Conjugacy and Iteration of Standard Interval Rank in Finite Ordered Sets*

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Abstract

In order theory, a rank function measures the vertical "level" of a poset element. It is an integer-valued function on a poset which increments with the covering relation, and is only available on a graded poset. Defining a vertical measure to an arbitrary finite poset can be accomplished by extending a rank function to be interval-valued [10]. This establishes an order homomorphism from a base poset to a poset over real intervals, and a standard (canonical) specific interval rank function is available as an extreme case. Various ordering relations are available over intervals, and we begin in this paper by considering conjugate orders which "partition" the space of pairwise comparisons of order elements. For us, these elements are real intervals, and we consider the weak and subset interval orders as (near) conjugates. It is also natural to ask about interval rank functions applied reflexively on whatever poset of intervals we have chosen, and thereby a general iterative strategy for interval ranks. We explore the convergence properties of standard and conjugate interval ranks, and conclude with a discussion of the experimental mathematics needed to support this work.

Contents

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1 Introduction

A characteristic of partial orders and partially ordered sets (posets) as used in order theory [6] is that they are amongst the simplest structures which can be called "hierarchical" in the sense of admitting to descriptions in terms of *levels*. These levels are typically identified via a rank function as an integer-valued function on a poset which increments with the covering relation. But rank functions are only available on graded posets, where all saturated chains connecting two elements are the same length. Effectively, each element in a graded poset can be assigned a unique, singular vertical level.

We have extended the concept of a rank function to all finite posets by making it interval-valued [10], and establishing it as an order homomorphism from a base poset to a poset over real intervals, for one of many possible order relations on intervals. An interval-valued rank reflects information about chain lengths to an element dually from both the top and the bottom of the poset, and a standard, or canonical, interval rank function is available as the largest such function, measuring specifically the maximum length maximum chain from both top and bottom.

As mentioned above, various ordering relations are available over intervals. The order ordinarily identified on intervals we call the strong order, where comparability is equivalent to disjointness. But we have identified [10] two other orders as being much more useful for interval rank function available on intervals: a weak order (actually the product order on endpoints) and ordering by subset. These are (nearly) conjugate orders, in that they "partition" the space of pairwise comparisons of order elements. We begin this paper by considering standard interval rank for the conjugate to the weak order. And since interval rank functions are order homomorphisms to an interval order, it is also natural to ask about interval rank functions applied reflexively on whatever poset of intervals we have chosen. This motivates studying the structure of the homomorphic image of a poset under the standard interval rank operator, both singly and in repeated iterations.

We close with a discussion of how the use of experimental mathematics, using computers as a tool to explore problems and form conjectures, influenced this work.

2 Preliminaries

Throughout this paper we will use $\mathbb{N}$ to denote $\{0, 1, \ldots\}$, the set of integers greater than or equal to 0, and for $N \in \mathbb{N}$, the set of integers between 0 and N will be denoted $\mathbb{N}_N := \{0, 1, \ldots, N\}$.

2.1 Ordered Sets

See e.g. [6, 16, 18] for the basics of order theory, the following is primarily for notational purposes.

Let P be a finite set of elements with $|P| \geq 2$, and $\leq$ be a binary relation on P (a subset of P^2) which is reflexive, transitive, and antisymmetric. Then $\leq$ is a **partial order**, and the structure $\mathcal{P} = \langle P, \leq \rangle$ is a **partially ordered set** (poset). Denote $a < b$ to mean that $a \leq b$ and $a \neq b$. $<$ is its own binary relation, a **strict order** $<$ on P, which is an irreflexive partial order. A strict order $<$ can be turned into a partial order through **reflexive closure**: $\leq : = < \cup \{\langle a, a\rangle \ : \ a \in P\}$.

For any pair of elements $a, b \in P$, we say $a \leq b \in P$ to mean that $a, b \in P$ and $a \leq b$. And for $a \in P, Q \subseteq P$, we say $a \leq Q$ to mean that $\forall b \in Q, a \leq b$. If $a \leq b \in P$ or $b \leq a \in P$ then we say that a and b are **comparable**, denoted $a \sim b$. If not, then they are **incomparable**, denoted $a \| b$. For $a, b \in P$, let $a \prec b$ be the **covering relation** where $a \leq b$ and $\nexists c \in P$ with $a < c < b$.

A set of elements $C \subseteq P$ is a **chain** if $\forall a, b \in C, a \sim b$. If P is a chain, then $\mathcal{P}$ is called a **total order**. A chain $C \subseteq P$ is **maximal** if there is no other chain $C' \subseteq P$ with $C \subseteq C'$. Naturally all maximal chains are **saturated**, meaning that $C = \{a_i\}_{i=1}^{|C|} \subseteq P$ can be sorted by $\leq$ and written as $C = a_1 \prec a_2 \prec \ldots \prec a_{|C|}$. The **height** $\mathcal{H}(\mathcal{P})$ of a poset is the size of its largest chain. Below we will use $\mathcal{H}$ alone for $\mathcal{H}(\mathcal{P})$ when clear from context.

A set of elements $A \subseteq P$ is an **antichain** if $\forall a, b \in A, a \| b$. The **width** $\mathcal{W}(\mathcal{P})$ of a poset is the size of its largest antichain.

A partial order $\leq$ generates a unique **dual** partial order $\geq$, where $a \geq b \in P$ iff $b \leq a \in P$. Given $\mathcal{P} = \langle P, \leq\rangle$ we denote by $\mathcal{P}^* = \langle P, \geq\rangle$ the dual of $\mathcal{P}$. Two partial orders $\leq_1, \leq_2$ on the same set P are said to be **conjugate** if every pair of distinct elements of P are comparable in exactly one of $\leq_1$ or $\leq_2$, i.e. that:

$$\forall a \neq b \in P, \quad (a \sim_1 b \text{ and } a \|_2 b) \text{ or } (a \|_1 b \text{ and } a \sim_2 b).$$

For any subset of elements $Q \subseteq P$, let $\mathcal{P}|_Q = \langle Q, \leq_Q\rangle$ be the sub-poset determined by Q, so that for $a, b \in Q$, $a \leq_Q b \in Q$ if $a \leq b \in P$.

For any element $a \in P$, define the **up-set** or **principal filter** $\uparrow a \ : = \{b \in P \ : \ b \geq a\}$, **down-set** or **principal ideal** $\downarrow a \ : = \{b \in P \ : \ b \leq a\}$, and **hourglass** $\Xi(a) \ : = \uparrow a \cup \downarrow a$. For $a \leq b \in P$, define the **interval** $[a, b] \ : = \{c \in P \ : \ a \leq c \leq b\} = \uparrow a \cap \downarrow b$.

For any subset of elements $Q \subseteq P$, define its maximal and minimal elements as

$$\mathrm{Max}(Q) \ : = \{a \in Q \ : \ \nexists b \in Q, a < b\} \subseteq Q$$

$$\mathrm{Min}(Q) \ : = \{a \in Q \ : \ \nexists b \in Q, b < a\} \subseteq Q,$$

called the **roots** and **leaves** respectively. Except where noted, in this paper we will assume that our posets $\mathcal{P}$ are bounded, so that $\bot \leq \top \in P$ with $\mathrm{Max}(P) = \{\top\}, \mathrm{Min}(P) = \{\bot\}$. Since we've disallowed the degenerate case of $|P| = 1$, we have $\bot < \top \in P$. All intervals $[a, b]$ are bounded sub-posets, and since $\mathcal{P}$ is bounded, $\forall a \in P, \uparrow a = [a, \top], \downarrow a = [\bot, a]$, and thus $\bot, \top \in \Xi(a) \subseteq \mathcal{P} = [\bot, \top]$. An example of a bounded poset and a sub-poset expressed as an hourglass is shown in Fig. 1.

The following properties are prominent in lattice theory, and are available for some of the highly regular lattices and posets that appear there (see e.g. Aigner [1]).

Definition 1: (Rank Function and Graded Posets) For a top-bounded poset $\mathcal{P}$ with $\top \in P$, a function $\rho \colon P \to \mathbb{N}_{\mathcal{H}-1}$ is a **rank function** when $\rho(\top) = 0$ and $\forall a \prec b \in P, \rho(a) = \rho(b) - 1$. A poset $\mathcal{P}$ is **graded**, or **fully graded**, if it has a rank function.

