Additional congruence conditions on the number of terms in sums of consecutive squared integers equal to squared integers

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Abstract

The problem of finding all the integer solutions in a, M and s of sums of M consecutive integer squares starting at $a^2 \geq 1$ equal to squared integers s^2, has no solutions if $M \equiv 3, 5, 6, 7, 8$ or $10 \, (mod\, 12)$ and has integer solutions if $M \equiv 0, 9, 24$ or $33 \, (mod\, 72)$; or $M \equiv 1, 2$ or $16 \, (mod\, 24)$; or $M \equiv 11 \, (mod\, 12)$. In this paper, additional congruence conditions are demonstrated on the allowed values of M that yield solutions to the problem by using Beeckmans' eight necessary conditions, refining further the possible values of M for which the sums of M consecutive integer squares equal integer squares.

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1 Introduction

The general problem of sums of M consecutive integer squares starting from $a^2 \geq 1$ being equal to integer squares s^2 involves solving a single Diophantine quadratic equation in three variables M, a and s that reads

$$\sum_{i=0}^{M-1} (a+i)^2 = M \left[\left(a + \frac{M-1}{2} \right)^2 + \frac{M^2-1}{12} \right] = s^2 \qquad (1)$$

where $M > 1, a, s \in \mathbb{Z}^+$, $i \in \mathbb{Z}^*$.

With the notations of this paper, Alfred investigated [1] several necessary conditions using basic congruence equations of M. Philipp [3] extended Alfred's work to confirm some of his findings. Beeckmans showed [2] that with eight necessary conditions on M, all values of M could be found, given in a Table with values of $M < 1000$, and noted two cases for $M = 25$ and 842 that fulfilled the eight necessary conditions but did not solve the problem.

The eight necessary conditions given by Beeckmans [2] on the value of M for (1) to hold can be summarized as follows, with $e, \alpha \in \mathbb{Z}^+$:
(C1.1) If $M \equiv 0\,(mod\, 2^e)$, then $e \equiv 1\,(mod\, 2)$.
(C1.2) If $M \equiv 0\,(mod\, 3^e)$, then $e \equiv 1\,(mod\, 2)$.
(C1.3) If $M \equiv -1\,(mod\, 3^e)$, then $e \equiv 1\,(mod\, 2)$.
(C2) If $p > 3$ is prime, $M \equiv 0\,(mod\, p^e)$, $e \equiv 1\,(mod\, 2)$, then $p \equiv \pm 1\,(mod\, 12)$.
(C3) If $p \equiv 3\,(mod\, 4)\,, p > 3$ is prime, $M \equiv -1\,(mod\, p^e)$, then $e \equiv 0\,(mod\, 2)$.
(C4.1) $M \neq 3\,(mod\, 9)$.
(C4.1) $\forall \alpha \geq 2$, $M \neq (2^\alpha - 1)\left(mod\, 2^{\alpha+2}\right)$.
(C4.1) $\forall \alpha \geq 2$, $M \neq 2^\alpha\left(mod\, 2^{\alpha+2}\right)$.

Extending Beeckmans' work, it was demonstrated [4] using these conditions that, for (1) to hold, M cannot be congruent to $3, 5, 6, 7, 8$ or $10\,(mod\, 12)$.
It was further demonstrated [4], independently from Beeckmans' conditions, that for (1) to hold, $M \equiv 0, 1, 2, 4, 9$ or $11\,(mod\, 12)$. Furthermore,
- if $M \equiv 0\,(mod\, 12)$, then $M \equiv 0$ or $24\,(mod\, 72)$;
- if $M \equiv 1\,(mod\, 12)$, then $M \equiv 1\,(mod\, 24)$;
- if $M \equiv 2\,(mod\, 12)$, then $M \equiv 2\,(mod\, 24)$;
- if $M \equiv 4\,(mod\, 12)$, then $M \equiv 16\,(mod\, 24)$;
- if $M \equiv 9\,(mod\, 12)$, then $M \equiv 9$ or $33\,(mod\, 72)$.
These are called allowed values of M. The values of M yielding solutions to (1) are given in [5].
In this paper, additional congruence conditions on the allowed values of M such that (1) holds are demonstrated by using Beeckmans' necessary conditions, refining further the possible values of M for which the sums of M consecutive integer squares equal integer squares.
In this paper, the notation $A\,(mod\, B) \equiv C$ is equivalent to $A \equiv C\,(mod\, B)$ and $A \equiv C\,(mod\, B) \Rightarrow A = Bk + C$ means that, if $A \equiv C\,(mod\, B)$, then $\exists k \in \mathbb{Z}^+$ such that $A = Bk + C$. By convention, $\sum_{j=inf}^{sup} f(j) = 0$ if $sup < inf$.

