A Schur-Weyl Duality Approach to
Walking on Cubes

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Abstract

Walks on the representation graph $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ determined by a group G and a G-module V are related to the centralizer algebras of the action of G on the tensor powers $\mathsf{V}^{\otimes k}$ via Schur-Weyl duality. This paper explores that connection when the group is $\mathbb{Z}_2^n$ and the module V is chosen so the representation graph is the n-cube. We describe a basis for the centralizer algebras in terms of labeled partition diagrams. We obtain an expression for the number of walks by counting certain partitions and determine the exponential generating functions for the number of walks.

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1   Introduction

Walks on graphs have widespread applications in modeling networks, particle interactions, biological and random processes, and many other phenomena. Typically, the walker (an impulse, physical or biological quantity, or person) transitions from one node to another along an edge, which may have an assigned probability. Some natural questions that arise in this context are: How many different walks of k steps are there from node a to node b on the graph? What is the probability that a particle moves from a to b in k steps?

Walks on graphs are also related to chip-firing games, or to what is often referred to in physics as the (abelian) sandpile model. In a chip-firing game, each node starts with a pile of chips. A step consists of selecting a node with at least

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as many chips as its degree and moving one chip from that node to each of its adjacent neighbors. The game continues indefinitely or terminates when no more firings are possible. In the latter case, the number of steps is related to the least positive eigenvalue of the Laplace operator of the graph [BLS], and it is bounded by an expression in the Dirichlet eigenvalues [CE].

The graphs considered here arise from the representation theory of groups in the following way: Let G be a finite group and V be a finite-dimensional G-module over the complex field $\mathbb{C}$. The representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$ of G associated to V has nodes corresponding to the irreducible G-modules $\{\mathsf{G}^\lambda \mid \lambda \in \Lambda(\mathsf{G})\}$ over $\mathbb{C}$. For $\mu \in \Lambda(\mathsf{G})$, there are $a_{\mu,\lambda}$ edges from μ to λ in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ if

$$\mathsf{G}^\mu \otimes \mathsf{V} = \bigoplus_{\lambda \in \Lambda(\mathsf{G})} a_{\mu,\lambda} \mathsf{G}^\lambda.$$

Thus, the number of edges $a_{\mu,\lambda}$ from μ to λ in $\mathcal{R}_\mathsf{V}(\mathsf{G})$ is the multiplicity of G^λ as a summand of $\mathsf{G}^\mu \otimes \mathsf{V}$.

Let $\mathsf{G}^\mathbf{0}$ be the trivial one-dimensional G-module on which every element of G acts as the identity transformation. Since each step on the graph is achieved by tensoring with V,

$$\mathsf{m}_k^\lambda : = \text{number of walks of } k \text{ steps from } \mathbf{0} \text{ to } \lambda$$
$$= \text{multiplicity of } \mathsf{G}^\lambda \text{ in } \mathsf{G}^\mathbf{0} \otimes \mathsf{V}^{\otimes k} \cong \mathsf{V}^{\otimes k}.$$

When the action of G on V is faithful, then every irreducible G-module G^λ occurs in some $\mathsf{V}^{\otimes k}$.

The *centralizer algebra*,

$$\mathsf{Z}_k(\mathsf{G}) = \{z \in \mathsf{End}(\mathsf{V}^{\otimes k}) \mid z(g.w) = g.z(w) \ \forall \, g \in \mathsf{G}, w \in \mathsf{V}^{\otimes k}\}, \qquad (1.1)$$

plays an essential role in studying $\mathsf{V}^{\otimes k}$, as it contains the projection maps onto the irreducible summands of $\mathsf{V}^{\otimes k}$.

