Writing π as sum of arcotangents with linear recurrent sequences, Golden mean and Lucas numbers

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Abstract

In this paper, we study the representation of π as sum of arcotangents. In particular, we obtain new identities by using linear recurrent sequences. Moreover, we provide a method in order to express π as sum of arcotangents involving the Golden mean, the Lucas numbers, and more in general any quadratic irrationality.

1 Expressions of π via arctangent function with linear recurrent sequences

The problem of expressing π as the sum of arctangents has been deeply studied during the years. The first expressions are due to Newton (1676), Machin (1706), Euler (1755), who expressed π using the following identities

$$\frac{\pi}{2} = 2 \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{4}{7}\right) + \arctan\left(\frac{1}{8}\right)$$

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)$$

$$\frac{\pi}{4} = 5 \arctan\left(\frac{1}{7}\right) + 2 \arctan\left(\frac{3}{79}\right),$$

respectively (see, e.g., [12] and [13]). Many other identities and methods to express and calculate π involving the arctangent function have been developed. Some recent results are obtained in [6] and [2].

In this section, we find a method to generate new expressions of π in terms of sum of arctangents, mainly using the properties of linear recurrent sequences. For the sake of simplicity, we will use the following notation:

$$A(x) = \arctan(x).$$

It is well–known that for $x, y \geq 0$, if $y \neq \dfrac{1}{x}$

$$A(x) + A(y) = \begin{cases} A(x \odot y) & \text{if} \quad xy < 1, \\ A(x \odot y) + \operatorname{sign}(x)\pi & \text{if} \quad xy > 1, \end{cases}$$

where

$$x \odot y = \frac{x+y}{1-xy}.$$

Let us denote by $x^{\odot n}$ the n–th power of x with respect to the product $\odot$.

Remark 1. *The product $\odot$ is associative, commutative and 0 is the identity.*

Definition 1. *We denote by $a = (a_n)_{n=0}^{+\infty} = \mathcal{W}(\alpha, \beta, p, q)$ the linear recurrent sequence of order 2 with characteristic polynomial $t^2 - pt + q$ and initial conditions α and β, i.e.,*

$$\begin{cases} a_0 = \alpha \\ a_1 = \beta \\ a_n = pa_{n-1} - qa_{n-2} \quad \forall n \geq 2\ . \end{cases}$$

Theorem 1. *Given $n \in \mathbb{N}$ and $x \in \mathbb{R}$, with $x \neq \pm 1$, we have*

$$\left(\frac{1}{x}\right)^{\odot n} = \frac{v_n(x)}{u_n(x)}, \quad \forall n \geq 1$$

where

$$(u_n(x))_{n=0}^{\infty} = \mathcal{W}(1, x, 2x, 1+x^2), \quad (v_n(x))_{n=0}^{\infty} = \mathcal{W}(0, 1, 2x, 1+x^2). \quad (1)$$

Proof. The matrix

$$M = \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$$

has characteristic polynomial $t^2 - 2xt + x^2 + 1$. Consequently, it is immediate to see that

$$M^n = \begin{pmatrix} u_n(x) & v_n(x) \\ -v_n(x) & u_n(x) \end{pmatrix}.$$

Using the matrix M we can observe that

$$\begin{pmatrix} u_{n-1}(x) & v_{n-1}(x) \\ -v_{n-1}(x) & u_{n-1}(x) \end{pmatrix} \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix} = \begin{pmatrix} u_n(x) & v_n(x) \\ -v_n(x) & u_n(x) \end{pmatrix},$$

i.e.,

$$\begin{cases} u_n(x) = xu_{n-1}(x) - v_{n-1}(x) \\ v_n(x) = u_{n-1}(x) + xv_{n-1}(x) \end{cases}, \quad \forall n \geq 1.$$

Now, we prove the theorem by induction. It is straightforward to check that

$$\frac{1}{x}=\frac{v_1(x)}{u_1(x)},\qquad \left(\frac{1}{x}\right)^{\odot 2}=\frac{\frac{1}{x}+\frac{1}{x}}{1-\frac{1}{x^2}}=\frac{2x}{x^2-1}=\frac{v_2(x)}{u_2(x)}.$$

