THE NUMBER OF SIMULTANEOUS CORE PARTITIONS

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ABSTRACT. Amdeberhan conjectured that the number of $(t, t+1, t+2)$-core partitions is $\sum_{0\leq k\leq [\frac{t}{2}]} \frac{1}{k+1}\binom{t}{2k}\binom{2k}{k}$. In this paper, we obtain the generating function of the numbers f_t of $(t, t+1, \ldots, t+p)$-core partitions. In particular, this verifies that Amdeberhan's conjecture is true. We also prove that the number of $(t_1, t_2, \ldots, t_m)$-core partitions is finite if and only if $\gcd(t_1, t_2, \ldots, t_m) = 1$, which extends Anderson's result on the finiteness of the number of (t_1, t_2)-core partitions for coprime positive integers t_1 and t_2 and thus rediscover a result of Keith and Nath with a different proof.

1. INTRODUCTION

Partitions of positive integers are widely studied in number theory and combinatorics. As we know, a *partition* of a positive integer n is a finite non increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ with $\sum_{1\leq i\leq r} \lambda_i = n$. In this case, n is called the *size* of λ, which is also be denoted by $\mid \lambda \mid$. We can associate a partition λ with its *Young diagram*, which is an array of boxes arranged in left-justified rows with λ_i boxes in the i-th row. To the (i, j)-box of the Young diagram, let $h(i, j)$ be its *hook length*, which is the number of boxes directly to the right, directly below, or the box itself. Let t be a positive integer. A partition λ is called a *t-core partition* if none of its hook lengths is a multiple of t. Finally, we say that λ is a $(t_1, t_2, \ldots, t_m)$-core partition if it is simultaneously a t_1-core, a t_2-core, ..., a t_m-core partition. For instance, Figure 1 shows the Young diagram and hook lengths of the partition $(5, 2, 2)$. It is easy to see that, the partition $(5, 2, 2)$ is a $(4, 5)$-core partition since non of its hook lengths is divisible by 4 or 5.

7	6	3	2	1
3	2			
2	1			

FIGURE 1. The Young diagram and hook lengths of the partition $(5, 2, 2)$.

For t-core partitions, Granville and Ono [7] proved that there always exists a t-core partition with size n for any $t \geq 4$ and $n \geq 1$. A very important result in the study of $(t_1, t_2, \ldots, t_m)$-core partitions was given by Anderson [2], that is, there are only finite (t_1, t_2)-core partitions when t_1 and t_2 are coprime to each other. Actually, Anderson showed that the number of (t_1, t_2)-core partitions is exactly $\frac{1}{t_1+t_2}\binom{t_1+t_2}{t_1}$ for relatively prime positive integers t_1 and t_2. Anderson's beautiful result attracts much attention and motives a lot of work in the study of simultaneous

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core partitions. Stanley and Zanello [12] showed that the average size of a $(t,t+1)$-core partition is $\binom{t+1}{3}/2$. In 2007, Olsson and Stanton [10] proved that the largest size of (t_1,t_2)-core partitions is $\frac{({t_1}^2-1)({t_2}^2-1)}{24}$ when t_1 and t_2 are coprime to each other. Ford, Mai, and Sze [5] showed that the number of self-conjugate (t_1,t_2)-core partitions is $\binom{[\frac{t_1}{2}]+[\frac{t_2}{2}]}{[\frac{t_1}{2}]}$ for relatively prime positive integers t_1 and t_2, where $[x]$ denotes the largest integer not greater than x.

Anderson [2] proved the finiteness of the number of (t_1,t_2)-core partitions for coprime positive integers t_1 and t_2. We will extend Anderson's this result to a more general case and thus rediscover Theorem 1 in [9] with a different proof:

Theorem 1.1. *The number of $(t_1,t_2,\dots,t_m)$-core partitions is finite if and only if gcd$(t_1,t_2,\dots,t_m)=1$, where gcd$(t_1,t_2,\dots,t_m)$ denotes the greatest common divisor of $t_1,t_2,\dots,t_m$.*

For the number of $(t,t+1,t+2)$-core partitions, Amdeberhan [1] gave the following conjecture, which we will prove in Section 3:

Theorem 1.2. *(Cf. Conjecture 11.1 of* [1]*.) The number f_t of $(t,t+1,t+2)$-core partitions is the $t-$th Motzkin number $\sum_{0\le k\le[\frac{t}{2}]}\frac{1}{k+1}\binom{t}{2k}\binom{2k}{k}$. The generating function of f_t is*

$$\sum_{t\ge 0} f_t x^t = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

2. Proof of Theorem 1.1

Suppose that $\lambda_1\ge\lambda_2\ge\cdots\ge\lambda_r\ge 1$ and $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_r)$ is a partition. The *β-set* of λ is denoted by

$$\beta(\lambda)=\{\lambda_i+r-i : 1\le i\le r\}.$$

It is obvious that $0\notin\beta(\lambda)$. Actually $\beta(\lambda)$ is just the set of hook lengths of boxes in the first column of the corresponding Young diagram. It is easy to see that a partition λ is uniquely determined by its β-set $\beta(\lambda)$. The following is a well-known result on β-sets of t-core partitions.

Lemma 2.1. *(*[8]*.) A partition λ is a t-core partition if and only if for any $x\in\beta(\lambda)$ such that $x\ge t$, we have $x-t\in\beta(\lambda)$.*

By Lemma 2.1, we can easily deduce the following result:

Lemma 2.2. *Let λ be a $(t_1,t_2,\dots,t_m)$-core partition and a_i be some non negative integers. Then $\sum_{1\le i\le m}a_it_i\notin\beta(\lambda)$.*

Proof. We will prove this result by induction on m. If $m=1$, by Lemma 2.1 we know $a_1t_1\notin\beta(\lambda)$ since $0\notin\beta(\lambda)$. Now we assume that $m\ge 2$ and the result is true for $m-1$, i.e., $\sum_{1\le i\le m-1}a_it_i\notin\beta(\lambda)$ if a_i are some non negative integers. Then by Lemma 2.1 we know $\sum_{1\le i\le m}a_it_i=a_mt_m+\sum_{1\le i\le m-1}a_it_i\notin\beta(\lambda)$. □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. $\Rightarrow$: Suppose that $\gcd(t_1,t_2,\dots,t_m)=d>1$. For every $n\in\mathbf{N}$, let λ_n be the partition whose β-set is

$$\beta(\lambda_n)=\{1,1+d,1+2d,\dots,1+nd\}.$$

Then for any $1 \leq i \leq m$ and $0 \leq j \leq n$ such that $1 + jd \geq t_i$, we have $1 + jd - t_i = 1 + j'd \in \beta(\lambda_n)$ for some non negative integer j' since $d \mid t_i$ and $d > 1$. Then by Lemma 2.1, λ_n is a $(t_1, t_2, \ldots, t_m)$-core partition for every $n \in \mathbf{N}$. This means that the number of $(t_1, t_2, \ldots, t_m)$-core partitions is infinite.

$\Leftarrow$: Suppose that $\gcd(t_1, t_2, \ldots, t_m) = 1$ and $1 \leq t_1 < t_2 < \cdots < t_m$. To show that the number of $(t_1, t_2, \ldots, t_m)$-core partitions is finite, we just need to show that for every $(t_1, t_2, \ldots, t_m)$-core partition λ and $x \geq (t_1 - 1)\sum_{2 \leq i \leq m} t_i$, we have $x \notin \beta(\lambda)$:

First we know there exist some $a_i \in \mathbf{Z}$ such that $x = \sum_{1 \leq i \leq m} a_i t_i$ since $\gcd(t_1, t_2, \ldots, t_m) = 1$. Furthermore, we can assume that $0 \leq a_i \leq t_1 - 1$ for $2 \leq i \leq m$ since

$$a_1 t_1 + a_i t_i = (a_1 - b t_i) t_1 + (a_i + b t_1) t_i$$

for every $b \in \mathbf{Z}$. Now we have

$$\sum_{1 \leq i \leq m} a_i t_i = x \geq (t_1 - 1) \sum_{2 \leq i \leq m} t_i$$

and $0 \leq a_i \leq t_1 - 1$ for $2 \leq i \leq m$. It follows that

$$a_1 t_1 = x - \sum_{2 \leq i \leq m} a_i t_i \geq x - (t_1 - 1) \sum_{2 \leq i \leq m} t_i \geq 0.$$