Let $\mathcal{C}(\mathcal{P}) \subseteq 2^P$ be the set of all maximal chains of $\mathcal{P}$. We assume that $\mathcal{P}$ is bounded, therefore $\forall C \in \mathcal{C}(\mathcal{P}), \bot, \top \in C$. We will refer to the **spindle chains** of a poset $\mathcal{P}$ as the set of its maximum length chains

$$\mathcal{I}(\mathcal{P}) \ : = \{C \in \mathcal{C}(\mathcal{P}) \ : \ |C| = \mathcal{H}\}.$$

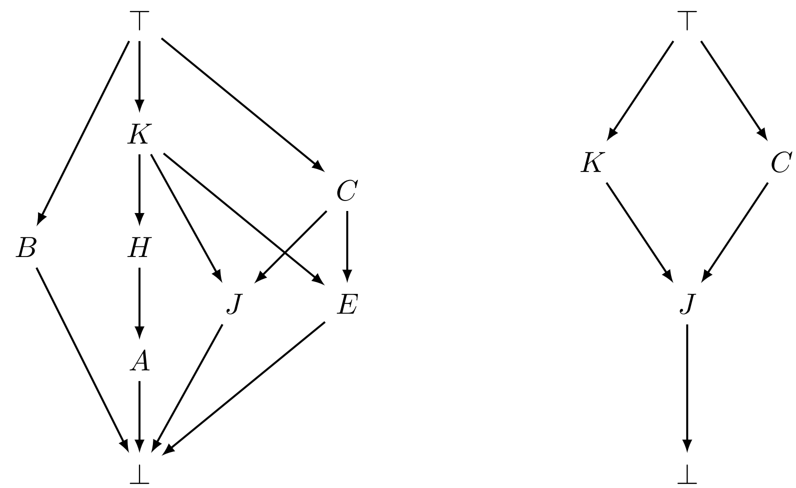


Figure 1: (Left) The Hasse diagram (canonical visual representation of the covering relation $\prec$) of an example bounded poset $\mathcal{P}$. (Right) The Hasse diagram of the sub-poset $\mathcal{P}|_{\Xi(J)}$ for the hourglass $\Xi(J) = \uparrow J \cup \downarrow J = [J, \top] \cup [\bot, J] = \{J, C, K, \top\} \cup \{\bot, J\} = \{\bot, C, J, K, \top\}$.

The **spindle set**

$$I(\mathcal{P}) := \bigcup_{C \in \mathcal{I}(\mathcal{P})} C \subseteq P$$

is then the set of **spindle elements**, including any elements which sit on a spindle chain. Note that if P is nonempty then there is always at least one spindle chain and thus at least one spindle element, so $\mathcal{I}(\mathcal{P}), I(\mathcal{P}) \neq \emptyset$. In our example in Fig. 1, we have $\mathcal{H} = 5$, $|\mathcal{C}(\mathcal{P})| = 6$, $\mathcal{I}(\mathcal{P}) = \{\bot \prec A \prec H \prec K \prec \top\}$, and $S(J) = \mathcal{H}(\uparrow J) + \mathcal{H}(\downarrow J) - 1 = 4$.

Given two posets $\mathcal{P}_1 = \langle P, \leq_P \rangle$ and $\mathcal{P}_2 = \langle Q, \leq_Q \rangle$, a function $f \colon P \to Q$ is an **order embedding** if $\forall a, b \in P$ we have $a \leq_P b \iff f(a) \leq_Q f(b)$. A weaker notion is that of f being an **order homomorphism** if $\forall a \leq_P b \in P, f(a) \leq_Q f(b)$. We can also say that f **preserves the order** $\leq_P$ into $\leq_Q$, and is an **isotone** mapping from $\mathcal{P}_1$ to $\mathcal{P}_2$. If $\forall a <_P b \in P, f(a) <_Q f(b) \in Q$ then we say that f does all this **strictly**. If instead f is an order homomorphism from $\mathcal{P}_1$ to the dual $\langle Q, \geq_Q \rangle$, then we say that f **reverses the order**, or f is an **antitone** mapping. If $f \colon P \to Q$ is an order homomorphism, then we can denote $f(\mathcal{P}_1) := \langle f(P), \leq_f \rangle$ as the **homomorphic image** of $\mathcal{P}_1$, with

$$f(P) := \{f(a) : a \in P\} \subseteq Q, \quad \leq_f := \leq_Q \big|_{f(P) \times f(P)}$$

When clear from context, we will simply re-use $\leq$ as the relevant order to its base set, e.g. for an isotone $f \colon P \to Q, a \leq b \in P \implies f(a) \leq f(b) \in Q$. A **linear extension** of a poset $\mathcal{P}$ is a chain, $\mathcal{C}$, for which there is an order homomorphism $f \colon P \to C$.

The **dimension** [3, 18] of a poset $\mathcal{P} = \langle P, \leq \rangle$, denoted $\dim(\mathcal{P})$, is the minimum number, m, of total orders, $\leq_i^T$, such that there is an order embedding of $\leq$ into $\bigcap_{i=1}^m \leq_i^T$. Dimension is related to width, in that $\dim(\mathcal{P}) \leq \mathcal{W}(\mathcal{P})$, but is a fundamentally different concept. The following theorem from [18], is taken to be the definition of dimension by some [14].

Theorem 2. *[18] Let $\mathcal{P} = \langle P, \leq \rangle$ be a poset. Then $\dim(\mathcal{P})$ is the least t for which $\mathcal{P}$ is isomorphic to an induced subposet of $\mathbb{R}^t = \langle \mathbb{R} \times \cdots \times \mathbb{R}, \leq \times \cdots \times \leq \rangle$.*

Finally, we recognize $\langle \mathbb{N}, \leq \rangle$ as a total order using the normal numeric order $\leq$, and observe that for any bounded poset $\mathcal{P}$, the functions $\mathcal{H}(\uparrow \cdot), \mathcal{H}(\downarrow \cdot) : P \to \mathbb{N}_{\mathcal{H}}$ induce strict antitone and isotone order morphisms, respectively. That is, if $a \prec b$ then

$$\mathcal{H}(\uparrow a) \geq \mathcal{H}(\uparrow b) + 1 > \mathcal{H}(\uparrow b) \qquad \text{and} \qquad \mathcal{H}(\downarrow a) < \mathcal{H}(\downarrow a) + 1 \leq \mathcal{H}(\downarrow b).$$

2.2 Interval Orders

Since our rank functions are interval-valued, we explicate the concepts surrounding the possible ordering relations among intervals (see [10] for a full discussion). $\langle \mathbb{R}, \leq \rangle$ is a total order, so for any $x_* \leq x^* \in \mathbb{R}$, we can denote the (real) interval $\bar{x} = [x_*, x^*]$, and $\overline{\mathbb{R}}$ as the set of all real intervals on $\mathbb{R}$, so that $\bar{x} \in \overline{\mathbb{R}}$. Additionally, for $N \in \mathbb{N}$ let $\overline{N}$ be the set of all intervals whose endpoints are nonnegative integers $\leq N$.

We now define the two most common interval orders (strong, subset), and a third less common order (weak) which we focus on in this paper.

Definition 3: (Strong Interval Order) Let $<_S$ be a strict order on $\overline{\mathbb{R}}$ where $\bar{x} <_S \bar{y}$ iff $x^* < y_*$. Let $\leq_S$ be the reflexive closure of $<_S$, so that $\bar{x} \leq_S \bar{y}$ iff $x^* < y_*$ or $\bar{x} = \bar{y}$.

Definition 4: (Subset Interval Order [17]) Let $\subseteq$ be a partial order on $\overline{\mathbb{R}}$ where $\bar{x} \subseteq \bar{y}$ iff $x_* \geq y_*$ and $x^* \leq y^*$.

Definition 5: (Weak Interval Order) Let $\leq_W$ be a partial order on $\overline{\mathbb{R}}$ where $\bar{x} \leq_W \bar{y}$ iff $x_* \leq y_*$ and $x^* \leq y^*$. This is equivalent to the product order $\leq \times \leq$ on $\mathbb{R}^2$.

Notice that the weak order $\leq_W$ and the subset order $\subseteq$ are nearly conjugate orders [3, 7]. First, Papadakis and Kaburlasos [15] observe that where our weak order $\leq_W$ on intervals is the product order on the reals $\leq \times \leq$, the subset order $\subseteq$ is also a product order, but conjugately as $\geq \times \leq$. But $\leq_W$ and $\subseteq$ are not truly conjugate, since it is possible for two distinct intervals to be comparable in both $\leq$ and $\subseteq$ if there is equality at one of the endpoints. As an example, $[3, 4] \subseteq [2, 4]$ and $[2, 4] \leq_W [3, 4]$.

One can ask whether the strong order $\leq_S$ also has a conjugate, and the answer appears to be yes, at least in small cases. We have experimentally identified numerous conjugates and pseudo-conjugates (in the sense in which $\leq_W$ and $\subseteq$ are pseudo-conjugates) for $\leq_S$ on the set of integer intervals with endpoints between 1 and 4, but none appear to have a compactly expressible structure, as both $\leq_W$ and $\subseteq$ do (see further discussion below in Sec. 6).

2.3 Interval Rank

For a full development of the concept of interval rank see [10], here we present the definitions needed for the rest of this paper. We begin by defining general interval rank functions, strict interval rank functions, and a standard interval rank function, a particularly significant strict interval rank function.

