Congruence conditions on the number of terms in sums of consecutive squared integers equal to squared integers

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Abstract

Considering the problem of finding all the integer solutions of the sum of M consecutive integer squares starting at a^2 being equal to a squared integer s^2, it is shown that this problem has no solutions if $M \equiv 3, 5, 6, 7, 8$ or $10 \,(mod\, 12)$ and has integer solutions if $M \equiv 0, 9, 24$ or $33 \,(mod\, 72)$; or $M \equiv 1, 2$ or $16 \,(mod\, 24)$; or $M \equiv 11 \,(mod\, 12)$. All the allowed values of M are characterized using necessary conditions. If M is a square itself, then $M \equiv 1 \,(mod\, 24)$ and $(M-1)/24$ are all pentagonal numbers, except the first two.

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1 Introduction

Lucas stated in 1873 [11] (see also [4]) that $\left(1^2 + ... + n^2\right)$ is a square only for $n =$ 1 and 24. He proposed further in 1875 [12] the well known cannonball problem, namely to find a square number of cannonballs stacked in a square pyramid. This problem can clearly be written as a Diophantine equation $\sum_{i=1}^{M}\left(i^2\right) = M(M+1)(2M+1)/6 = s^2$. The only solutions are $s^2 = 1$ and 4900, which correspond to the sum of the first M squared integers for $M = 1$ and $M = 24$. This was partially proven by Moret-Blanc [15] and Lucas [13], and entirely proven later on by Watson [24] (with elementary proofs in most cases and using elliptic functions for one case), Ljunggren [10], Ma [14] and Anglin [2] (both with only elementary proofs).

A more general problem is to find all values of a for which the sum of the M consecutive integer squares starting from $a^2 \geq 1$ is itself an integer square s^2. Different approaches have been proposed to solve this problem. Alfred studied [1] several necessary conditions on the values of M (with the notations of this paper), finding that $M = 2, 11, 23, 24, 26, ...$ until $M = 500$ by studying basic congruence equations of M, without being able to conclude if there were solutions for $M = 107, 193, 227, 275, 457$. This was further addressed by Philipp [16] who showed that solutions exist for $M = 107, 193, 457$ but not for $M = 227, 275$, and proving that there are a finite or an infinite number of solutions depending on whether M is or not a square integer. Laub showed [8] that the set of values of M yielding the sum of M consecutive squared integers being a squared integer, is infinite and has density zero. Beeckmans demonstrated [3] eight necessary conditions on M and gave a list of values of $M < 1000$ with the corresponding smallest value of $a > 0$, indicating two cases for $M = 25$ and 842 complying with the eight necessary conditions but not providing solutions to the problem.

In this paper, the method of determining the set of allowed values of M that yield the sum of M consecutive squared integers to be a squared integer is extended by expressing congruent $(mod\,12)$ conditions on M using Beeckmans' necessary conditions [3], showing that M cannot be congruent $(mod\,12)$ to $3, 5, 6, 7, 8$ or 10 and must be congruent $(mod\,12)$ to $0, 1, 2, 4, 9$ or 11, yielding that $M \equiv 0, 9, 24$ or $33\,(mod\,72)$; or $M \equiv 1, 2$ or $16\,(mod\,24)$; or $M \equiv 11\,(mod\,12)$. It is shown also that if M is a square itself, then M must be congruent to $1\,(mod\,24)$ and $(M-1)/24$ are all pentagonal numbers, except the first two.

Throughout the paper, the notation $A\,(mod\,B) \equiv C$ is equivalent to $A \equiv C\,(mod\,B)$ and $A \equiv C\,(mod\,B) \Rightarrow A = Bk + C$ means that, if $A \equiv C\,(mod\,B)$, then $\exists k \in \mathbb{Z}^+$ such that $A = Bk + C$. By convention, $\sum_{j=inf}^{sup} f(j) = 0$ if $sup < inf$.

2 Congruent $(mod\,12)$ values of M

A first theorem specifies the congruent $(mod\,12)$ values that M cannot take. In the demonstration of this theorem, several numerical series are encountered and the following lemma shows that these series take integer values for the indicated conditions.

Lemma 1. *For $n, \alpha \in \mathbb{Z}^+$ and $i, \delta \in \mathbb{Z}^*$:*
(i) $\left[\left(3^{2(n-1)} - 1\right)/4\right]$ *and* $\left[\left(3^{2n-1} + 1\right)/4\right] \in \mathbb{Z}^*$, $\forall n$;
furthermore, $\left[\left(3^{2n-1} + 1\right)/4\right] \equiv 1$ *or* $3\,(mod\,4)$ *for* $n \equiv 1$ *or* $0\,(mod\,2)$*;*
(ii) $\left[\left(3^{2n-1} - 2^{\alpha} + 1\right)/12\right] = \left[2\left(\sum_{i=0}^{n-2} 3^{2i}\right) - \left(2^{\alpha-2} - 1\right)/3\right] \in \mathbb{Z}^+$, $\forall n \geq 2$, $\forall \alpha \equiv 0\,(mod\,2)$*, and*
$\left[\left(3^{2n-1} - 5 \times 2^{\alpha} + 1\right)/12\right] = \left[2\left(\sum_{i=0}^{n-2} 3^{2i}\right) - \left(5 \times 2^{\alpha-2} - 1\right)/3\right] \in \mathbb{Z}^+$
$\forall n \geq 2$, $\forall \alpha \equiv 1\,(mod\,2)\,, \alpha > 1$*;*
(iii) $\left[\left(3^{2n-1} \times 13 + 1\right)/16\right] \in \mathbb{Z}^+$, $\forall n \equiv 0\,(mod\,2)$*;*
$\left[\left(3^{2n-1} \times 37 + 1\right)/16\right] \in \mathbb{Z}^+$, $\forall n \equiv 1\,(mod\,2)$*;*
$\left[\left(3^{2n-1} \times 25 - 11\right)/16\right] \in \mathbb{Z}^+$, $\forall n \equiv 1\,(mod\,2)$*;*
(iv) $\left[\left(3^{2n-1}\left(13 + 24\delta\right) - 23\right)/32\right] \in \mathbb{Z}^+$ *for* $\delta = 0$, $\forall n \equiv 3\,(mod\,4)$*; for* $\delta = 1$,

$\forall n \equiv 0\,(mod\,4)$; for $\delta = 2$, $\forall n \equiv 1\,(mod\,4)$; and for $\delta = 3$, $\forall n \equiv 2\,(mod\,4)$.

Proof. For $n, n', \alpha \in \mathbb{Z}^+$ and $i, \delta \in \mathbb{Z}^*$
(i) immediate as $\forall n$, $3^{2(n-1)} \equiv 1\,(mod\,8)$ and $3^{2n-1} \equiv 3\,(mod\,4)$ ([1]). Furthermore,
(i.1) if $n \equiv 1\,(mod\,2) \Rightarrow n = 2n'+1$, assume that $\left[\left(3^{2n-1}+1\right)/4\right] \equiv 1\,(mod\,4)$, then $\left(3^{2n-1}+1\right) \equiv 4\,(mod\,16) \Rightarrow \left(3\left(3^{4n'}-1\right)\right) \equiv 0\,(mod\,16)$, which is the case as $\forall n'$, $\left(3^{2n'}+1\right) \equiv 0\,(mod\,2)$ and $\left(3^{2n'}-1\right) \equiv 0\,(mod\,8)$;
(i.2) if now $n \equiv 0\,(mod\,2) \Rightarrow n = 2n'$, assume that $\left[\left(3^{2n-1}+1\right)/4\right] \equiv 3\,(mod\,4)$, then $\left(3^{2n-1}+1\right) \equiv 12\,(mod\,16) \Rightarrow \left(3\left(3^{4n'-2}-1\right)\right) \equiv 8\,(mod\,16)$, which is the case as $\forall n'$, $\left(3^{2n'-1}+1\right) \equiv 0\,(mod\,4)$ and $\left(3^{2n'-1}-1\right) \equiv 0\,(mod\,2)$.
(ii) As $\forall n \geq 2$, $\left[\left(3^{2(n-1)}-1\right)/8\right] = \sum_{i=0}^{n-2} 3^{2i}$ ([2]), then:

