text{for } x = 0 \\ x \land \neg x = x \land 0 = x \land 0 = 0 \quad \text{for } x \neq 0 \end{cases} \implies \neg x \text{ has non-contradiction property}$

Example B.17 (dual discrete negation) ⁸⁰ Let $L \triangleq (X, \lor, \land, \neg, 0, 1; \leq)$ be a *bounded lattice* (Definition A.19 page 25) with a function $\neg \in X^X$. The function $\neg x$ defined as

$$x \triangleq \begin{cases} 0 & \text{for } x = 1\\ 1 & \text{otherwise} \end{cases}$$

is a subminimal negation (Definition B.1 page 34) but it is not a minimal negation (Definition B.2 page 35) (and not any other negation defined here).

To be a subminimal negation, $\neg x$ must be antitone (Definition B.1 page 34). To be a minimal negation, $\neg x$ [®]Proof: must be *antitone* and have *weak double negation* (Definition B.2 page 35).

 $\begin{cases} \neg y \leq \neg x \iff 0 \leq 0 \quad \text{for } x = y = 1 \\ \neg y \leq \neg x \iff 0 \leq 1 \quad \text{for } x \leq y = 1 \\ \neg y \leq \neg x \iff 1 \leq 1 \quad \text{for } x \leq y \neq 1 \end{cases} \implies \neg x \text{ is antitone}$ $\begin{cases} \neg x = \neg 0 = 1 \geq x \quad \text{for } x = 1 \\ \neg x = \neg 1 = 0 \leq x \quad \text{for } x \neq 1 \end{cases} \implies \neg x \text{ does not have weak double negation}$

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⁷⁹ 🗢 [Fodor and Yager(2000)] page 128, 📃 [Yager(1980b)] pages 256–257, 📃 [Yager(1979)] (cf Fodor)

⁸⁰ 📼 [Fodor and Yager(2000)] page 128, 📃 [Ovchinnikov(1983)] page 235 (Example 4)

Example B.18

The function \neg illustrated to the right is an *ortho negation* (Definition B.3 page 35).

[®]Proof:

(1) Proof that \neg is *antitone*:

- $0 \le 1 \implies \neg 1 = 0 \le x = \neg 0 \implies \neg \text{ is antitone over } (0,1)$
- (2) Proof that \neg is *involutory*: $1 = \neg 0 = \neg \neg 1$
- (3) Proof that \neg has the *non-contradiction* property: $1 \land \neg 1 = 1 \land 0 = 0$ $0 \land \neg 0 = 0 \land 1 = 0$

Example B.19

The functions \neg illustrated to the right are *not* any negation defined here. In particular, none of them is *antitone*.

[®]Proof:

- 1. Proof that (a) is *not antitone*: $a \le 1 \implies \neg 1 = 1 \nleq a = \neg a$
- 2. Proof that (b) is *not antitone*: $a \le 1 \implies \neg 1 = a \nleq 0 = \neg a$
- 3. Proof that (c) is *not antitone*: $0 \le a \implies \neg a = 1 \le a = \neg 0$

Example B.20 The function \neg as illustrated to the right is *not* a *subminimal negation* (it is *not antitone*) and so is *not* any negation defined here. Note however that the problem is *not* the O_6 *lattice*—it is possible to define a negation on an O_6 *lattice* (Example B.31 page 43). PROOF: Proof that \neg is *not antitone*: $a \le c \implies \neg c = d \nleq b = \neg a$

Remark B.21 The concept of a *complement* (Definition A.35 page 28) and the concept of a *negation* are fundamentally different. A *complement* is a *relation* on a lattice *L* and a *negation* is a *function*. In Example B.20 (page 39), *b* and *d* are both complements of *a* (and so the lattice is *multiply complemented*), but yet \neg is *not* a negation. In the right side lattice of Example B.31 (page 43), both *b* and *d* are complements of *a* ($d = \neg a$). It can also be said that complementation is a property *of* a lattice, whereas negation is a function defined *on* a lattice.

Remark B.22 If a lattice is *complemented*, then by definition each element x in the lattice has a *complement* x' such that $x \wedge x' = 0$ (*non-contradiction* property) and $x \vee x' = 1$ (*excluded middle* property). If a lattice L is both *distributive* and *complemented*, then L is *uniquely complemented* (Definition A.41 page 30, Theorem A.42 page 30). If L is *uniquely complemented* and satisfies any one of *Huntington's properties* (L is *modular*, *atomic*, *ortho-complemented*, has *finite width*, or *de Morgan*), then L is *distributive* (Theorem A.40 page 29).

Example B.23 Each of the functions ¬ illustrated in Figure 10 (page 40) is a *subminimal negation* (Definition B.1 page 34); *none* of them is a *minimal negation* (each fails to have *weak double negation*).



01 = -1

- - 0

(a)

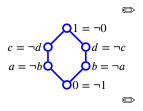
 $0^{-1} = -0^{-1}$

 $a = \neg 1$

(b)

 $= \neg a$





 $O1 = \neg a$

 $a = \neg 0$

0 = -1

(c)

01 = -0

0 = -1

 \square

$$\begin{array}{cccc}
01 & & 01 & & 01 = \neg 0 = \neg a \\
0a = \neg 0 & & a = \neg 1 = \neg a = \neg 0 & & a \\
0b = \neg 1 = \neg a & & 00 & & 00 = \neg 1 \\
(A) & & (B) & & (C)
\end{array}$$

Figure 10: *subminimal negations* on L₃ (Example B.23 page 39)

$$\begin{array}{c} 0 & 1 = \neg a = \neg 0 \\ 0 & a = \neg 1 \\ 0 & 0 \\ (A) \ minimal \ (but not fuzzy) \\ see \ Example \ B.24 \ page \ 40 \end{array} \begin{array}{c} 0 & 1 = \neg 0 \\ 0 & a = \neg a \\ 0 & 0 = \neg 1 \\ (B) \ Kleene \ and \ fuzzy \\ see \ Example \ B.25 \ page \ 41 \end{array} \begin{array}{c} 0 & 1 = \neg 0 \\ 0 & a = \neg a \\ 0 & 0 = \neg 1 = \neg a \\ (C) \ intuitionistic \ (and \ fuzzy) \\ see \ Example \ B.26 \ page \ 41 \end{array}$$

Figure 11: negations on L_3

[®]Proof:

(1) Proof that (A) \neg is *antitone*: $a \leq 1$ \implies $\neg 1 = 0 \leq 0 = \neg a \implies$ \neg is antitone over (a, 1) $\implies \neg 1 = 0 \leq a = \neg 0 \implies \neg$ is antitone over (0, 1)0 < 1 $0 \leq a$ $\implies \neg a = 0 \leq a = \neg 0 \implies \neg$ is antitone over (0, a)(2) Proof that (A) \neg *fails* to have *weak double negation*: $1 \leq a = \neg 0 = \neg \neg 1$ (3) Proof that (B) \neg is *antitone*: $1 \implies \neg 1 = a \leq a =$ \implies \neg is antitone over (a, 1)а < $\neg a$ 0 < 1 $\implies \neg 1 = a \leq a = \neg 0 \implies \neg$ is antitone over (0, 1) $\implies \neg a = a \leq a = \neg 0 \implies \neg$ is antitone over (0, a)0 $\leq a$ (4) Proof that (B) \neg *fails* to have *weak double negation*: $1 \leq a = \neg a = \neg 1$

(5) (C) is a special case of the *dual discrete negation* (Example B.17 page 38).