2 Additional congruence conditions on M

Additional congruence necessary conditions can be found using Beeckmans' eight necessary conditions written in all generality as follows.
For $M > 1, m, m_1, A, B, p, e, q, \alpha, j \in \mathbb{Z}^+$, $i, \mu \in \mathbb{Z}^*$, $0 \leq \mu \leq 11$ and noting $M = 12m + \mu = 12Am_1 + B\mu$, one has $\forall i \geq 0$ (unless indicated otherwise):
(C1.1) $M\,() \neq 0 mod\, 2^e \Rightarrow M \neq 2^e q$ with $e \equiv 0\,(mod\, 2) \Rightarrow e = 2i$, $q \equiv 1\,(mod\, 2)$
$\Rightarrow m_1 \neq \left(2^{2i} q - B\mu\right)/12A$, $\forall i > 1$.
(C1.2) $M \neq 0\,(mod\, 3^e) \Rightarrow M \neq 3^e q$ with $e \equiv 0\,(mod\, 2) \Rightarrow e = 2i$, $q \neq 0\,(mod\, 3)$
$\Rightarrow m_1 \neq \left(3^{2i} q - B\mu\right)/12A$, $\forall i > 0$.
(C1.3) $M \neq -1\,(mod\, 3^e) \Rightarrow M \neq 3^e q - 1$ with $e \equiv 0\,(mod\, 2) \Rightarrow e = 2i$,
$q \neq 0\,(mod\, 3) \Rightarrow m_1 \neq \left(3^{2i} q - B\mu - 1\right)/12A$, $\forall i > 0$.
(C2) If $p > 3$ is prime, then $M \neq p^e q \Rightarrow m_1 \neq \left(p^e q - B\mu\right)/12A$ with either

if $p \equiv \pm 1\,(mod\,12)$, $e \equiv 0\,(mod\,2) \Rightarrow e = 2i$, $\forall i > 0$, or
if $p \neq \pm 1\,(mod\,12)$, $e \equiv 1\,(mod\,2) \Rightarrow e = 2i+1$.
(C3) If $p \equiv 3\,(mod\,4)$, $p > 3$ is prime and $e \equiv 1\,(mod\,2) \Rightarrow e = 2i+1$
$\Rightarrow M+1 \neq p^{2i+1}q \Rightarrow m_1 \neq \left(p^{2i+1}q - B\mu - 1\right)/12A$.
(C4.1) $M \neq 3\,(mod\,9) \Rightarrow M \neq 9q+3 \Rightarrow m_1 \neq (9q - B\mu + 3)/12A$.
(C4.2) $M \neq (2^\alpha - 1)\left(mod\,2^{\alpha+2}\right) \Rightarrow M \neq 2^\alpha(4q+1) - 1$
$\Rightarrow m_1 \neq (2^\alpha(4q+1) - B\mu - 1)/12A$, $\forall \alpha \geq 2$.
(C4.3) $M \neq 2^\alpha\left(mod\,2^{\alpha+2}\right) \Rightarrow M \neq 2^\alpha(4q+1) \Rightarrow m_1 \neq (2^\alpha(4q+1) - B\mu)/12A$, $\forall \alpha \geq 2$.