$$\begin{aligned}\left(\frac{3^{2n-1}-2^{\alpha}+1}{12}\right) &= \left(\frac{3^{2n-1}-3}{12}\right) - \left(\frac{2^{\alpha}-4}{12}\right) \\ &= 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) - \left(\frac{2^{\alpha-2}-1}{3}\right) \in \mathbb{Z}^+ \qquad (1)\end{aligned}$$

as $\forall \alpha \equiv 0\,(mod\,2)$, $2^{\alpha-2} \equiv 1\,(mod\,3)$, and

$$\begin{aligned}\left(\frac{3^{2n-1}-5\times 2^{\alpha}+1}{12}\right) &= \left(\frac{3^{2n-1}-3}{12}\right) - \left(\frac{5\times 2^{\alpha}-4}{12}\right) \\ &= 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) - \left(\frac{5\times 2^{\alpha-2}-1}{3}\right) \in \mathbb{Z}^+ \qquad (2)\end{aligned}$$

as $\forall \alpha \equiv 1\,(mod\,2), \alpha > 1$, $2^{\alpha-2} \equiv 2\,(mod\,3) \Rightarrow \left(5\times 2^{\alpha-2}\right) \equiv 1\,(mod\,3)$.
(iii) Immediate as $\forall n \equiv 0\,(mod\,2)$, $3^{2n} \equiv 1\,(mod\,16)$ and $\forall n \equiv 1\,(mod\,2)$, $3^{2n-1} \equiv 3\,(mod\,16) \Rightarrow \left(3^{2n-1}\times 5\right) \equiv 15\,(mod\,16)$, yielding:
(iii.1) $\left(3^{2n-1}\times 13+1\right)(mod\,16) \equiv \left(-3^{2n}+1\right)(mod\,16) \equiv 0$;
(iii.2) $\left(3^{2n-1}\times 37+1\right)(mod\,16) \equiv \left(3^{2n-1}\times 5+1\right)(mod\,16) \equiv 0$;
(iii.3) $\left(3^{2n-1}\times 25-11\right)(mod\,16) \equiv \left(5\left(3^{2n-1}\times 5+1\right)\right)(mod\,16) \equiv 0$.
(iv) As $\left(3^{2n-1}(13+24\delta)-23\right)(mod\,32) \equiv 3^2\left(3^{2n-3}(13+24\delta)+1\right)(mod\,32)$,
$\delta = 0$: $3^2\left(3^{2n-3}\times 13+1\right)(mod\,32) \equiv 0$ as $\forall n \equiv 3\,(mod\,4)$, $3^{2n-3} \equiv 27\,(mod\,32)$ $\Rightarrow \left(3^{2n-3}\times 13\right) \equiv 31\,(mod\,32)$;
$\delta = 1$: $3^2\left(3^{2n-3}\times 37+1\right)(mod\,32) \equiv 3^2\left(3^{2n-3}\times 5+1\right)(mod\,32) \equiv 0$ as $\forall n \equiv 0\,(mod\,4)$, $3^{2n-3} \equiv 19\,(mod\,32) \Rightarrow \left(3^{2n-3}\times 5\right) \equiv 31\,(mod\,32)$;
$\delta = 2$: $3^2\left(3^{2n-3}\times 61+1\right)(mod\,32) \equiv 3^2\left(-3^{2n-2}+1\right)(mod\,32) \equiv 0$ as $\forall n \equiv 1\,(mod\,4)$, $3^{2n-2} \equiv 1\,(mod\,32)$;
$\delta = 3$: $3^2\left(3^{2n-3}\times 85+1\right)(mod\,32) \equiv 3^2\left(3^{2n-2}\times 7+1\right)(mod\,32) \equiv 0$ as $\forall n \equiv 2\,(mod\,4)$, $3^{2n-2} \equiv 9\,(mod\,32) \Rightarrow \left(3^{2n-2}\times 7\right) \equiv 31\,(mod\,32)$. □

[1] For increasing n, the series $\left[\left(3^{2n-1}+1\right)/4\right] = 1, 7, 61, 547, 4921, ...$ is given in [9].
[2] For increasing n, the series $\left[\left(3^{2(n-1)}-1\right)/8\right] = 1, 10, 91, 820, 7381, ...$ is given in [7, 20].

The following theorem can now be demonstrated with the eight necessary conditions given by Beeckmans [3] on the value of M for (24) to hold, that can be summarized as follows, with the notations of this paper and where $e, \alpha \in \mathbb{Z}^+$:
1) If $M \equiv 0 \,(mod\, 2^e)$ or if $M \equiv 0 \,(mod\, 3^e)$ or if $M \equiv -1 \,(mod\, 3^e)$, then $e \equiv 1 \,(mod\, 2)$; (C1.1, C1.2, C1.3)
2) If $p > 3$ is prime, $M \equiv 0 \,(mod\, p^e)$, $e \equiv 1 \,(mod\, 2)$, then $p \equiv \pm 1 \,(mod\, 12)$; (C2)
3) If $p \equiv 3 \,(mod\, 4)\,, p > 3$ is prime, $M \equiv -1 \,(mod\, p^e)$, then $e \equiv 0 \,(mod\, 2)$; (C3)
4) $M \neq 3 \,(mod\, 9)$, $M \neq (2^\alpha - 1) \left(mod\, 2^{\alpha+2}\right)$ and $M \neq 2^\alpha \left(mod\, 2^{\alpha+2}\right)$ $\forall \alpha \geq 2$. (C4.1, C4.2, C4.3)

Theorem 2. *For $M > 1, \in \mathbb{Z}^+$, the sum of squares of M consecutive integers cannot be an integer square if $M \equiv 3, 5, 6, 7, 8$ or $10 \,(mod\, 12)$.*

The demonstration is made in the order $M \equiv 5, 7, 6, 10, 8$ and $3 \,(mod\, 12)$.

Proof. For $M, \mu, i, k, K, m, m_i, e_i, p_i, n, \alpha, \beta, \epsilon, \gamma_n, \kappa, \xi, A, B \in \mathbb{Z}^+, \eta \in \mathbb{Z}$, $M > 1$, $3 \leq \mu \leq 10$, let $M \equiv \mu \,(mod\, 12) \Rightarrow M = 12m + \mu$.
(i) For $\mu = 5$ or 7, $M = 12m+5$ or $12m+7$, let $\prod (p_i^{e_i})$ be the decomposition of M in i prime factors p_i, with $\prod (p_i^{e_i}) \equiv 5$ or $7 \,(mod\, 12)$. Then one of the prime factors is $p_j \equiv 5$ or $7 \,(mod\, 12)$ with an exponent $e_j \equiv 1 \,(mod\, 2)$ (the remaining co-factor is $\left(\prod (p_i^{e_i}) / p_j^{e_j}\right) \equiv 1$ or $11 \,(mod\, 12)$), contradicting (C2) and these values of M must be rejected.
(ii) For $\mu = 6$ or 10, $M = 12m + 6$ or $12m + 10$, $M + 1 = 4\,(3m + 1) + 3$ or $4\,(3m + 2) + 3$, i.e. in both cases $(M + 1) \equiv 3 \,(mod\, 4)$. Let $\prod (p_i^{e_i})$ be the decomposition of $(M + 1)$ in i prime factors p_i. Then one of the prime factors is $p_j \equiv 3 \,(mod\, 4)$ with an exponent $e_j \equiv 1 \,(mod\, 2)$ (the remaining co-factor being $\left(\prod (p_i^{e_i}) / p_j^{e_j}\right) \equiv 1 \,(mod\, 4)$), contradicting (C3).
(iv) For $\mu = 8$, $M = 12m + 8$ and $M + 1 = 3\,(4m + 3)$, cases appear cyclically with values of $(M + 1)$ having either a factor 3 with an even exponent or a factor f such as $f \equiv 3 \,(mod\, 4)$. Indeed, let first $m \neq 0 \,(mod\, 3)$ and second $m \equiv 0 \,(mod\, 3) \Rightarrow m = 3m_1$. Let then first $m_1 \neq 2 \,(mod\, 3)$ and second $m_1 \equiv 2 \,(mod\, 3) \Rightarrow m_1 = 3m_2 + 2$. Let then again first $m_2 \neq 0 \,(mod\, 3)$ and second $m_2 \equiv 0 \,(mod\, 3) \Rightarrow m_2 = 3m_3$, and so on, yielding:

$M + 1 = 3\,(4m + 3)$,
$\Rightarrow$ if $m \neq 0 \,(mod\, 3) \Rightarrow (M + 1) \equiv 3 \,(mod\, 4)$,
$\Rightarrow$ if $m \equiv 0 \,(mod\, 3) \Rightarrow m = 3m_1 \Rightarrow M + 1 = 3^2\,(4m_1 + 1)$,
$\Rightarrow$ if $m_1 \neq 2 \,(mod\, 3) \Rightarrow (M + 1) \equiv 0 \left(mod\, 3^2\right)$,
$\Rightarrow$ if $m_1 \equiv 2 \,(mod\, 3) \Rightarrow m_1 = 3m_2 + 2 \Rightarrow M + 1 = 3^3\,(4m_2 + 3)$,
$\Rightarrow$ if $m_2 \neq 0 \,(mod\, 3) \Rightarrow (M + 1) \equiv 3 \,(mod\, 4)$,
$\Rightarrow$ if $m_2 \equiv 0 \,(mod\, 3) \Rightarrow m_2 = 3m_3 \Rightarrow M + 1 = 3^4\,(4m_3 + 1)$,

and so on. After n iterations, $\exists m_n \in \mathbb{Z}^+$ such as either $M + 1 = 3^n\,(4m_n + 1)$ if $n \equiv 0 \,(mod\, 2)$, contradicting (C1.3), or $M + 1 = 3^n\,(4m_n + 3)$ if $n \equiv 1 \,(mod\, 2)$. Then let $\prod (p_i^{e_i})$ be the decomposition of $[(M + 1) / 3^n]$ in i prime factors p_i, with $\prod (p_i^{e_i}) \equiv 3 \,(mod\, 4)$. Then one of the prime factors is $p_j \equiv 3 \,(mod\, 4)$ with an exponent e_j such as $e_j \equiv 1 \,(mod\, 2)$ (the remaining co-factor being such as $\left(\prod (p_i^{e_i}) / p_j^{e_j}\right) \equiv 1 \,(mod\, 4)$), contradicting (C3). Therefore, these values of M must be rejected in both cases.

(v) For $\mu = 3$, $M = 3\,(4m+1)$, cases appear cyclically with values of M being the product of a power of 3 and a factor which is $(mod\,12)$ congruent to either $1, 5, 7$ or 11.
(v.1) Let m be successively $(mod\,3)$ congruent to $0, 1$ and 2, and the $m \equiv 2\,(mod\,3)$ step is subdivided in $m \equiv 5, 8$ and $2\,(mod\,9)$ sub-steps; the process is then repeated, yielding respectively:
$M = 3\,(4m+1)$
$\Rightarrow$ if $m = 3m_1 \Rightarrow M = 3\,(12m_1+1)$,
$\Rightarrow$ if $m = 3m_1+1 \Rightarrow M = 3\,(12m_1+5)$,
$\Rightarrow$ if $m = 3^2m_1+5 \Rightarrow M = 3^2\,(12m_1+7)$,
$\Rightarrow$ if $m = 3^2m_1+8 \Rightarrow M = 3^2\,(12m_1+11)$,
$\Rightarrow$ if $m = 3^2m_1+2 \Rightarrow M = 3^3\,(4m_1+1)$,
$\Rightarrow$ if $m_1 = 3m_2 \Rightarrow m = 3^3m_2+2 \Rightarrow M = 3^3\,(12m_2+1)$,
$\Rightarrow$ if $m_1 = 3m_2+1 \Rightarrow m = 3^3m_2+11 \Rightarrow M = 3^3\,(12m_2+5)$,
$\Rightarrow$ if $m_1 = 3^2m_2+5 \Rightarrow m = 3^4m_2+47 \Rightarrow M = 3^4\,(12m_2+7)$,
$\Rightarrow$ if $m_1 = 3^2m_2+8 \Rightarrow m = 3^4m_2+74 \Rightarrow M = 3^4\,(12m_2+11)$,
$\Rightarrow$ if $m_1 = 3^2m_2+2 \Rightarrow m = 3^4m_2+20 \Rightarrow M = 3^5\,(4m_2+1)$.
Taking again $(mod\,3)$ and $(mod\,9)$ congruent values of m_2 yield new expressions of M as a product of a power of 3 and a factor $(mod\,12)$ congruent to either 1,5,7 or 11. One obtains then after n iterations, with $\left[\left(3^{2(n-1)}-1\right)/4\right] \in \mathbb{Z}^+$ (see Lemma 1):
if $m = 3^{2n-1}m_n + \left[\left(3^{2(n-1)}-1\right)/4\right] \Rightarrow M = 3^{2n-1}\,(12m_n+1)$,
if $m = 3^{2n-1}m_n + \left[3^{2(n-1)} + \left(3^{2(n-1)}-1\right)/4\right] \Rightarrow M = 3^{2n-1}\,(12m_n+5)$,
if $m = 3^{2n}m_n + \left[5\times 3^{2(n-1)} + \left(3^{2(n-1)}-1\right)/4\right] \Rightarrow M = 3^{2n}\,(12m_n+7)$,
if $m = 3^{2n}m_n + \left[8\times 3^{2(n-1)} + \left(3^{2(n-1)}-1\right)/4\right] \Rightarrow M = 3^{2n}\,(12m_n+11)$.
(v.2) For $M = 3^{2n-1}\,(12m_n+5)$, let $\prod\left(p_i^{e_i}\right)$ be the decomposition of $\left(M/3^{2n-1}\right)$ in i prime factors p_i, with $\prod\left(p_i^{e_i}\right) \equiv 5\,(mod\,12)$. Then one of the prime factors is either $p_j \equiv 5$ or $7\,(mod\,12)$ (the remaining co-factor being respectively either $\left(\prod\left(p_i^{e_i}\right)/p_j\right) \equiv 1$ or $11\,(mod\,12)$), contradicting (C2).
(v.3) For $M = 3^{2n}\,(12m_n+7)$ and $M = 3^{2n}\,(12m_n+11)$, both contradict (C1.2) as $(12m_n+7)$ and $(12m_n+11)$ cannot be $(mod\,3)$ congruent to 0.
(v.4) For $M = 3^{2n-1}\,(12m_n+1)$, if $n=1$, $M = 3\,(12m_1+1)$ contradicts (C4.1). For $n \geq 2$, (C4.2) is used first in (v.4.1) to reject some values of M, then (C.3) is used in (v.4.2) to reject those values of M that were not rejected by (C4.2).
(v.4.1) Condition (C4.2) for $M \equiv 3\,(mod\,12)$ and $\alpha \geq 2$ yields
$M \neq (2^\alpha - 1)\left(mod\left(3\times 2^{\alpha+2}\right)\right)$ if $2^\alpha \equiv 1\,(mod\,3)$, i.e. $\alpha \equiv 0\,(mod\,2)$, and
$M \neq (5\times 2^\alpha - 1)\left(mod\left(3\times 2^{\alpha+2}\right)\right)$ if $2^\alpha \not\equiv 1\,(mod\,3)$, i.e. $\alpha \equiv 1\,(mod\,2)$.
Those values of m_n yielding $M = 3^{2n-1}\,(12m_n+1)$ to be rejected are

$$\left(3^{2n-1}m_n\right) \equiv -\beta\,(mod\,2^\alpha) \tag{3}$$

with, for $\alpha \equiv 0\,(mod\,2)$ and Lemma 1,

$$\beta = \left(\frac{3^{2n-1}-2^\alpha+1}{12}\right) = 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) - \left(\frac{2^{\alpha-2}-1}{3}\right) \tag{4}$$

