Finding all squared integers expressible as the sum of consecutive squared integers using generalized Pell equation solutions with Chebyshev polynomials

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Abstract

Square roots s of sums of M consecutive integer squares starting from $a^2 \geq 1$ are integers if $M \equiv 0, 9, 24$ or $33 \,(mod\, 72)$; or $M \equiv 1, 2$ or $16 \,(mod\, 24)$; or $M \equiv 11 \,(mod\, 12)$ and cannot be integers if $M \equiv 3, 5, 6, 7, 8$ or $10 \,(mod\, 12)$. Finding all solutions with s integer requires to solve a Diophantine quadratic equation in variables a and s with M as a parameter. If M is not a square integer, the Diophantine quadratic equation in variables a and s is transformed into a generalized Pell equation whose form depends on the $M \,(mod\, 4)$ congruent value, and whose solutions, if existing, yield all the solutions in a and s for a given value of M. Depending on whether this generalized Pell equation admits one or several fundamental solution(s), there are one or several infinite branches of solutions in a and s that can be written simply in function of Chebyshev polynomials evaluated at the fundamental solutions of the related simple Pell equation. If M is a square integer, it is known that $M \equiv 1 \,(mod\, 24)$ and $M = (6n-1)^2$ for all integers n; then the Diophantine quadratic equation in variables a and s reduces to a simple difference of integer squares which yields a finite number of solutions in a and s to the initial problem.

Keywords: Sum of consecutive squared integers ; Generalized Pell equation ; Chebyshev polynomials
MSC2010 : 11D09 ; 11E25 ; 33D45

1 Introduction

Lucas' cannonball problem [17, 18] of finding a square number of cannonballs stacked in a square pyramid has only two solutions, 1 and 4900, the later corre-

sponding to the sum of the first 24 squared integers and was proven by several authors [25, 19, 36, 16, 20, 2].
More generally, finding all integers s equal to the sum of M consecutive integer squares starting from $a^2 \geq 1$ involves solving a single Diophantine quadratic equation in three variables, two independent (a and M) and one dependent (s). Philipp [28], extending the previous work of Alfred [1], proved that there are a finite or an infinite number of solutions depending on whether M is or not a square integer and in the later case, using a form of the generalized Pell equation. Beeckmans [3], after demonstrating eight necessary conditions on M with a table of values of $M < 1000$ and the smallest values of $a > 0$, developed a method based on solving generalized Pell equations to provide all solutions.
In two previous papers, this Author showed [30] that no solution exists if M is congruent to $3, 5, 6, 7, 8$ or $10\,(mod\,12)$ using Beeckmans necessary conditions, and that integer solutions exist if M is congruent to $0, 1, 2, 4, 9$ or $11\,(mod\,12)$, yielding M to be congruent to $0, 9, 24$ or $33\,(mod\,72)$, and M to be congruent to $1, 2$ or $16\,(mod\,24)$. These are called allowed values. Additional congruence conditions were demonstrated [31] on the allowed values of M using Beeckmans' necessary conditions. Furthermore, it was shown also [30] that if M is a square itself, M must be congruent to $1\,(mod\,24)$ and $(M-1)/24$ are all pentagonal numbers, except the first two. The values of M yielding integer solutions are given in [33].
In this paper, firstly for non-square integer values of M, the Diophantine quadratic equation expressing the sum of consecutive squared integers equaling a squared integer is transformed into a generalized Pell equation for which, depending on its number of fundamental solutions, one or several infinite branch(es) of solutions in a and s are found analytically, using Chebyshev polynomials. Secondly, for square values of M, the quadratic equation reduces to a difference of squares for which a finite number of solutions in a and s are found analytically.

