THE $1/k$-EULERIAN POLYNOMIALS AND k-STIRLING PERMUTATIONS

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ABSTRACT. In this paper, we establish a connection between the $1/k$-Eulerian polynomials introduced by Savage and Viswanathan (Electron. J. Combin. 19 (2012), #P9) and k-Stirling permutations. We also introduce the dual set of Stirling permutations.

Keywords: k-Stirling permutations; $1/k$-Eulerian polynomials; Ascent-plateau

1. INTRODUCTION

For $k \geq 1$, the $1/k$-*Eulerian polynomials* $A_n^{(k)}(x)$ are defined by

$$\sum_{n\geq 0} A_n^{(k)}(x)\frac{z^n}{n!} = \left(\frac{1-x}{e^{kz(x-1)}-x}\right)^{\frac{1}{k}}. \tag{1}$$

Let $e = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n$. Let $I_{n,k} = \{e|0 \leq e_i \leq (i-1)k\}$, which known as the set of n-dimensional k-*inversion sequences*. The number of *ascents* of e is defined by

$$\operatorname{asc}(e) = \#\left\{i : 1 \leq i \leq n-1 \mid \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1}\right\}.$$

Savage and Viswanathan [12] showed that

$$A_n^{(k)}(x) = \sum_{e\in I_{n,k}} x^{\operatorname{asc}(e)}. \tag{2}$$

Let $\mathfrak{S}_n$ be the symmetric group on the set $[n] = \{1, 2, \ldots, n\}$ and $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. The number of *excedances* of π is $\operatorname{exc}(\pi) := \#\{i : 1 \leq i \leq n-1|\pi_i > i\}$. Let $\operatorname{cyc}(\pi)$ be the number of *cycles* in the disjoint cycle representation of π. In [5], Foata and Schützenberger introduced a q-analog of the classical Eulerian polynomials defined by

$$A_n(x; q) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{exc}(\pi)}q^{\operatorname{cyc}(\pi)}. \tag{3}$$

The polynomials $A_n(x; q)$ satisfy the recurrence relation

$$A_{n+1}(x; q) = (nx + q)A_n(x; q) + x(1-x)\frac{d}{dx}A_n(x; q), \tag{4}$$

with the initial conditions $A_1(x; q) = 1$ and $A_2(x; q) = q$ (see [2, Proposition 7.2]). Savage and Viswanathan [12, Section 1.5] discovered that

$$A_n^{(k)}(x) = k^n A_n(x; 1/k) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{exc}(\pi)}k^{n-\operatorname{cyc}(\pi)}. \tag{5}$$

Let $A_n^{(k)}(x) = \sum_{j=0}^{n-1} a_{n,j}^{(k)}x^j$. It follows from (4) and (5) that

$$a_{n+1,j}^{(k)} = (1 + kj)a_{n,j}^{(k)} + k(n-j+1)a_{n,j-1}^{(k)}, \tag{6}$$

with the initial condition $a_{1,0}^{(k)} = 1$.

Let $\left[{n \atop k}\right]$ be the Stirling number of the first kind, i.e., the number of permutations in $\mathfrak{S}_n$ with precisely k cycles. It is well known that

$$\sum_{k=0}^{n}\left[{n \atop k}\right]x^k=\prod_{i=0}^{n-1}(x+i).$$

Thus it follows from (5) that

$$A_n^{(k)}(1)=\prod_{i=1}^{n-1}(ik+1)\quad \text{for } n\geq 1.$$

Since $\prod_{i=1}^{n-1}(ik+1)$ also count k-Stirling permutations of order n (see [7, 8]), it is natural to consider the following question: Is there existing a connection between $A_n^{(k)}(x)$ and k-Stirling permutations? The main object of this paper is to provide a solution to this problem.

2. k-Stirling permutations and their longest ascent-plateau

In the following discussion, we always let $j^i=\underbrace{j,j,\ldots,j}_{i}$ for $i,j\geq 1$. Stirling permutations were defined by Gessel and Stanley [6]. A *Stirling permutation* of order n is a permutation of the multiset $\{1^2,2^2,\ldots,n^2\}$ such that for each i, $1\leq i\leq n$, all entries between the two occurrences of i are larger than i. We call a permutation of the multiset $\{1^k,2^k,\ldots,n^k\}$ a *k-Stirling permutation* of order n if for each i, $1\leq i\leq n$, all entries between the two occurrences of i are at least i. Denote by $\mathcal{Q}_n(k)$ the set of *k-Stirling permutation* of order n. Clearly, $\mathcal{Q}_n(1)=\mathfrak{S}_n$ and $\mathcal{Q}_n(2)$ is the set of ordinary Stirling permutations of order n.