Table 1: Values of m_{n0}

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$
$\alpha=2$	2	0	2	0	2
$\alpha=3$	3	5	7	1	3
$\alpha=4$	13	15	1	3	5
$\alpha=5$	17	3	5	23	25
$\alpha=6$	57	11	13	63	33
$\alpha=7$	73	27	93	15	49
$\alpha=8$	105	59	253	47	81

and, for $\alpha \equiv 1\,(mod\,2)$ and Lemma 1,

$$\beta = \left(\frac{3^{2n-1} - 5 \times 2^{\alpha} + 1}{12}\right) = 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) - \left(\frac{5 \times 2^{\alpha-2} - 1}{3}\right) \tag{5}$$

Then the values of m_n yielding $M = 3^{2n-1}\,(12m_n + 1)$ to be rejected are

$$m_n \equiv m_{n0}\,(mod\,2^{\alpha}) \Rightarrow m_n = 2^{\alpha} i + m_{n0} \tag{6}$$

where m_{n0} is the smallest value of m_n for (3) to hold, i.e.$\exists m_{n0} \in \mathbb{Z}^*$ such as

$$K = \left(\frac{3^{2n-1} m_{n0} + \beta}{2^{\alpha}}\right) \in \mathbb{Z}^+ \tag{7}$$

Table 1 shows the first values of m_{n0}. For $\alpha = 2$ and $\alpha = 3$, the values of m_{n0} repeat themselves. Taking the $(mod\,2^{\alpha})$ congruence of 3^{2n-1} and β in (7) yield $\left[\left(\left(3^{2n-1}\,(mod\,2^{\alpha})\right) m_{n0} + \beta\,(mod\,2^{\alpha})\right)/2^{\alpha}\right] \in \mathbb{Z}^+$, meaning that for $\alpha = 2$ and 3 and $\forall n$, $3^{2n-1} \equiv 3\,(mod\,4)$ and $3\,(mod\,8)$. Furthermore, from (4), for $\alpha = 2$, $\beta = 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) \equiv 2$ or $0\,(mod\,4)$ for $n \equiv 0$ or $1\,(mod\,2)$, while for $\alpha = 3$, $\beta = 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) - 3 \equiv 7, 1, 3$ or $5\,(mod\,8)$ respectively for $n \equiv 2, 3, 0$ or $1\,(mod\,4)$. Therefore, the values of m_{n0} for $\alpha = 2$ and $\alpha = 3$ appear cyclically, respectively $m_{n0} = 2$ and 0 for $n \equiv 0$ and $1\,(mod\,2)$, and $m_{n0} = 3, 5, 7$ and 1 for $n \equiv 2, 3, 0$ and $1\,(mod\,4)$.

(v.4.2) Those values of $M = 3^{2n-1}\,(12m_n + 1)$ with $n \geq 2$ that are not rejected by (C4.2) in the previous section (v.4), can be rejected by (C3). It is sufficient to show as above that $(M + 1)$ has a factor f such as $f \equiv 3\,(mod\,4)$, as the decomposition of f in product of prime factors includes then a prime factor $p_j^{e_j}$ such as $p_j^{e_j} \equiv 3\,(mod\,4)$ with $e_j \equiv 1\,(mod\,2)$. One has then generally

$$M + 1 = 3^{2n-1}\,(12m_n + 1) + 1 = 4\left(3^{2n} m_n + \frac{3^{2n-1} + 1}{4}\right) \tag{8}$$

with $\left[\left(3^{2n-1} + 1\right)/4\right] \equiv 1$ or $3\,(mod\,4)$ for $n \equiv 1$ or $0\,(mod\,2)$ (see Lemma 1). Let then $\left[\left(3^{2n-1} + 1\right)/4\right] = 4\gamma_n + 1$ or $4\gamma_n + 3$ for $n \equiv 1$ or $0\,(mod\,2)$.

(v.4.2.1) For an even number of iterations, i.e. $n \equiv 0\,(mod\,2)$, as the values of m_n from (6) yielding $M = 3^{2n-1}(12m_n + 1)$ to be rejected for $\alpha = 2$ are $m_n \equiv 2\,(mod\,4)$, let us show that $M = 3^{2n-1}(12m_n + 1)$ can also be rejected by (C.3) for $m_n \equiv 0, 3$ and $1\,(mod\,4)$.
(v.4.2.1.1) Let first $m_n \equiv 0\,(mod\,4) \Rightarrow m_n = 4m'_n$ and (8) yields $M + 1 = 4\left[4\left(3^{2n}m'_n + \gamma_n\right) + 3\right]$, contradicting (C3).
(v.4.2.1.2) Let now $m_n \equiv 3\,(mod\,4)$ and two cases are considered.
First, as the values of m_n to be rejected for $\alpha = 3$ and $\forall n \equiv 0\,(mod\,4)$ are $m_n \equiv 7\,(mod\,8)$, let $m_n \equiv 3\,(mod\,8) \Rightarrow m_n = 8m'_n + 3$, yielding with Lemma 1,

$$\begin{aligned} M+1 &= 4\left(3^{2n} \times 8m'_n + 3^{2n+1} + \frac{3^{2n-1}+1}{4}\right) \\ &= 8\left[4\left(3^{2n}m'_n + \frac{3^{2n-1} \times 37 - 23}{32}\right) + 3\right] \end{aligned} \tag{9}$$

contradicting again (C3).
Second, as the values of m_n to be rejected for $\alpha = 3$ and $\forall n \equiv 0\,(mod\,4)$ are $m_n \equiv 3\,(mod\,8)$, let $m_n \equiv 7\,(mod\,8) \Rightarrow m_n = 8m'_n + 7$, yielding with Lemma 1,

$$\begin{aligned} M+1 &= 4\left(3^{2n} \times 8m'_n + 7 \times 3^{2n} + \frac{3^{2n-1}+1}{4}\right) \\ &= 8\left[4\left(3^{2n}m'_n + \frac{3^{2n-1} \times 85 - 23}{32}\right) + 3\right] \end{aligned} \tag{10}$$

contradicting again (C3).
(v.4.2.1.3) Let now $m_n \equiv 1\,(mod\,4)$ and consider more generally the case

$$m_n = 4\left(2^{\kappa}m'_n + \xi\right) + 1 = 2^{\kappa+2}m'_n + (4\xi + 1) \tag{11}$$

with $\kappa \geq 2$ and $0 \leq \xi \leq 2^{\kappa} - 1$, yielding from (8)

$$M + 1 = 2^{\kappa+2}\left[4\left(3^{2n}m'_n + A\right) + 3\right] \tag{12}$$

with $A = \left(3^{2n-1}(48\xi + 13) - \left(3 \times 2^{\kappa+2} - 1\right)\right)/2^{\kappa+4}$. The values of $\xi \in \mathbb{Z}^+$ that renders $A \in \mathbb{Z}^+$ $\forall \kappa \geq 2$ and $n \equiv 0\,(mod\,2)$ are, with Lemma 1,