Thus we know $a_1 \geq 0$. Then by Lemma 2.2, we have

$$x = \sum_{1 \leq i \leq m} a_i t_i \notin \beta(\lambda).$$

This means that $x \notin \beta(\lambda)$ if λ is a $(t_1, t_2, \ldots, t_m)$-core partition and $x \geq (t_1 - 1)\sum_{2 \leq i \leq m} t_i$. Now we know for a $(t_1, t_2, \ldots, t_m)$-core partition λ, its β-set $\beta(\lambda)$ must be a subset of $\{1, 2, \ldots, (t_1 - 1)\sum_{2 \leq i \leq m} t_i - 1\}$. This implies that the number of $(t_1, t_2, \ldots, t_m)$-core partitions must be finite. □

3. Main results

Throughout this section, let p be a given positive integer.

Let $S_{t,i} = \{x \in \mathbf{Z} : (i-1)(t+p) + 1 \leq x \leq it - 1\}$. The following result is a characterization of β-sets of $(t, t+1, \ldots, t+p)$-core partitions.

Lemma 3.1. *Suppose that λ is a $(t, t+1, \ldots, t+p)$-core partition. Then*

$$\beta(\lambda) \subseteq \bigcup_{1 \leq i \leq [\frac{t+p-2}{p}]} S_{t,i}.$$

Proof. By Lemma 2.2, we have $\sum_{0 \leq k \leq p} a_k(t+k) \notin \beta(\lambda)$ for non negative integers a_k. Let

$$T_{t,i} = \{\sum_{0 \leq k \leq p} a_k(t+k) : a_k \in \mathbf{Z},\ a_k \geq 0,\ \sum_{0 \leq k \leq p} a_k = i\}.$$

Then $T_{t,i} \bigcap \beta(\lambda) = \emptyset$ for $i \geq 0$. It is easy too see that

$$T_{t,i} = \{x \in \mathbf{Z} : it \leq x \leq i(t+p)\}$$

and

$$\bigcup_{i \geq [\frac{t+p-2}{p}]} T_{t,i} = \{x \in \mathbf{Z} : x \geq [\frac{t+p-2}{p}]t\}$$

since $(i+1)t-1 \leq it+[\frac{t+p-2}{p}]p \leq i(t+p)$ for $i \geq [\frac{t+p-2}{p}]$. Thus $\beta(\lambda)$ must be a subset of

$$\{x \in \mathbf{Z} : 1 \leq x \leq [\frac{t+p-2}{p}]t-1\} \setminus (\bigcup_{1 \leq i \leq [\frac{t+p-2}{p}]-1} \{x \in \mathbf{Z} : it \leq x \leq i(t+p)\}),$$

which equals to $\bigcup_{1\leq i \leq [\frac{t+p-2}{p}]} S_{t,i}$. □

We can define a partial order relation on $\bigcup_{1\leq i \leq [\frac{t+p-2}{p}]} S_{t,i}$. That is, for every $x, y \in \bigcup_{1\leq i \leq [\frac{t+p-2}{p}]} S_{t,i}$, we define $y \preceq x$ if and only if $x - y = \sum_{0 \leq k \leq p} a_k(t+k)$ for some non negative integers a_k. It is easy to verify that $\preceq$ is indeed a partial order relation. We say that a subset S of a partially ordered set T is *good* if for every $x \in S$, $y \in T$ such that $y \preceq x$ in T, we always have $y \in S$.

By the definition of $S_{t,i}$, It is easy to see that

$$S_{t,i} = \{x - (t+k) : x \in S_{t,i+1},\ 0 \leq k \leq p\}$$

for $1 \leq i \leq [\frac{t+p-2}{p}]-1$. Then by Lemma 2.1 and Lemma 3.1 the following result is obvious:

Lemma 3.2. *A partition λ is a $(t, t+1, \ldots, t+p)$-core partition if and only if $\beta(\lambda)$ is a good subset of $\bigcup_{1\leq i \leq [\frac{t+p-2}{p}]} S_{t,i}$.*

Let $R_{t,j}$ be the set of $(t, t+1, \ldots, t+p)$-core partitions whose β-sets contain every positive integer smaller than j but don't contain j. Let $r_{t,j} = \#R_{t,j}$ be the number of elements in $R_{t,j}$.