For $\sigma=\sigma_1\sigma_2\cdots\sigma_{2n}\in\mathcal{Q}_n(2)$, an occurrence of an *ascent* (resp. *plateau*) is an index i such that $\sigma_i<\sigma_{i+1}$ (resp. $\sigma_i=\sigma_{i+1}$). The reader is referred to [1, 7, 8, 11] for recent progress on the study of patterns in Stirling permutations.

Definition 1. *Let $\sigma=\sigma_1\sigma_2\cdots\sigma_{kn}\in\mathcal{Q}_n(k)$. We say that an index $i\in\{2,3,\ldots,nk-k+1\}$ is a longest ascent-plateau if*

$$\sigma_{i-1}<\sigma_i=\sigma_{i+1}=\sigma_{i+2}=\cdots=\sigma_{i+k-1}.$$

Let $\operatorname{ap}(\sigma)$ be the number of the longest ascent-plateau of σ. For example, $\operatorname{ap}(1122\mathbf{3}3321)=1$.

Now we present the main results of this paper.

Theorem 2. *For $n\geq 1$ and $k\geq 2$, we have*

$$A_n^{(k)}(x)=\sum_{\sigma\in\mathcal{Q}_n(k)}x^{\operatorname{ap}(\sigma)}.$$

Proof. Let

$$T(n,j;k)=\#\{\sigma\in\mathcal{Q}_n(k):\operatorname{ap}(\sigma)=j\}.$$

There are two ways in which a permutation $\widetilde{\sigma}\in\mathcal{Q}_{n+1}(k)$ with the number of the longest ascent-plateau equals j can be obtained from a permutation $\sigma\in\mathcal{Q}_n(k)$.

(a) If the number of the longest ascent-plateau of σ equals j, then we can insert k copies of $(n+1)$ into σ without increasing the number of the longest ascent-plateau. Let i be one of the longest ascent-plateau of σ. Then we can insert k copies of $(n+1)$ before σ_i or after σ_t, where $i\leq t\leq i+k-2$. Moreover, the k copies of $(n+1)$ can also be inserted into the front of σ. This accounts for $(1+kj)T(n,j;k)$ possibilities.

(b) If the number of the longest ascent-plateau of σ equals $j-1$, then we insert k copies of $(n+1)$ into the remaining $1+kn-(1+k(j-1))=k(n-j+1)$ positions. This gives $k(n-j+1)T(n,j-1;k)$ possibilities.

Hence

$$T(n+1,j;k)=(1+kj)T(n,j;k)+k(n-j+1)T(n,j-1;k).$$

Clearly, $T(n,0;k)=1$, corresponding to the permutation $n^k(n-1)^k\cdots 1^k$. Therefore, the numbers $T(n,j;k)$ satisfy the same recurrence relation and initial conditions as $a_{n,j}^{(k)}$, so they agree. □

Define

$$\mathcal{Q}_n^0(k)=\{0\sigma:\sigma\in\mathcal{Q}_n(k)\}.$$

Therefore, for $\sigma\in\mathcal{Q}_n^0(k)$, we let $\sigma_0=0$ and the indices of the longest ascent-plateau belong to $\{1,2,3,\ldots,nk-k+1\}$. For example, $\operatorname{ap}(0\mathbf{1}1\mathbf{2}\mathbf{3}32)=2$.

Define

$$x^nA_n^{(k)}\left(\frac{1}{x}\right)=\sum_{j=1}^{n}b_{n,j}^{(k)}x^j.$$

Then $b_{n,j}^{(k)}=a_{n,n-j}^{(k)}$. It follows from (6) that

$$b_{n+1,j}^{(k)}=kjb_{n,j}^{(k)}+(kn-kj+k+1)b_{n,j-1}^{(k)}.$$

Along the same lines of the proof of Theorem 2, we get the following result.