$$\xi = \left((A+3)\,2^{\kappa-2} - \left(\frac{3^{2n-1} \times 13 + 1}{16}\right)\right)3^{-2n} \tag{13}$$

and shown in Table 2. These values of ξ can be represented as polynomials in $n' = n/2$ with the independent term (i.e. the value of ξ for $n \equiv 0\,\left(mod\,2^{\kappa+1}\right)$) being either $\sum_{i=0}^{(\kappa-4)/2}\left(2^{2i}\right)$ if $\kappa \equiv 0\,(mod\,2)$ or $\left(2^{\kappa-2} + \sum_{i=0}^{(\kappa-3)/2}\left(2^{2i}\right)\right)$ if $\kappa \equiv 1\,(mod\,2)$. The coefficients c_i of the powers of n' in the polynomials $P(n') = \sum c_i n'^i$ can be chosen to fit the congruence $\xi \equiv P(n')\,(mod\,2^{\kappa})$ and the polynomials with the smallest $c_i \in \mathbb{Z}^+$ are shown in Table 3. Replacing these values of ξ in $m_n = 2^{\kappa+2}m'_n + (4\xi + 1)$ yields $(M + 1)$ to have a factor f such as

Table 2: Values of ξ for $0 \leq n < 32$ with $n \equiv 0\,(mod\,2)$ and $2 \leq \kappa \leq 10$

n	κ								
	2	3	4	5	6	7	8	9	10
0	0	3	1	13	5	53	21	213	85
2	1	0	6	2	58	42	138	74	970
4	2	5	11	7	31	15	111	47	943
6	3	2	0	28	52	100	196	388	260
8	0	7	5	1	57	41	137	329	201
10	1	4	10	22	46	94	190	126	1022
12	2	1	15	27	19	3	99	35	931
14	3	6	4	16	40	24	120	312	184
16	0	3	9	21	45	29	253	189	61
18	1	0	14	10	34	18	242	434	818
20	2	5	3	15	7	119	87	279	663
22	3	2	8	4	28	76	44	492	876
24	0	7	13	9	33	17	113	305	689
26	1	4	2	30	22	70	38	486	358
28	2	1	7	3	59	107	75	267	139
30	3	6	12	24	16	0	224	416	288

Table 3: Polynomials $P(n')$ such as $\xi \equiv P(n')\,(mod\,2^{\kappa})$ with $n' = n/2$

κ	$P(n')\,(mod\,2^{\kappa})$
2	$(n')\,(mod\,4)$
3	$(5n'+3)\,(mod\,8)$
4	$(5n'+1)\,(mod\,16)$
5	$\left(8n'^2+13n'+13\right)(mod\,32)$
6	$\left(24n'^2+29n'+15\right)(mod\,64)$
7	$\left(56n'^2+61n'+53\right)(mod\,128)$
8	$\left(56n'^2+61n'+21\right)(mod\,256)$
9	$\left(384n'^3+440n'^2+61n'+213\right)(mod\,512)$
10	$\left(384n'^3+952n'^2+573n'+85\right)(mod\,1024)$

$f \equiv 3\,(mod\,4)$, contradicting again (C3). Therefore, all values of $n \equiv 0\,(mod\,2)$ yield M to be rejected.
(v.4.2.2) For an odd number of iterations, i.e. $n \equiv 1\,(mod\,2)$, as the values of m_n from (6) yielding $M = 3^{2n-1}\,(12m_n + 1)$ to be rejected for $\alpha = 2$ are $m_n \equiv 0\,(mod\,4)$, let us show that $M = 3^{2n-1}\,(12m_n + 1)$ can also be rejected by (C.3) for $m_n \equiv 2, 1$ and $3\,(mod\,4)$.
(v.4.2.2.1) Let first $m_n \equiv 2\,(mod\,4) \Rightarrow m_n = 4m'_n + 2$ and (8) yields then

$$\begin{aligned} M+1 &= 4\left(3^{2n}m_n + \gamma_n + 1\right) \\ &= 4\left[4\left(3^{2n}m'_n + \frac{3^{2n-1}\times 25 - 11}{16}\right) + 3\right] \end{aligned} \tag{14}$$

with Lemma 1, contradicting again (C3).
(v.4.2.2.2) Let now $m_n \equiv 1\,(mod\,4)$ and two cases are again considered.
First, as the values of m_n to be rejected for $\alpha = 3$ and $\forall n \equiv 3\,(mod\,4)$ are $m_n \equiv 5\,(mod\,8)$, let $m_n \equiv 1\,(mod\,8) \Rightarrow m_n = 8m'_n + 1$, yielding with Lemma 1,

$$\begin{aligned} M+1 &= 4\left(3^{2n}\times 8m'_n + 3^{2n} + \frac{3^{2n-1}+1}{4}\right) \\ &= 8\left[4\left(3^{2n}m'_n + \frac{3^{2n-1}\times 13 - 23}{32}\right) + 3\right] \end{aligned} \tag{15}$$

contradicting again (C3).
Second, as the values of m_n to be rejected for $\alpha = 3$ and $\forall n \equiv 1\,(mod\,4)$ are $m_n \equiv 1\,(mod\,8)$, let $m_n \equiv 5\,(mod\,8) \Rightarrow m_n = 8m'_n + 5$, yielding with Lemma 1,

$$\begin{aligned} M+1 &= 4\left(3^{2n}\times 8m'_n + 5\times 3^{2n} + \frac{3^{2n-1}+1}{4}\right) \\ &= 8\left[4\left(3^{2n}m'_n + \frac{3^{2n-1}\times 61 - 23}{32}\right) + 3\right] \end{aligned} \tag{16}$$

contradicting again (C3).
(v.4.2.2.3) Let now $m_n \equiv 3\,(mod\,4)$ and consider more generally the case

$$m_n = 4\left(2^\kappa m'_n + \xi\right) + 3 = 2^{\kappa+2}m'_n + (4\xi + 3) \tag{17}$$

with $\kappa \geq 2$ and $0 \leq \xi \leq 2^\kappa - 1$, yielding

$$M + 1 = 2^{\kappa+2}\left[4\left(3^{2n}m'_n + B\right) + 3\right] \tag{18}$$

with $B = \left(3^{2n-1}\,(48\xi + 37) - \left(3\times 2^{\kappa+2} - 1\right)\right)/2^{\kappa+4}$. The values of $\xi \in \mathbb{Z}^+$ that renders $B \in \mathbb{Z}^+$ $\forall\kappa \geq 2$ and $n \equiv 1\,(mod\,2)$ are, with Lemma 1,

$$\xi = \left((B+3)\,2^{\kappa-2} - \left(\frac{3^{2n-1}\times 37 + 1}{16}\right)\right)3^{-2n} \tag{19}$$

and shown in Table 4. These values of ξ can be represented as polynomials

Table 4: Values of ξ for $0 < n < 32$ with $n \equiv 1\,(mod\,2)$ and $2 \leq \kappa \leq 10$

n	κ								
	2	3	4	5	6	7	8	9	10
1	0	7	13	9	33	81	49	497	881
3	1	4	10	22	46	94	62	254	638
5	2	1	7	19	43	91	59	251	635
7	3	6	4	0	24	72	40	232	104
9	0	3	1	29	53	37	5	453	325
11	1	0	14	10	2	114	210	146	530
13	2	5	11	7	63	47	143	79	975
15	3	2	8	20	44	92	60	508	892
17	0	7	5	17	9	121	217	153	537
19	1	4	2	30	22	6	102	294	166
21	2	1	15	27	19	3	227	163	35
23	3	6	12	8	0	112	80	16	400
25	0	3	9	5	29	77	173	109	493
27	1	0	6	18	42	26	250	186	570
29	2	5	3	15	39	87	55	503	887
31	3	2	0	28	20	4	100	292	676