Moreover, let us suppose

$$\left(\frac{1}{x}\right)^{\odot(n-1)}=\frac{v_{n-1}(x)}{u_{n-1}(x)}$$

for a given integer $n\geq 1$, then

$$\left(\frac{1}{x}\right)^{\odot n}=\frac{1}{x}\odot\left(\frac{1}{x}\right)^{\odot(n-1)}=\frac{1}{x}\odot\frac{v_{n-1}(x)}{u_{n-1}(x)}=\frac{u_{n-1}(x)+xv_{n-1}(x)}{xu_{n-1}(x)-v_{n-1}(x)}=\frac{v_n(x)}{u_n(x)}.$$

$\square$

Theorem 2. *Given $n\in\mathbb{N}$ and $x\in\mathbb{R}$, with $x\neq\pm 1$, we have*

$$x^{\odot n}=(-1)^{n+1}\left(\frac{v_n(x)}{u_n(x)}\right)^{(-1)^n},\quad \forall n\geq 1$$

where $u_n(x)$ and $v_n(x)$ are given by Eq.(1).

Proof. By using the same arguments of Theorem 1, we can write

$$x=\frac{u_1(x)}{v_1(x)}\qquad\text{and}\qquad x^{\odot 2}=\frac{2x}{1-\frac{1}{x^2}}=-\frac{v_2(x)}{u_2(x)}.$$

Let us suppose by induction that $x^{\odot(n-1)}=(-1)^n\left(\frac{v_{n-1}(x)}{u_{n-1}(x)}\right)^{(-1)^{n-1}}$, then if n is even

$$x^{\odot n}=\frac{x-\dfrac{v_{n-1}(x)}{u_{n-1}(x)}}{1+x\dfrac{v_{n-1}(x)}{u_{n-1}(x)}}=\frac{xu_{n-1}(x)-v_{n-1}(x)}{u_{n-1}(x)+xv_{n-1}(x)}=\frac{u_n(x)}{v_n(x)},$$

if n is odd

$$x^{\odot n}=\frac{x+\dfrac{u_{n-1}(x)}{v_{n-1}(x)}}{1-x\dfrac{u_{n-1}(x)}{v_{n-1}(x)}}=\frac{xv_{n-1}(x)+u_{n-1}(x)}{v_{n-1}(x)-xu_{n-1}(x)}=-\frac{v_n(x)}{u_n(x)}.$$

$\square$

Let us highlight the matrix representation of the sequences $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$ used in the previous theorem. Given the matrix

$$M = \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix}$$

we have

$$M^n = \begin{pmatrix} u_n(x) & v_n(x) \\ -v_n(x) & u_n(x) \end{pmatrix}$$

$$M^n \begin{pmatrix} v_m(x) \\ u_m(x) \end{pmatrix} = \begin{pmatrix} v_{n+m}(x) \\ u_{n+m}(x) \end{pmatrix}$$

The sequences $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$ are particular cases of the Rédei polynomials $N_n(d, z)$ and $D_n(d, z)$, introduced by Rédei [10] from the expansion of $(z + \sqrt{d})^n = N_n(d, z) + D_n(d, z)\sqrt{d}$. The rational functions $\dfrac{N_n(d, z)}{D_n(d, z)}$ have many interesting properties, e.g. , they are permutations of finite fields, as described in the book of Lidl [7]. In [1], the authors showed that Rédei polynomials are linear recurrent sequences of degree 2:

$$(N_n(d, z))_{n=0}^{\infty} = \mathcal{W}(1, z, 2z, z^2 - d), \quad (D_n(d, z))_{n=0}^{\infty} = \mathcal{W}(0, 1, 2z, z^2 - d).$$

Thus, we can observe that

$$u_n(x) = N_n(-1, x), \quad v_n(x) = D_n(-1, x), \quad \forall n \geq 0.$$

Moreover, a closed expression of Rédei polynomials is well–known (see, e.g., [1]). In this way, we can derive a closed expression for the sequences $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$:

$$\begin{cases} u_n(x) = \displaystyle\sum_{k=0}^{[n/2]} \binom{n}{2k} (-1)^k x^{n-2k} \\ v_n(x) = \displaystyle\sum_{k=0}^{[n/2]} \binom{n}{2k+1} (-1)^k x^{n-2k-1} \end{cases} . \tag{2}$$