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Example B.24 Consider the function \neg on L_3 illustrated in Figure 11 page 40 (A):

- 1. ¬ is a **minimal negation** (Definition B.2 page 35);
- 2. \neg is not an intuitionistic negation and it is not a de Morgan negation.

[®]Proof:

(1) Proof that \neg is *antitone*:

(2) Proof that \neg is a *weak double negation* (and so is a *minimal negation*, but is *not* a *de Morgan negation*):

 $1 = 1 = \neg a = \neg \neg 1 \implies \neg \text{ is involutory at } 1$

- $a = a = \neg 1 = \neg \neg a \implies \neg$ is involutory at a
- $0 \le a = \neg 1 = \neg \neg 0 \implies \neg$ is a *weak double negation* at 0
- (3) Proof that \neg does *not* have the *non-contradiction* property (and so is not an *intuitionistic negation*): $1 \land \neg 1 = 1 \land a = a \neq 0$
- (4) Proof that \neg is not a *fuzzy negation*: $\neg 1 = a \neq 0$

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Example B.25 ⁸¹ Consider the function \neg on L_3 illustrated in Figure 11 page 40 (B).

- 1. ¬ is a **Kleene negation** (Definition B.3 page 35) and is also a **fuzzy negation** (Definition B.2 page 35, Example 1.31 page 16).
- 2. ¬ is *not* an *ortho negation* and is *not* an *intuitionistic negation* (it does not have the *non-contradiction* property).
- 3. This negation on *L*₃ is used with an *implication* function to construct the *Kleene 3-valued logic* in Example C.7 (page 51), with another *implication* to construct the *Łukasiewicz 3-valued logic* in Example C.8 (page 52), and with yet another *implication* to construct the *RM*₃ *logic* in Example C.9 (page 52).

[®]Proof:

(1) Proof that \neg is *antitone*:

 \implies \neg is antitone over (a, 1) $a \leq 1$ = 0 $\leq a =$ $\neg 1$ $\neg a$ $\neg 1 = 0 \leq$ \implies \neg is antitone over (0, 1) 0 1 $\neg 0$ ≤ 1 = $\neg 0 \implies \neg$ is *antitone* over (0, a)0 $\leq a$ $\neg a = a \leq$ 1 = (2) Proof that \neg is *involutory* (and so is a *de Morgan negation*):

 $1 = \neg 0 = \neg \neg 1 \implies \neg \text{ is involutory at } 1$ $a = \neg a = \neg \neg a \implies \neg \text{ is involutory at } a$ $0 = \neg 0 = \neg \neg 0 \implies \neg \text{ is involutory at } 0$

- (3) Proof that \neg does *not* have the *non-contradiction* property (and so is not an *ortho negation*): $x \land \neg x = x \land x = x \neq 0$
- (4) Proof that \neg satisfies the *Kleene condition* (and so is a *Kleene negation*):

1	\wedge	٦1	=	1	\wedge	0	=	0	\leq	а	=	а	\vee	а	=	а	\vee	$\neg a$
1	٨	٦1	=	1	\wedge	0	=	0	\leq	1	=	0	V	1	=	0	V	$\neg 0$
а	٨	$\neg a$	=	1	\wedge	а	=	а	\leq	1	=	1	V	0	=	1	V	71
а	٨	$\neg a$	=	1	\wedge	а	=	а	\leq	1	=	0	V	1	=	0	V	$\neg 0$
0	٨	¬0	=	0	\wedge	1	=	0	\leq	1	=	1	V	0	=	1	V	71
0	٨	$\neg 0$	=	0	\wedge	1	=	0	\leq	а	=	а	V	а	=	а	V	$\neg a$

Example B.26 (Heyting 3-valued logic/Jaśkowski's first matrix) ⁸² Consider the function \neg on L_3 illustrated in Figure 11 page 40 (C):

1. \neg is an intuitionistic negation (Definition B.2 page 35) (and thus is also a fuzzy negation).

- 2. \neg is not a de Morgan negation.
- 3. This negation on L_3 is used with an *implication* function to construct the *Heyting 3-valued logic* in Example C.10 (page 52).

PROOF: This is simply a special case of the *discrete negation* (Example B.16 page 38).





⁸¹ ■ [Łukasiewicz(1920)], ■ [Avron(1991)] pages 277–278, ■ [Kleene(1938)] page 153, ➡ [Kleene(1952)], pages 332–339 (§64. The 3-valued logic), ■ [Sobociński(1952)]

⁸² **(Karpenko**(2006)] page 45, **(Johnstone**(1982)] page 9 (§1.12), **(Heyting**(1930a)], **(Heyting**(1930b)], **(Heyting**(1930c)], **(Heyting**(1930d)], **(Heyting**(1930d)), **(Heyting**(1930d)), **(Heyting**(1930d)), **(Heyting**(1930d)), **(Heyting**(1930d)), **(State**), **(State**), **(Heyting**(1930d)), **(Heyting**(1930

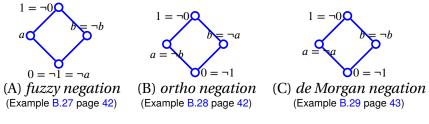
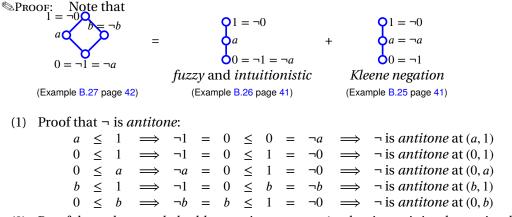


Figure 12: negations on M_2

Example B.27 The function \neg illustrated in Figure 12 page 42 (A) is a **fuzzy negation** (Definition B.2 page 35). It is not an *intuitionistic negation* (it does not have the *non-contradiction* property) and it is *not* a *de Morgan negation* (it is not *involutory*).



(2) Proof that ¬ has *weak double negation* property (and so is a *minimal negation*, but *not* a *de Morgan nega-tion*):

1	=	¬ 0	=	רר1			\implies	\neg is <i>involutory</i> at 1
а	\leq	1	=	$\neg 0$	=	$\neg \neg a$	\implies	\neg has weak double negation at a
0	=	٦1	=	0			\implies	\neg is <i>involutory</i> at 0
b	=	$\neg b$	=	$\neg \neg b$	=		\Rightarrow	\neg is <i>involutory</i> at <i>b</i>

- (3) Proof that \neg does *not* have the *non-contradiction* property (and so is *not* an *intuitionistic negation*): $b \land \neg b = b \land b = b \neq 0$
- (4) Proof that \neg is has *boundary conditions* (and so is a *fuzzy negation*): $\neg 1 = 0$, $\neg 0 = 1$

Example B.28 ⁸³ The function \neg illustrated in Figure 12 page 42 (B) is an **ortho negation** (Definition B.3 page 35).