in $n' = (n-1)/2$ with the independent term (i.e. the value of ξ for $n \equiv 1\,(mod\,2^{\kappa+1})$) being either $\left(\sum_{i=0}^{(\kappa-4)/2}\left(2^{2i}\right) + 4\sigma_{\kappa/2}\right)(mod\,2^{\kappa})$ or $\left(\sum_{i=0}^{(\kappa-3)/2}\left(2^{2i}\right) + 4\sigma_{(\kappa-1)/2} + 2^{\kappa-2}\right)(mod\,2^{\kappa})$ if respectively $\kappa \equiv 0$ or $1\,(mod\,2)$, where the integer sequence $\sigma_j = 1, 3, 7, 71, 199, ...$ is given in [6, 21]. The coefficients c_i of the powers of n' in the polynomials $P(n') = \sum c_i n'^i$ can also be chosen to fit the congruence $\xi \equiv P(n')\,(mod\,2^{\kappa})$ and the polynomials with the smallest $c_i \in \mathbb{Z}^+$ are shown in Table 5. Replacing these values of ξ in $m_n = 2^{\kappa+2}m'_n + (4\xi+3)$ yields $(M+1)$ to have a factor f such as $f \equiv 3\,(mod\,4)$, contradicting again (C3).
Therefore, all values of $n \equiv 1\,(mod\,2)$ yield M to be rejected.
(v.5) It follows that the sum of squares of M consecutive integers cannot be an integer square if $M \equiv 3, 5, 6, 7, 8$ or $10\,(mod\,12)$. □

Example 3. For an even number of iterations $n \equiv 0\,(mod\,2)$ in the case (v.4.2.1.3) above, the following example for $n = 2$ shows that there are no $m_n \equiv 1\,(mod\,4)$ values such that the sum of squares of $M = (12m_n + 3)$ consecutive integers can be an integer square as the following values of m_n have to be rejected:

$m_n \equiv 1\,(mod\,32)$, i.e, $1, 33, 65, ...$, by (C3), $\kappa = 3$, $\xi = 0$ in (11);
$m_n \equiv 5\,(mod\,16)$, i.e. $5, 21, 37, ...$, by (C3), $\kappa = 2$, $\xi = 1$ in (11);
$m_n \equiv 9\,(mod\,128)$, i.e. $9, 137, 245, ...$, by (C3), $\kappa = 5$, $\xi = 2$ in (11);
$m_n \equiv 13\,(mod\,16)$, i.e. $13, 29, 45, ...$, by (C4.2), $\alpha = 4$, $m_{n0} = 13$ in (6);

Table 5: Polynomials $P(n')$ such as $\xi \equiv P(n') \, (mod \, 2^\kappa)$ with $n' = (n-1)/2$

κ	$P(n') \, (mod \, 2^\kappa)$
2	$(n') \, (mod \, 4)$
3	$(5n' + 7) \, (mod \, 8)$
4	$(13n' + 13) \, (mod \, 16)$
5	$\left(8n'^2 + 5n' + 9\right) (mod \, 32)$
6	$\left(24n'^2 + 53n' + 33\right) (mod \, 64)$
7	$\left(56n'^2 + 85n' + 81\right) (mod \, 128)$
8	$\left(120n'^2 + 149n' + 49\right) (mod \, 256)$
9	$\left(384n'^3 + 504n'^2 + 405n' + 457\right) (mod \, 512)$
10	$\left(384n'^3 + 1016n'^2 + 425n' + 881\right) (mod \, 1024)$

$m_n \equiv 17 \, (mod \, 32)$, i.e. $17, 49, 81, ...$, by (C4.2), $\alpha = 5$, $m_{n0} = 17$ in (6);
$m_n \equiv 25 \, (mod \, 64)$, i.e. $25, 89, 153, ...$, by (C3), $\kappa = 4$, $\xi = 6$ in (11);
$m_n \equiv 41 \, (mod \, 1024)$, i.e. $41, 1065, ...$, by (C4.2), $\alpha = 10$, $m_{n0} = 41$ in (6);
$m_n \equiv 57 \, (mod \, 64)$, i.e. $57, 121, 185, ...$, by (C4.2), $\alpha = 6$, $m_{n0} = 57$ in (6); etc.
For an odd number of iterations (i.e. $n \equiv 1 \, (mod \, 2)$) in the case (v.4.2.2.3) above, the following example for $n = 3$ shows that there are no $m_n \equiv 3 \, (mod \, 4)$ values such that the sum of squares of $M = (12m_n + 3)$ consecutive integers can be an integer square as the following values of m_n have to be rejected:
$m_n \equiv 3 \, (mod \, 32)$, i.e. $3, 35, 67, ...$, by (C4.2), $\alpha = 5$, $m_{n0} = 3$ in (6);
$m_n \equiv 7 \, (mod \, 16)$, i.e. $7, 23, 39, ...$, by (C3), $\kappa = 2$, $\xi = 1$ in (17);
$m_n \equiv 11 \, (mod \, 64)$, i.e. $11, 75, 139, ...$, by (C4.2), $\alpha = 6$, $m_{n0} = 11$ in (6);
$m_n \equiv 15 \, (mod \, 16)$, i.e. $15, 31, 47, ...$, by (C4.2), $\alpha = 4$, $m_{n0} = 15$ in (6);
$m_n \equiv 19 \, (mod \, 32)$, i.e. $19, 51, 83, ...$, by (C3), $\kappa = 3$, $\xi = 4$ in (17);
$m_n \equiv 27 \, (mod \, 128)$, i.e. $27, 155, 283, ...$, by (C4.2), $\alpha = 7$, $m_{n0} = 27$ in (6);
$m_n \equiv 43 \, (mod \, 64)$, i.e. $43, 107, 171, ...$, by (C3), $\kappa = 4$, $\xi = 10$ in (17); etc.

Remark 4. Note that the values of m_{n0} in section (v.4.1) above are not independent and within the same n^{th} iteration, the value $m_{n0,\alpha}$ of m_{n0} for a given value of α is related to the preceding value $m_{n0,(\alpha-1)}$ for $(\alpha - 1)$ by

$$m_{n0,\alpha} \, (mod \, 2^\alpha) \equiv \left(m_{n0,(\alpha-1)} + \epsilon \times 2^{\alpha-1} + 2^{\alpha-3}\right) \tag{20}$$

with either $\epsilon = -1$, or 0, or +1. From (7), $m_{n0} = (2^\alpha K - \beta)/3^{2n-1}$ and one has respectively

$$m_{n0} = \frac{2^\alpha K - 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) + \left(\frac{2^{\alpha-2}-1}{3}\right)}{3^{2n-1}} \text{ if } \alpha \equiv 0 \, (mod \, 2) \tag{21}$$

$$m_{n0} = \frac{2^\alpha K - 2\left(\sum_{i=0}^{n-2} 3^{2i}\right) + \left(\frac{5\times 2^{\alpha-2}-1}{3}\right)}{3^{2n-1}} \text{ if } \alpha \equiv 1 \, (mod \, 2) \tag{22}$$

Forming now the difference $m_{n0,\alpha} - m_{n0,(\alpha-1)}$, one obtains $m_{n0,\alpha} - m_{n0,(\alpha-1)} =$

$\epsilon \times 2^{\alpha-1} + 2^{\alpha-3}$ with

$$\epsilon = \frac{\left(2K_\alpha - K_{(\alpha-1)}\right) - \left(\frac{3^{2n-1}+1}{4} + \eta\right)}{3^{2n-1}} \tag{23}$$

with $\eta = 0$ or -1 if $\alpha \equiv 0$ or $1\,(mod\,2)$ and where K_α and $K_{(\alpha-1)}$ are the values of K corresponding to $m_{n0,\alpha}$ and $m_{n0,(\alpha-1)}$ in (7). $\epsilon = 1$ or 0 or -1 if $2K_\alpha - K_{(\alpha-1)} = \left(\eta + \left(5 \times 3^{2n-1} + 1\right)/4\right)$ or $\left(\eta + \left(3^{2n-1}+1\right)/4\right)$ or $\left(\eta + \left(1 - 3^{2n}\right)/4\right)$ (see also Lemma 1).