2 Simple and generalized Pell equations

Pell equations of the general form

$$X^2 - DY^2 = N \tag{1}$$

with $X, Y, N \in \mathbb{Z}$ and squarefree $D \in \mathbb{Z}^+$, i.e. $\sqrt{D} \notin \mathbb{Z}^+$, have been investigated in various forms since long (see historical accounts in [7, 15, 37, 14]) and are treated in several classical text books (see e.g. [26, 27, 38] and references therein). A simple reminder is given here and further details can be found in the references.
For $N = 1$, the simple Pell equation reads classically

$$X^2 - DY^2 = 1 \tag{2}$$

which has, beside the trivial solution $(X_t, Y_t) = (1, 0)$, a whole infinite branch of solutions for $k \in \mathbb{Z}^+$ given by

$$X_k = \frac{\left(X_1 + \sqrt{D}Y_1\right)^k + \left(X_1 - \sqrt{D}Y_1\right)^k}{2} \quad (3)$$

$$Y_k = \frac{\left(X_1 + \sqrt{D}Y_1\right)^k - \left(X_1 - \sqrt{D}Y_1\right)^k}{2\sqrt{D}} \quad (4)$$

where (X_1, Y_1) is the fundamental solution to (2), i.e. the smallest integer solution $(X_1 > 1, Y_1 > 0, \in \mathbb{Z}^+)$ different from the trivial solution. Among the five methods listed by Robertson [32] to find the fundamental solution (X_1, Y_1), the classical method introduced by Lagrange [13], based on the continued fraction expansion of the quadratic irrational $\sqrt{D}$, is central to several other methods.
For $N = n^2$ an integer square, the generalized Pell equation (1) admits always integer solutions. The variable change $(X', Y') = ((X/n), (Y/n))$ transforms the generalized Pell equation in a simple Pell equation $X'^2 - DY'^2 = 1$ which has integer solutions (X'_k, Y'_k). The integer solutions to the generalized Pell equation can then be found as $(X_k, Y_k) = (nX'_k, nY'_k)$. Note however that not all solutions in (X, Y) may be found in this way (see e.g. [38]).
For the case where N is not an integer square, the generalized Pell equation (1) can have either no solution at all, or one or several fundamental solutions (X_1, Y_1), and all integer solutions, if they exist, can be expressed in function of the fundamental solution(s) (X_1, Y_1). Several authors (see e.g. [13, 4, 26, 24, 32, 7, 21, 22] and references therein) discussed how to find the fundamental solution(s) of the generalized Pell equation, based on Lagrange's method of continued fractions with various modifications (see e.g. [29]), and further how to find additional solutions from the fundamental solution(s).
Noting now (x_f, y_f) the fundamental solutions of the related simple Pell equation (2), the other solutions (X_k, Y_k) can be found from the fundamental solution(s) (X_1, Y_1) by

$$X_k + \sqrt{D}Y_k = \pm\left(X_1 + \sqrt{D}Y_1\right)\left(x_f + \sqrt{D}y_f\right)^k \quad (5)$$

for a proper choice of sign $\pm$ [32].
It is less known that Chebyshev polynomials can be used to find the additional solutions of the generalized Pell equation once the fundamental solutions (X_1, Y_1) have been found. In fact, Chebyshev polynomials $T_k(x)$ and $U_k(x)$ of the first and second kinds [35, 12] can be defined as solutions of the simple Pell equation

$$T_k(x)^2 - \left(x^2 - 1\right)U_{k-1}(x)^2 = 1 \quad (6)$$

on a ring $R(x)$ [6, 5]. The following lemma shows how to find the additional solutions of the generalized Pell equation.

Lemma 1. *For $X, Y, D, N, k \in \mathbb{Z}^+$ and D not a perfect square (i.e. $\sqrt{D} \notin \mathbb{Z}$), if the generalized Pell equation*

$$X^2 - DY^2 = N \tag{7}$$

admits one or several fundamental solution(s) (X_1, Y_1), then it admits one or several infinite branch(es) of solutions and these can be written as

$$X_k = X_1 T_{k-1}(x_f) + DY_1 y_f U_{k-2}(x_f) \tag{8}$$
$$Y_k = X_1 y_f U_{k-2}(x_f) + Y_1 T_{k-1}(x_f) \tag{9}$$

in function of the fundamental solution(s) (X_1, Y_1) and of Chebyshev polynomials of the first and second kinds, $T_{k-1}(x_f)$ and $U_{k-2}(x_f)$ evaluated at the fundamental solution (x_f, y_f) of the related simple Pell equation $X^2 - DY^2 = 1$.