[∞]Proof:

Proof that \neg is <i>antitone</i> :	а	\leq	1	\implies	٦1	=	0	\leq	b	=	$\neg a$
	0	\leq	1	\Rightarrow	٦1	=	0	\leq	1	=	$\neg 0$
	0	\leq	а	\Rightarrow	$\neg a$	=	b	\leq	1	=	$\neg 0$
	b	\leq	1	\Rightarrow	٦1	=	0	\leq	а	=	$\neg b$
	0	\leq	b	\Rightarrow	$\neg b$	=	а	\leq	1	=	$\neg 0$
	Proof that ¬ is <i>antitone</i> :	0 0 <i>b</i>	$\begin{array}{rrrr} 0 & \leq \\ 0 & \leq \\ b & \leq \end{array}$	$\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr$	$\begin{array}{ccccc} 0 & \leq & 1 & \Longrightarrow \\ 0 & \leq & a & \Longrightarrow \\ b & \leq & 1 & \Longrightarrow \end{array}$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$0 \leq 1 \implies \neg 1 = 0$ $0 \leq a \implies \neg a = b$ $b \leq 1 \implies \neg 1 = 0$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$0 \leq 1 \implies \neg 1 = 0 \leq 1$ $0 \leq a \implies \neg a = b \leq 1$ $b \leq 1 \implies \neg 1 = 0 \leq a$	Proof that \neg is antitone: $a \leq 1 \implies \neg 1 = 0 \leq b =$ $0 \leq 1 \implies \neg 1 = 0 \leq 1 =$ $0 \leq a \implies \neg a = b \leq 1 =$ $b \leq 1 \implies \neg 1 = 0 \leq a =$ $0 \leq b \implies \neg b = a \leq 1 =$

⁸³ [[Belnap(1977)] page 13, [∞] [Restall(2000)] page 177 (Example 8.44), [[Pavičić and Megill(2009)] page 28 (Definition 2, *classical implication*)

(2) Proof that \neg is *involutory* (and so is a *de Morgan negation*): $1 = \neg 0 = \neg 1$ $a = \neg a = \neg a$ $b = \neg b = \neg b$ $0 = \neg 0 = \neg 0$

(3) Proof that \neg is has the *non-contradiction* property (and so is an *ortho negation*):

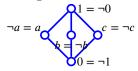
 $\wedge \neg 1 = 1 \wedge 0$ 1 = 00 а Λ $\neg a = a \wedge b$ = h ٨ $\neg h$ $= b \wedge a$ = 0 $\neg 0$ $= 0 \wedge$ 1 = 0 ۸

Example B.29 (BN₄) ⁸⁴ The function \neg illustrated in Figure 12 page 42 (C) is a **de Morgan negation** (Definition B.3 page 35), but it is *not* a *Kleene negation* and not an *ortho negation* (it does *not* satisfy the *Kleene condition*).

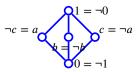
[®]Proof:

- (1) Proof that \neg is *antitone*: $a \leq 1$ $\neg 1$ = 0 \rightarrow $\leq b$ $\neg a$ 0 ≤ 0 \leq 1 71 = 1 = $\neg 0$ \implies $0 \leq a$ $a \leq$ = 1 $\neg 0$ $\neg a$ = b \leq 1 $\neg 1$ 0 \leq b = = $\neg b$ 0 \leq b $\neg b = b \leq$ 1 \implies = $\neg 0$ (2) Proof that \neg is *involutory* (and so is a *de Morgan negation*): 1 $\neg 0$ --1 а = = $\neg \neg a$ $\neg a$ b = $\neg b$ = $\neg \neg b$ 0 = $\neg 0$ = $\neg \neg 0$
- (3) Proof that \neg does *not* have the *non-contradiction* property (and so is *not* an *ortho negation*): $a \land \neg a = a \land a = a \neq 0$ $b \land \neg b = b \land b = b \neq 0$
- (4) Proof that \neg does *not* satisfy the *Kleene condition* (and so is a *de Morgan negation*): $a \land \neg a = a \land a = a \nleq b \land \neg b = b$

Example B.30

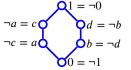


The function \neg illustrated to the left is a **de Morgan negation**, but it is *not* a *Kleene negation* and not an *ortho negation*. The *negation* illustrated to the right is a **Kleene negation**, but it is *not* an *ortho negation*.

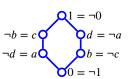


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Example B.31



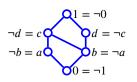
The function \neg illustrated to the left is a **de Morgan nega**tion (Definition B.3 page 35); it is *not* a *Kleene negation* (it does $\neg b = c$ not satisfy the Kleene condition). The *negation* illustrated $\neg d = a$ to the right is an **ortho negation** (Definition B.3 page 35).



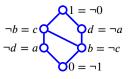
⁸⁴ \blacksquare [Cignoli(1975)] page 270, \heartsuit [Restall(2000)] page 171 (Example 8.39), \blacksquare [de Vries(2007)] pages 15–16 (Example 26), \blacksquare [Dunn(1976)], \blacksquare [Belnap(1977)]



Example B.32



The function \neg illustrated to the left is **not antitone** and therefore is not a *negation* (Definition B.2 page 35). The function \neg illustrated to the right is a **Kleene negation** (Definition B.3 page 35); it is *not* an *ortho negation* (it does not have the *non-contradiction* property).



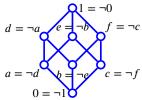
[®]Proof:

- (1) Proof that left \neg is *not antitone*: $a \le c$ but $\neg c \nleq \neg a$.
- (2) Proof that right \neg satisfies the *Kleene condition*:

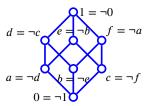
$$x \wedge \neg x = \begin{cases} b & \text{for } x = b \\ 0 & \text{otherwise} \end{cases} \quad \forall x \in X \qquad \text{and} \qquad y \wedge \neg y = \begin{cases} c & \text{for } y = c \\ 0 & \text{otherwise} \end{cases} \quad \forall y \in X \end{cases}$$
$$\implies x \wedge \neg x \leq y \vee \neg y \quad \forall x, y \in X \end{cases}$$

(3) Proof that right \neg does not have the *non-contradiction* property: $b \land \neg b = b \land c = b \neq 0$

Example B.33



The lattices illustrated to the left and right are Boolean (Definition A.41 page 30). The function \neg illustrated $d = \neg c$ to the left is a **Kleene negation** (Definition B.3 page 35), but it is not an ortho negation (it does not have the noncontradiction property). The negation illustrated to the right is an **ortho negation** (Definition B.3 page 35).



[®]Proof:

- (1) Proof that left side negation does *not* have *non-contradiction* property (and so is *not* an *ortho negation*): $a \wedge \neg a = a \wedge d = a \neq 0$
- (2) Proof that left side negation does *not* satisfy *Kleene condition* (and so is *not* a *Kleene negation*): $a \wedge \neg a = a \wedge d = a \nleq f = c \lor f = c \lor \neg c$

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Appendix C New implication functions for non-Boolean logics

C.1 Implication functions

This paper deals with how to construct a *fuzzy subset logic* not only on a *Boolean lattice*, but more generally on other types of *lattices* as well. However, any logic (fuzzy or otherwise) is arguably not complete without the inclusion of an *implication* function \rightarrow . If we were only concerned with logics on *Boolean lattices*, then arguably the *classical implication* $x \stackrel{c}{\rightarrow} y \triangleq \neg x \lor y$ would suffice. However, for some *non-Boolean* lattices, we may do well to have other options. Two common properties of *classical implication* are *entailment* and *modus ponens*. However, these properties do not always support well known logic systems that are constructed on *non-orthocomplented* (and hence also *non-Boolean*) lattices. For example,

- the *RM*₃ *logic* does not support the *strong entailment* property,
- uncerted by the *Lukasiewicz 3-valued logic* does not support the *strong modus ponens* property, and
- $\overset{\textbf{\tiny (4)}}{=}$ the *Kleene 3-valued logic* and *BN*₄ *logic* do not support either property.