Now we can give the main result in this paper.

Theorem 3.3. *Suppose that p is a given positive integer. The number f_t of $(t, t+1, \ldots, t+p)$-core partitions is computed recursively by*

$$f_t = 0 \text{ for } t < 0;\ f_0 = 1;\ f_t = \sum_{i=1}^{p-1} f_{t-i} + \sum_{j=0}^{t-p} f_j f_{t-p-j} \text{ for } t \geq 1.$$

The generating function of f_t is

$$\sum_{t\geq 0} f_t x^t = \frac{1 - \sum_{1\leq i\leq p-1} x^i - \sqrt{(1 - \sum_{1\leq i \leq p-1} x^i)^2 - 4x^p}}{2x^p}.$$

Proof. For convenience, let $f_t = 0$ for $t < 0$ and $f_0 = 1$. Now suppose that $t \geq 1$. First we know $r_{t,j} = 0$ for $j \geq t+1$ since $t \notin \beta(\lambda)$ and thus $f_t = \sum_{1\leq j \leq t} r_{t,j}$.

Step 1. We claim that $r_{t,j} = f_{t-j}$ for $1 \leq j \leq p-1$:

Notice that $r_{t,j} = f_{t-j} = 0$ is true if $t+1 \leq j \leq p-1$ since we already assume that $f_t = 0$ for $t < 0$. Now we can assume that $1 \leq j \leq p-1$ and $j \leq t$. Let λ be a partition such that $1, 2, \ldots, j-1 \in \beta(\lambda)$ and $j \notin \beta(\lambda)$. If $\lambda \in R_{t,j}$, i.e., λ is a $(t, t+1, \ldots, t+p)$-core partition, then by Lemma 2.1, we have $x \notin \beta(\lambda)$ for $i \geq 2$ and $(i-1)(t+p)+1 \leq x \leq (i-1)(t+p)+j$ since $j \notin \beta(\lambda)$ and $t \leq t+p+1-j \leq t+p$. Let

$$\begin{aligned} S'_{t,i} &= S_{t,i} \setminus \{x \in \mathbf{Z} : (i-1)(t+p)+1 \leq x \leq (i-1)(t+p)+j\} \\ &= \{x \in \mathbf{Z} : (i-1)(t+p)+j+1 \leq x \leq it-1\}. \end{aligned}$$

Notice that $S'_{t,i}=\emptyset$ when $i>[\frac{t-j+p-2}{p}]$. Thus it is easy to see that

$$\{1,2,\ldots,j-1\}\subseteq\beta(\lambda)\subseteq(\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i})\bigcup\{1,2,\ldots,j-1\}$$

if $\lambda\in R_{t,j}$. We can define a partial order relation on $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}$ induced by the partial order relation $\preceq$ on $\bigcup_{1\leq i\leq[\frac{t+p-2}{p}]}S_{t,i}$. That is, for every two integers x,y in $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}$, we have $y\preceq x$ if and only if $x-y=\sum_{0\leq k\leq p}a_k(t+k)$ for some non negative integers a_k.

Let λ' be a partition such that

$$\beta(\lambda')=\beta(\lambda)\setminus\{1,2,\ldots,j-1\}.$$

By the definition of $S'_{t,i}$, we know for $1\leq i\leq[\frac{t-j+p-2}{p}]-1$,

$$S'_{t,i}=\{x-(t+k):x\in S'_{t,i+1},\ 0\leq k\leq p\}.$$

Then by Lemma 2.1, it is easy to see that $\lambda\in R_{t,j}$ if and only if λ' is a $(t,t+1,\ldots,t+p)$-core partition with $\beta(\lambda')\subseteq\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}$, which is equivalent to $\beta(\lambda')$ is a good subset of $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}$.