These necessary conditions are applied to $M = 12m + \mu$ for each case of $\mu = 0, 1, 2, 4, 9, 11$. For brevity, conditions that are not applicable, i.e. with which M is always compliant, are not indicated.
For $\mu = 0$, $A = 2$ and taking in all generality $M \equiv 0\,(mod\,24) \Rightarrow M = 24m_1$:
(C1.1) $q \equiv 3\,(mod\,6)$, $m_1 \neq 2^{2i-3}\left(mod\,2^{2i-2}\right)$, $\forall i > 1$.
(C1.2) if $q \equiv 1\,(mod\,3) \Rightarrow q \equiv 16\,(mod\,24)$, $m_1 \neq \left(2 \times 3^{2i-1}\right)\left(mod\,3^{2i}\right)$,
if $q \equiv 2\,(mod\,3) \Rightarrow q \equiv 8\,(mod\,24)$, $m_1 \neq 3^{2i-1}\left(mod\,3^{2i}\right)$.
(C2) $q \equiv 0\,(mod\,24)$, $m_1 \neq 0\,(mod\,p^e)$ with either
if $p \equiv \pm 1\,(mod\,12)$, $e \equiv 0\,(mod\,2) \Rightarrow e = 2i > 0$, $\forall i > 0$, or
if $p \neq \pm 1\,(mod\,12)$, $e \equiv 1\,(mod\,2) \Rightarrow e = 2i+1$.
(C3) $q \equiv p\,(mod\,24)$, $m_1 \neq \left[\left(p^{2i+2} - 1\right)/24\right]\left(mod\,p^{2i+1}\right)$.
(C4.1) $q \equiv 5\,(mod\,8)$, $m_1 \neq 2\,(mod\,3)$.
(C4.3) $q \equiv 2\,(mod\,3)$, $m_1 \neq \left(3 \times 2^{\alpha-3}\right)\left(mod\,2^{\alpha-1}\right)$, $\forall \alpha \geq 3$.
For $\mu = 1$, $A = 2$, $B = 1$, yielding $M \equiv 1\,(mod\,24) \Rightarrow M = 24m_1 + 1$:
(C2) if $p \equiv \pm 1\,(mod\,12) \Rightarrow q \equiv 1\,(mod\,24)$,
$m_1 \neq \left[\left(p^{2i} - 1\right)/24\right]\left(mod\,p^{2i}\right)$,
if $p \neq \pm 1\,(mod\,12) \Rightarrow q \equiv p\,(mod\,24)$,
$m_1 \neq \left[\left(p^{2i+2} - 1\right)/24\right]\left(mod\,p^{2i+1}\right)$ ([1]).
(C3) $q \equiv 2p\,(mod\,24)$, $m_1 \neq \left[\left(p^{2i+2} - 1\right)/12\right]\left(mod\,p^{2i+1}\right)$.

For $\mu = 2$, $A = 2$, $B = 1$, yielding $M \equiv 2\,(mod\,24) \Rightarrow M = 24m_1 + 2$:
(C1.3) if $q \equiv 1\,(mod\,3) \Rightarrow q \equiv 19\,(mod\,24)$, $m_1 \neq \left[\left(19 \times 3^{2i-1} - 1\right)/8\right]\left(mod\,3^{2i}\right)$,
if $q \equiv 2\,(mod\,3) \Rightarrow q \equiv 11\,(mod\,24)$, $m_1 \neq \left[\left(11 \times 3^{2i-1} - 1\right)/8\right]\left(mod\,3^{2i}\right)$.
(C2) if $p \equiv \pm 1\,(mod\,12) \Rightarrow q \equiv 2\,(mod\,24)$, $m_1 \neq \left[\left(p^{2i} - 1\right)/12\right]\left(mod\,p^{2i}\right)$, $\forall i > 0$,
if $p \neq \pm 1\,(mod\,12) \Rightarrow q \equiv 2p\,(mod\,24)$, $m_1 \neq \left[\left(p^{2i+2} - 1\right)/12\right]\left(mod\,p^{2i+1}\right)$.
(C3) $q \equiv 3p\,(mod\,24)$, $m_1 \neq \left[\left(p^{2i+2} - 1\right)/8\right]\left(mod\,p^{2i+1}\right)$.