Rational powers with respect to the product $\odot$ can also be considered by defining the n–th root as usual by

$$z = x^{\odot \frac{1}{n}} \qquad \text{iff} \qquad z^{\odot n} = x. \tag{3}$$

Moreover, by means of Theorem 2, we have that Eqs. (3) are equivalent to

$$x = (-1)^{n+1} \left(\frac{v_n(z)}{u_n(z)} \right)^{(-1)^n},$$

i.e., by Eqs. (2), the n–th root of x with respect to the product $\odot$ is a root of the polynomial

$$P_n(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{\lfloor \frac{k+1}{2} \rfloor} x^{\frac{1+(-1)^{k+1}}{2}} z^k.$$

Let us consider the equation

$$nA\left(\frac{1}{x}\right) + A\left(\frac{1}{y}\right) = \frac{\pi}{4}, \tag{4}$$

we want to solve it when n and x are integer values. We point out that Eq. (4) is equivalent to

$$\left(\frac{1}{x}\right)^{\odot n} \odot \frac{1}{y} = 1 \tag{5}$$

By Theorem 1 we have

$$\left(\frac{1}{x}\right)^{\odot n} \odot \frac{1}{y} = \frac{v_n(x)}{u_n(x)} \odot \frac{1}{y} = \frac{u_n(x) + v_n(x)y}{-v_n(x) + u_n(x)y}.$$

Thus

$$y = \frac{u_n(x) + v_n(x)}{u_n(x) - v_n(x)}$$

solves Eq. (5), i.e.,

$$\left(\frac{1}{x}\right)^{\odot n} \odot \frac{u_n(x) + v_n(x)}{u_n(x) - v_n(x)} = 1, \quad \forall x \in \mathbb{Z}$$

and consequently we can solve Eq. (4), i.e.,

$$nA\left(\frac{1}{x}\right) + A\left(\frac{u_n(x) - v_n(x)}{u_n(x) + v_n(x)}\right) = \frac{\pi}{4} + k(n,x)\pi, \quad \forall x \in \mathbb{Z}, \tag{6}$$

where k is a certain integer number depending on n and x. Precisely, we have

$$k(n,x) = \operatorname{sign}\left(nA\left(\frac{1}{x}\right) - \frac{\pi}{4}\right)\left(\lfloor T \rfloor + \chi_{\left(\frac{1}{2},1\right)}(\{T\})\right), \tag{7}$$

where $\chi_{\left(\frac{1}{2},1\right)}$ is the characteristic function of the set $\left(\frac{1}{2}, 1\right)$ and

$$T = \frac{\left|\frac{\pi}{4} - nA\left(\frac{1}{x}\right)\right|}{\pi}.$$

In order to obtain Eq. (7), we can rewrite Eq. (6) as

$$A\left(\frac{u_n(x)-v_n(x)}{u_n(x)+v_n(x)}\right)=\frac{\pi}{4}-nA\left(\frac{1}{x}\right)+k(n,x)\pi.$$

Let us consider the case in which the first member lies in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. If $\frac{\pi}{4}-nA\left(\frac{1}{x}\right)\geq 0$, then $k(n,x)$ must be negative so that $\frac{\pi}{4}-nA\left(\frac{1}{x}\right)+k(n,x)\pi$ lies in the correct interval. Since

$$\frac{\pi}{4}-nA\left(\frac{1}{x}\right)=\pi\left(\lfloor T\rfloor+\{T\}\right),$$

it follows that if $0\leq\{T\}\leq\frac{1}{2}$, then $0\leq\pi\cdot\{T\}\leq\frac{\pi}{2}$ and consequently $k=-\lfloor T\rfloor$. Conversely, if $\frac{1}{2}<\{T\}<1$, then $\frac{\pi}{2}<\pi\cdot\{T\}<\pi$ and, observing that

$$\frac{\pi}{4}-nA\left(\frac{1}{x}\right)=\pi\left(\lfloor T\rfloor+1\right)+\pi\left(\{T\}-1\right),$$

we obtain $-\frac{\pi}{2}<\pi(\{T\}-1)<0$, that is $k(n,x)=-(\lfloor T\rfloor+1)$.
Similar considerations apply to $\frac{\pi}{4}-nA\left(\frac{1}{x}\right)<0$, obtaining Eq. (7).