Notice that $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S_{t-j,i}$ is a partially ordered set and for every two integers x',y' in $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S_{t-j,i}$, we know $y'\preceq x'$ if and only if $x'-y'=\sum_{0\leq k\leq p}a_k(t-j+k)$ for some non negative integers a_k. Now we can build a function

$$\phi:\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}\to\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S_{t-j,i},$$

that is, for every $x\in S'_{t,i}$, let $\phi(x)=x-ij$. Then it is obvious that ϕ is a bijection. Let $x\in S'_{t,i+1}$ and $y\in S'_{t,i}$. We have $\phi(x)-\phi(y)=x-y-j$. Thus we know $t-j\leq\phi(x)-\phi(y)\leq t-j+p$ if and only if $t\leq x-y\leq t+p$, which implies that $\phi(y)\preceq\phi(x)$ in $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S_{t-j,i}$ if and only if $y\preceq x$ in $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}$. This means that ϕ is an isomorphism of partially ordered sets. Then $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}$ and $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S_{t-j,i}$ has the same number of good subsets and thus by Lemma 3.2 we have $r_{t,j}=f_{t-j}$. We mention that if $j=t\leq p-1$, then $r_{t,t}=f_0=1$ is true since in this case, we have $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S'_{t,i}=\emptyset$ and the empty subset of a partially ordered set is always a good subset.

Step 2. We claim that $r_{t,j}=f_{j-p}f_{t-j}$ for $p\leq j\leq t$:

Let $\lambda\in R_{t,j}$, i.e., λ is a $(t,t+1,\ldots,t+p)$-core partition such that $1,2,\ldots,j-1\in\beta(\lambda)$ and $j\notin\beta(\lambda)$. If $i\geq 0$ and $it+j\leq x\leq i(t+p)+j$, by Lemma 2.1 we have $x\notin\beta(\lambda)$. Let

$$S'_{t,i}=\{x\in\mathbf{Z}:i(t+p)+1\leq x\leq it+j-1\}$$

and

$$S''_{t,i}=\{x\in\mathbf{Z}:(i-1)(t+p)+j+1\leq x\leq it-1\}.$$

Then

$$S'_{t,i}\bigcup S''_{t,i+1}=S_{t,i+1}\setminus\{x\in\mathbf{Z}:it+j\leq x\leq i(t+p)+j\}$$

and

$$S'_{t,0}=\{1,2,\ldots,j-1\}\subseteq\beta(\lambda).$$

Notice that $S'_{t,i}=\emptyset$ when $i>[\frac{j-2}{p}]$ and $S''_{t,i}=\emptyset$ when $i>[\frac{t-j+p-2}{p}]$. Thus it is easy to see

$$\beta(\lambda)\subseteq(\bigcup_{1\leq i\leq[\frac{j-2}{p}]}S'_{t,i})\bigcup(\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S''_{t,i})\bigcup S'_{t,0}.$$

We can define partial order relations on $\bigcup_{1\leq i\leq[\frac{j-2}{p}]}S'_{t,i}$ and $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S''_{t,i}$ induced by the partial order relation $\preceq$ on $\bigcup_{1\leq i\leq[\frac{t+p-2}{p}]}S_{t,i}$ as in Step 1.

Let λ' be the partition such that

$$\beta(\lambda')=(\beta(\lambda)\bigcap(\bigcup_{1\leq i\leq[\frac{j-2}{p}]}S'_{t,i}))\bigcup S'_{t,0}$$

and λ'' be the partition such that

$$\beta(\lambda'')=\beta(\lambda)\bigcap(\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S''_{t,i}).$$

By the definition of $S'_{t,i}$ and $S''_{t,i}$, we know

$$S'_{t,i}=\{x-(t+k):x\in S'_{t,i+1},\ 0\leq k\leq p\}$$

for $0\leq i\leq[\frac{j-2}{p}]-1$ and

$$S''_{t,i}=\{x-(t+k):x\in S''_{t,i+1},\ 0\leq k\leq p\}$$

for $1\leq i\leq[\frac{t-j+p-2}{p}]-1$. Then by Lemma 2.1 it is easy to see that λ' and λ'' are $(t,t+1,\ldots,t+p)$-core partitions since λ is a $(t,t+1,\ldots,t+p)$-core partition. On the other hand, if λ' and λ'' are $(t,t+1,\ldots,t+p)$-core partitions such that

