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A NEW THEOREM ON THE PRIME-COUNTING FUNCTION

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Abstract. For $x > 0$ let $\pi(x)$ denote the number of primes not exceeding x. For integers a and $m > 0$, we determine when there is an integer $n > 1$ with $\pi(n) = (n + a)/m$. In particular, we show that for any integers $m > 2$ and $a \leqslant \lceil e^{m-1}/(m-1)\rceil$ there is an integer $n > 1$ with $\pi(n) = (n+a)/m$. Consequently, for any integer $m > 4$ there is a positive integer n with $\pi(mn) = m+n$. We also pose several conjectures for further research; for example, we conjecture that for each $m = 1, 2, 3, \ldots$ there is a positive integer n such that $m + n$ divides $p_m + p_n$, where p_k denotes the k-th prime.

1. Introduction

For $x > 0$ let $\pi(x)$ denote the number of primes not exceeding x. The function $\pi(x)$ is usually called the *prime-counting function*. For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, let p_n stand for the n-th prime. By the Prime Number Theorem,

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to +\infty;$$

equivalently, $p_n \sim n \log n$ as $n \to +\infty$. The asymptotic behaviors of $\pi(x)$ and p_n have been intensively investigated by analytic number theorists. Recently, the author [S15] formulated many conjectures on arithmetic properties of $\pi(x)$ and p_n which depend on exact values of $\pi(x)$ or p_n. For example, he conjectured that for any integer $n > 1$, the number $\pi(kn)$ is prime for some $k = 1, \ldots, n$.

In 1962, S. Golomb [G] found the following surprising property of $\pi(x)$: For any integer $k > 1$ there is an integer $n > 1$ with $n/\pi(n) = k$. Along this line, we obtain the following general result.

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Theorem 1.1. (i) *Let m be any positive integer. For the set*

$$S_m := \left\{a \in \mathbb{Z}:\ \pi(n) = \frac{n+a}{m} \text{ for some integer } n > 1\right\}, \tag{1.1}$$

we have

$$S_m = \{\dots, -2, -1, \dots, S(m)\}, \tag{1.2}$$

where

$$S(m) := \max\{km - p_k :\ k \in \mathbb{Z}^+\} = \max\{km - p_k :\ k = 1, 2, \dots, \lfloor e^{m+1} \rfloor\}. \tag{1.3}$$

(ii) *We have*

$$(m-1)S(m+1) > mS(m) \quad \text{for any } m \in \mathbb{Z}^+. \tag{1.4}$$

Also,

$$\frac{e^{m-1}}{m-1} < S(m) < (m-1)e^{m+1} \quad \text{for all } m = 3, 4, \dots, \tag{1.5}$$

and hence

$$\lim_{m \to +\infty} \sqrt[m]{S(m)} = e. \tag{1.6}$$

Remark 1.1. For any integer $m \geqslant 2$, we have $S(m) \geqslant m - p_1 \geqslant 0$ and hence Theorem 1.1 yields Golomb's result $0 \in S_m$. In view of (1.5), for each $m = 3, 4, \dots$, the least $k \in \mathbb{Z}^+$ with $km - p_k = S(m)$ is greater than $e^{m-1}/(m-1)^2$.

Corollary 1.1. *Let $m > 0$ and $a \leqslant m^2 - m - 1$ be integers. Then there is an integer $n > 1$ with $\pi(n) = (n+a)/m$, i.e.,*

$$\pi(mn - a) = n \quad \text{for some } n \in \mathbb{Z}^+. \tag{1.7}$$

Remark 1.2. For any positive integer m, if we let n be the number of primes not exceeding the m-th composite number, then $\pi(m+n) = n$.

Corollary 1.2. *For any integer $m > 4$, there is a positive integer n such that*

$$\pi(mn) = m + n. \tag{1.8}$$

Remark 1.3. Let n be any positive integer. Clearly $\pi(n) < n + 1$ and $\pi(2n) \leqslant n < n + 2$. Observe that

$$\begin{aligned}
2n = &\left\lfloor \frac{3n}{2} \right\rfloor + n - \left\lfloor \frac{n}{2} \right\rfloor \\
=&|\{1 \leqslant k \leqslant 3n :\ 2 \mid k\}| + |\{1 \leqslant k \leqslant 3n :\ 3 \mid k\}| - |\{1 \leqslant k \leqslant 3n :\ 6 \mid k\}| \\
=&|\{1 \leqslant k \leqslant 3n :\ \gcd(k, 6) > 1\}| \\
\leqslant&|\{1 \leqslant k \leqslant 3n :\ k \text{ is not prime}\}| + 1 = 3n - \pi(3n) + 1
\end{aligned}$$