The next theorem gives additional conditions on the allowed $(mod\,12)$ congruent values that M can take.

Theorem 5. *For $M > 1, a, s \in \mathbb{Z}^+$, $i \in \mathbb{Z}^*$, there exist M satisfying $M \equiv 0,1,2,4,9$ or $11\,(mod\,12)$ such as the sums of M consecutive squared integers $(a+i)^2$ equal integer squares s^2. Furthermore, if $M \equiv 0\,(mod\,12)$, then $M \equiv 0$ or $24\,(mod\,72)$; if $M \equiv 1\,(mod\,12)$, then $M \equiv 1\,(mod\,24)$; if $M \equiv 2\,(mod\,12)$, then $M \equiv 2\,(mod\,24)$; if $M \equiv 4\,(mod\,12)$, then $M \equiv 16\,(mod\,24)$; if $M \equiv 9\,(mod\,12)$, then $M \equiv 9$ or $33\,(mod\,72)$; and the corresponding congruent values of a and s are given in Table 6.*

Proof. For $M > 1, m, a, s \in \mathbb{Z}^+$ and $\mu, i \in \mathbb{Z}^*$, $0 \le \mu \le 11$, let $M \equiv \mu\,(mod\,12)$ $\Rightarrow M = 12m + \mu$.
Expressing the sum of M consecutive integer squares starting from a^2 equal to an integer square s^2 as

$$\sum_{i=0}^{M-1} (a+i)^2 = M\left[\left(a + \frac{M-1}{2}\right)^2 + \frac{M^2-1}{12}\right] = s^2 \tag{24}$$

and replacing M by $12m + \mu$ in (24) yields

$$(12m+\mu)\left[a^2 + a\,(12m+\mu-1) + 48m^2 + 2m\,(4\mu-3) + \frac{2\mu^2 - 3\mu + 1}{6}\right] = s^2 \tag{25}$$

Recalling that integer squares are congruent to either $0, 1, 4$ or $9\,(mod\,12)$, replacing the values of $\mu = 0, 1, 2, 4, 9, 11$ in (25) and reducing $(mod\,12)$ yield:
(i) for $\mu = 0$, $\left(2m\left(6a^2 - 6a + 1\right) - s^2\right) \equiv 0\,(mod\,12)$.
As $\forall a$, $\left(6a^2 - 6a\right) \equiv 0\,(mod\,12)$, it yields $\left(2m - s^2\right) \equiv 0\,(mod\,12) \Rightarrow s \equiv 0\,(mod\,6)$ for $m \equiv 0\,(mod\,6)$ and $s \equiv 2$ or $4\,(mod\,6)$ for $m \equiv 2\,(mod\,6)$.
Therefore, $M \equiv 0\,(mod\,72)$ with $s \equiv 0\,(mod\,6)$ or $M \equiv 24\,(mod\,72)$ with $s \equiv 2$ or $4\,(mod\,6)$ and a can take any value.
(ii) for $\mu = 1$, $\left(a^2 + 2m - s^2\right) \equiv 0\,(mod\,12)$.
For $a^2 \equiv \{0,1,4,9\}\,(mod\,12)$, $2m \equiv \{(0 \text{ or } 4), (0 \text{ or } 8), (0 \text{ or } 8), (0 \text{ or } 4)\}\,(mod\,12)$ respectively for $s^2 \equiv \{(0 \text{ or } 4), (1 \text{ or } 9), (4 \text{ or } 0), (9 \text{ or } 1)\}\,(mod\,12)$, yielding $m \equiv 0\,(mod\,2)$ and $M \equiv 1\,(mod\,24)$. Furthermore,
- if $m \equiv 0\,(mod\,6)$, a and s can take any values;

Table 6: Congruent values of M, m, a, and s

μ	$M \equiv$	$m \equiv$	$a \equiv$	$s \equiv$
0	$0\,(mod\,72)$	$0\,(mod\,6)$	$\forall$	$0\,(mod\,6)$
	$24\,(mod\,72)$	$2\,(mod\,6)$	$\forall$	2 or $4\,(mod\,6)$
1	$1\,(mod\,24)$	$0\,(mod\,6)$	$\forall$	$\forall$
		$2\,(mod\,6)$	$0\,(mod\,6)$	2 or $4\,(mod\,6)$
			$3\,(mod\,6)$	1 or $5\,(mod\,6)$
		$4\,(mod\,6)$	$1\,(mod\,2)$	$3\,(mod\,6)$
			$0\,(mod\,2)$	$0\,(mod\,6)$
2	$2\,(mod\,24)$	$0\,(mod\,6)$	$0, 2\,(mod\,3)$	1 or $5\,(mod\,6)$
		$2\,(mod\,6)$	$1\,(mod\,3)$	$3\,(mod\,6)$
		$4\,(mod\,6)$	$\forall$	$1\,(mod\,2)$
4	$16\,(mod\,24)$	$1\,(mod\,6)$	$0\,(mod\,3)$	2 or $4\,(mod\,6)$
		$3\,(mod\,6)$	$1, 2\,(mod\,3)$	$0\,(mod\,6)$
		$5\,(mod\,6)$	$\forall$	$0\,(mod\,2)$
9	$9\,(mod\,72)$	$0\,(mod\,6)$	$0\,(mod\,2)$	$0\,(mod\,6)$
			$1\,(mod\,2)$	$3\,(mod\,6)$
	$33\,(mod\,72)$	$2\,(mod\,6)$	$0\,(mod\,2)$	2 or $4\,(mod\,6)$
			$1\,(mod\,2)$	1 or $5\,(mod\,6)$
11	$11\,(mod\,12)$	$0\,(mod\,6)$	$0, 2\,(mod\,6)$	1 or $5\,(mod\,6)$
		$1\,(mod\,6)$	$1\,(mod\,6)$	2 or $4\,(mod\,6)$
			$3, 5\,(mod\,6)$	$0\,(mod\,6)$
		$2\,(mod\,6)$	$4\,(mod\,6)$	$3\,(mod\,6)$
		$3\,(mod\,6)$	$3, 5\,(mod\,6)$	2 or $4\,(mod\,6)$
		$4\,(mod\,6)$	$0, 2\,(mod\,6)$	$3\,(mod\,6)$
			$4\,(mod\,6)$	1 or $5\,(mod\,6)$
		$5\,(mod\,6)$	$1\,(mod\,6)$	$0\,(mod\,6)$

- if $m \equiv 2\,(mod\,6)$, either $a \equiv 0\,(mod\,6)$ and $s \equiv 2$ or $4\,(mod\,6)$, or $a \equiv 3\,(mod\,6)$ and $s \equiv 1$ or $5\,(mod\,6)$; and
- if $m \equiv 4\,(mod\,6)$, either $a \equiv 1\,(mod\,2)$ and $s \equiv 3\,(mod\,6)$, or $a \equiv 0\,(mod\,2)$ and $s \equiv 0\,(mod\,6)$.