Let $\Lambda_k(\mathsf{G})$ denote the subset of $\Lambda(\mathsf{G})$ corresponding to the irreducible G-modules which occur in $\mathsf{V}^{\otimes k}$ with multiplicity at least one. *Schur-Weyl duality* establishes important connections between the representation theories of G and $\mathsf{Z}_k(\mathsf{G})$:

- the irreducible $\mathsf{Z}_k(\mathsf{G})$-modules are in bijection with the elements of $\Lambda_k(\mathsf{G})$;

- the G-module decomposition of $\mathsf{V}^{\otimes k}$ into irreducible summands is given by $\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} \mathsf{m}_k^\lambda \mathsf{G}^\lambda$, where m_k^λ is the number of walks of k steps from $\mathbf{0}$ to λ on the representation graph $\mathcal{R}_\mathsf{V}(\mathsf{G})$;

- the $\mathsf{Z}_k(\mathsf{G})$-module decomposition of $\mathsf{V}^{\otimes k}$ into irreducible $\mathsf{Z}_k(\mathsf{G})$-modules $\mathsf{Z}_k^\lambda, \lambda \in \Lambda_k(\mathsf{G})$, is given by $\mathsf{V}^{\otimes k} \cong \bigoplus_{\lambda \in \Lambda_k(\mathsf{G})} \mathsf{d}^\lambda \mathsf{Z}_k^\lambda$, where

$$\mathsf{d}^\lambda = \dim \mathsf{G}^\lambda \quad \text{and} \quad \mathsf{m}_k^\lambda = \dim \mathsf{Z}_k^\lambda.$$

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- $\dim Z_k(G) = \sum_{\lambda \in \Lambda_k(G)} (\dim Z_k^\lambda)^2 = \sum_{\lambda \in \Lambda_k(G)} (m_k^\lambda)^2 = m_{2k}^0$,
  (the number of walks of $2k$ steps from $\mathbf{0}$ to $\mathbf{0}$ on $\mathcal{R}_V(G)$).

The following result, which was shown in [B], gives an efficient way of computing the Poincaré series

$$m^\mu(t) = \sum_{k \geq 0} m_k^\mu t^k$$

for the multiplicities m_k^μ, $\mu \in \Lambda(G)$ (that is, for the number of walks of k steps from $\mathbf{0}$ to μ on the representation graph $\mathcal{R}_V(G)$, or equivalently, for the dimension of the centralizer modules Z_k^μ, $k \geq 0$). We assume $V^{\otimes 0} = G^{\mathbf{0}}$ and the columns of the adjacency matrix have been indexed so that the one corresponding to $\mathbf{0}$ is the first.

Theorem 1.2. ([B, Thm. 2.1]) *Let G be a finite group with irreducible modules G^λ, $\lambda \in \Lambda(G)$, over $\mathbb{C}$, and let V be a finite-dimensional G-module such that the action of G on V is faithful. Assume $m^\mu(t) = \sum_{k \geq 0} m_k^\mu t^k$ is the Poincaré series for the multiplicities m_k^μ ($k \geq 0$) of G^μ in $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$. Let A be the adjacency matrix of the representation graph $\mathcal{R}_V(G)$, and let M^μ be the matrix $I - tA$ with the column indexed by μ replaced by $\underline{\delta} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then*

$$m^\mu(t) = \frac{\det(M^\mu)}{\det(I - tA)} = \frac{\det(M^\mu)}{\prod_{g \in \Gamma} (1 - \chi_V(g)t)}, \tag{1.3}$$

where Γ is a set of conjugacy class representatives of G and $\chi_V(g)$ is the value of the character χ_V of V on g.

Remark 1.4. *The space of G-invariants in $V^{\otimes k}$ is the sum of the copies of the trivial G-module $G^{\mathbf{0}}$ in $V^{\otimes k}$. Hence, the dimension of the space of G-invariants in $V^{\otimes k}$ is $m_k^{\mathbf{0}}$, and the Poincaré series $m^{\mathbf{0}}(t)$ is the generating function for those dimensions. Under the assumptions of Theorem 1.2, it follows that*

$$m^{\mathbf{0}}(t) = \frac{\det\left(I - t\mathring{A}\right)}{\det\left(I - tA\right)} = \frac{\det\left(I - t\mathring{A}\right)}{\prod_{g \in \Gamma} (1 - \chi_V(g)t)}, \tag{1.5}$$

where A is the adjacency matrix of the representation graph $\mathcal{R}_V(G)$ and $\mathring{A}$ is the adjacency matrix of the graph obtained from $\mathcal{R}_V(G)$ by removing the node $\mathbf{0}$ and all its incident edges.

Consideration of the minimum polynomial of the adjacency matrix of a graph with finitely many vertices leads to the next result.