Proposition 1. *The sequences $(u_n(x)+v_n(x))_{n=0}^{\infty}$ and $(u_n(x)-v_n(x))_{n=0}^{\infty}$ are linear recurrent sequences of order 2 and precisely*

$$(u_n(x)+v_n(x))_{n=0}^{\infty}=\mathcal{W}(1,x+1,2x,1+x^2),\quad (u_n(x)-v_n(x))_{n=0}^{\infty}=\mathcal{W}(1,x-1,2x,1+x^2)$$

Proof. It immediately follows from the definition of the sequences $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$. □

Eq. (6) provides infinitely many identities that express π as sum of arctangents.

Example 1. Taking $n=7$ and $x=3$ in Eq. (6) we have

$$7A\left(\frac{1}{3}\right)+A\left(\frac{u_7(3)-v_7(3)}{u_7(3)+v_7(3)}\right)=\frac{\pi}{4},$$

i.e.,

$$7\arctan\left(\frac{1}{3}\right)-\arctan\left(\frac{278}{29}\right)=\frac{\pi}{4}.$$

For $n = 8$ and $x = 3$, we have

$$8 \arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{863}{191}\right) = \frac{\pi}{4} + \pi.$$

For $n = 5$ and $x = 2$, we have

$$5 \arctan\left(\frac{1}{2}\right) - \arctan\left(\frac{79}{3}\right) = \frac{\pi}{4}.$$

For $n = 2$ and $x = 7$, we have

$$2 \arctan\left(\frac{1}{7}\right) + \arctan\left(\frac{17}{31}\right) = \frac{\pi}{4}.$$

2 Golden mean and π

In Mathematics the most famous numbers are π and the Golden mean. Thus, it is very interesting to find identities involving these special numbers. In particular, many expressions for π in terms of the Golden mean have been found. For example, using the Machin formula of π via arctangents, the following equalities arise

$$\frac{\pi}{4} = \arctan\left(\frac{1}{\phi}\right) + \arctan\left(\frac{1}{\phi^3}\right)$$

$$\frac{\pi}{4} = 2 \arctan\left(\frac{1}{\phi^2}\right) + \arctan\left(\frac{1}{\phi^6}\right)$$

$$\frac{\pi}{4} = 3 \arctan\left(\frac{1}{\phi^3}\right) + \arctan\left(\frac{1}{\phi^5}\right)$$

$$\pi = 12 \arctan\left(\frac{1}{\phi^3}\right) + 4 \arctan\left(\frac{1}{\phi^5}\right),$$

see [3], [4], [5]. Moreover, in [8], the authors found all possible relations of the form

$$\frac{\pi}{4} = a \arctan(\phi^k) + b \arctan(\phi^l),$$

where a, b are rational numbers and k, l integers.

In this section, we find new expressions of π as sum of arctangents involving ϕ. When $n = 2$, from Eq. (5) we find

$$y = \frac{x^2 + 2x - 1}{x^2 - 2x - 1}. \tag{8}$$

It is well–known that the minimal polynomial of ϕ^m is

$$f_m(t) = t^2 - L_m t + (-1)^m,$$

where $(L_m)_{m=0}^{\infty} = \mathcal{W}(2, 1, 1, -1)$ is the sequence of Lucas numbers (A000032 in OEIS [11]). If we set $x = \phi^m$ in (8), then it is equivalent to replace $x^2 + 2x - 1$ and $x^2 - 2x - 1$ with

$$x^2 + 2x - 1 \pmod{f_m(x)}, \quad x^2 - 2x - 1 \pmod{f_m(x)},$$

respectively. When m is odd, dividing by $x^2 - L_m x - 1$, we obtain

$$y = \frac{(L_m + 2)x}{(L_m - 2)x} = \frac{L_m + 2}{L_m - 2}$$

and when m is even, we have

$$y = \frac{-2 + (2 + L_m)x}{-2 + (-2 + L_m)x},$$

and therefore

$$y = \frac{-2 + (2 + L_m)\phi^m}{-2 + (-2 + L_m)\phi^m}.$$

We find the following identities

$$\frac{\pi}{4} = 2\arctan\left(\frac{1}{\phi^{2k+1}}\right) + \arctan\left(\frac{L_{2k+1} - 2}{L_{2k+1} + 2}\right) \tag{9}$$

$$\frac{\pi}{4} = 2\arctan\left(\frac{1}{\phi^{2k}}\right) + \arctan\left(\frac{-2 + (L_{2k} - 2)\phi^{2k}}{-2 + (L_{2k} + 2)\phi^{2k}}\right).$$

The above procedure can be reproduced for any root α of a polynomial $x^2 - hx + k$, finding expression of π as the sum of arctangents involving quadratic irrationalities.