Theorem 3. *For $n\geq 1$ and $k\geq 2$, we have*

$$x^nA_n^{(k)}\left(\frac{1}{x}\right)=\sum_{\sigma\in\mathcal{Q}_n^0(k)}x^{\operatorname{ap}(\sigma)}.$$

3. The dual set of Stirling permutations

For convenience, we let $\mathcal{Q}_n=\mathcal{Q}_n(2)$. Let $\sigma=\sigma_1\sigma_2\cdots\sigma_{2n}\in\mathcal{Q}_n$. Let Φ be the bijection which map each first occurrence of letter j in σ to $2j$ and the second occurrence of letter j in σ to $2j-1$, where $j\in[n]$. For example, $\Phi(221331)=432651$. The *dual set* $\Phi(\mathcal{Q}_n)$ of $\mathcal{Q}_n$ is defined by

$$\Phi(\mathcal{Q}_n)=\{\pi:\sigma\in\mathcal{Q}_n,\Phi(\sigma)=\pi\}.$$

Clearly, $\Phi(\mathcal{Q}_n)$ is a subset of $\mathfrak{S}_{2n}$. Let ab be an ascent in σ, so $a<b$. Using Φ, we see that ab is maps into $(2a-1)(2b-1)$, $(2a-1)(2b)$, $(2a)(2b-1)$ or $(2a)(2b)$, and vice versa. Let $\operatorname{as}(\sigma)$ (resp. $\operatorname{as}(\pi)$) be the number of ascents of σ (resp. π). Then Φ preserving ascents, i.e., $\operatorname{as}(\sigma)=\operatorname{as}(\Phi(\sigma))=\operatorname{as}(\pi)$. Hence the well known *Eulerian polynomial of second kind* $P_n(x)$ (see [13, A008517]) has the expression

$$P_n(x)=\sum_{\pi\in\Phi(\mathcal{Q}_n)}x^{\operatorname{as}(\pi)}.$$

Perhaps one of the most important permutation statistics is the peaks statistic; see, e.g., [4, 9] and the references contained therein. Let $\pi=\pi_1\pi_2\cdots\pi_n\in\mathfrak{S}_n$. An *interior peak* in π is an index $i\in\{2,3,\ldots,n-1\}$ such that $\pi_{i-1}<\pi_i>\pi_{i+1}$. Let $\operatorname{ipk}(\pi)$ denote the number of interior peaks in π. A *left peak* in π is an index $i\in[n-1]$ such that $\pi_{i-1}<\pi_i>\pi_{i+1}$, where we take $\pi_0=0$. Denote by $\operatorname{lpk}(\pi)$ the number of left peaks in π. For example, $\operatorname{ipk}(21435)=1$ and $\operatorname{lpk}(21435)=2$.

As pointed out by Savage and Viswanathan [12, Section 4] that the numbers $a_{n,j}^{(2)}$ appear as A185410 in [13], and the numbers $a_{n,n-j}^{(2)}$ appear as A156919 in [13]. We can now present a unified characterization of these numbers.

Theorem 4. *For $n \geq 1$, we have*

$$A_n^{(2)}(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\mathrm{ipk}\,(\pi)}, \tag{7}$$

$$x^n A_n^{(2)}\left(\frac{1}{x}\right) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\mathrm{lpk}\,(\pi)}. \tag{8}$$

Proof. Recall that an occurrence of a *pattern* τ in a sequence π is defined as a subword in π whose letters are in the same relative order as those in τ.

Let $\sigma \in \mathcal{Q}_n$ and let $\Phi(\sigma) = \pi$. Let

$$C = \{112, 211, 122, 221, 213, 312, 123, 321\}.$$

For all $\sigma \in \mathcal{Q}_n$, we see that all patterns of length three of σ are belong to C. Let abb be an occurrence of the pattern 122 in σ, so $a < b$. Using Φ, we see that abb is maps to either $(2a-1)(2b)(2b-1)$ or $(2a)(2b)(2b-1)$, which is an interior peak of the pattern 132. Moreover, one can easily verify that interior peaks can not be generated by the other patterns. Recall that an occurrence of the longest ascent-plateau in Stirling permutations is an occurrence of the pattern 122. Then we get (7) by using Theorem 2. Similarly, from Theorem 3, we get (8). □