$$S'_{t,0}\subseteq\beta(\lambda')\subseteq(\bigcup_{1\leq i\leq[\frac{j-2}{p}]}S'_{t,i})\bigcup S'_{t,0}$$

and

$$\beta(\lambda'')\subseteq\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S''_{t,i},$$

by Lemma 2.1 we can reconstruct the $(t,t+1,\ldots,t+p)$-core partition $\lambda\in R_{t,j}$ by letting

$$\beta(\lambda)=\beta(\lambda')\bigcup\beta(\lambda''),$$

which implies that

$$(\beta(\lambda)\bigcap(\bigcup_{1\leq i\leq[\frac{j-2}{p}]}S'_{t,i}))\bigcup S'_{t,0}=\beta(\lambda')$$

and

$$\beta(\lambda)\bigcap(\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S''_{t,i})=\beta(\lambda'').$$

Thus the number of $(t,t+1,\ldots,t+p)$-core partitions in $R_{t,j}$ equals to the number of pairs $(\lambda',\ \lambda'')$ such that λ' and λ'' are $(t,t+1,\ldots,t+p)$-core partitions, $S'_{t,0}\subseteq\beta(\lambda')\subseteq(\bigcup_{1\leq i\leq[\frac{j-2}{p}]}S'_{t,i})\bigcup S'_{t,0}$, and $\beta(\lambda'')\subseteq\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S''_{t,i}$, which equals to the product of the number of good subsets of $\bigcup_{1\leq i\leq[\frac{j-2}{p}]}S'_{t,i}$ and the number of good subsets of $\bigcup_{1\leq i\leq[\frac{t-j+p-2}{p}]}S''_{t,i}$ by Lemma 2.1.

First we compute the number of good subsets of $\bigcup_{1\le i\le [\frac{j-2}{p}]} S'_{t,i}$. Notice that for every two integers x', y' in $\bigcup_{1\le i\le [\frac{j-2}{p}]} S_{j-p,i}$, we have $y' \preceq x'$ if and only if $x'-y' = \sum_{0\le k\le p} a_k(j-p+k)$ for some non negative integers a_k. We define a function

$$\phi : \bigcup_{1\le i\le [\frac{j-2}{p}]} S'_{t,i} \to \bigcup_{1\le i\le [\frac{j-2}{p}]} S_{j-p,i}$$

such that for every $x \in S'_{t,i}$, let $\phi(x) = x - i(t+p-j) - j$. Then it is easy to see that ϕ is a bijection. Let $x \in S'_{t,i+1}$ and $y \in S'_{t,i}$. We have $\phi(x) - \phi(y) = x - y - (t+p-j)$. Thus $\phi(y) \preceq \phi(x)$ if and only if $y \preceq x$ since both of them are equivalent to $t \le x - y \le t+p$. This means that ϕ is an isomorphism of partially ordered sets. Then $\bigcup_{1\le i\le [\frac{j-2}{p}]} S'_{t,i}$ and $\bigcup_{1\le i\le [\frac{j-2}{p}]} S_{j-p,i}$ has the same number of good subsets, which equals to f_{j-p} by Lemma 3.2. We mention that $\bigcup_{1\le i\le [\frac{j-2}{p}]} S'_{t,i}$ has f_{j-p} good subsets is true for $j-p=0$ since the empty subset of a partially ordered set is always a good subset and we already assume that $f_0 = 1$.

Next we compute the number of good subsets of $\bigcup_{1\le i\le [\frac{t-j+p-2}{p}]} S''_{t,i}$. Notice that for every two integers x', y' in $\bigcup_{1\le i\le [\frac{t-j+p-2}{p}]} S_{t-j,i}$, we have $y' \preceq x'$ if and only if $x'-y' = \sum_{0\le k\le p} a_k(t-j+k)$ for some non negative integers a_k. We define a function

$$\varphi : \bigcup_{1\le i\le [\frac{t-j+p-2}{p}]} S''_{t,i} \to \bigcup_{1\le i\le [\frac{t-j+p-2}{p}]} S_{t-j,i}$$