Example 2. Let us express π in terms of $\sqrt{2}$. Its minimal polynomial is $x^2 - 2$ and

$$x^2 + 2x - 1 \pmod{x^2 - 2} = 1 + 2x, \quad x^2 - 2x - 1 \pmod{x^2 - 2} = 1 - 2x.$$

We have

$$\frac{\pi}{4} = 2\arctan\left(\frac{1}{\sqrt{2}}\right) + \arctan\left(\frac{1 - 2\sqrt{2}}{1 + 2\sqrt{2}}\right).$$

In general, if k is odd the minimal polynomial of $\sqrt{2^k}$ is $x^2 - 2^k$ and

$$x^2+2x-1 \pmod{x^2-2^k} = 2^k-1+2x, \quad x^2-2x-1 \pmod{x^2-2^k} = 2^k-1-2x.$$

We have the following identity

$$\frac{\pi}{4} = 2\arctan\left(\frac{1}{\sqrt{2^k}}\right) + \arctan\left(\frac{2^k - 1 - 2^{\frac{k}{2}+1}}{2^k - 1 + 2^{\frac{k}{2}+1}}\right).$$

Example 3. Let us consider $\alpha = \frac{1}{2}(5+\sqrt{29})$. The minimal polynomial of α^3 is $x^2 - 140x - 1$ and

$$x^2+2x-1 \pmod{x^2-140x-1} = 142x, \quad x^2-2x-1 \pmod{x^2-140x-1} = 138x.$$

Thus, we have

$$\frac{\pi}{4} = 2\arctan\left(\frac{8}{(5+\sqrt{29})^3}\right) + \arctan\left(\frac{69}{71}\right).$$

We can find different identities involving π and the Golden mean considering the equation

$$x^{\odot\frac{1}{2}} \odot y = 1. \tag{10}$$

Proposition 2. *For any real number x, the following equalities hold*

$$2A(-x \pm \sqrt{1+x^2}) + A(x) = \pm\frac{\pi}{2}. \tag{11}$$

Proof. By Theorem 2 we know that the roots of the polynomial $P_2(z) = xz^2 + 2z - x$ are the values of $x^{\odot\frac{1}{2}}$. Hence, from Eq. (10) we obtain

$$z_i \odot y = 1, \quad i = 1, 2, \tag{12}$$

where

$$z_1 = \frac{-1+\sqrt{1+x^2}}{x} \quad \text{and} \quad z_2 = \frac{-1-\sqrt{1+x^2}}{x}.$$

Finally, solving Eq. (10) with respect to y we get

$$y_1 = -x + \sqrt{1+x^2} \qquad \text{or} \qquad y_2 = -x - \sqrt{1+x^2}.$$

It should be noted that if x is positive then $y_2 < 0$ and $z_2 \cdot y_2 > 1$ so that

$$\frac{1}{2}A(x) + A(y_2) = A\left(x^{\odot\frac{1}{2}} + y_2\right) - \frac{\pi}{2},$$

similar reasoning can be applied if x is negative.

Now, substituting in Eqs. (12) we have

$$\frac{1}{2}A(x) + A(-x \pm \sqrt{1+x^2}) = \pm\frac{\pi}{4},$$

or equivalently

$$2A(-x \pm \sqrt{1+x^2}) + A(x) = \pm\frac{\pi}{2}.$$

$\square$

Eqs. (11) yield to other interesting formulas involving π, ϕ and Lucas numbers. To show this, we need some identities about Lucas numbers, Fibonacci numbers and the Golden mean:

$$\phi^m = \frac{L_m + F_m\sqrt{5}}{2}, \quad L_m^2 - 5F_m^2 = 4(-1)^m,$$

see, e.g., [9]. Considering m odd, if we set

$$x = \frac{L_m}{2}$$

it follows

$$-x - \sqrt{1+x^2} = \frac{-L_m - \sqrt{4+L_m^2}}{2} = \frac{-L_m - F_m\sqrt{5}}{2} = -\phi^m. \tag{13}$$