For $\mu = 4$, $A = 2$, $B = 4$, yielding $M \equiv 16\,(mod\,24) \Rightarrow M = 24m_1 + 16$:
(C1.1) $q \equiv 1\,(mod\,6)$, $m_1 \neq \left[2\left(2^{2i-4} - 1\right)/3\right]\left(mod\,2^{2i-2}\right)$ ([2]) or
$m_1 \neq \left(2\sum_{j=3}^{i} 2^{2(j-3)}\right)\left(mod\,2^{2i-2}\right)$, $\forall i > 1$.
(C2) if $p \equiv \pm 1\,(mod\,12) \Rightarrow q \equiv 32\,(mod\,48)$, $m_1 \neq \left[2\left(p^{2i} - 1\right)/3\right]\left(mod\,p^{2i}\right)$,
if $p \neq \pm 1\,(mod\,12) \Rightarrow q \equiv 16\,(mod\,48)$, $m_1 \neq \left[2\left(p^{2i+1} - 1\right)/3\right]\left(mod\,p^{2i+1}\right)$.

[1] Beeckmans notes [2] that $M = 25$ cannot be rejected using his necessary conditions. It is found here that for $p = 5$ and $i = 0$, $m \neq 1$ and $M = 25$ can be rejected.
[2] The sequence $\left(2^{2n} - 1\right)/3 = 1, 5, 21, 85, 341, 1365, \ldots$ is given in [6]

(C3) $q\equiv 17p\,(mod\,24)$, $m_1\neq\left[17\left(p^{2i+2}-1\right)/24\right]\left(mod\,p^{2i+1}\right)$.
(C4.3) if $\alpha\equiv 0\,(mod\,2)\Rightarrow q\equiv 0\,(mod\,3)$,
$m_1\neq\left[2\left(2^{\alpha-4}-1\right)/3\right]\left(mod\,2^{\alpha-1}\right)$ or $m_1\neq\left(2\sum_{j=3}^{\alpha/2}2^{2(j-3)}\right)\left(mod\,2^{\alpha-1}\right)$,
if $\alpha\equiv 1\,(mod\,2)\Rightarrow q\equiv 1\,(mod\,3)$,
$m_1\neq\left[2\left(5\times 2^{\alpha-4}-1\right)/3\right]\left(mod\,2^{\alpha-1}\right)$ ([3]) or
$m_1\neq\left(2^{\alpha-3}+\sum_{j=2}^{(\alpha-1)/2}2^{2j-3}\right)$, $\forall\alpha\geq 3$.

For $\mu=9$, $A=2$, $B=1$ and taking in all generality $M\equiv 9\,(mod\,24)\Rightarrow M=24m_1+9$:
(C1.2) if $q\equiv 1\,(mod\,3)\Rightarrow q\equiv 1\,(mod\,24)$, $m_1\neq\left[3\left(3^{2i-2}-1\right)/8\right]\left(mod\,3^{2i}\right)$ ([4]),
or $m_1\neq\left(\sum_{j=0}^{i-2}3^{2j}\right)\left(mod\,3^{2i}\right)$,
if $q\equiv 2\,(mod\,3)\Rightarrow q\equiv 17\,(mod\,24)$,
$m_1\neq\left(2\times 3^{2i-1}+\left[3\left(3^{2i-2}-1\right)/8\right]\right)\left(mod\,3^{2i}\right)$.
(C2) if $p\equiv\pm 1\,(mod\,12)\Rightarrow q\equiv 9\,(mod\,24)$, $m_1\neq\left[3\left(p^{2i}-1\right)/8\right]\left(mod\,p^{2i}\right)$;
if $p\neq\pm 1\,(mod\,12)$,
if $p\equiv 5\,(mod\,24)\Rightarrow q\equiv 21\,(mod\,24)$, $m_1\neq\left[\left(7p^{2i+1}-3\right)/8\right]\left(mod\,p^{2i+1}\right)$,
if $p\equiv 7\,(mod\,24)\Rightarrow q\equiv 15\,(mod\,24)$, $m_1\neq\left[\left(5p^{2i+1}-3\right)/8\right]\left(mod\,p^{2i+1}\right)$,
if $p\equiv 17\,(mod\,24)\Rightarrow q\equiv 9\,(mod\,24)$, $m_1\neq\left[3\left(p^{2i+1}-1\right)/8\right]\left(mod\,p^{2i+1}\right)$,
if $p\equiv 19\,(mod\,24)\Rightarrow q\equiv 3\,(mod\,24)$, $m_1\neq\left[\left(p^{2i+1}-3\right)/8\right]\left(mod\,p^{2i+1}\right)$.
(C3) if $p\equiv 1\,(mod\,6)\Rightarrow q\equiv 22\,(mod\,24)$, $m_1\neq\left[11\left(p^{2i+1}-5\right)/12\right]\left(mod\,p^{2i+1}\right)$,
if $p\equiv 5\,(mod\,6)\Rightarrow q\equiv 14\,(mod\,24)$, $m_1\neq\left[\left(7p^{2i+1}-5\right)/12\right]\left(mod\,p^{2i+1}\right)$
(C4.1) $q\equiv 6\,(mod\,8)$, $m_1\neq 2\,(mod\,3)$.