Definition 6: (Interval Rank Function) Let $\mathcal{P} = \langle P, \leq \rangle$ be a poset and $\sqsubseteq$ an order on real intervals. Then a function, $R_{\sqsubseteq} \colon P \to \overline{\mathbb{N}}$, with $R_{\sqsubseteq}(a) = [r_*(a), r^*(a)]$ for $a \in P$, is an **interval rank function for** $\sqsubseteq$ if $R_{\sqsubseteq}$ is a strict order homomorphism $\mathcal{P} \mapsto \langle \overline{\mathbb{N}}, \sqsubseteq \rangle$, i.e., for all $a < b$ in $\mathcal{P}$ we have $R_{\sqsubseteq}(a) \sqsubset R_{\sqsubseteq}(b)$. Let $\mathbf{R}_{\sqsubseteq}(\mathcal{P})$ be the set of all interval rank functions $R_{\sqsubseteq}$ for the interval order $\sqsubseteq$ on a poset $\mathcal{P}$.

In [10] we showed that the endpoints of these interval rank functions are monotonic iff the interval order is one of $\{\leq_W, \geq_W, \subseteq, \supseteq\}$. In order to ensure that both endpoints of the intervals are *strictly* monotonic we must introduce the strict interval rank function.

Definition 7: (Strict Interval Rank Function) An interval rank function $R_{\sqsubseteq} \in \mathbf{R}_{\sqsubseteq}(\mathcal{P})$ is **strict** if r_* and r^* are strictly monotonic. Let $\mathbf{S}_{\sqsubseteq}(\mathcal{P})$ be the set of all strict interval rank functions $R_{\sqsubseteq}$ for the interval order $\sqsubseteq$ on a poset $\mathcal{P}$.

Strict interval rank functions have the distinction of being the only interval-valued functions which are strictly monotonic on both interval endpoints. Each of the four types of monotonicity corresponds to an interval order.

Proposition 8. *[10] Let $F : P \to \overline{\mathbb{N}}$ be an interval-valued function on poset $\langle \mathcal{P}, \leq \rangle$, so that $F(a) = [F_*(a), F^*(a)] \in \overline{\mathbb{N}}$. Then, F_* and F^* are strictly monotonic functions iff F is a strict interval rank function for $\leq_W, \geq_W, \subseteq$, or $\supseteq$. In particular:*

(i) F_* and F^* are both strictly isotone iff $\sqsubseteq = \leq_W$.

(ii) F_* and F^* are both strictly antitone iff $\sqsubseteq = \geq_W$.

(iii) F_* is strictly antitone and F^* is strictly isotone iff $\sqsubseteq = \subseteq$.

(iv) F_* is strictly isotone and F^* is strictly antitone iff $\sqsubseteq = \supseteq$.

We point out one particular strict interval rank function that is especially useful, and call it the standard interval rank function.

Definition 9: (Standard Interval Rank) Let

$$R^+(a) := [\mathcal{H}(\uparrow a) - 1, \mathcal{H} - \mathcal{H}(\downarrow a)] \in \overline{\mathcal{H} - 1}$$

be called the **standard interval rank function**. For convenience we also denote $R^+(a) = \left[r^t(a), r^b(a)\right]$, where $r^t(a) := \mathcal{H}(\uparrow a) - 1$ is called the **top rank** and $r^b(a) := \mathcal{H} - \mathcal{H}(\downarrow a)$ is called the **bottom rank**.

The following proposition shows why we prefer R^+ over any other strict interval rank function.

Proposition 10. *[10] For a finite bounded poset $\mathcal{P}$, R^+ is maximal w.r.t. $\subseteq$ in the sense that $\forall R \in \mathbf{S}_{\geq_W}(\mathcal{P})$ with $R(\mathcal{P}) \subseteq \overline{\mathcal{H} - 1}$, $\forall a \in P, R(a) \subseteq R^+(a)$.*

3 Conjugate Interval Rank

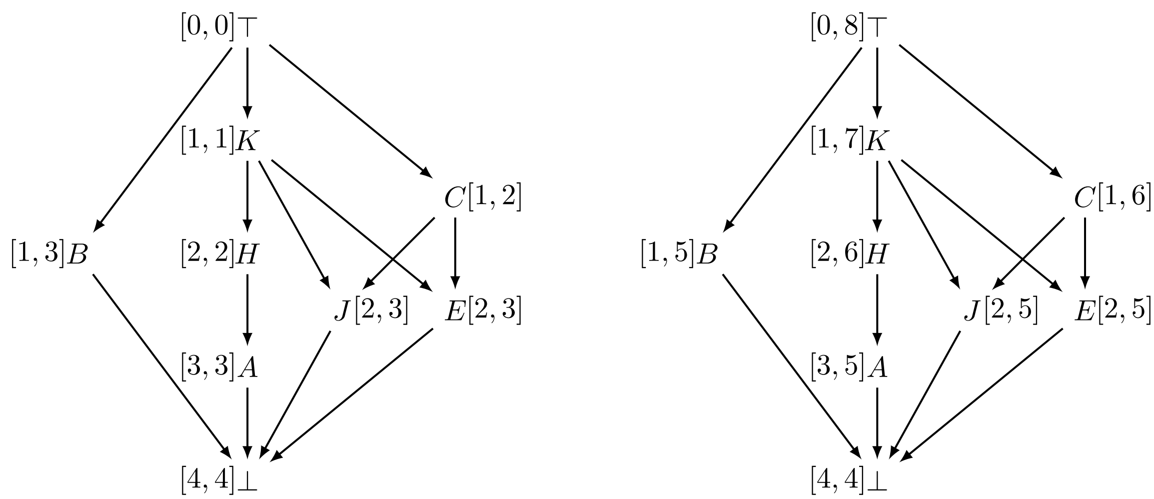
Just as there is a privileged strict interval rank function for the weak interval order $\geq_W$, there is also one for the subset interval order $\subseteq$. We will call this the **conjugate standard interval rank function** because it is an interval rank function for the interval order $\subseteq$, a conjugate interval order to $\geq_W$ for which the standard interval rank is defined, and because its properties closely mirror those of the standard interval rank function.

Definition 11: (Conjugate Standard Interval Rank) Let

$$R^\circ(a) = [\mathring{r}_*(a), \mathring{r}^*(a)] := [\mathcal{H}(\uparrow a) - 1, \mathcal{H} + \mathcal{H}(\downarrow a) - 2]$$

be called the **conjugate standard interval rank function**.

In Fig. 2 we show the standard interval rank values along with the conjugate standard interval rank values for the same example poset. We can prove a result for R° analogous to that found in Proposition 10 for R^+, plus two other properties.



(a) Example poset with $R^+(p)$ for each $p \in \mathcal{P}$.

(b) Example poset with $R^\circ(p)$ for each $p \in \mathcal{P}$.

Figure 2

Proposition 12. *For a finite bounded poset* $\mathcal{P}$*,*

(i) $R^\circ \in \mathbf{S}_{\subseteq}(\mathcal{P})$ *is a strict interval rank function for the subset interval order* $\subseteq$*;*

(ii) $R^\circ(\top) = [0, 2(\mathcal{H}-1)]$*;* $R^\circ(\bot) = [\mathcal{H}-1, \mathcal{H}-1]$*;*

(iii) R° *is minimal w.r.t.* $\leq_W$ *in the sense that* $\forall R \in \mathbf{S}_{\subseteq}(\mathcal{P})$ *with* $R(\mathcal{P}) \subseteq \overline{2(\mathcal{H}-1)}$*,* $\forall a \in P, R(a) \geq_W R^\circ(a)$*.*

Proof.

(i) We first show that $\mathring{r}_*(a) \leq \mathring{r}^*(a)$ so that R° is an interval-valued function. We claim that $\mathcal{H}(\uparrow a) - 1 \leq \mathcal{H} - \mathcal{H}(\downarrow a)$. Indeed, $\forall a \in P$ we have that

$$\mathcal{H}(\downarrow a) + \mathcal{H}(\uparrow a) \leq \mathcal{H}([0,1]) + 1 = \mathcal{H} + 1$$

with equality iff $a \in I(\mathcal{P})$ is a spindle element. Rearranging we have the desired $\mathcal{H}(\uparrow a) - 1 \leq \mathcal{H} - \mathcal{H}(\downarrow a)$. Then, if we increase only the right side by a positive value we retain the inequality. Since $\mathcal{H}(\downarrow a) \geq 1$ we can add $2\mathcal{H}(\downarrow a) - 2 > 0$ to the right side. This yields

$$\mathcal{H}(\uparrow a) - 1 \leq \mathcal{H} - \mathcal{H}(\downarrow a) + 2\mathcal{H}(\downarrow a) - 2 = \mathcal{H} + \mathcal{H}(\downarrow a) - 2.$$

Therefore, R° is an interval-valued function.