Thus, substituting Eq. (13) into Eqs. (11) we find the formula

$$-\frac{\pi}{2} = \arctan\left(\frac{L_{2k+1}}{2}\right) - 2\arctan\left(\phi^{2k+1}\right). \tag{14}$$

On the other hand, if we consider $y = -x + \sqrt{1+x^2}$ we have

$$-x + \sqrt{1+x^2} = \frac{-L_m + \sqrt{4+L_m^2}}{2} = \frac{-L_m + F_m\sqrt{5}}{2}. \tag{15}$$

Moreover,

$$\phi^m \cdot \frac{-L_m + F_m\sqrt{5}}{2} = \frac{-L_m^2 + 5F_m^2}{4} = 1,$$

and substituting in Eqs. (11) another interesting formula arises

$$\frac{\pi}{2} = \arctan\left(\frac{L_{2k+1}}{2}\right) + 2\arctan\left(\frac{1}{\phi^{2k+1}}\right). \tag{16}$$

Furthermore, by Eq. (9) we obtain an identity that only involves the Lucas numbers

$$\frac{\pi}{4} = \arctan\left(\frac{L_{2k+1}}{2}\right) - \arctan\left(\frac{L_{2k+1}-2}{L_{2k+1}+2}\right). \tag{17}$$

The previous identity corresponds to a special case of the following proposition.

Proposition 3. *Let f, g be real functions. If*

$$g(x) = \frac{f(x)-1}{f(x)+1},$$

then

$$A(f(x)) - A(g(x)) = \frac{\pi}{4} + k\pi, \tag{18}$$

for some integer k.

Proof. We use the product $\odot$ for solving $A(f(x)) - A(g(x)) = \dfrac{\pi}{4}$. We have

$$A\left(\frac{f(x)-g(x)}{1+f(x)g(x)}\right) = \frac{\pi}{4}$$

and

$$\frac{f(x)-g(x)}{1+f(x)g(x)} = 1$$

from which

$$g(x) = \frac{f(x)-1}{f(x)+1}.$$

$\square$

Remark 2. *Eq.* (18) *has been found by means of only elementary algebraic considerations. The same result could be derived from analysis. Observe that given the functions f and g satisfying the hypothesis of the previous proposition, then* $(\arctan f(x))' = (\arctan g(x))'$.

When $f(x)$ and $g(x)$ are specified in Eq. (18), the value of k can be retrieved as in Eq. (7) with analogous considerations.

The previous proposition allows to determine new beautiful identities. For example, the function $f(x) = \dfrac{ax}{b}$ determines the function $g(x) = \dfrac{ax-b}{ax+b}$ and

$$A\left(\frac{ax}{b}\right) - A\left(\frac{ax-b}{ax+b}\right) = \frac{\pi}{4} + k\pi.$$

For $a = 1$ and $b = 2$, we obtain the following interesting formulas

$$\frac{\pi}{4} = \arctan\left(\frac{x}{2}\right) - \arctan\left(\frac{x-2}{x+2}\right), \tag{19}$$

which holds for any real number $x > -2$ and

$$-\frac{3\pi}{4} = \arctan\left(\frac{x}{2}\right) - \arctan\left(\frac{x-2}{x+2}\right), \tag{20}$$

valid for any real number $x < -2$. Eqs. (19) and (20) provide infinitely many interesting identities, like Eq. (17) and, e.g., the following ones

$$\frac{\pi}{4} = \arctan\left(\frac{\phi}{2}\right) - \arctan\left(\frac{\phi-2}{\phi+2}\right)$$

$$\frac{\pi}{4} = \arctan\left(\frac{F_m}{2}\right) - \arctan\left(\frac{F_m-2}{F_m+2}\right)$$

$$\frac{\pi}{4} = \arctan\left(\frac{\sqrt{2}}{2}\right) - \arctan\left(\frac{\sqrt{2}-2}{\sqrt{2}+2}\right).$$

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