and hence $\pi(3n) \leqslant n+1 < n+3$. As $k := \pi(4n) \geqslant 2$, we have $4n \geqslant p_k \geqslant k(\log k + \log\log k - 1)$ by [D]. If $n \geqslant 45$, then $\log k + \log\log k \geqslant 5$ and hence $\pi(4n) = k \leqslant n < n+4$. We can easily verify that $\pi(4n) < n+4$ if $n \leqslant 44$.

Recall that the well-known Fibonacci numbers F_n $(n \in \mathbb{N} = \{0,1,2,\ldots\})$ are given by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_{k+1} = F_k + F_{k-1}\ (k = 1,2,3,\ldots).$$

Corollary 1.3. *For any integer $m > 3$, there is a positive integer n such that*

$$\pi(mn) = F_m + n. \tag{1.9}$$

A positive integer n is called a *practical number* if every $m = 1,\ldots,n$ can be expressed as a sum of some distinct (positive) divisors of n. The only odd practical number is 1. The distribution of practical numbers is quite similar to that of prime numbers. For $x > 0$ let $P(x)$ denote the number of practical numbers not exceeding x. Similar to the Prime Number Theorem, we have

$$P(x) \sim c\frac{x}{\log x} \quad \text{for some constant } c > 0,$$

which was conjectured by M. Margenstern [M] in 1991 and proved by A. Weingartner [W] in 2014. In view of this, our method to prove Theorem 1.1(i) allows us to deduce for any positive integer m the equality

$$\left\{a \in \mathbb{Z}:\ P(n) = \frac{n+a}{m} \quad \text{for some } n \in \mathbb{Z}^+\right\} = \{\ldots,-2,-1,0,\ldots,T(m)\}, \tag{1.10}$$

where $T(m) = \max\{km - q_k :\ k \in \mathbb{Z}^+\}$ with q_k the k-th practical number.

We are going to show Theorem 1.1 in the next section. Section 3 contains our proofs of Corollaries 1.1-1.3 and related numerical tables. In Section 4 we pose several conjectures for further research.

2. Proof of Theorem 1.1

Proof of Theorem 1.1(i). By [D],

$$p_k \geqslant k(\log k + \log\log k - 1) \quad \text{for all } k = 2,3,\ldots.$$

So, for any integer $k > e^{m+1}$, we have

$$km - p_k \leqslant k(m - \log(k\log k) + 1) < 0 \quad \text{and hence } km - p_k \leqslant -1 \leqslant m - p_1.$$

Therefore $S(m) = \max\{km - p_k :\ k = 1,2,\ldots,\lfloor e^{m+1}\rfloor\}$.

For any $a \in S_m$, there is an integer $n > 1$ such that $k := \pi(n) = (n+a)/m$ and hence $a = km - n \leqslant km - p_k \leqslant S(m)$.

Define $I_k := \{km - p_{k+1} + 1, \ldots, km - p_{k+1} + m\}$ for all $k \in \mathbb{N}$. As $\min I_0 = -1$, and $\min I_{k+1} \leqslant \max I_k$ for all $k \in \mathbb{N}$, we see that $\bigcup_{k\in\mathbb{N}} I_k \supseteq \{-1, \ldots, S(m)\}$. Note that $\max I_k \leqslant S(m)$ and $km - p_{k+1} \to -\infty$. If a is an integer with $\max I_{k+1} < a < \min I_k$, then for $n = (k+1)m - a$ we have $p_{k+1} + m - 1 < n < p_{k+2} - m$, hence $a \in S_m$ since $\pi(n) = k + 1 = (n+a)/m$. Therefore

$$\left(\bigcup_{k\in\mathbb{N}} I_k\right) \cup S_m = \{\ldots, -2, -1, \ldots, S(m)\}. \tag{2.1}$$