Proof. For $X, Y, D, N, k, i \in \mathbb{Z}^+$ and square free D, let (X_1, Y_1) be one of the fundamental solutions of (7) if they exist, and let (x_f, y_f) be the fundamental solution of the related simple Pell equation $X^2 - DY^2 = 1$ (i.e. $x_f > 1, y_f > 0$).
(i) Additional solutions (X_k, Y_k) of (7) can then be found by the recurrence relations

$$X_k = x_f X_{k-1} + Dy_f Y_{k-1} \tag{10}$$
$$Y_k = x_f Y_{k-1} + y_f X_{k-1} \tag{11}$$

which can be demonstrated by induction.
For $k = 2$, as (X_1, Y_1) is a fundamental solution of (7), (X_2, Y_2) obtained from (10) and (11) verify also (7) as $x_f^2 - Dy_f^2 = 1$.
Let (X_{k-1}, Y_{k-1}) be a solution of (7), i.e. $X_{k-1}^2 - DY_{k-1}^2 = N$. Then multiplying the two terms on the left of this equation by $1 = x_f^2 - Dy_f^2$, adding and subtracting $2Dx_f y_f X_{k-1} Y_{k-1}$, rearranging and replacing by (10) and (11) yield $X_k^2 - DY_k^2 = N$, i.e. (X_k, Y_k) is also a solution of (7).
(ii) Further, to express X_k and Y_k in function of X_1, Y_1, x_f and y_f only, one replaces successively for $3 \leq i \leq k$, X_{i-1} and Y_{i-1} in function of X_1 and Y_1 in the expressions (10) and (11) of X_i, Y_i (with the substitution $x_f^2 + Dy_f^2 = 2x_f^2 - 1$ whenever needed) to obtain successively Chebyshev polynomials of the first and second kinds evaluated at x_f and of increasing indices, respectively $i - 1$ and $i - 2$, i.e. $T_{i-1}(x_f)$ and $U_{i-2}(x_f)$, yielding eventually (8) and (9).
One can verify by induction that (8) and (9) yield all solutions to (7).
As (X_1, Y_1) is a fundamental solution of (7), for $k = 2$, one has $T_1(x_f) = x_f$ and $U_0(x_f) = 1$ in (8) and (9), yielding directly (10) and (11).
Further, let us assume that (X_{k-1}, Y_{k-1}) with

$$X_{k-1} = X_1 T_{k-2}(x_f) + DY_1 y_f U_{k-3}(x_f) \tag{12}$$
$$Y_{k-1} = X_1 y_f U_{k-3}(x_f) + Y_1 T_{k-2}(x_f) \tag{13}$$

are a solution of (7); then replacing (12) and (13) in (10) and (11) yield

$$
\begin{aligned}
X_k &= x_f \left[X_1 T_{k-2}(x_f) + DY_1 y_f U_{k-3}(x_f)\right] + \\
&\quad Dy_f \left[X_1 y_f U_{k-3}(x_f) + Y_1 T_{k-2}(x_f)\right] \\
&= X_1 \left[x_f T_{k-2}(x_f) + \left(x_f^2 - 1\right) U_{k-3}(x_f)\right] + \\
&\quad DY_1 y_f \left[x_f U_{k-3}(x_f) + T_{k-2}(x_f)\right] \qquad (14) \\
Y_k &= x_f \left[X_1 y_f U_{k-3}(x_f) + Y_1 T_{k-2}(x_f)\right] + \\
&\quad y_f \left[X_1 T_{k-2}(x_f) + DY_1 y_f U_{k-3}(x_f)\right] \\
&= X_1 y_f \left[x_f U_{k-3}(x_f) + T_{k-2}(x_f)\right] + \\
&\quad Y_1 \left[x_f T_{k-2}(x_f) + \left(x_f^2 - 1\right) U_{k-3}(x_f)\right] \qquad (15)
\end{aligned}
$$

where Dy_f^2 has been replaced by $Dy_f^2 = x_f^2 - 1$ in (14) and (15). As

$$
\begin{aligned}
T_{k-1}(x_f) &= x_f T_{k-2}(x_f) + \left(x_f^2 - 1\right) U_{k-3}(x_f) \qquad (16) \\
U_{k-2}(x_f) &= x_f U_{k-3}(x_f) + T_{k-2}(x_f) \qquad (17)
\end{aligned}
$$