Proposition 1.6. *Suppose*

$$p(t) = t^d + p_{d-1}t^{d-1} + \cdots + p_1 t + p_0 \in \mathbb{C}[t]$$

is the minimum polynomial of the adjacency matrix A *of a finite graph* $\mathcal{G}$*, and* $\mathsf{m}^{\mu}_{\nu,k}$ *is the number of walks on* $\mathcal{G}$ *of* k *steps from a node* ν *to a node* μ*.*

(i) *The following recursion relation holds for all* $k \geq 0$*:*

$$\mathsf{m}^{\mu}_{\nu,k+d} + p_{d-1}\mathsf{m}^{\mu}_{\nu,k+d-1} + \cdots + p_1 \mathsf{m}^{\mu}_{\nu,k+1} + p_0\mathsf{m}^{\mu}_{\nu,k} = 0. \qquad (1.7)$$

(ii) *The corresponding exponential generating function,* $\mathsf{g}^{\mu}_{\nu}(t) = \sum_{k \geq 0} \mathsf{m}^{\mu}_{\nu,k} \dfrac{t^k}{k!}$,

satisfies the differential equation

$$y^{(d)} + p_{d-1}y^{(d-1)} + \cdots p_1 y^{(1)} + p_0 y = 0. \qquad (1.8)$$

Proof. If $p(t)$ is as above, then

$$\mathrm{A}^{k+d} + p_{d-1}\mathrm{A}^{k+d-1} + \cdots + p_1 \mathrm{A}^{k+1} + p_0\mathrm{A}^k = \mathrm{A}^k p(\mathrm{A}) = 0 \qquad (1.9)$$

for all $k \geq 0$. Since $(\mathrm{A}^{\ell})_{\nu,\mu} = \mathsf{m}^{\mu}_{\nu,\ell}$ for all $\ell \geq 0$, taking the (ν, μ) entry of (1.9) gives the desired result in (1.7). It follows from (1.7) that $\mathsf{g}^{\mu}_{\nu}(t)$ satisfies (1.8), where $y^{(r)} = \left(\frac{d}{dt}\right)^r (y)$ for all $r \geq 0$. $\qquad \square$

Theorem 1.10. *Let* G *be a finite group with irreducible modules* G^{λ}*,* $\lambda \in \Lambda(\mathsf{G})$*, over* $\mathbb{C}$*, and let* V *be a faithful finite-dimensional* G*-module. Assume* $\mathsf{g}^{\mu}(t) = \sum_{k \geq 0} \mathsf{m}^{\mu}_k \dfrac{t^k}{k!}$ *is the exponential generating function for the multiplicity* m^{μ}_k *of* G^{μ} *in* $\mathsf{V}^{\otimes k}$ *for* $k \geq 0$ *(equivalently for the number of walks of* k *steps on* $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$ *from* $\mathbf{0}$ *to* μ*). Then*

(i) $\mathsf{g}^{\mu}(t)$ *satisfies the differential equation* $y^{(d)} + p_{d-1}y^{(d-1)} + \cdots + p_1 y^{(1)} + p_0 y = 0.$ *where* $p(t) = t^d + p_{d-1}t^{d-1} + \cdots + p_1 t + p_0$ *is the minimum polynomial of the adjacency matrix* A *of the representation graph* $\mathcal{R}_{\mathsf{V}}(\mathsf{G})$*.*

(ii) *The roots of* $p(t)$ *are the distinct character values in* $\{\chi_{\mathsf{V}}(g) \mid g \in \Gamma\}$*, where* Γ *is a set of conjugacy class representatives of* G*. When* $p(t)$ *has distinct roots* ξ_j*,* $j = 1, \ldots, d$*, then* $\mathsf{g}^{\mu}(t)$ *is a linear combination of the exponential functions* $e^{\xi_j t}$*.*

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Proof. The assertion in (i) is an immediate consequence of Proposition 1.6. That the roots of $p(t)$ are given by character values follows as in (1.3) (compare [St, Sec. 1]). □