For $\mu=11$, $A=1$, $B=1$, yielding $M\equiv 11\,(mod\,12)\Rightarrow M=12m_1+11$:
(C1.3) if $q\equiv 1\,(mod\,3)\Rightarrow q\equiv 4\,(mod\,12)$, $m_1\neq\left(3^{2i-1}-1\right)\left(mod\,3^{2i}\right)$,
if $q\equiv 2\,(mod\,3)\Rightarrow q\equiv 8\,(mod\,12)$, $m_1\neq\left(2\times 3^{2i-1}-1\right)\left(mod\,3^{2i}\right)$.
(C2) if $p\equiv\pm 1\,(mod\,12)\Rightarrow q\equiv 11\,(mod\,12)$, $m_1\neq\left[11\left(p^{2i}-1\right)/12\right]\left(mod\,p^{2i}\right)$,
if $p\equiv 5\,(mod\,12)\Rightarrow q\equiv 7\,(mod\,12)$, $m_1\neq\left[\left(7p^{2i+1}-11\right)/12\right]\left(mod\,p^{2i+1}\right)$
or $m_1\neq\left[\left(7\,(p-5)/12\right)\left(\sum_{j=0}^{2i}C_j^{2i+1}\,(p-5)^{2i-j}\,5^j\right)+\left(7\times 5^{2i+1}-11\right)/12\right]$
$\left(mod\,p^{2i+1}\right)$,
if $p\equiv 7\,(mod\,12)\Rightarrow q\equiv 5\,(mod\,12)$, $m_1\neq\left[\left(5p^{2i+1}-11\right)/12\right]\left(mod\,p^{2i+1}\right)$
or $m_1\neq\left[\left(5\,(p-7)/12\right)\left(\sum_{j=0}^{2i}C_j^{2i+1}\,(p-7)^{2i-j}\,7^j\right)+\left(5\times 7^{2i+1}-11\right)/12\right]$
$\left(mod\,p^{2i+1}\right)$.
(C3) $q\equiv 0\,(mod\,12)$, $m_1\neq -1\left(mod\,p^{2i+1}\right)$.
(C4.2) $q\equiv 2\,(mod\,3)$, $m_1\neq\left(3\times 2^{\alpha-2}-1\right)(mod\,2^{\alpha})$, $\forall\alpha\geq 2$.

Note that these are still only necessary conditions and that they do not exclude all values of M that do not give solutions to (1). For example, as signaled by Beeckmans, the value $M=842=24\times 35+2$, although complying with the conditions (C1.3), (C2) and (C3) for $\mu=2$, does not yield solutions to (1).

[3] The sequence $\left(5\times 2^{2n}-2\right)/3=1,6,26,106,426,1706,...$ is given in [7].
[4] The sequence $\left(3^{2n}-1\right)/8=1,10,91,820,7381,...$ is given in [8].

3 Conclusion

The allowed congruent values of M for which sums of M consecutive integer squares equal squared integers, namely$M \equiv 0, 9, 24$ or $33\,(mod\,72)$, $M \equiv 1, 2$ or $16\,(mod\,24)$, $M \equiv 11\,(mod\,12)$, were further characterized with additional congruence conditions using Beeckmans' eight necessary conditions.

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