(see e.g. [35]), (14) and (15) yield directly (8) and (9). Replacing now (8) and (9) in (7) gives

$$
X_k^2 - DY_k^2 = \left(X_1^2 - DY_1^2\right)\left(T_{k-1}(x_f)^2 - Dy_f^2 U_{k-2}(x_f)^2\right) = N \qquad (18)
$$

by (6) with $Dy_f^2 = x_f^2 - 1$, showing that (X_k, Y_k) (8, 9) also solve (7).
Finally, as k is unbound, there is an infinity of solutions (8) and (9). □

3 General method to find all solutions

The sum of $M > 1$ consecutive integer squares starting from $a^2 \geq 1$ being equal to an integer square s^2 can be written in all generality as [30]

$$
\sum_{i=0}^{M-1} (a+i)^2 = M\left[\left(a + \frac{M-1}{2}\right)^2 + \frac{M^2-1}{12}\right] = s^2 \qquad (19)
$$

where M are allowed values (see [30, 31]). To find all integer solutions of (19), two cases are considered and treated separately: first, M is not a squared integer, and second, M is a squared integer.

3.1 M not a squared integer

The next theorem allows to find all the solutions to (19) in a and s for allowed values of M not being squared integers.

Theorem 2. *For $M > 1, \sigma, j, k, a_{k,j}, s_{k,j}, x_f, y_f \in \mathbb{Z}^+, \lambda \in \mathbb{Q}$, for all allowed square free values of M (i.e. $\sqrt{M} \notin \mathbb{Z}$), there is a number $\sigma \geq 1$ of infinite branch(es) of values of $a_{k,j}$, $1 \leq j \leq \sigma$, such that the sums of squares of M consecutive integers starting from $a_{k,j}$ are equal to squared positive integers $s_{k,j}^2$*

and these can be written in function of Chebyshev polynomials of the first and second kinds, $T_{k-1}(x_f)$ *and* $U_{k-2}(x_f)$ *as*

$$
\begin{aligned}
a_{k,j} &= \frac{2\lambda s_{1,j} y_f U_{k-2}(x_f) + (2a_{1,j} + M - 1) T_{k-1}(x_f) - (M-1)}{2} && (20)\\
s_{k,j} &= s_{1,j} T_{k-1}(x_f) + \frac{\lambda M}{2} y_f (2a_{1,j} + M - 1) U_{k-2}(x_f) && (21)
\end{aligned}
$$

with $\lambda = 1$ *for* $M \equiv 1 \,(mod\, 2)$ *or* $M \equiv 2 \,(mod\, 4)$*, and* $\lambda = 1/2$ *for* $M \equiv 0 \,(mod\, 4)$*, and where* $(a_{1,j}, s_{1,j})$ *are the smallest positive values of* $(a_{k,j}, s_{k,j})$ *solutions of (19) and* (x_f, y_f) *is the fundamental solution of the simple Pell equation* $X^2 - (\lambda^2 M) Y^2 = 1$*.*