In this work, we focus on the abelian group $\mathsf{G} = \mathbb{Z}_2^n$, where $\mathbb{Z}_2$ denotes the integers mod 2. This group appears in many different settings including important ones in computing, where the elements $\mathsf{a} = (\mathsf{a}_1, \ldots, \mathsf{a}_n)$, $\mathsf{a}_i \in \{0, 1\}$ for all $i \in [1, n] := \{1, 2, \ldots, n\}$, of $\mathbb{Z}_2^n$ are regarded as n bits. Here it is convenient to think of $\mathbb{Z}_2^n$ as a multiplicative group and write e^{a} rather than a, so that the group operation is given by $\mathsf{e}^{\mathsf{a}}\mathsf{e}^{\mathsf{b}} = \mathsf{e}^{\mathsf{a}+\mathsf{b}}$, $\mathsf{a}, \mathsf{b} \in \mathbb{Z}_2^n$, where the sum $\mathsf{a} + \mathsf{b}$ is componentwise addition in $\mathbb{Z}_2$. Since $\mathbb{Z}_2^n$ is abelian, the irreducible $\mathbb{Z}_2^n$-modules are all one-dimensional, and we label them with the elements of $\mathbb{Z}_2^n$. Thus, for $\mathsf{b} \in \mathbb{Z}_2^n$, let $\mathsf{X}^{\mathsf{b}} = \mathbb{C}x_{\mathsf{b}}$, where

$$\mathsf{e}^{\mathsf{a}}x_{\mathsf{b}} = (-1)^{\mathsf{a}\cdot\mathsf{b}}x_{\mathsf{b}},$$

and $\mathsf{a} \cdot \mathsf{b}$ is the usual dot product. Observe that

$$\mathsf{e}^{\mathsf{a}+\mathsf{a}'}x_{\mathsf{b}} = (-1)^{(\mathsf{a}+\mathsf{a}')\cdot\mathsf{b}}x_{\mathsf{b}} = (-1)^{\mathsf{a}\cdot\mathsf{b}}(-1)^{\mathsf{a}'\cdot\mathsf{b}}x_{\mathsf{b}} = \mathsf{e}^{\mathsf{a}}(\mathsf{e}^{\mathsf{a}'}x_{\mathsf{b}}),$$

so this does in fact define a $\mathbb{Z}_2^n$-module action on X^{b}, and the corresponding character χ_{b} of X^{b} is given by

$$\chi_{\mathsf{b}}(\mathsf{a}) = \mathsf{tr}_{\mathsf{X}^{\mathsf{b}}}(\mathsf{e}^{\mathsf{a}}) = (-1)^{\mathsf{a}\cdot\mathsf{b}}. \tag{1.11}$$

Moreover, since

$$\mathsf{e}^{\mathsf{a}}(x_{\mathsf{b}} \otimes x_{\mathsf{c}}) = (-1)^{\mathsf{a}\cdot\mathsf{b}}(-1)^{\mathsf{a}\cdot\mathsf{c}}x_{\mathsf{b}} \otimes x_{\mathsf{c}} = (-1)^{\mathsf{a}\cdot(\mathsf{b}+\mathsf{c})}x_{\mathsf{b}} \otimes x_{\mathsf{c}},$$

we have that

$$\mathsf{X}^{\mathsf{b}} \otimes \mathsf{X}^{\mathsf{c}} \cong \mathsf{X}^{\mathsf{b}+\mathsf{c}} \tag{1.12}$$

for all $\mathsf{b}, \mathsf{c} \in \mathbb{Z}_2^n$.

For $i \in [1, n]$, let ε_i denote the n-tuple in $\mathbb{Z}_2^n$ with 1 as its ith component and 0 for all other components. Set

$$\mathsf{V} = \mathsf{X}^{\varepsilon_1} \oplus \mathsf{X}^{\varepsilon_2} \oplus \cdots \oplus \mathsf{X}^{\varepsilon_n}.$$

The nodes of the representation graph $\mathcal{R}_{\mathsf{V}}(\mathbb{Z}_2^n)$ are just the elements of $\mathbb{Z}_2^n$, hence, are the vertices $\mathsf{a} = (\mathsf{a}_1, \ldots, \mathsf{a}_n)$, $\mathsf{a}_i \in \{0, 1\}$ for all i, of the n-cube (hypercube). By (1.12), $\mathsf{X}^{\mathsf{a}} \otimes \mathsf{V} = \sum_{i=1}^n \mathsf{X}^{\mathsf{a}+\varepsilon_i}$, so that tensoring X^{a} with $\mathsf{V}^{\otimes k}$ amounts to taking a walk of k steps on the n-cube starting from node a.