Now suppose that a is an integer with $a \leqslant S(m)$ and $a \notin S_m$. We want to deduce a contradiction. In light of (2.1), for some $k \in \mathbb{N}$ we have

$$a \in I_k = \{km - p_{k+1} + 1, \ldots, km - p_{k+1} + m\}. \tag{2.2}$$

Write $a = km + r$ with $1 - p_{k+1} \leqslant r \leqslant m - p_{k+1}$. We claim that

$$\frac{n+r}{\pi(n) - k} = m \quad \text{for some integer } n \geqslant p_{k+1}. \tag{2.3}$$

This is obvious for $m = p_{k+1} + r$ since

$$\frac{p_{k+1} + r}{\pi(p_{k+1}) - k} = p_{k+1} + r.$$

Below we assume $m > p_{k+1} + r$. As $\pi(n) \sim n/\log n$, we see that

$$\lim_{n\to+\infty} \frac{n+r}{\pi(n) - k} = +\infty.$$

So, we may choose the least integer $n \geqslant p_{k+1}$ with $(n+r)/(\pi(n) - k) \geqslant m$. Clearly $n \neq p_{k+1}$, thus $n - 1 \geqslant p_{k+1}$ and hence

$$\frac{n+r}{\pi(n) - k} \geqslant m > \frac{(n-1) + r}{\pi(n-1) - k} \tag{2.4}$$

by the choice of n. Set

$$s = n - 1 + r \quad \text{and} \quad t = \pi(n-1) - k.$$

As $n - 1 \geqslant p_{k+1}$, we have $t \geqslant 1$. Note also that

$$\begin{aligned}
s - t =& n - 1 + r - (\pi(n-1) - k) \\
\geqslant& n - 1 + (1 - p_{k+1}) - \pi(n-1) + k \\
=& (n - 1 - p_{k+1}) - (\pi(n-1) - \pi(p_{k+1})) \\
=& |\{p_{k+1} < d \leqslant n - 1 :\ d \text{ is composite}\}| \\
\geqslant& 0.
\end{aligned}$$

If n is prime, then $\pi(n) = \pi(n-1) + 1$ and hence

$$\frac{n+r}{\pi(n)-k} = \frac{s+1}{t+1} \leqslant \frac{s}{t} = \frac{n-1+r}{\pi(n-1)-k}$$

which contradicts (2.4). Thus n is not prime and hence

$$n + r \geqslant m(\pi(n) - k) = m(\pi(n-1) - k) > n - 1 + r.$$

It follows that

$$\frac{n+r}{\pi(n)-k} = m.$$

By the claim (2.3), for some integer $n \geqslant p_{k+1}$ we have

$$\pi(n) = k + \frac{n+r}{m} = \frac{n+a}{m}.$$

Therefore $a \in S_m$, which contradicts the supposition.

In view of the above, we have completed the proof of Theorem 1.1(i). $\square$

Proof of Theorem 1.1(ii). For any given $m \in \mathbb{Z}^+$, we may choose $k \in \mathbb{Z}^+$ with $km - p_k = S(m)$, and hence

$$\begin{aligned}(m-1)S(m+1) \geqslant &(m-1)(k(m+1) - p_k) = (m-1)S(m) + k(m-1) \\ > &(m-1)S(m) + km - p_k = mS(m).\end{aligned}$$

This proves (1.4).

Clearly (1.6) follows from (1.5). Let $m > 2$ be an integer. As $p_k > k$ for $k \in \mathbb{Z}^+$, we have $S(m) < (m-1)e^{m+1}$ by (1.3). So it remains to show $j := \lfloor e^{m-1}/(m-1) \rfloor < S(m)$.

For $m = 3$, we clearly have $j = 3 < 3 \times 3 - p_3 \leqslant S(3)$.

Below we assume $m \geqslant 4$. Then $j \geqslant 6$ and hence

$$p_j \leqslant j(\log j + \log\log j)$$

by [RS, (3.13)] and [D, Lemma 1]. Clearly

$$\log j \leqslant \log \frac{e^{m-1}}{m-1} = m - 1 - \log(m-1) < m - 1,$$

and thus

$$jm - p_j \geqslant j(m - \log j) - j\log\log j > j(1 + \log(m-1)) - j\log(m-1) = j.$$

Therefore $j < S(m)$ as desired. $\square$

3. Proofs of Corollaries 1.1-1.3 and related data

Proof of Corollary 1.1. By Theorem 1.1, it suffices to show that $m^2 - m - 1 \leqslant S(m)$.