This section introduces a new definition for an *implication* function with weakened forms of *entailment* and *modus ponens* (herein called *weak entailment* and *weak modus ponens*), and that supports logics constructed on a large class of lattices including *non-orthocomplemented* (and *non-Boolean*) ones.

Definition C.1 Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25). The function \rightarrow in X^X is an **implication** on L if

1. $\{x \le y\} \implies x \to y \ge x \lor y \quad \forall x, y \in X \quad (weak entailment)$ and 2. $x \land (x \to y) \le \neg x \lor y \quad \forall x, y \in X \quad (weak modus ponens)$

Proposition C.2 Let \rightarrow be an IMPLICATION (Definition C.1 page 45) On a BOUNDED LATTICE $L \triangleq (X, \lor, \land, 0, 1; \leq)$ (Definition A.19 page 25).

 $\{x \le y\} \quad \Longleftrightarrow \quad \{x \to y \ge x \lor y\} \quad \forall x, y \in X$

[®]Proof:

- (1) Proof for \implies case: by *weak entailment* property of *implications* (Definition C.1 page 45).
- (2) Proof for \Leftarrow case:

$y \ge x \land (x \to y)$	by right hypothesis
$\geq x \land (x \lor y)$	by modus ponens property of \rightarrow (Definition C.1 page 45)
= x	by <i>absorptive</i> property of <i>lattices</i> (Definition A.11 page 24)

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Remark C.3 Let $L \triangleq (X, \lor, \land, 0, 1; \le)$ be a *bounded lattice* (Definition A.19 page 25). In the context of *orthocomplemented lattices*, a more common (and stronger) definition of *implication* \rightarrow might be⁸⁵

1. $x \le y \implies x \to y = 1 \quad \forall x, y \in X \quad (entailment \mid strong entailment)$ and 2. $x \land (x \to y) \le y \quad \forall x, y \in X \quad (modus \ ponens \mid strong \ modus \ ponens)$

This definition yields a result stronger than that of Proposition C.2 (page 45):

 $\{x \le y\} \quad \Longleftrightarrow \quad \{x \to y = 1\} \quad \forall x, y \in X$

The Heyting 3-valued logic (Example C.10 page 52) and Boolean 4-valued logic (Example C.12 page 53) have both strong entailment and strong modus ponens. However, for non-Boolean logics in general, these two properties seem inappropriate to serve as a definition for *implication*. For example, the Kleene 3-valued logic (Example C.7 page 51), RM_3 logic (Example C.9 page 52), and BN_4 logic (Example C.13 page 54) do not have the strong entailment property; and the Kleene 3-valued logic, Łukasiewicz 3-valued logic (Example C.8 page 52), and BN_4 logic do not have the strong modus ponens property.

Proof:

(1) Proof for \implies case: by *entailment* property of *implications* (Definition C.1 page 45).

⁸⁵ [[Hardegree(1979)] page 59 \langle (E),(MP),(E*) \rangle , [] [Kalmbach(1973)] page 498, \square [Kalmbach(1983)] pages 238–239 \langle Chapter 4 §15 \rangle , [] [Pavičić and Megill(2009)] page 24, \square [Xu et al.(2003)Xu, Ruan, Qin, and Liu] page 27 \langle Definition 2.1.1 \rangle , [] [Xu(1999)] page 25, [] [Jun et al.(1998)Jun, Xu, and Qin] page 54





(2) Proof for \Leftarrow case:

page 46

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x \rightarrow y = 1 \implies x \land 1 \le yby modus ponens property (Definition C.1 page 45)\implies x \le yby definition of 1 (least upper bound) (Definition A.8 page 24)
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Example C.4 ⁸⁶ Let $L \triangleq (X, \lor, \land, \neg, 0, 1; \le)$ be a *lattice with negation* (Definition B.5 page 35). If L is an **orthocomplemented lattice**, then under Definition C.1, functions (1)–(5) below are valid *implication* functions with *strong entailment* and *weak modus ponens*. The *relevance implication* (6) in this lattice is *not* a valid implication: It does have *weak modus ponens*, but it does not have weak or strong entailment. However, if L is an **orthomodular lattice** (Definition A.23 page 26, a special case of an orthocomplemented lattice), then (6) is also a valid implication function with *strong entailment*.

1.	$x \xrightarrow{c} y$	≜	$\neg x \lor y$	$\forall x, y \in X$	(classical im	plicatio	n/ materia	l implication/horseshoe)
2.	$x \xrightarrow{s} y$	≜	$\neg x \lor (x \land$	<i>y</i>)			$\forall x, y \in X$	(Sasaki hook / quantum implication)
3.	$x \stackrel{d}{\rightarrow} y$	≜	$y \lor (\neg x \land$	$\neg y$)			$\forall x, y \in X$	(Dishkant implication)
4.	$x \stackrel{k}{\rightarrow} y$	≜	$(\neg x \land y) \lor$	$/(\neg x \land \neg y)$	$\vee (x \wedge (\neg x))$	$\lor y))$	$\forall x, y \in X$	(Kalmbach implication)
5.	$x \xrightarrow{n} y$	≜	$(\neg x \land y) \lor$	$(x \land y) \lor ($	$(\neg x \lor y) \land \neg$	¬ <i>y</i>)	$\forall x, y \in X$	(non-tollens implication)
6.	$x \xrightarrow{r} y$	≜	$(\neg x \land y) \lor$	$(x \land y) \lor ($	$\neg x \land \neg y)$		$\forall x, y \in X$	(relevance implication)
			. 1 1		11 0.1	•	1	• 1 • • • • 1 1

Moreover, if *L* is a **Boolean lattice**, then all of these implications are equivalent to $\stackrel{c}{\rightarrow}$, and all of them have *strong entailment* and *strong modus ponens*.

Note that $\forall x, y \in X$, $x \stackrel{d}{\to} y = \neg y \stackrel{s}{\to} \neg x$ and $x \stackrel{n}{\to} y = \neg y \stackrel{k}{\to} \neg x$. The values for the six implications on an orthocomplemented O_6 *lattice* (Definition A.45 page 31) are listed in Example C.14 (page 54).

[®]Proof:

- (1) Proofs for the *classical implication* $\stackrel{c}{\rightarrow}$:
 - (a) Proof that on an *orthocomplemented lattice*, $\stackrel{c}{\rightarrow}$ is an *implication*:

$x \le y \implies x \xrightarrow{c} y \triangleq \neg x \lor y$	by definition of \xrightarrow{c}
$\geq \neg y \lor y$	by $x \le y$ and <i>antitone</i> property of \neg (Definition B.3 page 35)
= 1	by <i>excluded middle</i> property of \neg (Theorem B.15 page 37)
\implies strong entailment	by definition of strong entailment
$x \land (\neg x \lor y) \le \neg x \lor y$	by definition of \land (Definition A.9 page 24)
\implies weak modus ponens	by definition of weak modus ponens

Note that in general for an *orthocomplemented lattice*, the bound cannot be tightened to *strong modus ponens* because, for example in the O_6 *lattice* (Definition A.45 page 31) illustrated to the right



 $x \land (\neg x \lor y) = x \land 1 = x \nleq y \implies not strong modus ponens$ (b) Proof that on a *Boolean lattice*, $\stackrel{c}{\rightarrow}$ is an *implication*:

$x \land (\neg x \lor y) = (x \land \neg x) \lor (x \land y)$	by <i>distributive</i> property (Definition A.41 page 30)
$= 1 \lor (x \land y)$	by excluded middle property of Boolean lattices
$= x \wedge y$	by definition of 1
$\leq y$	by $definition \ of \land$ (Definition A.9 page 24)
\implies strong modus ponens	by definition of strong modus ponens