We apply Schur-Weyl duality results to relate walks of k steps on the n-cube to the centralizer algebra $\mathsf{Z}_k(\mathbb{Z}_2^n) = \mathsf{End}_{\mathbb{Z}_2^n}(\mathsf{V}^{\otimes k})$ and its irreducible modules $\mathsf{Z}_k^{\mathsf{a}}$.

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That connection enables us to give an expression for the dimension of $\mathsf{Z}_k(\mathbb{Z}_2^n)$ and for the dimension of the module $\mathsf{Z}_k^{\mathsf{a}}$ for all $\mathsf{a} \in \mathbb{Z}_2^n$. The group $\mathbb{Z}_2^n$ is a normal subgroup of the reflection group $\mathsf{G}(2,1,n) \cong \mathbb{Z}_2 \wr \mathrm{S}_n$ (the Weyl group of type B_n). We identify V with the irreducible $\mathsf{G}(2,1,n)$-module on which elements of $\mathsf{G}(2,1,n)$ act as $n \times n$ signed permutation matrices relative to the basis $x_i = x_{\varepsilon_i}$, $i \in [1,n]$. Tanabe [T] has described a basis for the centralizer algebra $\mathsf{Z}_k\big(\mathsf{G}(2,1,n)\big) = \mathsf{End}_{\mathsf{G}(2,1,n)}(\mathsf{V}^{\otimes k})$ in terms of diagrams corresponding to set partitions. Since $\mathbb{Z}_2^n \subseteq \mathsf{G}(2,1,n)$, there is a reverse inclusion of centralizers

$$\mathsf{Z}_k\big(\mathsf{G}(2,1,n)\big) \subseteq \mathsf{Z}_k(\mathbb{Z}_2^n).$$

We use that relationship to index a basis for $\mathsf{Z}_k(\mathbb{Z}_2^n)$ by labeled partition diagrams and to give a formula for $\dim \mathsf{Z}_k(\mathbb{Z}_2^n)$ and for $\dim \mathsf{Z}_k^{\mathsf{a}}$ by counting certain partitions. Theorem 1.2 can be used to obtain the Poincaré series (generating function) for the dimensions of the irreducible modules $\mathsf{Z}_k^{\mathsf{a}}$, $k \geq 0$ (hence, for the number of walks of k steps from $\mathbf{0} = (0,\dots,0)$ to a on the n-cube). In the final section, we discuss these series and also determine the exponential generating functions for these dimensions. The main result of that section is Theorem 4.24, which says that for $\mathsf{a} \in \mathbb{Z}_2^n$,

$$\mathsf{g}^{\mathsf{a}}(t) = \sum_{k \geq 0} \mathsf{m}_k^{\mathsf{a}} \frac{t^k}{k!} = (\cosh t)^{n-h} \, (\sinh t)^h$$

where $h = h(\mathsf{a})$, the Hamming weight (the number of ones) of a, and $\cosh t$ and $\sinh t$ are hyperbolic cosine and sine.

The group $\mathsf{G}(2,1,n)$ also contains the symmetric group S_n as a subgroup, and there is a reverse inclusion of centralizer algebras $\mathsf{Z}_k\big(\mathsf{G}(2,1,n)\big) \subseteq \mathsf{Z}_k(\mathrm{S}_n)$, which has provided a number of interesting results and motivated the study of the corresponding Hecke algebras (see for example, [A1, A2, AK, HR1]).

2   Walks on graphs and on the n-cube

Assume A is the adjacency matrix of a finite graph $\mathcal{G}$ with undirected edges so that A is a real symmetric matrix. Let $\mathcal{V}$ be the vertex set of $\mathcal{G}$. Then A has real eigenvalues λ_v, $v \in \mathcal{V}$. Let $\mathcal{E}_v = (\mathcal{E}_{u,v})_{u \in \mathcal{V}}$ be the orthonormal eigenvectors of A so that $\mathrm{A}\mathcal{E}_v = \lambda_v \mathcal{E}_v$, and let $\mathcal{E} = (\mathcal{E}_{u,v})$ be the matrix whose vth column is the column vector $\mathcal{E}_v$. Then the transpose $\mathcal{E}^{\mathsf{t}} = \mathcal{E}^{-1}$, and $\mathcal{E}\mathcal{E}^{\mathsf{t}} = \mathcal{E}^{\mathsf{t}}\mathcal{E} = \mathrm{I}_{|\mathcal{V}|}$.