For $m \leqslant 5$, we have $m^2 - m - 1 \leqslant km - p_k$ for some $k \in \mathbb{Z}^+$. In fact,

$$1^2 - 1 - 1 = 1 \times 1 - p_1,\ 2^2 - 2 - 1 = 2 \times 2 - p_2,\ 3^2 - 3 - 1 = 5 = 4 \times 3 - p_4,$$
$$4^2 - 4 - 1 = 11 = 6 \times 4 - p_6 \ \text{ and } \ 5^2 - 5 - 1 = 19 < 8 \times 5 - p_8 = 21.$$

For $m \geqslant 6$, we have $m^2 - m - 1 < e^{m-1}/(m-1)$ and hence $m^2 - m - 1 < S(m)$ by (1.5). This concludes the proof. $\square$

As $S(m) = \max\{km - p_k :\ k = 1, \ldots, \lfloor e^{m+1} \rfloor\}$, we can determine the exact values of $S(m)$ for smaller positive integers m.

Table 3.1: Values of $S(m)$ for $m = 1, \ldots, 17$

m	1	2	3	4	5	6	7	8	9	10
$S(m)$	-1	1	5	13	37	83	194	469	1111	2743

m	11	12	13	14	15	16	17
$S(m)$	6698	16379	40543	101251	254053	640483	1622840

In the following table, for each $m = 2, \ldots, 20$ we give the least integer $n > 1$ with $\pi(n) = (n-1)/m$ as well as the least integer $n > 1$ with $\pi(n) = (n+m-1)/m$.

Table 3.2

m	Least $n > 1$ with $\pi(n) = \frac{n-1}{m}$	Least $n > 1$ with $\pi(n) = \frac{n+m-1}{m}$
2	9	3
3	28	4
4	121	93
5	336	306
6	1081	1003
7	3060	2997
8	8409	8361
9	23527	23518
10	64541	64531
11	175198	175187
12	480865	480817
13	1304499	1303004
14	3523885	3523871
15	9557956	9557746
16	25874753	25874737
17	70115413	70115311
18	189961183	189961075
19	514272412	514272393
20	1394193581	1394193361

Proof of Corollary 1.2. Note that $\pi(5 \times 9) = 5 + 9$ and $\pi(6 \times 7) = 6 + 7$.

Now we assume $m \geqslant 7$. Then $m^2 < e^{m-1}/(m-1)$. By Theorem 1.1, there is a positive integer N with $\pi(N) = (N + m^2)/m$. Clearly $n = N/m \in \mathbb{Z}^+$ and $\pi(mn) = (mn + m^2)/m = m + n$. This concludes the proof. $\square$

Table 3.3: Smallest $n = s(m)$ with $\pi(mn) = m + n$ for $5 \leqslant m \leqslant 21$

m	5	6	7	8	$9 \sim 14$	15	16
$s(m)$	9	7	6	998	5	636787	1617099

m	17	18	19	20	21
$s(m)$	4124188	10553076	5	5	179992154

Proof of Corollary 1.3. Observe that

$$\pi(4 \times 5) = F_4 + 5,\ \pi(5 \times 9) = F_5 + 9,\ \pi(6 \times 12) = F_6 + 12,$$
$$\pi(7 \times 16) = F_7 + 16 \ \text{ and } \ \pi(8 \times 25) = F_8 + 25.$$

Now we assume $m \geqslant 9$. Then $mF_m < e^{m-1}/(m-1)$. By Theorem 1.1, there is a positive integer N with $\pi(N) = (N + mF_m)/m$. Note that $n = N/m \in \mathbb{Z}^+$ and $\pi(mn) = (mn + mF_m)/m = F_m + n$. This concludes the proof. $\square$

Table 3.4: Least $n = f(m)$ with $\pi(mn) = F_m + n$ for $4 \leqslant m \leqslant 22$

m	4	5	6	7	8	9	10	11	12	13	14
$f(m)$	5	9	12	16	25	45	68	116	183	287	457

m	15	16	17	18	19	20	21	22
$f(m)$	628346	1600659	1942	3133	5028	8131	13100	21142

4. Some conjectures

In view of Theorem 1.1, we pose the following conjecture.