⁸⁶ [[Kalmbach(1973)] page 499, [[Kalmbach(1974)], [] [Mittelstaedt(1970)] ⟨Sasaki hook⟩, [] [Finch(1970)] page 102 ⟨Sasaki hook (1.1)⟩, [Sasaki hook (1.1)⟩, [Kalmbach(1983)] page 239 ⟨Chapter 4 §15, 3. ТНЕОКЕМ⟩

by excluded middle prop. (Theorem B.15 page 37)

by definition of strong entailment

by definition of \land (Definition A.9 page 24) by definition of \land (Definition A.9 page 24)

(2) Proofs for Sasaki implication $\stackrel{s}{\rightarrow}$:

(a) Proof that on an *orthocomplemented lattice*, $\stackrel{s}{\rightarrow}$ is an *implication*:

$$x \le y \implies x \xrightarrow{s} y$$

$$\triangleq \neg x \lor (x \land y)$$

$$= \neg x \lor x$$

$$= 1$$

$$\Longrightarrow strong entailment$$

$$x \land (x \xrightarrow{s} y) \triangleq x \land [\neg x \lor (x \land y)]$$

$$\leq [\neg x \lor (x \land y)]$$

$$\leq \neg x \lor y$$

$$\Longrightarrow weak modus ponens$$

(b) Proof that on a *Boolean lattice*, $\stackrel{s}{\rightarrow} = \stackrel{c}{\rightarrow}$:

$$x \xrightarrow{s} y \triangleq \neg x \lor (x \land y)$$
by definition of \xrightarrow{s} $= \neg x \lor y$ by Lemma A.43 (page 30) $= x \xrightarrow{c} y$ by definition of \xrightarrow{c}

by definition of $\stackrel{k}{\rightarrow}$ by $x \le y$ hypothesis

by definition of \xrightarrow{s}

- (3) Proofs for *Dishkant implication* $\stackrel{d}{\rightarrow}$:
 - (a) Proof that $x \stackrel{d}{\rightarrow} y \equiv \neg y \stackrel{s}{\rightarrow} \neg x$:

$x \stackrel{d}{\rightarrow} y \triangleq y \lor (\neg x \land \neg y)$	by definition of \xrightarrow{d}
$= y \lor (\neg y \land \neg x)$	by <i>commutative</i> property of <i>lattices</i> (Theorem A.14 page 25)
$= \neg \neg y \lor (\neg y \land \neg x)$	by <i>involutory</i> property of <i>ortho negations</i> (Definition B.3 page 35)
$\triangleq \neg y \xrightarrow{s} \neg x$	by definition of \xrightarrow{s}

(b) Proof that on an *orthocomplemented lattice*, $\stackrel{d}{\rightarrow}$ is an *implication*:

$x \le y \implies x \stackrel{d}{\to} y$	
$\triangleq y \lor (\neg x \land \neg y)$	by definition of $\stackrel{d}{\rightarrow}$
$= y \vee \neg y$	by $x \leq y$ hypothesis and <i>antitone</i> property
= 1	by excluded middle property of ortho negation
\implies strong entailment	by definition of strong entailment
$x \land (x \stackrel{d}{\rightarrow} y) \triangleq y \lor (\neg x \land \neg y)$	by definition of $\stackrel{d}{\rightarrow}$
$= y \vee \neg x$	by definition of \land (Definition A.9 page 24)
\implies weak modus ponens	

(c) Proof that on a *Boolean lattice*, $\stackrel{d}{\rightarrow} = \stackrel{c}{\rightarrow}$:

$$x \xrightarrow{d} y \triangleq y \lor (\neg x \land \neg y)$$
by definition of \xrightarrow{d} $= \neg x \lor y$ by Lemma A.43 (page 30) $= x \xrightarrow{c} y$ by definition of \xrightarrow{c}

(4) Proofs for the *Kalmbach implication* $\stackrel{k}{\rightarrow}$:



(a) Proof that on an *orthocomplemented lattice*, $\stackrel{k}{\rightarrow}$ is an *implication*:

 $x \leq y \implies x \stackrel{k}{\rightarrow} y$ $\triangleq (\neg x \land y) \lor (\neg x \land \neg y) \lor [x \land (\neg x \lor y)]$ by definition of $\stackrel{k}{\rightarrow}$ $= (\neg x \land y) \lor (\neg y) \lor [x \land (\neg x \lor y)]$ by antitone property (Definition B.3 page 35) $= (\neg x \land y) \lor \neg y \lor [x \land (1)]$ $= (\neg x \land y) \lor (x \lor \neg y)$ by definition of 1 (Definition A.8 page 24) $= \neg \neg (\neg x \land y) \lor (x \lor \neg y)$ by *involutory* property (Definition B.3 page 35) $= \neg(\neg \neg x \lor \neg y) \lor (x \lor \neg y)$ by *de Morgan* property (Theorem B.15 page 37) $= \neg (x \lor \neg y) \lor (x \lor \neg y)$ by *involutory* property (Definition B.3 page 35) = 1by *excluded middle* prop. (Theorem B.15 page 37) \implies strong entailment

 $x \wedge (x \xrightarrow{k} y)$ $\triangleq x \land [(\neg x \land y) \lor (\neg x \land \neg y) \lor [x \land (\neg x \lor y)]]$ by definition of $\stackrel{k}{\rightarrow}$ by definition of \land (Definition A.9 page 24) $\leq (\neg x \land y) \lor (\neg x \land \neg y) \lor [x \land (\neg x \lor y)]$ by definition of \land (Definition A.9 page 24) $\leq (\neg x \land y) \lor (\neg x \land \neg y) \lor (\neg x \lor y)$ $\leq y \lor (\neg x \land \neg y) \lor \neg x \lor y$ by definition of \land (Definition A.9 page 24) $= y \lor \neg x \lor (\neg x \land \neg y)$ by *idempotent* p. (Theorem A.14 page 25) by definition of \land (Definition A.9 page 24) $\leq y \lor \neg x \lor \neg x$ by *idempotent* p. (Theorem A.14 page 25) $= \neg x \lor y$ \implies weak modus ponens

(b) Proof that on a *Boolean lattice*, $\stackrel{k}{\rightarrow} = \stackrel{c}{\rightarrow}$:

 $x \stackrel{k}{\rightarrow} y$ $\triangleq (\neg x \land y) \lor (\neg x \land \neg y) \lor [x \land (\neg x \lor y)]$ $= (\neg x \land y) \lor (\neg x \land \neg y) \lor [(x \land \neg x) \lor (x \land y)]$ $= (\neg x \land y) \lor (\neg x \land \neg y) \lor [(0) \lor (x \land y)]$ $= (\neg x \land y) \lor (\neg x \land \neg y) \lor (x \land y)$ $= \neg x \land (y \lor \neg y) \lor (x \land y)$ $= \neg x \land 1 \lor (x \land y)$ $= \neg x \lor y$ $\triangleq x \stackrel{c}{\rightarrow} y$

by definition of $\stackrel{k}{\rightarrow}$ by *distributive* property (Definition A.41 page 30) by *non-contradiction* property by *bounded* property (Definition A.19 page 25) by *distributive* property (Definition A.41 page 30) by *excluded middle* property by definition of 1 (Definition A.8 page 24) by Lemma A.43 (page 30) by definition of $\stackrel{c}{\rightarrow}$