For $n \geq 1$, we define $C_n(x)$ by

$$(1+x)C_n(x) = xA_n^{(2)}(x^2) + x^{2n}A_n^{(2)}\left(\frac{1}{x^2}\right). \tag{9}$$

Set $C_0(x) = 1$. It follows from (1) that

$$C(x,z) = \sum_{n\geq 0} C_n(x)\frac{z^n}{n!} = \frac{e^{z(x-1)(1+x)} + x}{1+x}\sqrt{\frac{1-x^2}{e^{2\,z(x-1)(1+x)} - x^2}}.$$

The first few $C_n(x)$ are given as follows:

$$\begin{aligned}
C_1(x) &= x,\\
C_2(x) &= x + x^2 + x^3,\\
C_3(x) &= x + 3x^2 + 7x^3 + 3x^4 + x^5,\\
C_4(x) &= x + 7x^2 + 29x^3 + 31x^4 + 29x^5 + 7x^6 + x^7,\\
C_5(x) &= x + 15x^2 + 101x^3 + 195x^4 + 321x^5 + 195x^6 + 101x^7 + 15x^8 + x^9.
\end{aligned}$$

Let $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$. We say that π changes direction at position i if either $\pi_{i-1} < \pi_i > \pi_{i+1}$, or $\pi_{i-1} > \pi_i < \pi_{i+1}$, where $i \in \{2, 3, \ldots, n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i where π changes direction (see [13, A059427]). Let $\mathrm{run}\,(\pi)$ denote the number of alternating runs of π. For example, $\mathrm{run}\,(214653) = 3$. There is a large literature devoted to the distribution of alternating runs. The reader is referred to [3, 10] for recent results on this subject.

We can now conclude the following result.

Theorem 5. *For $n \geq 1$, we have*

$$C_n(x) = \sum_{\pi \in \Phi(\mathcal{Q}_n)} x^{\mathrm{run}\,(\pi)}. \tag{10}$$

Proof. Define

$$S_1 = \{\pi \in \Phi(\mathcal{Q}_n) : \mathrm{lpk}\,(\pi) = \mathrm{ipk}\,(\pi)\},$$
$$S_2 = \{\pi \in \Phi(\mathcal{Q}_n) : \mathrm{lpk}\,(\pi) = \mathrm{ipk}\,(\pi) + 1\}.$$

Then $\Phi(\mathcal{Q}_n)$ can be partitioned into subsets S_1 and S_2.

From (9), we have

$$\begin{aligned}(1+x)C_n(x) &= \sum_{\pi\in\Phi(\mathcal{Q}_n)} x^{2\mathrm{ipk}\,(\pi)+1} + \sum_{\pi\in\Phi(\mathcal{Q}_n)} x^{2\mathrm{lpk}\,(\pi)} \\ &= x\sum_{\pi\in S_1} x^{\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)} + \sum_{\pi\in S_2} x^{\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)} + \sum_{\pi\in S_1} x^{\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)} + \\ & x\sum_{\pi\in S_2} x^{\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)} \\ &= (1+x)\sum_{\pi\in S_1} x^{\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)} + (1+x)\sum_{\pi\in S_2} x^{\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)}.\end{aligned}$$

Thus

$$C_n(x) = \sum_{\pi\in\Phi(\mathcal{Q}_n)} x^{\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)}.$$

Note that all $\pi\in\Phi(\mathcal{Q}_n)$ ends with a descent, i.e., $\pi_{2n-1}>\pi_{2n}$. Hence (10) follows from the fact that $\mathrm{run}\,(\pi)=\mathrm{ipk}\,(\pi)+\mathrm{lpk}\,(\pi)$. □

4. Concluding remarks

It follows from (2) and Theorem 2, we have

$$\sum_{e\in I_{n,k}} x^{\mathrm{asc}\,(e)} = \sum_{\sigma\in\mathcal{Q}_n(k)} x^{\mathrm{ap}\,(\sigma)}. \tag{11}$$

Combining (5) and Theorem 2, we have

$$\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} k^{n-\mathrm{cyc}\,(\pi)} = \sum_{\sigma\in\mathcal{Q}_n(k)} x^{\mathrm{ap}\,(\sigma)}. \tag{12}$$

It would be interesting to present a combinatorial proof of (11) or (12).

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