Proof. For $M > 1, \sigma, j, k, a, s, a_{k,j}, s_{k,j}, x_f, y_f, X, Y, N, D \in \mathbb{Z}^+, \lambda \in \mathbb{Q}$, for the allowed square free values of M, rewriting (19) for $M \equiv 1 \,(mod\, 2)$ as

$$s^2 - M\left(a + \frac{M-1}{2}\right)^2 = \frac{M\left(M^2-1\right)}{12} \qquad (22)$$

or for $M \equiv 0 \,(mod\, 4)$ as

$$s^2 - \frac{M}{4}(2a + M - 1)^2 = \frac{M\left(M^2-1\right)}{12} \qquad (23)$$

or for $M \equiv 2 \,(mod\, 4)$ as

$$(2s)^2 - M(2a + M - 1)^2 = \frac{M\left(M^2-1\right)}{3} \qquad (24)$$

transform (19) in generalized Pell equations (1) in $X = s$ or $2s$ and $Y = (a + (M-1)/2)$ or $(2a + M - 1)$, with $N = M\left(M^2-1\right)/12$ or $M\left(M^2-1\right)/3$ and $D = M$ or $M/4$.
If these generalized Pell equations (22) to (24) admit σ solution(s), then for $1 \le j \le \sigma$,
(i) for $M \equiv 1 \,(mod\, 2)$, let $(s_{1,j}, (a_{1,j} + (M-1)/2))$ be the j^{th} fundamental solution of (22) and let (x_f, y_f) be the fundamental solution of the related simple Pell equation $X^2 - MY^2 = 1$, i.e. $x_f > 1$ and $y_f > 0$. Then, (8) and (9) yield

$$
\begin{aligned}
a_{k,j} &= s_{1,j} y_f U_{k-2}(x_f) + \left(a_{1,j} + \frac{M-1}{2}\right) T_{k-1}(x_f) - \left(\frac{M-1}{2}\right) && (25)\\
s_{k,j} &= s_{1,j} T_{k-1}(x_f) + M y_f \left(a_{1,j} + \frac{M-1}{2}\right) U_{k-2}(x_f) && (26)
\end{aligned}
$$

(ii) for $M \equiv 0 \,(mod\, 4)$, similarly let $(s_{1,j}, (2a_{1,j} + M - 1))$ be the j^{th} fundamental solution of (23) and let (x_f, y_f) be the fundamental solution of the related simple Pell equation $X^2 - (M/4) Y^2 = 1$. Then, (8) and (9) yield

$$
\begin{aligned}
a_{k,j} &= \frac{s_{1,j} y_f U_{k-2}(x_f) + (2a_{1,j} + M - 1) T_{k-1}(x_f) - (M-1)}{2} && (27)\\
s_{k,j} &= s_{1,j} T_{k-1}(x_f) + \frac{M}{4} y_f (2a_{1,j} + M - 1) U_{k-2}(x_f) && (28)
\end{aligned}
$$

Table 1: First solutions $(a_{k,j}, s_{k,j})$ for $M = 11$, $1 \leq j \leq 2$ and $1 \leq k \leq 6$ of the $\sigma = 2$ infinite branches of solutions of $s^2 - 11\,(a+5)^2 = 110$

k	$a_{k,1}$	$s_{k,1}$	$a_{k,2}$	$s_{k,2}$
1	[-4]	[11]	18	77
2	38	143	456	1529
3	854	2849	9192	30503
4	17132	56837	183474	608531
5	341876	1133891	3660378	12140117
6	6820478	22620983	73024176	242193809

$[a_{1,1}]$: solution rejected as $a_{1,1} \leq 0$

(iii) for $M \equiv 2\,(mod\,4)$, similarly let $(2s_{1,j}, (2a_{1,j} + M - 1))$ be the j^{th} fundamental solution of (24) and let (x_f, y_f) be the fundamental solution of the related simple Pell equation $X^2 - MY^2 = 1$. Then, (8) and (9) yield

$$a_{k,j} = \frac{2s_{1,j}y_fU_{k-2}(x_f) + (2a_{1,j} + M - 1)\,T_{k-1}(x_f) - (M-1)}{2} \qquad (29)$$

$$s_{k,j} = s_{1,j}T_{k-1}(x_f) + \frac{M}{2}y_f\,(2a_{1,j} + M - 1)\,U_{k-2}(x_f) \qquad (30)$$