Then, it is evident from $\mathcal{H}(\uparrow a)$ being strictly antitone, and $\mathcal{H}(\downarrow a)$ being strictly isotone that $\mathring{r}_*$ is strictly antitone and $\mathring{r}^*$ is strictly isotone. Therefore, by Proposition 8 we have that R° is a strict interval rank function for $\subseteq$.

(ii) Follows from $\mathcal{H}(\uparrow\top) = \mathcal{H}(\downarrow\bot) = 1$ and $\mathcal{H}(\downarrow\top) = \mathcal{H}(\uparrow\bot) = \mathcal{H}$.

(iii) Since any $R \in \mathbf{S}_{\subseteq}(\mathcal{P})$ is a strict interval rank function for $\subseteq$ we know that if $a < b \in P$, $R(a) \subset R(b)$. Therefore

$$r_*(a) > r_*(b) \text{ and } r^*(a) < r^*(b)$$

so that r_* is strictly antitone and r^* is strictly isotone. In addition, we are restricting to the case where $0 \le r_* \le r^* \le 2(\mathcal{H}-1)$. Under these assumptions we must show that $\forall a \in P$

$$r_*(a) \ge \mathcal{H}(\uparrow a) - 1$$
$$r^*(a) \ge \mathcal{H} + \mathcal{H}(\downarrow a) - 2.$$

First notice that, by definition of $\mathcal{H}$, there must be a chain $C_{\mathcal{H}} \subseteq P$ of length $\mathcal{H}$ with greatest element $\top$ and least element $\bot$. Since $R(\mathcal{P}) \subseteq \overline{2(\mathcal{H}-1)}$, r_* is strictly antitone, r^* is strictly isotone, and $r_* \le r^*$ we must have $R(\bot) = [\mathcal{H}-1, \mathcal{H}-1]$ (so that we can decrease r_* by one along $C_{\mathcal{H}}$ and stay positive, and increase r^* by one and stay less than $2(\mathcal{H}-1)$ as we go from $\bot$ to $\top$) and $R(\top) = [0, 2(\mathcal{H}-1)]$.

Now, let $a \in P$, by definition of $\mathcal{H}(\cdot)$ we know that there must be a chain $C \subseteq P$ of length $\mathcal{H}(\uparrow a)$ with greatest element $\top$ and least element a. We already showed that $r_*(\top) = 0$. Then, in order for r_* to be strictly antitone we need $\forall c_1 < c_2 \in C$, $r_*(c_1) > r_*(c_2)$. Therefore $r_*(c)$ must be at least the chain distance from $\top$ to c along C, less one, for all $c \in C$. In particular, $r_*(a) \ge \mathcal{H}(\uparrow a) - 1$.

Dually, there must be a chain $D \subseteq P$ of length $\mathcal{H}(\downarrow a)$ with greatest element a and least element $\bot$. We already know $r^*(\bot) = \mathcal{H} - 1$. In order for r^* to be strictly isotone it must be true that $\forall d_1 < d_2 \in D$, $r^*(d_1) < r^*(d_2)$. Therefore, $r^*(d)$ must be at least $\mathcal{H} - 1$ plus the chain distance from $\bot$ to d along D, less one, for all $d \in D$. In particular

$$r^*(a) \ge \mathcal{H} - 1 + (\mathcal{H}(\downarrow a) - 1) = \mathcal{H} + \mathcal{H}(\downarrow a) - 2.$$

□

One might ask why R^+ is called "standard" when its conjugate R° is available. First, the behavior of R^+ is much more natural and meets our criteria of advancing monotonically with level, where the intervals for R° "nest". We will see in Proposition 13 below that the homomorphic image under the conjugate interval rank R° is isomorphic to that of the standard interval rank R^+, and thus does not bring any particular value compared to R^+. And finally, R° does not conform to our desired criteria of ranks being in the set $\{0, 1, \dots, \mathcal{H}-1\}$, but rather being in $\{0, 1, \dots, 2(\mathcal{H}-1)\}$.

4 The Homomorphic Image of Standard Interval Ranks

A general interval rank function R takes elements $p \in P$ of a poset to intervals $R(p) \in \overline{\mathcal{H}-1}$. But from the discussion in Sec. 2.2, we know that these intervals $R(p)$ can also be ordered. In this section we consider the behavior and properties of standard interval rank R^+ as an order morphism.

The rank intervals $R^+(a)$ are themselves elements in a poset, that is in the homomorphic image $R^+(\mathcal{P}) = \langle R^+(P), \le_W \rangle$. Thus they also have an interval rank structure within $R^+(\mathcal{P})$. We can analyze the structure of this homomorphic image, including changes in height, width, and dimension compared to the underlying poset $\langle P, \leq \rangle$. And finally, R^+ as an order morphism can be iteratively applied to derive $R^+(R^+(\mathcal{P}))$, etc., and we can examine the long-term properties of this iterated application of R^+.

4.1 The Interval Rank Poset

Recalling the definition of the homomorphic image of a poset given in Section 2.1, we will refer to $R^+(\mathcal{P})$ as the *interval rank poset of* $\mathcal{P}$. From Definition 6 we know that R^+ is a strict order homomorphism into $\langle \overline{N}, \geq_W \rangle$. Therefore, $R^+(\mathcal{P})$ is an induced subposet of $\overline{\mathcal{H}-1}$ with $\geq_W$ as its ordering. Fig. 3 shows the homomorphic image of the poset found in Fig. 1. It is easily verified that there is a strict order homomorphism from $\mathcal{P}$ to $R^+(\mathcal{P})$. But notice that two of the elements of $\mathcal{P}$, J and E, had the same standard interval rank, so they collapse into a single element, $[2,3]$, in $R^+(\mathcal{P})$. These behaviors will be considered in detail below in Sec. 4.3.

The structure of $R^+(\mathcal{P})$ allows us to identify elements $a, b \in P$ which are either comparable $R^+(a) \sim_W R^+(b)$ or noncomparable $R^+(a) \|_W R^+(b)$ in terms of the weak interval order relation $\leq_W$ between their interval ranks. If they are noncomparable in the weak order they are thereby comparable in the conjugate subset order, so that $R^+(a) \subseteq R^+(b)$ or $R^+(a) \supseteq R^+(b)$.

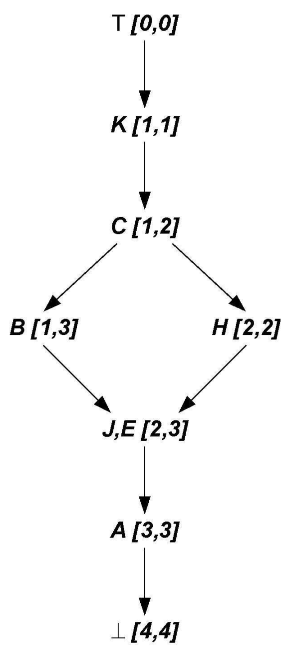


Figure 3: The homomorphic image $R^+(\mathcal{P})$ induced from the example in Fig. 1 showing the weak interval order $\leq_W$ on the standard interval ranks $R^+(a) \in \overline{N}$.

Fig. 4 now shows the example in Fig. 2 equipped with both standard interval rank and the edges in the interval rank poset $R^+(\mathcal{P})$, shown in dashed lines, with J and E identified as a new contracted element in $R^+(\mathcal{P})$ with a dashed oval. Note how the dashed edges of the interval rank poset proceed vertically very tightly from top to bottom, linking elements with the closest standard interval ranks, whether those element pairs are in $\mathcal{P}$ or not.

As mentioned in Section 3, we now prove that the homomorphic image of a poset, $\mathcal{P}$, under the

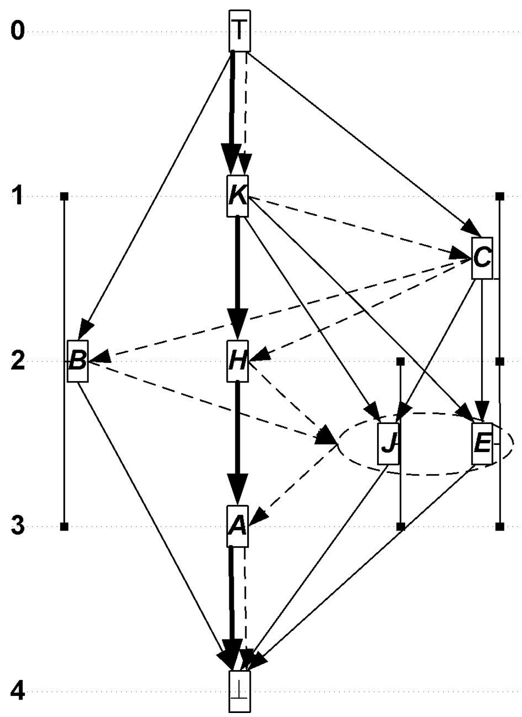


Figure 4: Example poset from Fig. 1 equipped with interval ranks $R^+(a)$, homomorphic image links in dashed lines, and separations and widths on all links.

conjugate interval rank R° is isomorphic to that of the standard interval rank R^+. Therefore, it is enough to study the structure of just the homomorphic image of R^+.