(5) Proofs for the *non-tollens implication* ^{*n*}→:
(a) Proof that x ^{*n*}→ y ≡ ¬y ^{*k*}→ ¬x:

$$x \stackrel{n}{\rightarrow} y \triangleq (\neg x \land y) \lor (x \land y) \lor [(\neg x \lor y) \land \neg y]$$
 by definition of $\stackrel{n}{\rightarrow}$
$$= (y \land \neg x) \lor (y \land x) \lor [\neg y \land (y \lor \neg x)]$$

$$= (\neg y \land \neg x) \lor (\neg y \land \neg x) \lor [\neg y \land (\neg y \lor \neg x)]$$

$$\triangleq \neg y \stackrel{k}{\rightarrow} \neg x$$
 by definition of $\stackrel{k}{\rightarrow}$

(b) Proof that on an *orthocomplemented lattice*, \xrightarrow{n} is an *implication*:

$$x \le y \implies x \stackrel{n}{\to} y$$

$$\equiv \neg y \stackrel{k}{\to} \neg x \qquad by item (5a) page 48$$

$$= 1 \qquad by item (4a) page 48$$

$$\implies strong entailment$$

$$x \land (x \stackrel{n}{\to} y) = x \land (\neg y \stackrel{k}{\to} \neg x) \qquad by item (5a) page 48$$

$$\leq \neg \neg y \lor \neg x \qquad by item (4a) page 48$$

$$= y \lor \neg x \qquad by involutory property of \neg (Definition B.3 page 35)$$

$$= \neg x \lor y \qquad by commutative property of lattices$$

$$\implies weak modus ponens$$

(c) Proof that on a *Boolean lattice*, $\xrightarrow{n} = \xrightarrow{c}$:

$x \xrightarrow{n} y = \neg y \xrightarrow{k} \neg x$	by item (5a) page 48
$= \neg \neg y \lor \neg x$	by item (4b) page 48
$= y \lor \neg x$	by <i>involutory</i> property of \neg (Definition B.3 page 35)
$= \neg x \lor y$	by <i>commutative</i> property of <i>lattices</i> (Definition A.11 page 24)
$\triangleq x \xrightarrow{c} y$	by definition of $\stackrel{c}{\rightarrow}$

- (6) Proofs for the *relevance implication* \xrightarrow{r} :
 - (a) Proof that on an *orthocomplemented lattice*, → does *not* have *weak entailment*: In the *orthocomplemented lattice* to the right...

$$x \le y \implies x \xrightarrow{r} y$$

$$\triangleq (\neg x \land y) \lor (x \land y) \lor (\neg x \land \neg y) \qquad \text{by definition of } \xrightarrow{r}$$

$$= 0 \lor x \lor \neg y$$

$$= x \lor \neg y$$

$$\neq x \lor y$$

(b) Proof that on an *orthomodular lattice*, \xrightarrow{r} *does* have *strong entailment*:

$x \le y \implies x \xrightarrow{r} y$	
$\triangleq (\neg x \land y) \lor (x \land y) \lor (\neg x \land \neg y)$	by definition of \xrightarrow{r}
$= (\neg x \land y) \lor x \lor (\neg x \land \neg y)$	by $x \le y$ hypothesis
$= (\neg x \land y) \lor x \lor \neg y$	by $x \le y$ and <i>antitone</i> property (Definition B.3 page 35)
$= y \vee \neg y$	by orthomodular identity (Definition B.3 page 35)
= 1	by <i>excluded middle</i> property of ¬ (Theorem B.15 page 37)

(c) Proof that on an *orthocomplemented lattice*, \xrightarrow{r} *does* have *weak modus ponens*:

 $\begin{array}{ll} x \wedge (x \xrightarrow{r} y) \triangleq x \wedge [(\neg x \wedge y) \lor (x \wedge y) \lor (\neg x \wedge \neg y)] & \text{by definition of } \xrightarrow{r} \\ \leq [(\neg x \wedge y) \lor (x \wedge y) \lor (\neg x \wedge \neg y)] & \text{by definition of } \wedge (\text{Definition A.9 page 24}) \\ \leq \neg x \lor (x \wedge y) \lor (\neg x \wedge \neg y) & \text{by definition of } \wedge (\text{Definition A.9 page 24}) \\ \leq \neg x \lor y \lor (\neg x \wedge \neg y) & \text{by definition of } \wedge (\text{Definition A.9 page 24}) \\ \leq \neg x \lor y & \text{by definition of } \wedge (\text{Definition A.9 page 24}) \\ \leq \neg x \lor y & \text{by definition of } \wedge (\text{Definition A.9 page 24}) \\ \leq \neg x \lor y & \text{by definition property (Theorem A.14 page 25)} \\ \implies weak modus ponens \end{array}$



(d) Proof that on a *Boolean lattice*, $\xrightarrow{r} = \xrightarrow{c}$:

$x \xrightarrow{r} y \triangleq (\neg x \land y) \lor (x \land y) \lor (\neg x \land \neg y)$	by definition of \xrightarrow{r}
$= [\neg x \land (y \lor \neg y)] \lor (x \land y)$	by <i>distributive</i> property (Definition A.41 page 30)
$= [\neg x \land 1] \lor (x \land y)$	by <i>excluded middle</i> property of ¬ (Theorem B.15 page 37)
$= \neg x \lor (x \land y)$	by definition of 1 and \land (Definition A.9 page 24)
$= \neg x \lor y$	by property of <i>Boolean lattices</i> (Lemma A.43 page 30)
$\triangleq x \stackrel{c}{\to} y$	by definition of $\stackrel{c}{\rightarrow}$

C.2 Logics

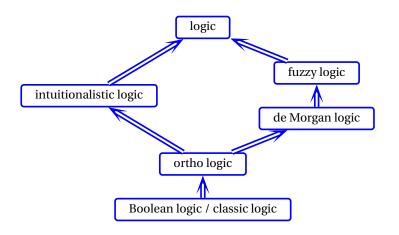


Figure 13: lattice of logics

Definition C.5 ⁸⁷ Let \rightarrow be an *implication* (Definition C.1 page 45) defined on a *lattice with negation* $L \triangleq (X, \lor, \land, \neg, 0, 1; \leq)$ (Definition B.5 page 35).

$(X, \lor, \land, \neg, 0, 1; \leq, \rightarrow)$ is a logic	if ¬ is a <i>minimal negation</i> .
$(X, \lor, \land, \neg, 0, 1; \leq, \rightarrow)$ is a fuzzy logic	if ¬ is a <i>fuzzy negation</i> .
$(X, \lor, \land, \neg, 0, 1; \leq, \rightarrow)$ is an intuitionalistic logic	if ¬ is an <i>intuitionalistic negation</i> .
$(X, \lor, \land, \neg, 0, 1; \leq, \rightarrow)$ is a de Morgan logic	if ¬ is a <i>de Morgan negation</i> .
$(X, \lor, \land, \neg, 0, 1; \leq, \rightarrow)$ is a Kleene logic	if ¬ is a <i>Kleene negation</i> .
$(X, \lor, \land, \neg, 0, 1; \leq, \rightarrow)$ is an ortho logic	if ¬ is an <i>ortho negation</i> .
$(X, \lor, \land, \neg, 0, 1; \leq, \rightarrow)$ is a Boolean logic	if \neg is an <i>ortho negation</i> and <i>L</i> is <i>Boolean</i> .