(iii) for $\mu = 2$, $\left(2\left(a^2+a\right)+2m+1-s^2\right) \equiv 0\,(mod\,12)$.
For $\left(2\left(a^2+a\right)+1\right) \equiv \{1,5\}\,(mod\,12)$, $2m \equiv \{(0\,\mathrm{or}\,8),(4\,\mathrm{or}\,8)\}\,(mod\,12)$ respectively for $s^2 \equiv \{(1\,\mathrm{or}\,9),(9\,\mathrm{or}\,1)\}\,(mod\,12)$, yielding $m \equiv 0\,(mod\,2)$ and $M \equiv 2\,(mod\,24)$. Furthermore,
- if $m \equiv 0\,(mod\,6)$, $a \equiv 0$ or $2\,(mod\,3)$ and $s \equiv 1$ or $5\,(mod\,6)$;
- if $m \equiv 2\,(mod\,6)$, $a \equiv 1\,(mod\,3)$ and $s \equiv 3\,(mod\,6)$; and
- if $m \equiv 4\,(mod\,6)$, a can take any value and $s \equiv 1\,(mod\,2)$.

(iv) for $\mu = 4$, $\left(2\left(2a^2+1\right)+2m-s^2\right) \equiv 0\,(mod\,12)$.
For $\left(2\left(2a^2+1\right)\right) \equiv \{2,6\}\,(mod\,12)$, $2m \equiv \{(2\,\mathrm{or}\,10),(6\,\mathrm{or}\,10)\}\,(mod\,12)$ respectively for $s^2 \equiv \{(4\,\mathrm{or}\,0),(0\,\mathrm{or}\,4)\}\,(mod\,12)$, yielding $m \equiv 1\,(mod\,2)$ and $M \equiv 16\,(mod\,24)$. Furthermore,
- if $m \equiv 1\,(mod\,6)$, $a \equiv 0\,(mod\,3)$ and $s \equiv 2$ or $4\,(mod\,6)$;
- if $m \equiv 3\,(mod\,6)$, $a \equiv 1$ or $2\,(mod\,3)$ and $s \equiv 0\,(mod\,6)$; and
- if $m \equiv 5\,(mod\,6)$, a can take any value and $s \equiv 0\,(mod\,2)$.

(v) for $\mu = 9$, $\left(9a^2+2m-s^2\right) \equiv 0\,(mod\,12)$.
For $\left(9a^2\right) \equiv \{0,9\}\,(mod\,12)$, $2m \equiv \{(0\,\mathrm{or}\,4),(0\,\mathrm{or}\,4)\}\,(mod\,12)$ respectively for $s^2 \equiv \{(0\,\mathrm{or}\,4),(9\,\mathrm{or}\,1)\}\,(mod\,12)$, yielding $m \equiv 0$ or $2\,(mod\,6)$ and $M \equiv 9$ or $33\,(mod\,72)$. Furthermore,
- if $m \equiv 0\,(mod\,6)$, either $a \equiv 0\,(mod\,2)$ and $s \equiv 0\,(mod\,6)$, or $a \equiv 1\,(mod\,2)$ and $s \equiv 3\,(mod\,6)$; and
- if $m \equiv 2\,(mod\,6)$, either $a \equiv 0\,(mod\,2)$ and $s \equiv 2$ or $4\,(mod\,6)$, or $a \equiv 1\,(mod\,2)$ and $s \equiv 1$ or $5\,(mod\,6)$.

(vi) for $\mu = 11$, $\left(11a^2+2a+1+2m-s^2\right) \equiv 0\,(mod\,12)$.
For $\left(11a^2+2a+1\right) \equiv \{1,2,5,10\}\,(mod\,12)$, $2m \equiv \{(0\,\mathrm{or}\,8),(2\,\mathrm{or}\,10),(4\,\mathrm{or}\,8),(2\,\mathrm{or}\,6)\}\,(mod\,12)$ respectively for $s^2 \equiv \{(1\,\mathrm{or}\,9),(4\,\mathrm{or}\,0),(9\,\mathrm{or}\,1),(0\,\mathrm{or}\,4)\}\,(mod\,12)$ yielding
- if $m \equiv 0\,(mod\,6)$, $a \equiv 0$ or $2\,(mod\,6)$ and $s \equiv 1$ or $5\,(mod\,6)$;
- if $m \equiv 1\,(mod\,6)$, either $a \equiv 1\,(mod\,6)$ and $s \equiv 2$ or $4\,(mod\,6)$, or $a \equiv 3$ or $5\,(mod\,6)$ and $s \equiv 0\,(mod\,6)$;
- if $m \equiv 2\,(mod\,6)$, $a \equiv 4\,(mod\,6)$ and $s \equiv 3\,(mod\,6)$;
- if $m \equiv 3\,(mod\,6)$, $a \equiv 3$ or $5\,(mod\,6)$ and $s \equiv 2$ or $4\,(mod\,6)$;
- if $m \equiv 4\,(mod\,6)$, either $a \equiv 0$ or $2\,(mod\,6)$ and $s \equiv 3\,(mod\,6)$, or $a \equiv 4\,(mod\,6)$ and $s \equiv 1$ or $5\,(mod\,6)$; and
- if $m \equiv 5\,(mod\,6)$, $a \equiv 1\,(mod\,6)$ and $s \equiv 0\,(mod\,6)$.

Therefore, the congruences of Table 6 hold. □

Additional necessary conditions can be found using Beeckmans' necessary conditions and are given in [17]. Theorem 5 yields also that M can only be congruent to $0,1,2,9,11,16,23,24,25,26,33,35,40,47,49,50,59,64$ or $71\,(mod\,72)$.
The values of M yielding solutions to (24) are given in [19].

3 Case of M being square

An interesting case occurs when M is itself a squared integer as shown in the following theorem.

Theorem 6. *For $M > 1 \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, if M is a square integer, then there exist M satisfying $M \equiv 1\,(mod\,24)$ such as the sums of M consecutive squared integers $(a+i)^2$ equal integer squares s^2; furthermore $M = (6n-1)^2$, i.e $(M-1)/24$ are all generalized pentagonal numbers $n(3n-1)/2$.*

Proof. For $M > 1, m, m_1, m_2 \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, let $M = m^2$; then $m \neq 0\,(mod\,2)$ and $m \neq 0\,(mod\,3)$ by (C1.1) and (C1.2). Therefore, $m \equiv \pm 1\,(mod\,6) \Rightarrow m = 6m_1 \pm 1$, yielding $M = 12m_1(3m_1 \pm 1) + 1$ or $M \equiv 1\,(mod\,12)$. Then, by Theorem 5, $M \equiv 1\,(mod\,24) \Rightarrow M = 24m_2 + 1$, and $24m_2 + 1 = (6m_1 \pm 1)^2$, or $m_2 = m_1(3m_1 \pm 1)/2$ which is equivalent to $n(3n-1)/2$, $\forall n \in \mathbb{Z}$. □

The generalized pentagonal numbers $n(3n-1)/2$ [5, 25] take the values $0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, ...$ [22], which then yields $M = 1, 25, 49, 121, 169, 289, 361, 529, 625, 841, 961, 1225, 1369, ...$ The first two values, $M = 1, 25$, should be rejected, the first one because M must be greater than 1, and the second one because for $M = 25$, one finds the unique solution $a = 0$ and $s = 70$ and a must be positive, although it is obviously equivalent to the solution with $a = 1$ and $s = 70$ for $M = 24$ of Lucas' cannonball problem (see also [18]).

4 Conclusions

It was shown that the problem of finding all the integer solutions of the sum of M consecutive integer squares starting at $a^2 \geq 1$ being equal to a squared integer s^2 has no solutions if M is congruent to 3, 5, 6, 7, 8 or 10 $(mod\,12)$ using Beeckmans necessary conditions. It was further proven that the problem has integer solutions if M is congruent to 0, 9, 24 or 33 $(mod\,72)$; or to 1, 2 or 16 $(mod\,24)$; or to 11 $(mod\,12)$. If M is a square itself, then M must be congruent to 1 $(mod\,24)$ and $(M-1)/24$ are all pentagonal numbers, except the first two.
In a second paper [18], the Diophantine quadratic equation (24) in variables a and s with M as a parameter is solved generally.

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