Proposition 13. *$R^+(\mathcal{P}) \cong R^\circ(\mathcal{P})$.*

Proof. To show that these two posets are isomorphic we will show that there is an order embedding, $\varphi : R^+(\mathcal{P}) \to R^\circ(\mathcal{P})$, that is surjective. Define

$$\begin{aligned}\varphi : & R^+(\mathcal{P}) \longrightarrow R^\circ(\mathcal{P}) \\ & [x, y] \quad \longmapsto [x, 2(\mathcal{H} - 1) - y].\end{aligned}$$

We will first show that φ is surjective, i.e., that for every $[a, b] \in R^\circ(\mathcal{P})$ there is an $[x, y] \in R^+(\mathcal{P})$ so that $\varphi([x, y]) = [a, b]$. Consider $[a, b] \in R^\circ(\mathcal{P})$. Then there is some $p \in \mathcal{P}$ such that $\mathcal{H}(\uparrow p)-1 = a$ and $\mathcal{H} + \mathcal{H}(\downarrow p) - 2 = b$. Let $[x, y] = [\mathcal{H}(\uparrow p) - 1, \mathcal{H} - \mathcal{H}(\downarrow p)]$. Clearly this is an element of $R^+(\mathcal{P})$ by the definition of the R^+ operator; in fact, it is $R^+(p)$. Then the image of $[x, y]$ under φ is

$$\begin{aligned}\varphi([x, y]) &= \varphi([\mathcal{H}(\uparrow p) - 1, \mathcal{H} - \mathcal{H}(\downarrow p)]) \\ &= [\mathcal{H}(\uparrow p) - 1, 2(\mathcal{H} - 1) - (\mathcal{H} - \mathcal{H}(\downarrow p))] = [a, b]\end{aligned}$$

So, for every $[a, b] \in R^\circ(\mathcal{P})$ we have an $[x, y] \in R^+(\mathcal{P})$ such that $\varphi([x, y]) = [a, b]$. Therefore, φ is surjective.

To show that φ is an order embedding we must show that for all $[x, y], [z, w] \in R^+(\mathcal{P})$ we have $[x, y] \geq_W [z, w] \iff \varphi([x, y]) \subseteq \varphi([z, w])$. First we show the forward direction. Assume we have $[x, y] \geq_W [z, w]$, then $x \geq z$ and $y \geq w$. We map these intervals to $\varphi([x, y]) = [x, 2(\mathcal{H}-1)-y] = [a, b]$ and $\varphi([z, w]) = [z, 2(\mathcal{H} - 1) - w] = [c, d]$. Obviously $a \geq c$ since $a = x$ and $c = z$. Then, since $y \geq w$ we have $-y \leq -w$ which means $b \leq d$. Putting this together we have $\varphi([x, y]) = [a, b] \subseteq [c, d] = \varphi([z, w])$ which proves the forward implication.

Now, we assume that $[a, b] = \varphi([x, y]) \subseteq \varphi([z, w]) = [c, d]$, so $a \geq c$ and $b \leq d$. Because $a = x$ and $z = c$ we have $x \geq z$. Then, $b = 2(\mathcal{H} - 1) - y \leq 2(\mathcal{H} - 1) - w = d$ so $-y \leq -w$ and then $y \geq w$. Putting this together we get $[x, y] \geq_W [z, w]$ as desired.

□

4.2 Properties of the Interval Rank Poset

We now consider the interval rank structure of $R^+(\mathcal{P})$, the homomorphic image of $\mathcal{P}$, itself, and further iterations, $R^+(R^+(\ldots R^+(\mathcal{P}) \ldots))$, thereof.

First, when we compare $\mathcal{P}$ to its homomorphic image $R^+(\mathcal{P})$ we observe that the height always increases while the width decreases.

Proposition 14. *The height of the interval rank poset is greater than the height of the ordered set itself, i.e., $\mathcal{H}(R^+(\mathcal{P})) \geq \mathcal{H}(\mathcal{P})$.*

Proof. Let S be a spindle chain in $\mathcal{P}$. The elements in S have rank intervals $[i, i]$ where i takes all integer values between 0 and $\mathcal{H}(\mathcal{P}) - 1$ (inclusively). The image of this spindle chain S under R^+ is a (not necessarily saturated) chain, S', in $R^+(\mathcal{P})$. From the definition of the height of a poset we have that

$$\forall C \in \mathcal{C}(R^+(\mathcal{P})), \quad \mathcal{H}(R^+(\mathcal{P})) \geq |C|.$$

Thus, $\mathcal{H}(R^+(\mathcal{P})) \geq |S'| = |S| = \mathcal{H}(\mathcal{P})$. □

Proposition 15. *The width of the interval rank poset is less than the width of the ordered set itself, i.e.,* $\mathcal{W}(R^+(\mathcal{P})) \leq \mathcal{W}(\mathcal{P})$*.*

Proof. Let $\mathcal{A}$ be the set of all antichains in $\mathcal{P}$, not necessarily maximal; and $\mathcal{A}^+$ the set of all antichains in $R^+(\mathcal{P})$, also not necessarily maximal. Let $A \in \mathcal{A}^+$. Consider the set of preimages of elements in A w.r.t. the map R^+,

$$(R^+)^{-1}(A) = \{p \in \mathcal{P} : R^+(p) \in A\}.$$

Note that it is of course possible that $|(R^+)^{-1}(A)| \geq |A|$. We claim that $(R^+)^{-1}(A) \in \mathcal{A}$. If not then there are two elements, $p, q \in (R^+)^{-1}(A)$ such that $p < q$. Since R^+ is an order homomorphism we have that $R^+(p) < R^+(q)$. But this is a contradiction to A being an antichain, so $(R^+)^{-1}(A)$ must be an antichain in $\mathcal{P}$. Therefore we have the following chain of inequalities on width:

$$\begin{aligned}\mathcal{W}(R^+(\mathcal{P})) = \max_{A \in \mathcal{A}^+} |A| &\leq \max_{A \in \mathcal{A}^+} |(R^+)^{-1}(A)| \\ &\leq \max_{A' \in \mathcal{A}} |A'| = \mathcal{W}(\mathcal{P}).\end{aligned}$$

□

Notice that in the proof of Proposition 14 we were able to start with a chain (namely the spindle chain) in $\mathcal{P}$ and use the fact that its image in $R^+(\mathcal{P})$ is a chain. This is because R^+ is an order morphism to the reversed interval order, so the images of comparable elements are comparable. However, there is no equivalent property that we could use in the proof of Proposition 15. Every antichain in $\mathcal{A}^+$ must come from an antichain in $\mathcal{A}$, but some antichains in $\mathcal{A}$ map to non-antichains in $R^+(\mathcal{P})$.

Given Theorem 2, which states that an n dimensional poset is one which is an induced subposet of $\mathbb{R}^n$ with ordering relation $\leq^n$, but not an induced subposet of $\mathbb{R}^{n-1}$, we can now easily see that the dimension of $R^+(\mathcal{P})$ is at most 2.

Corollary 16. *For any poset* $\mathcal{P} = \langle P, \leq \rangle$ *its homomorphic image under the standard interval rank function,* $R^+(\mathcal{P})$*, has*

$$\dim(R^+(\mathcal{P})) \leq 2$$

Proof. By definition, $R^+(\mathcal{P})$ is a subposet of $\mathbb{R}^2$ with the reversed product order, so by Theorem 2 we see that $R^+(\mathcal{P})^*$ (the dual of $R^+(\mathcal{P})$) has dimension at most 2. It's clear that a poset and its dual have the same dimension, so we have that $\dim(R^+(\mathcal{P})) \leq 2$. □

Note that it may be the case that $R^+(\mathcal{P})$ has dimension one (i.e., it is a chain). For example, the following proposition gives a particular sufficient condition on $\mathcal{P}$ for $R^+(\mathcal{P})$ to be a chain.

Proposition 17. *If* $\mathcal{P}$ *is graded then* $R^+(\mathcal{P})$ *is a chain.*

In order to prove this we must cite a part of a proposition proved in [10].