Example C.6 (Aristotelian logic/classical logic) ⁸⁸ The *classical bi-variate logic* is defined below. It is a 2 element *Boolean logic* (Definition C.5 page 50). with $L \triangleq (\{1,0\}, \land, \neg, 0, 1, \le; \lor)$ and a *classical implication* \rightarrow with *strong entailment* and *strong modus ponens*. The value 1 represents "*true*" and 0 represents

⁸⁷ [] [Straßburger(2005)] page 136 (Definition 2.1), [] [de Vries(2007)] page 11 (Definition 16)

⁸⁸ S [Novák et al.(1999)Novák, Perfilieva, and Močkoř] pages 17–18 (EXAMPLE 2.1)

"false".

$$\begin{array}{c} \mathbf{0}_{1} = \neg 0 \\ \mathbf{0}_{0} = \neg 1 \end{array} \qquad x \to y \triangleq \left\{ \begin{array}{cc} 1 & \forall x \leq y \\ y & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{c} \overrightarrow{-1} & 1 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 1 & 1 \end{array} \right. \qquad \forall x, y \in X \\ \end{array} \right\} = \neg x \lor y$$

[®]Proof:

- (1) Proof that \neg is an *ortho negation*: by Definition B.3 (page 35)
- (2) Proof that \rightarrow is an *implication* with *strong entailment* and *strong modus ponens*:
 - (a) *L* is *Boolean* and therefore is *orthocomplemented*.
 - (b) \rightarrow is equivalent to the *classical implication* $\stackrel{c}{\rightarrow}$ (Example C.4 page 46).
 - (c) By Example C.4 (page 46), \rightarrow has strong entailment and strong modus ponens.

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The *classical logic* (previous example) can be generalized in several ways. Arguably one of the simplest of these is the 3-valued logic due to Kleene (next example).

Example C.7 ⁸⁹ The **Kleene 3-valued logic** $(X, \lor, \land, \neg, 0, 1; \le, \rightarrow)$ is defined below. The function \neg is a *Kleene negation* (Definition B.3 page 35) and is presented in Example B.25 (page 41). The function \rightarrow is the *classic implication* $x \rightarrow y \triangleq \neg x \lor y$. The values 1 represents "*true*", 0 represents "*false*", and *n* represents "*neutral*" or "*undecided*".

[∞]Proof:

- (1) Proof that \neg is a *Kleene negation*: see Example B.25 (page 41)
- (2) Proof that \rightarrow is an *implication*: This follows directly from the definition of \rightarrow and the definition of an *implication* (Definition C.1 page 45).
- (3) Proof that \rightarrow does not have *strong entailment*: $n \rightarrow n = n = n \lor n \neq 1$.
- (4) Proof that \rightarrow does not have *strong modus ponens*: $n \rightarrow 0 = n = \neg n \lor 0 \nleq 0$.

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A lattice and negation alone do not uniquely define a logic. Łukasiewicz also introduced a 3-valued logic with identical lattice structure to Kleene, but with a different implication relation (next example). Historically, Łukasiewicz's logic was introduced before Kleene's.



⁸⁹ [[[Kleene(1938)] page 153, [Kleene(1952)], pages 332–339 (§64. The 3-valued logic), [[[Avron(1991)] page 277

Example C.8 90

The **Łukasiewicz 3-valued logic** $(X, \lor, \land, \neg, 0, 1; \le, \rightarrow)$ is defined to the right and below. The function \neg is a *Kleene negation* (Definition B.3 page 35) and is presented in Example B.25 (page 41). The implication has *strong entailment* but *weak modus ponens*. In the implication table below, values that differ from the classical $x \rightarrow y \triangleq \neg x \lor y$ are shaded.

 $0 = \neg 0$ $0 = \neg n$ $0 = \neg 1$

$$x \to y \triangleq \left\{ \begin{array}{ccc} 1 & \forall x \le y \\ \neg x \lor y & \text{otherwise} \end{array} \right\} = \left\{ \begin{array}{ccc} \hline 1 & n & 0 \\ \hline 1 & 1 & n & 0 \\ n & 1 & 1 & n \\ 0 & 1 & 1 & 1 \end{array} \right\} \forall x, y \in X = \left\{ \begin{array}{ccc} 1 & \text{for } x = y = n \\ \neg x \lor y & \text{otherwise} \end{array} \right\}$$

[®]Proof:

- (1) Proof that \neg is a *Kleene negation*: see Example B.25 (page 41)
- (2) Proof that \rightarrow is an *implication*: This follows directly from the definition of \rightarrow and the definition of an *implication* (Definition C.1 page 45).
- (3) Proof that \rightarrow does not have *strong modus ponens*: $n \rightarrow 0 = n = \neg n \lor 0 \nleq 0$.

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Example C.9 ⁹¹ The **RM**₃ **logic** $(X, \lor, \land, \neg, 0, 1; \le, \rightarrow)$ is defined below. The function \neg is a *Kleene negation* (Definition B.3 page 35) and is presented in Example B.25 (page 41). The implication function has *weak entailment* but *strong modus ponens*. In the implication table below, values that differ from the classical $x \rightarrow y \triangleq \neg x \lor y$ are **shaded**.

$$\begin{cases} 0 \\ n = \neg n \\ 0 = \neg 1 \end{cases} x \rightarrow y \triangleq \begin{cases} 1 & \forall x < y \\ n & \forall x = y \\ 0 & \forall x > y \end{cases} = \begin{cases} \frac{\rightarrow}{1} & n & 0 \\ 1 & 1 & 0 & 0 \\ n & 1 & n & 0 \\ 0 & 1 & 1 & 1 \end{cases} \quad \forall x, y \in X \end{cases}$$

[©]Proof:

- (1) Proof that \neg is a *Kleene negation*: see Example B.25 (page 41)
- (2) Proof that \rightarrow is an *implication*: This follows directly from the definition of \rightarrow and the definition of an *implication* (Definition C.1 page 45).
- (3) Proof that \rightarrow does not have *strong entailment*: $n \rightarrow n = n = n \lor n \neq 1$.

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In a 3-valued logic, the negation does not necessarily have to be as in the previous three examples. The next example offers a different negation.

Example C.10 (Heyting 3-valued logic/Jaśkowski's first matrix) ⁹²

The **Heyting 3-valued logic** $(X, \lor, \land, \neg, 0, 1; \le, \rightarrow)$ is defined below. The negation \neg is both *intuition-istic* and *fuzzy* (Definition B.2 page 35), and is defined on a 3 element *linearly ordered lattice* (Definition A.3 page 23).

 $^{^{90}}$ \blacksquare [Łukasiewicz(1920)] page 17 (II. The principles of consequence), \blacksquare [Avron(1991)] page 277 (Łukasiewicz.)

⁹¹ [[[Avron(1991)] pages 277–278, [] [Sobociński(1952)]

⁹² ► [Karpenko(2006)] page 45, ► [Johnstone(1982)] page 9 (§1.12), [[Heyting(1930a)], [[Heyting(1930b)], [] [Heyting(1930c)], [] [Heyting(1930d)], [] [Jaskowski(1936)], [] [Mancosu(1998)]

The implication function has both *strong entailment* and *strong modus ponens*. In the implication table below, values that differ from the classical $x \to y \triangleq \neg x \lor y$ are shaded.

$$\begin{cases} 0 \ 1 = \neg 0 \\ n \\ n \\ 0 = \neg n = \neg 1 \end{cases} x \rightarrow y \triangleq \begin{cases} 1 & \forall x \le y \\ y & \text{otherwise} \end{cases} = \begin{cases} \frac{\rightarrow}{1} & \frac{1}{n} & \frac{n}{0} \\ 1 & 1 & n & 0 \\ n & 1 & \frac{1}{1} & 0 \\ 0 & 1 & 1 & 1 \end{cases} \forall x, y \in X \end{cases}$$

[®]Proof:

- (1) Proof that \neg is a *Kleene negation*: see Example B.26 (page 41)
- (2) Proof that \rightarrow is an *implication*: by definition of *implication* (Definition C.1 page 45)

Of course it is possible to generalize to more than 3 values (next example).