Finally, as k is unbound, there is in each case and for each $1 \leq j \leq \sigma$ an infinity of solutions $(s_{k,j}, a_{k,j})$. □

Note that some of the first solutions $a_{1,j}$ may be rejected if the j^{th} fundamental solution of (22) (or (23) or (24)) is such that $(a_{1,j} + (M-1)/2) < (M-1)/2$, yielding a non-positive value of $a_{1,j}$.
In the following examples, the method indicated by Matthews [22] based on an algorithm by Frattini [9, 10, 11] using Nagell's bounds [26, 23] is used to find the fundamental solution(s) of the generalized Pell equation.
A first example for the case $M \equiv 11\,(mod\,12)$, let $M = 11$. Then, (22) reads $s^2 - 11\,(a+5)^2 = 110$, which has $\sigma = 2$ fundamental solutions, yielding, with $1 \leq j \leq 2$, $(s_{1,j}, (a_{1,j} + 5)) = (11, 1), (77, 23)$ and the fundamental solution of the related simple Pell equation $X^2 - 11Y^2 = 1$ is $(x_f, y_f) = (10, 3)$. Replacing in (25) and (26) yield then the solutions given in Table 1. The first solution $(a_{1,1}, s_{1,1})$ is rejected as $a_{1,1} < 0$. The solutions are then ordered as $a_{1,2} < a_{2,1} < a_{2,2} < a_{3,1} < ....$
A second example for the case $M \equiv 0\,(mod\,24)$, let $M = 24$. Then, (23) reads $s^2 - 6\,(2a+23)^2 = 1150$, having $\sigma = 6$ fundamental solutions, $(s_{1,j}, (2a_{1,j} + 23))$ $= (34, 1), (38, 7), (50, 15)\,(70, 25), (106, 41), (158, 63)$ and the fundamental solution of the related simple Pell equation $X^2 - 6Y^2 = 1$ is $(x_f, y_f) = (5, 2)$. Replacing in (27) and (28) yield then the solutions given in Table 2. The first three solutions $(a_{1,j}, s_{1,j})$ for $1 \leq j \leq 3$ are rejected as $a_{1,j} < 0$. The solutions are then ordered as $a_{1,4} < a_{1,5} < a_{1,6} < a_{2,1} < a_{2,3} < a_{2,4} < ....$ Note that the first solution of the fourth branch ($a_{1,4} = 1$, $s_{1,4} = 70$) gives the second solution of Lucas' cannonball problem.

Table 2: First solutions $(a_{k,j}, s_{k,j})$ for $M = 24$, $1 \leq j \leq 6$ and $1 \leq k \leq 6$ of the $\sigma = 6$ infinite branches of solutions of $s^2 - 6\,(2a + 23)^2 = 1150$

k	$a_{k,1}$	$s_{k,1}$	$a_{k,2}$	$s_{k,2}$	$a_{k,3}$	$s_{k,3}$
1	[-11]	[34]	[-8]	[38]	[-4]	[50]
2	25	182	44	274	76	430
3	353	1786	540	2702	856	4250
4	3597	17678	5448	26746	8576	42070
5	35709	174994	54032	264758	84996	416450
6	353585	1732262	534964	2620834	841476	4122430

k	$a_{k,4}$	$s_{k,4}$	$a_{k,5}$	$s_{k,5}$	$a_{k,6}$	$s_{k,6}$
1	1	70	9	106	20	158
2	121	650	197	1022	304	1546
3	1301	6430	2053	10114	3112	15302
4	12981	63650	20425	100118	30908	151474
5	128601	630070	202289	991066	306060	1499438
6	1273121	6237050	2002557	9810542	29991872	146929622