Proposition 18. *Let* $\mathcal{P} = \langle P, \leq \rangle$ *be a poset such that* $\mathcal{P}$ *is bounded, and* $|P| \geq 2$*. For an element* $a \in P$*, the width of its standard interval rank is zero iff* a *is a spindle element. I.e.,* $W(R^+(a)) = 0 \iff a \in I(\mathcal{P})$*.*

Proof of Proposition 17. Every element in a graded poset sits on a spindle chain. This is simply because in a graded poset every maximal chain is the same length. Now, from Proposition 18 we know that if $a \in I(\mathcal{P})$ then $W(R^+(a)) = 0$. Therefore, the only elements of $R^+(\mathcal{P})$ are the trivial intervals $[i, i]$ for $0 \leq i \leq \mathcal{H} - 1$. This set forms a chain under $\geq_W$ (the dual of the product order on $\mathbb{R}^2$). □

However, there are ungraded posets, $\mathcal{P}$, for which $R^+(\mathcal{P})$ is a chain. For example, the poset N_5, consisting of a length 4 chain and a length 3 chain which share their top and bottom elements, is ungraded and $R^+(N_5)$ is a chain. Also, whether the dimension of $R^+(\mathcal{P})$ is 2 or 1 does not depend on the dimension of $\mathcal{P}$. There are posets of dimension greater than 2 whose interval rank poset has dimension 1: any boolean n-cube has dimension n and is graded, thus its interval rank poset is a chain. In addition, there are posets of dimension 2 whose interval rank poset also has dimension 2.

4.3 Iterating R^+

The fact that height is non-decreasing and width non-increasing from $\mathcal{P}$ to R^+ leads us to ask the following question: what happens when we repeatedly apply the R^+ operator? If height *strictly* increases and width *strictly* decreases then it's clear that we end up with a chain if we apply R^+ enough times. However, Propositions 14 and 15 cannot be reformulated with strict inequalities so this chain conjecture is not obvious. Based on experimental evidence (see Section 5) it appeared that when we apply R^+ enough times the result is a chain. This turned out to be true, which we now prove.

For Lemma 20 we will need to define the poset $R^+_{All}(\mathcal{P})$. See Fig. 5 for an example, and compare to Figure 3 which contains $R^+(\mathcal{P})$ for the same $\mathcal{P}$.

Definition 19: Given a poset $\mathcal{P} = \langle P, \leq \rangle$ define $R^+_{All}(\mathcal{P}) = \langle P, \leq_{R_A} \rangle$ where $p <_{R_A} q$ iff $R^+(p) >_W R^+(q)$. Notice that $<_{R_A}$ is a strict order, so we must take its reflexive closure to create $\leq_A$. This is just $R^+(\mathcal{P})$ without identifying elements that have the same interval rank. So if $R^+(p) = R^+(q)$ for $p, q \in \mathcal{P}$ with $p \neq q$ we retain both elements and make them incomparable.

Given this definition we will now prove two lemmas that will lead us to the proof that iterating R^+ enough times yields a chain.

Lemma 20. *Given a poset, $\mathcal{P} = \langle P, \leq \rangle$, there exists an m such that $(R^+_{All})^m(\mathcal{P})$ is graded.*

Proof. Fix an ungraded poset, $\mathcal{P}$, and consider the set of graded posets that extend $\mathcal{P}$,

$$G(\mathcal{P}) = \{\mathcal{Q} = \langle P, \leq_{\mathcal{Q}} \rangle : \ \leq \ \subset \ \leq_{\mathcal{Q}}, \mathcal{Q} \text{ graded}\}.$$

That is, if $p \leq q$ then $p \leq_{\mathcal{Q}} q$. Clearly this set is nonempty since there is at least one linear extension of $\mathcal{P}$ (recall the definition of a linear extension from Section 2.1), and linear extensions, being chains, are graded. Also, the number of comparisons (ordered pairs (a, b) such that $a \leq b$) in a total order only depends on the number of elements in the chain. All pairs of elements are comparable, and we have the reflexive comparisons, so

$$\max_{\mathcal{Q} \in G(\mathcal{P})} |\leq_{\mathcal{Q}}| = \binom{|P|}{2} + |P| = \frac{|P|^2 + |P|}{2}.$$

Also, any poset $\mathcal{L} = \langle P, \leq_{\mathcal{L}} \rangle$ such that $|\leq_{\mathcal{L}}| = \frac{|P|^2+|P|}{2}$ is a total order.

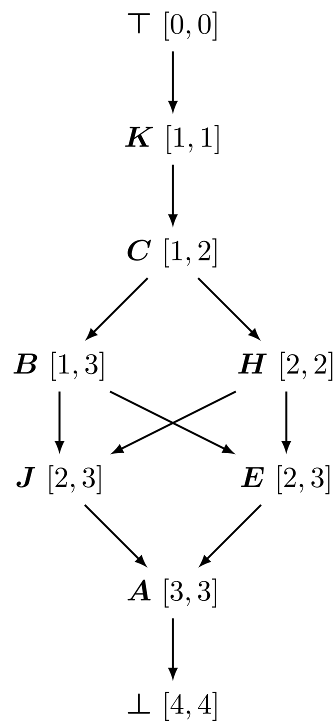


Figure 5: An example of $R^{+}_{All}(\mathcal{P})$ where $\mathcal{P}$ is the poset found in Fig. 1. Compare to $R^{+}(\mathcal{P})$ found in Fig. 3.

Now we claim that $\mathcal{P}$ being ungraded implies $|\leq| < |\leq_{R_A}|$. So when we apply the R^+_{All} operation we always end up with strictly more comparable pairs of elements. If this is true then either

1. at some point, iterating R^+_{All} yields a graded poset whose partial order has strictly less than $\frac{|P|^2+|P|}{2}$ elements, or

2. after iterating R^+_{All} enough times we will get a partial order with exactly $\frac{|P|^2+|P|}{2}$ comparisons.

We will show that every time we iterate R^+_{All} on an ungraded poset we add at least one comparison. If we iterate R^+_{All} and get to $\frac{|P|^2+|P|}{2}$ without hitting a graded poset up to this point, then we are in case 2. Otherwise, we got to a graded poset with strictly less than $\frac{|P|^2+|P|}{2}$ elements, and are in case 1. Either way, there is an m such that $(R^+_{All})^m(\mathcal{P})$ is graded.

Finally, we must prove that $\mathcal{P}$ being ungraded implies $|\leq| < |\leq_{R_A}|$, i.e., that we gain at least one comparison. Choose $p \in P$ with $R^+(p) = [x, y]$ for some $x, y \in \mathbb{N}$ such that $x \neq y$. There must be at least one p (if not, then $\mathcal{P}$ is graded). Then, choose $q \in P$ with $R^+(q) = [x, x]$. Again, there must be at least one since $0 \leq x \leq \mathcal{H}(\mathcal{P}) - 1$ and all $[z, z]$ with $0 \leq z \leq \mathcal{H}(\mathcal{P}) - 1$ are represented on a spindle chain. Clearly, $p \nsim q$ in $\mathcal{P}$ because $\mathcal{H}(\uparrow p) = \mathcal{H}(\uparrow q)$. But in $R^+_{All}(\mathcal{P})$ we have $q >_{R_A} p$ since $[x, x] \leq_W [x, y]$. So going from $\leq$ to $\leq_{R_A}$ we added at least one comparison. $\square$

Lemma 21. *If $(R^+_{All})^m(\mathcal{P})$ is graded then $(R^+)^m(\mathcal{P})$ is graded.*

Proof. Assume that $(R^+_{All})^m(\mathcal{P})$ is graded. Then there is a rank function, $r : P \to \mathbb{N}$ such that if $p \prec q$ in $(R^+_{All})^m(\mathcal{P})$ then $r(p) = r(q) + 1$. Now, there may be some elements, p_i in P such that $(R^+)^m(p_i)$ are all equal. So in $(R^+_{All})^m(\mathcal{P})$ they all have the same immediate parents $(\{a_j\})$ and children $(\{c_k\})$. Thus, $r(p_i)$ are all equal. So when we collapse all p_i into one element, p, in $R^+(\mathcal{P})$ we have a non-ambiguous rank, $r(p) = r(p_i)$ for all i, for p. $\square$

Proposition 22. *Given a poset, $\mathcal{P} = \langle P, \leq \rangle$, there exists an n such that $(R^+)^n(\mathcal{P})$ is a chain.*

Proof. Fix a poset $\mathcal{P} = \langle P, \leq \rangle$. By Lemma 20 there is an m such that $(R^+_{All})^m(\mathcal{P})$ is graded. Then by Lemma 21, $(R^+)^m(\mathcal{P})$ is graded. Finally, by Proposition 17, since $(R^+)^m(\mathcal{P})$ is graded we have that $(R^+)^{m+1}(\mathcal{P})$ is a chain. $\square$

The chain that we arrive at can be thought of as a *total preorder* – a reflexive and transitive relation in which every pair of elements is comparable [6] – which extends $\mathcal{P}$. Preorders are not antisymmetric, so when two elements p and q are identified in the final chain we will say that $p \leq q$ and $q \leq p$, but $p \neq q$. This resulting total preorder is easily computable, and clearly calls out our concept of "levels" in an ungraded poset. Figure 6 shows the final total preorder for the example from Fig. 1.