such that for every $x \in S''_{t,i}$, let $\varphi(x) = x - ij$. Then it is easy to see that φ is a bijection. Let $x \in S''_{t,i+1}$ and $y \in S''_{t,i}$. We have $\varphi(x) - \varphi(y) = x - y - j$. Thus $\varphi(y) \preceq \varphi(x)$ if and only if $y \preceq x$ since both of them are equivalent to $t \le x-y \le t+p$. This means that φ is an isomorphism of partially ordered sets. Then $\bigcup_{1\le i\le [\frac{t-j+p-2}{p}]} S''_{t,i}$ and $\bigcup_{1\le i\le [\frac{t-j+p-2}{p}]} S_{t-j,i}$ has the same number of good subsets, which equals to f_{t-j} by Lemma 3.2. We mention that $\bigcup_{1\le i\le [\frac{t-j+p-2}{p}]} S''_{t,i}$ has f_{t-j} good subsets is true for $t-j=0$ since the empty subset of a partially ordered set is always a good subset and we already assume that $f_0 = 1$.

Now we have $r_{t,j} = f_{j-p} f_{t-j}$ for $p \le j \le t$ by Lemma 3.2 and prove the claim.

Step 3. Put Step 1 and Step 2 together, we have

$$f_t = \sum_{j=1}^{t} r_{t,j} = \sum_{i=1}^{p-1} f_{t-i} + \sum_{j=p}^{t} f_{j-p} f_{t-j} = \sum_{i=1}^{p-1} f_{t-i} + \sum_{j=0}^{t-p} f_j f_{t-p-j}$$

for $t \ge 1$.

Let $F(x) = \sum_{t\ge 0} f_t x^t$ be the generating function of f_t. Then we have

$$\begin{aligned} F(x) - 1 &= \sum_{t\ge 1} f_t x^t = \sum_{t\ge 1} \Big(\sum_{i=1}^{p-1} f_{t-i} + \sum_{j=0}^{t-p} f_j f_{t-p-j}\Big) x^t \\ &= \sum_{i=1}^{p-1} \sum_{t\ge 1} f_{t-i} x^t + \sum_{t\ge 1} \sum_{j=0}^{t-p} f_j f_{t-p-j} x^t \\ &= \sum_{i=1}^{p-1} x^i F(x) + x^p (F(x))^2. \end{aligned}$$

Then $F(x) = \frac{1-\sum_{1\leq i\leq p-1} x^i - \sqrt{(1-\sum_{1\leq i\leq p-1} x^i)^2 - 4x^p}}{2x^p}$. We finish the proof. □

Suppose that $p = 1$ in Theorem 3.3. We can give a new proof of Anderson's result on the number of (t_1, t_2)-core partitions in [2] for the case $t_2 = t_1 + 1$.

Corollary 3.4. *The number f_t of $(t, t+1)$-core partitions is $f_t = \frac{1}{2t+1}\binom{2t+1}{t}$. The generating function of f_t is*

$$\sum_{t\geq 0} f_t x^t = \frac{1-\sqrt{1-4x}}{2x}.$$

Proof. Let $p = 1$ in Theorem 3.3. We have the generating function of f_t is

$$\sum_{t\geq 0} f_t x^t = \frac{1-\sqrt{1-4x}}{2x}.$$

This is the generating function of Catalan numbers. Then it is easy to see that $f_t = \frac{1}{t+1}\binom{2t}{t} = \frac{1}{2t+1}\binom{2t+1}{t}$. □

Suppose that $p = 2$ in Theorem 3.3. Then it is easy to see that Theorem 1.2 is a direct corollary of Theorem 3.3.

Proof of Theorem 1.2. Let $p = 2$ in Theorem 3.3. We have the generating function of f_t is

$$\sum_{t\geq 0} f_t x^t = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

By $A001006$ in [11] we know this is the generating function of Motzkin numbers. It is well-known that Bernhart [4] proved that the $t-$th Motzkin number equals to $\sum_{0\leq k\leq [\frac{t}{2}]} \frac{1}{k+1}\binom{t}{2k}\binom{2k}{k}$. We finish the proof. □

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