$[a_{1,j}]$: solutions rejected as $a_{1,j} \leq 0$ for $1 \leq j \leq 3$

A third example for the case $M \equiv 2\,(mod\,24)$, let $M = 2$. Then, (24) reads $(2s)^2 - 2\,(2a+1)^2 = 2$, having $\sigma = 1$ fundamental solution $(2s_{1,j}, (2a_{1,j}+1)) = (2, 1)$ and the fundamental solution of the related simple Pell equation $X^2 - 2Y^2 = 1$ is $(x_f, y_f) = (3, 2)$. Replacing in (29) and (30) yield then the solutions given in Table 3, where the first solution is again to be rejected (it corresponds to the identity relation $0^2 + 1^2 = 1^2$) and the second solution is the Pythagorean relation $3^2 + 4^2 = 5^2$.
Still for the case $M = 2\,(mod\,24)$, let $M = 842$ which does not yield solutions to (19). Indeed, although the related simple Pell equation $X^2 - 842Y^2 = 1$ has the fundamental solution $(x_f, y_f) = (1683, 58)$, the generalized Pell equation from (24) $(2s)^2 - 842\,(2a + 841)^2 = 198982282$ has no fundamental solution

Table 3: First solutions $(a_{k,1}, s_{k,1})$ for $M = 2$ and $1 \leq k \leq 6$ of the single infinite branch of solutions of $(2s)^2 - 2\,(2a+1)^2 = 2$

k	$a_{k,1}$	$s_{k,1}$
1	[0]	[1]
2	3	5
3	20	29
4	119	169
5	696	985
6	4059	5741

$[a_{1,1}]$: solution rejected as $a_{1,1} \leq 0$

($\sigma = 0$). This case was already signaled by Beeckmans [3]: the value of $M = 842 = 24 \times 35 + 2$, although complying with Beeckmans' conditions does not yield solutions to (19) (see also [31]).

3.2 M is a squared integer

It was demonstrated [30] that, if M is a square integer, then for the sums of M consecutive squared integers to equal integer squares, $M \equiv 1\,(mod\,24)$ and $\exists g_i \in \mathbb{Z}^+$ such that $M = 24g_n + 1$ where $g_n = n\,(3n-1)\,/2$ are all generalized pentagonal numbers $\forall n \in \mathbb{Z}$ [8, 39], yielding $M = (6n-1)^2$, i.e. $g_n = 0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, ...$ [34], yielding
$M = 1, 25, 49, 121, 169, 289, 361, 529, 625, 841, 961, 1225, 1369, ...$ of which the first two $M = 1, 25$, should be rejected as $M > 1$ and $a > 0$ (see further).
For M an integer square, the above method with solutions of the Pell equation can clearly not be followed as Pell equations are not defined for $D = M$ being a squared integer. Instead, another method (see e.g. [4] p. 486, and [24]) is used in the following theorem showing how to find the finite number of solutions for M being a squared integer.

Theorem 3. *For $M > 1, \varphi, k, a_k, s_k \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, for all allowed squared integer values of $M = (6n-1)^2$, there is a finite number φ of values of a_k such that the sums of squares of M consecutive integers starting from a_k are equal to squared positive integers s_k^2, that can be written as*

$$s_k = (6n-1)\left(\frac{v_k + u_k}{2}\right) \tag{31}$$

$$a_k = \frac{v_k - u_k}{2} - 6n\,(3n-1) \tag{32}$$

where u_k and v_k are the factor and co-factor of $[2n\,(3n-1)\,(6n\,(3n-1)+1)]$, with $u_k < v_k$, $u_k \equiv v_k \equiv 0\,(mod\,2)$ and $1 \le k \le \varphi$.

Proof. For $M > 1, \varphi, k, a, s, a_k, s_k \in \mathbb{Z}^+$, $n \in \mathbb{Z}$, from (19), s must be such as $s \equiv 0\,(mod\ (6n-1))$. Replacing in (22) yields then

$$\left(\frac{s}{6n-1}\right)^2 - (a + 6n\,(3n-1))^2 = 2n\,(3n-1)\,(6n\,(3n-1)+1) \tag{33}$$