Now that we have established that we always end up with a chain after iterating $R^+(\mathcal{P})$ enough times, three interesting questions arise: how many iterations does it take to end up with a chain, what is the final chain length, and how much does the height increase. We were able to answer these questions exactly for all bounded posets of size 3 to 9 (see Section 5 for more on this). The averages[1] are collected in Table 1.

[1] Note that there is only one bounded poset of height 9 when we restrict to posets of size between 3 and 9, the chain of length 9. So, the average final height of chain for bounded posets of size 9 is the average taken over only one sample.

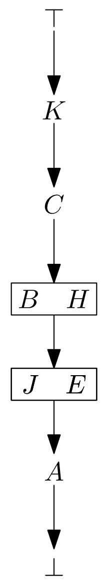


Figure 6: The canonical total preorder for the poset found in Fig. 1. Notice that B and H are identified as well as J and E. Looking at Fig. 1 it's easy to see why J and E are identified, but not clear why B and H are.

Bounded Poset Size	3	4	5	6	7	8	9
Avg. # of elements in chain	3	3.5	4.2	4.75	5.381	5.959	6.517
Avg. # of iterations to chain	0	0.5	0.8	1.00	1.127	1.236	1.335

Bounded Poset Height	3	4	5	6	7	8	9
Avg. final height of chain	3	4.348	6.068	7.092	7.806	8.409	9

Table 1: Averages of iteration data for all bounded posets of size 3 to 9. Note that we are only looking here at posets of size between 3 and 9. In particular, there is only one poset of size ≤ 9 with height 9, the chain of length 9. This gives us a misleading average final height of chain when the starting poset is height 9.

For bounded posets of size greater than 9 we cannot generate all posets with the computing resources available (there are only 2045 bounded posets of size 9 but 16999 of size 10, see A000112 in [9] for values for larger posets). Therefore, we generated many random bounded posets of sizes between 10 and 25 (200 posets of each size) to get similar data. See Section 5 for a discussion on the method used to generate random posets.

Figures 7, 8, and 9 show the box plot and regression for the size of $\mathcal{P}$ vs. size of chain, size of $\mathcal{P}$ vs. number of iterations to a chain, and height of $\mathcal{P}$ vs. height of chain respectively. From the limited amount of data it appears that the the number of iterations that must be done to get to the total preorder is roughly linear in the number of elements in the poset. The compression that we get in the end is approximately $\ln(n)$.

5 Experimental Math

Throughout Section 4.1 we mentioned the use of experiments to make conjectures or gather data. "Experimental mathematics" was once an oxymoron, however with the advent of computer algebra systems (e.g., Maple and Mathematica) and more powerful computers, it has gained much more acceptability. An article in the Notices of the American Mathematical Society [2] discusses the many historical and current uses of experimentation in mathematics. In a list of eight interpretations of experimental math [2, 4] we use it in our work as a tool to: (a) gain insight and intuition, (b) test conjectures, and (c) suggest approaches for formal proof.

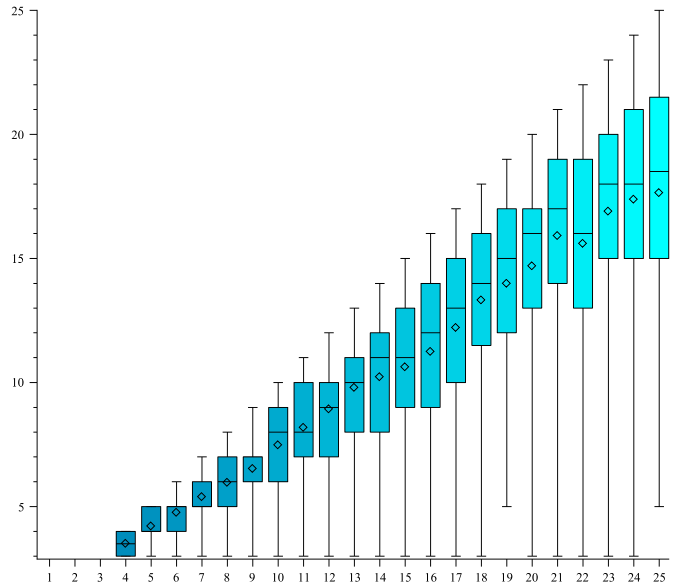
5.1 Generating Small Posets

As we began to investigate properties of the interval rank poset, $R^+(\mathcal{P})$, we found it necessary to construct many examples. Rather than construct arbitrary posets we constructed the set of all bounded posets of size less than or equal to 9 using the computer algebra system Maple on a laptop with an Intel Core i5-2520M processor and 4GB RAM. We did this by first generating all reflexive $\{0,1\}$-matrices of size between 1 and 7. We considered these matrices as binary relations, i.e., if the $(i,j)^{th}$ entry is 1 then $x_i \leq x_j$. Then we checked each matrix for antisymmetry and transitivity: if $a_{i,j} = a_{j,i} = 1$ then $i = j$, and if $a_{i,j} = a_{j,k} = 1$ then $a_{i,k} = 1$. We then had to check for isomorphic copies of the same poset, i.e., checking that permuting the rows and columns of one matrix didn't yield another matrix in the set. Lastly, we added a top and bottom bound (a row of all 1's and a column of all 1's). We generated all of the posets of size up to 6 rather quickly, even checking for isomorphic copies, there are only 318 of size 6. However, it took many days to generate the 2045 posets of size 7. Bounding each was then a trivial step. We then had a test set of all 2450 bounded posets of size between 3 and 9. Of course, any real-world posets would be considerably larger than size 9, however this test set allowed us to get some intuition about behavior of the R^+ operator.

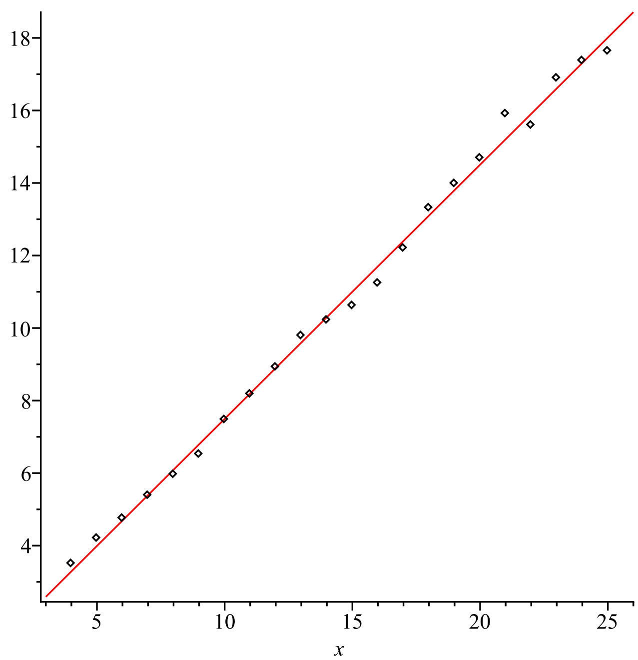
5.2 Generating Larger Posets

The test set of 2450 posets was enough for us to get a feeling for the structure of $R^+(\mathcal{P})$, and formulate the conjectures that would become Propositions 14, 15, 17, and 22. However, for the work in Section 4.3 we wanted to gather data for larger posets. Because the number of posets of size n, P_n, is $2^{n^2/4+o(n^2)}$ [11] it is difficult to get a representative sample of posets of size n.

Our strategy was to generate random posets using two different algorithms, both found in [5]. The first is the random graph model. Given some $n \in \mathbb{N}$ and $0 \leq p \leq 1$ generate the Erdős-Rényi