Example C.11 ⁹³ The **Łukasiewicz 5-valued logic** $(X, \lor, \land, \neg, 0, 1; \le, \rightarrow)$ is defined below. The implication function has *strong entailment* but *weak modus ponens*. In the implication table below, values that differ from the classical $x \to y \triangleq \neg x \lor y$ are shaded.

$\mathbf{O}1 = \neg 0$		\rightarrow	1	р	n	т	0	
$b p = \neg m$		1	1	р	п	т	0	-
$0 = \neg 0$ $p = \neg m$ $n = \neg n$ $m = \neg p$ $0 = \neg 1$	$x \to y \triangleq \langle$	р	1	1	п	т	т	$\forall \dots \neg \nabla$
$\int m = \neg p$	$x \rightarrow y = y$	n	1	1	1	т	п	$\forall x, y \in X$
$\int_{0}^{1} - 1$		т	1	1	1	1	р	
		0	1	1	1	1	1	

[®]Proof:

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All the previous examples in this section are *linearly ordered*. The following examples employ logics that are not.

Example C.12 ⁹⁴ The **Boolean 4-valued logic** is defined below. The negation function \neg is an *ortho negation* (Example B.28 page 42) defined on an M_2 lattice. The value 1 represents "*true*", 0 represents "*false*", and *m* and *n* represent some intermediate values.

⁹³ Sample 2.1.3), International (2003) Xu, Ruan, Qin, and Liu] page 29 (Example 2.1.3), International (1998) Jun, Xu, and Qin] page 54 (Example 2.2)

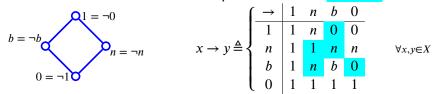


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 $[\]mathbb{P}_{4}$ [Belnap(1977)] page 13, (Restall(2000)] page 177 (Example 8.44), (Pavičić and Megill(2009)] page 28 (Definition 2, *classical implication*), (Mittelstaedt(1970)], (Finch(1970)) page 102 ((1.1)), (Smets(2006))] page 270

All the previous examples in this section are *distributive*; the previous example was *Boolean*. The next example is *non-distributive*, and *de Morgan* (but *non-Boolean*). Note for a given order structure, the method of negation may not be unique; in the previous and following examples both have identical lattices, but are negated differently.

Example C.13 ⁹⁵ The **BN**₄ **logic** is defined below. The function \neg is a *de Morgan negation* (Example B.29 page 43) defined on a 4 element M_2 *lattice*. The value 1 represents "*true*", 0 represents "*false*", *b* represents "*both*" (both true and false), and *n* represents "*neither*". In the implication table below, the values that differ from those of the *classical implication* $\stackrel{c}{\rightarrow}$ are shaded.



Example C.14

The tables that follow are the 6 implications defined in Example C.4 (page 46) on the O_6 lattice with ortho negation (Definition B.3 page 35), or the O_6 orthocomplemented lattice (Definition A.45 page 31), illustrated to the right. In the tables, the $\neg d = a$ values that differ from those of the *classical implication* $\stackrel{c}{\rightarrow}$ are shaded.

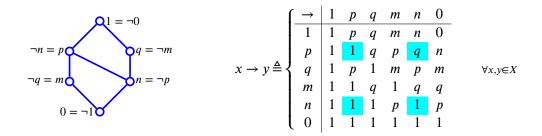
												•			_							
\xrightarrow{c}	1	d	С	b	а	0		\xrightarrow{s}	1	d	С	b	а	0		\xrightarrow{d}	1	d	С	b	а	0
1	1	d	С	b	а	0	-	1	1	d	С	b	а	0		1	1	d	С	b	а	0
d	1	1	С	1	а	а		d	1	1	а	1	а	а		d	1	1	С	1	а	а
$c \mid$	1	d	1	b	1	b		с	1	b	1	b	1	b		с	1	d	1	b	1	b
$b \mid$	1	1	с	1	С	С		b	1	1	с	1	с	с		b	1	1	С	1	а	С
a	1	d	1	d	1	d		а	1	1	1	d	1	d		a	1	d	1	b	1	d
0	1	1	1	1	1	1		0	1	1	1	1	1	1		0	1	1	1	1	1	1
\xrightarrow{k}	1	d	с	b	а	0]	\xrightarrow{n}	1	d	с	b	а	0		\xrightarrow{r}	1	d	С	b	а	0
1	1	d	С	b	а	0		1	1	d	С	b	а	0]	1	1	d	С	b	а	0
d	1	1	а	1	а	а		d	1	1	a	1	а	а		d	1	1	а	1	а	a
c	1	b	1	b	1	b		c	1	b	1	b	1	b		c	1	b	1	b	1	b
b	1	1	с	1	а	с		b	1	1	с	1	а	с		b	1	1	с	1	а	с
a	1	d	1	b	1	d		a	1	d	1	b	1	d		a	1	d	1	b	1	d
0	1	1	1	1	1	1		0	1	1	1	1	1	1		0	1	1	1	1	1	1
	$ \begin{array}{c} 1\\ d\\ c\\ b\\ a\\ 0\\ \hline \\ \hline \\ 1\\ d\\ c\\ b\\ a\\ a\\ \end{array} $	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{c ccccccccccccccccccccccccccccccccccc$	$\begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	$\begin{array}{c c c c c c c c c c c c c c c c c c c $	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$	1 1	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$						

Example C.15 ⁹⁶ A 6 element logic is defined below. The function \neg is a *Kleene negation* (Example B.32 page 44). The implication has *strong entailment* but *weak modus ponens*. In the implication table below, the values that differ from those of the *classical implication* $\stackrel{c}{\rightarrow}$ are shaded.

⁹⁵ ☜ [Restall(2000)] page 171 (Example 8.39)

^{96 📼 [}Xu et al.(2003)Xu, Ruan, Qin, and Liu] pages 29–30 (Example 2.1.4)

page 55



[®]Proof:

- (1) Proof that \neg is a *Kleene negation*: see Example B.32 (page 44)
- (2) Proof that \rightarrow is an *implication*: This follows directly from the definition of \rightarrow and the definition of an *implication* (Definition C.1 page 45).
- (3) Proof that \rightarrow does not have *strong modus ponens*:

$\neg p \land (p \rightarrow m)$	=	$n \wedge p$	=	n	\leq	р	=	$\neg p \lor m$	≰	т
$\neg n \land (n \rightarrow m)$	=	$n \wedge p$	=	n	\leq	р	=	$\neg p \lor m$	≰	т
$\neg p \wedge (p \to 0)$	=	$n \wedge n$	=	n	\leq	п	=	$\neg p \lor 0$	≰	0
$\neg n \land (n \to 0)$	=	$p \wedge n$	=	n	\leq	р	=	$\neg n \lor 0$	≰	0

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For an example of an 8-valued logic, see [[Kamide(2013)]. For examples of 16-valued logics, see [[Shramko and Wansing(2005)].

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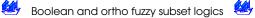
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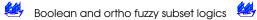
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