i.e. the difference of two integer squares must be an even integer.
One has then to determine all the integer values of X_k and Y_k solutions of the equation $X^2 - Y^2 = N$, with $X = s/\,(6n-1)$, $Y = (a + 6n\,(3n-1))$ and $N = 2n\,(3n-1)\,(6n\,(3n-1)+1)$. For this, let $N = u_k v_k$ and only both even factor and co-factor u_k and v_k are considered as $N \equiv 0\,(mod\,4)$ [24]. As N is finite, there is a finite number φ of ways of decomposing N in product of two even factors. Then, with $u_k < v_k$ and $1 \le k \le \varphi$, $X_k = (v_k + u_k)\,/2$ and $Y_k = (v_k - u_k)\,/2$, yielding $s_k = (6n-1)\,(v_k + u_k)\,/2$ and $a_k = ((v_k - u_k)\,/2) - 6n\,(3n-1)$. □

Table 4: All $\varphi = 12$ solutions for $M = 289$ with $N = u_k v_k = 6960$

k	$u_k \times v_k$	X_k	Y_k	s_k	a_k
1	60×116	88	28	[1496]	[-116]
2	58×120	89	31	[1513]	[-113]
3	40×174	107	67	[1819]	[-77]
4	30×232	131	101	[2227]	[-43]
5	24×290	157	133	[2669]	[-11]
6	20×348	184	164	3128	20
7	12×580	296	284	5032	140
8	10×696	353	343	6001	199
9	8×870	439	431	7463	287
10	6×1160	583	577	9911	433
11	4×1740	872	868	14824	724
12	2×3480	1741	1739	29597	1595

$[a_k]$: solutions to be rejected as $a_k < 0$

Note that here also some of the first solutions a_k may be rejected if half the difference of the factor and co factor of N is such that $((v_k - u_k)/2) < 6n(3n-1)$, yielding a non-positive value of a_k.
As a first example, let $M = 25$ with $n = 1$. Then $X = s/5$, $Y = a + 12$ and there is only one way to decompose $N = 52$ in the product of two even integer factors, $N = 52 = 2 \times 26 = u_1 v_1$, yielding then $\varphi = 1$ and there is only one solution, given by $X_1 = 14$ and $Y_1 = 12$, or $s_1 = 70$ and $a_1 = 0$. This case for $M = 25$ must be rejected as it has no solution with $a > 0$. Note however that this solution with $s = 70$ and $a = 0$ for the case $M = 25$ is obviously equivalent to the solution with $s = 70$ and $a = 1$ for the case $M = 24$ of Lucas' cannonball problem.
A second example, let $M = 289$ with $n = 3$. Then $X = s/17$, $Y = a + 144$ and $N = 6960$. As there are twelve ways to decompose $N = 6960$ in products of two even integer factors, there are $\varphi = 12$ solutions in X and Y given in Table 4, five of which have to be rejected as the corresponding values of a_k are negative.

4 Conclusion

The problem of finding all the integer solutions of the sum of M consecutive integer squares starting at $a^2 \geq 1$ being equal to a squared integer s^2 can be written as a Diophantine quadratic equation $M\left[(a + (M-1)/2)^2 + \left(M^2 - 1\right)/12\right] = s^2$ in variables a and s. Based on previous results, it is known that integer solutions exist only if $M \equiv 0, 9, 24$ or $33\,(mod\,72)$; or $M \equiv 1, 2$ or $16\,(mod\,24)$; or $M \equiv 11\,(mod\,12)$.
If M is different from a square integer, the Diophantine quadratic equation is solved generally by transforming it into a generalized Pell equation whose form depends on the $(mod\,4)$ congruent value of M, and whose solutions, if

existing, yield all the solutions in a and s for a given value of M. Depending on whether this generalized Pell equation admits one or several fundamental solution(s), there are one or several infinite branches of solutions in a and s that can be written simply in function of Chebyshev polynomials evaluated at the fundamental solutions of the related simple Pell equation.
If M is a square integer, for $M \equiv 1\,(mod\,24)$ and $M = (6n-1)^2$, $\forall n \in \mathbb{Z}$, then the Diophantine quadratic equation in variables a and s reduces to a simple difference of integer squares which admits a finite number of solutions, yielding a finite number solutions in a and s to the initial problem.

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