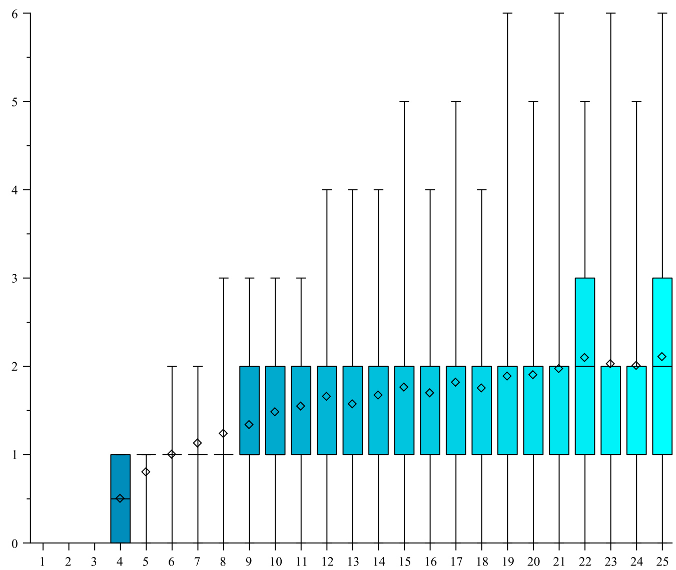


(a) Box Plot

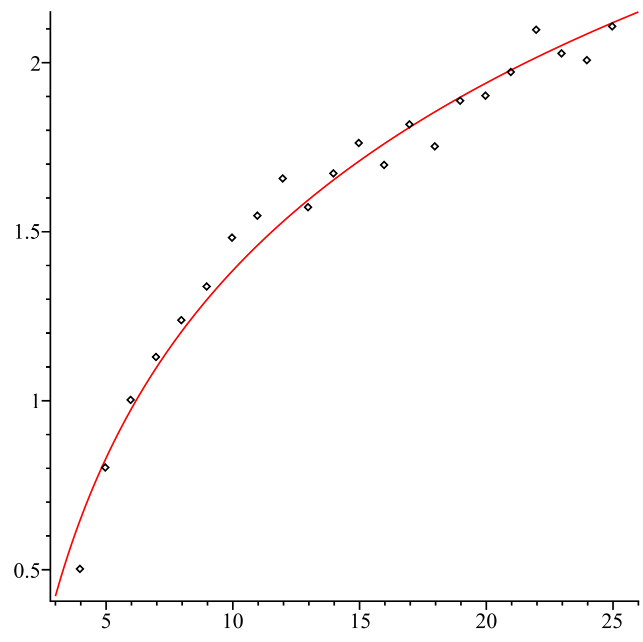


(b) Linear fit for average size of resulting chain as a function of size of original poset: $y = 0.7010x + 0.4854$, $R^2 = 0.9964$

Figure 7: Statistics for the "size of poset vs. size of resulting chain" data

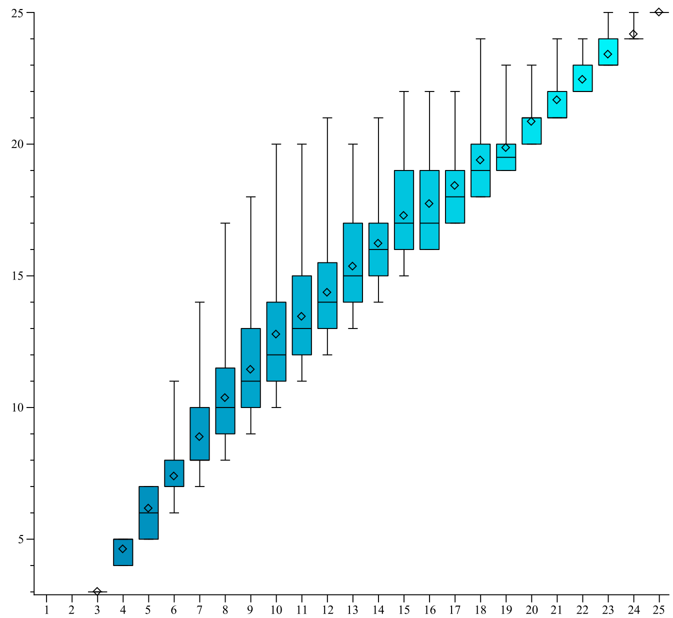


(a) Box Plot

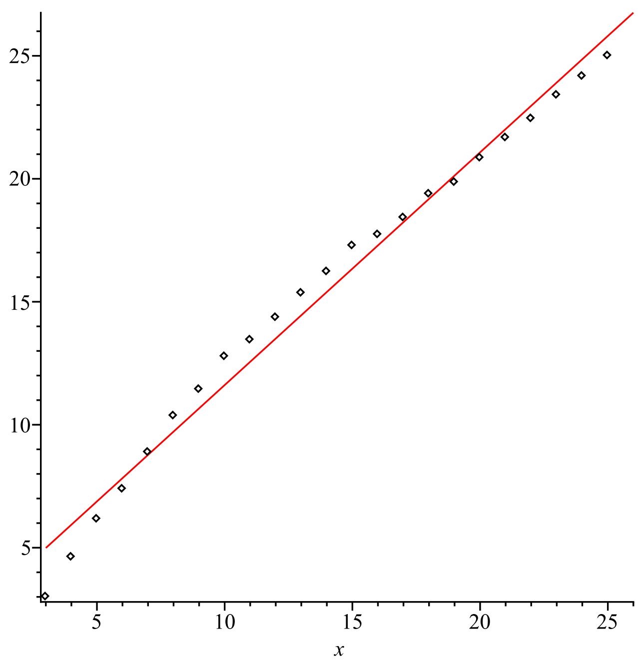


(b) Logarithmic fit for average number of iterations as a function of size of original poset: $y = 0.8003\ln(x) - 0.4574$, $R^2 = 0.9762$

Figure 8: Statistics for the "size of poset vs. number of iterations to arrive at a chain" data



(a) Box Plot



(b) Linear fit for height of resulting chain as a function of height of original poset: $y = 0.9463x + 2.1496$, $R^2 = 0.9839$

Figure 9: Statistics for the "height of poset vs. height of resulting chain" data

random graph $G_{n,p}$. This is a graph on n labeled vertices where each edge, (i, j), is included with probability p. We then create a directed graph by directing each edge from smaller to larger, i.e., if $\{1, 2\}$ is an edge then we direct it from 1 to 2. Notice that this graph must be acyclic since there can be no decreasing edge. From here we take the transitive closure to form a random partial order.

The second model is the random k-dimensional model. Here, we choose k random linear orders (i.e., k random permutations of $[n]$) and take their intersection. By the definition of dimension, the resulting partial order has dimension at most k.

6 Conclusion and Future Work

The results reported here point towards a number of continuing efforts.

Canonical Strong Conjugate: In this paper we did not pursue the problem of finding a canonical conjugate order to the strong interval order (if one exists). While the weak interval order $\leq_W$ and the subset order $\subseteq$ stand as conjugates, in Sec. 2.2 we discussed the availability of conjugates for the strong interval order $\leq_S$. We have identified a number of possibilities experimentally, and would like to cast the question more generally. To do so we need to consider the comparability graph of the strong order [8], that is, the graph $G = \langle P, E\rangle$ where $\langle a, b\rangle \in E \subseteq P^2$ if $a \sim_S b$. The complement of G should be the comparability graph for any conjugate order to $\leq_S$, if they exist. We also know that the complement of the comparability graph of P must be an interval graph, that is, the intersection graph of the intervals $a \in P$. So we are brought to the question of whether an interval graph can, or must, also be a comparability graph. To our knowledge, this is an open question.

Characterizing Iterative Interval Rank Convergence: In Sec. 4.3 we analyzed the iterative behavior of the R^+ mapping, and determined that this converges to a chain. A natural open question is how structural properties of $\mathcal{P}$ can affect the convergence of $\{(R^+)^n(\mathcal{P})\}$ to a chain. Another natural question is whether any of the resulting pre-order chains can be equal (even with the same sets of elements to be identified), for two different order relations on the set $\mathcal{P}$.

It is known that the number of labeled partial orders on n elements is asymptotically $P_n \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+O(ln(n))}$ [12], and the number of labeled preorders is asymptotically $R_n \sim \frac{n!}{2\ln(2)^{n+1}}$ [13]. We deal more with unlabeled posets, but note that if there are two labeled posets, $\mathcal{P}_1 \neq \mathcal{P}_2$ with $(R^+)^m(\mathcal{P}_1) = (R^+)^\ell(\mathcal{P}_2)$ then there are two unlabeled posets with the same property. It's clear that $P_n > R_n$ as n goes to ∞ (2^{n^2} beats $n!$). Therefore, since there are asymptotically more partial orders than preorders, we must have cases in which $\mathcal{P}_1 \neq \mathcal{P}_2$ but $R^+(\mathcal{P}_1) = R^+(\mathcal{P}_2)$. Additionally, it's not too difficult to show that for any preorder, E, there is a corresponding poset, $\mathcal{P}$, for which $(R^+)^k(\mathcal{P}) = E$ for $k = 1$. So, we ask, what structural properties of $\mathcal{P}_1 \neq \mathcal{P}_2$ would allow $(R^+)^m(\mathcal{P}_1) = (R^+)^\ell(\mathcal{P}_2)$?

Measures of Gradedness: Just as we hold that rank in posets is naturally and profitably extended to an interval-valued concept, so this work suggests that we should consider extending gradedness from a qualitative to a quantitative concept. A graded poset is all spindle, with all elements being precisely ranked with width 0, and *vice versa*. Therefore there should be a concept of posets which fail that criteria to a greater or lesser extent, that is, being more or less graded. In fact, we have sought such measures of gradedness as non-decreasing monotonically with iterations of R^+. Candidate measures we have considered have included the avarege interval rank width, the proportion of the spindle to the whole poset, and various distributional properties of the set of the lengths of the maximal chains. While our efforts have been so far unsuccessful, counterexamples were sometimes very difficult to find, and exploring the possibilities has been greatly illuminating.

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