Conjecture 4.1. (i) *Let m be any positive integer. Then $km - p_k$ is a square for some $k \in \mathbb{Z}^+$, and $p_k - km$ is a square for some $k \in \mathbb{Z}^+$. Also, $km - p_k$ is prime for some $k \in \mathbb{Z}^+$, and $p_k - km$ is prime for some $k \in \mathbb{Z}^+$.*

(ii) *The sequence $\sqrt[m]{S(m)}$ $(m = 1, 2, 3, \ldots)$ is strictly increasing.*

Remark 4.1. See [S14, A247278, A247893 and A247895] for some sequences related to part (i); for example, $29 \times 5 - p_{29} = 145 - 109 = 6^2$ and $p_{12} - 12 \times 3 = 1^2$. The second part of Conjecture 4.1 arises naturally in the spirit of [S13].

Golomb's result [G] indicates that for any integer $m \geqslant 2$ we have $\pi(mn) = n\,(= mn/m)$ for some $n \in \mathbb{Z}^+$. Motivated by this and Corollary 1.2, we pose the following conjecture related to Euler's totient function φ.

Conjecture 4.2. *Let m be any positive integer. Then $\pi(mn) = \varphi(n)$ for some $n \in \mathbb{Z}^+$. Also, $\pi(mn) = \varphi(m) + \varphi(n)$ for some $n \in \mathbb{Z}^+$, and $\pi(mn) = \varphi(m+n)$ for some $n \in \mathbb{Z}^+$.*

Remark 4.2. Our method to establish Theorem 1.1 does not work for this conjecture.

Table 4.1: Least $n \in \mathbb{Z}^+$ with $\pi(mn) = \varphi(n)$ for $m \leqslant 18$

m	1	2	3	4	5	6	7	8	9	10	11
n	2	1	13	31	73	181	443	2249	238839	6473	3001

m	12	13	14	15	16	17	18
n	40123	108539	251707	637321	7554079	4124437	241895689

Table 4.2: Least $n \in \mathbb{Z}^+$ with $\pi(mn) = \varphi(m) + \varphi(n)$ for $m \leqslant 18$

m	1	2	3	4	5	6	7	8	9	10	11
n	6	2	2	23	3	1	3	1033	2	6449	15887

m	12	13	14	15	16	17	18
n	1	100169	268393	636917	2113589	70324093	1

Table 4.3: Least $n \in \mathbb{Z}^+$ with $\pi(mn) = \varphi(m + n)$ for $m \leqslant 20$

m	1	2	3	4	5	6	7	8	9	10	11	12	13
n	3	2	1	91	6	5	1	5	1	8041	15870	39865	1

m	14	15	16	17	18	19	20
n	251625	637064	1829661	4124240	10553093	1	69709253

For $n \in \mathbb{Z}^+$ let $\sigma(n)$ denote the number of (positive) divisors of n. We also formulate the following conjecture motivated by Conjecture 4.2.

Conjecture 4.3. *For any integer $m > 1$, there is a positive integer n with $\pi(mn) = \sigma(n)$. Also, for any integer $m > 4$, $\pi(mn) = \sigma(m) + \sigma(n)$ for some $n \in \mathbb{Z}^+$, and $\pi(mn) = \sigma(m + n)$ for some $n \in \mathbb{Z}^+$.*

Example 4.1. The least $n \in \mathbb{Z}^+$ with $\pi(23n) = \sigma(n)$ is 8131355, the least $n \in \mathbb{Z}^+$ with $\pi(39n) = \sigma(39) + \sigma(n)$ is 75999272, and the least $n \in \mathbb{Z}^+$ with $\pi(30n) = \sigma(30 + n)$ is 39298437.

Now we pose one more conjecture which is motivated by Corollary 1.2.

Conjecture 4.4. *Let m be any positive integer. Then $m + n$ divides $p_m + p_n$ for some $n \in \mathbb{Z}^+$. Moreover, we may require $n < m(m - 1)$ if $m > 2$.*

Remark 4.3. We have verified this for all $m = 1, \ldots, 10^5$, see [S14, A247824] for related data. We also conjecture that for any $m \in \mathbb{Z}^+$ there is a positive integer n such that $\pi(mn)$ divides $p_m + p_n$, see [S14, A247793] for related data.

Example 4.2. The least $n \in \mathbb{Z}^+$ with $2 + n$ dividing $p_2 + p_n$ is 5. For $m = 79276$, the least $n \in \mathbb{Z}^+$ with $m + n$ dividing $p_m + p_n$ is $3141281384 > 3 \times 10^9$.

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