The next results are from [S, Chaps. 1,2]. We include the proofs, in part to establish our notation.

Proposition 2.1. *The number of walks on the graph $\mathcal{G}$ of k-steps from node v to node w is* $(\mathrm{A}^k)_{v,w} = \sum_{u \in \mathcal{V}} \mathcal{E}_{v,u} \mathcal{E}_{w,u} \lambda_u^k.$

Proof. We have $\mathcal{E}^{-1}A\mathcal{E} = \text{diag}\{\lambda_u\}_{u\in\mathcal{V}}$, so that $\mathcal{E}^{-1}A^k\mathcal{E} = (\mathcal{E}^{-1}A\mathcal{E})^k = \text{diag}\{\lambda_u^k\}_{u\in\mathcal{V}}$. Therefore, $A^k = \mathcal{E}\,\text{diag}\{\lambda_u^k\}\,\mathcal{E}^t$, and

$$(A^k)_{v,w} = \sum_{u\in\mathcal{V}} \mathcal{E}_{v,u}\lambda_u^k(\mathcal{E}^t)_{u,w} = \sum_{u\in\mathcal{V}} \mathcal{E}_{v,u}\mathcal{E}_{w,u}\lambda_u^k. \qquad \square$$

We specialize now to the case that G is the group $\mathbb{Z}_2^n$ and set $\mathbf{0} = (0,\dots,0)$. For $\mathsf{a} \in \mathbb{Z}_2^n$, let $h(\mathsf{a})$ be the *Hamming weight* of a, i.e., the number of ones in a. We consider the case that the graph $\mathcal{G}$ is the representation graph $\mathcal{R}_{V_S}(\mathbb{Z}_2^n)$ obtained from the $\mathbb{Z}_2^n$-module $V_S = \bigoplus_{s\in S} X^s$, where S is a nonempty subset of $\mathbb{Z}_2^n$. The next result gives the eigenvalues and corresponding eigenvectors of the adjacency matrix A_S for $\mathcal{R}_{V_S}(\mathbb{Z}_2^n)$. Note that $X^a \otimes V_S = \bigoplus_{s\in S} X^{a+s}$. For $\mathsf{b} \in \mathbb{Z}_2^n$, we will write $\underline{\mathsf{b}}$ for $2^n \times 1$ column vector with 1 in the row corresponding to b and 0 for all its other components. The argument in [S, Chap. 2] involves the discrete Radon transform on $\mathbb{Z}_2^n$ (see also [DG1]), which we don't use here.

Proposition 2.2. *Let S be a nonempty subset of $\mathbb{Z}_2^n$, and let A_S be the adjacency matrix of the representation graph $\mathcal{R}_{V_S}(\mathbb{Z}_2^n)$, where $V_S = \bigoplus_{s\in S} X^s$. Then the character values $\chi_{V_S}(\mathsf{a}) := \chi_{V_S}(e^{\mathsf{a}}) = \sum_{s\in S}(-1)^{\mathsf{a}\cdot\mathsf{s}}$ for $\mathsf{a} \in \mathbb{Z}_2^n$ are the eigenvalues of A_S, and the vector $\mathcal{E}_\mathsf{a} = \sum_{\mathsf{b}\in\mathbb{Z}_2^n}(-1)^{\mathsf{a}\cdot\mathsf{b}}\underline{\mathsf{b}}$ is an eigenvector for A_S corresponding to the eigenvalue $\chi_{V_S}(\mathsf{a})$. The vectors $\mathcal{E}_\mathsf{a}, \mathsf{a} \in \mathbb{Z}_2^n$, give a basis for $\mathbb{C}^{2^n}$.*

Proof. Observe that

$$A_S\mathcal{E}_\mathsf{a} = \sum_{\mathsf{b}\in\mathbb{Z}_2^n}(-1)^{\mathsf{a}\cdot\mathsf{b}}\left(\sum_{s\in S}\underline{\mathsf{b}+\mathsf{s}}\right) = \sum_{\mathsf{c}\in\mathbb{Z}_2^n}\left(\sum_{s\in S}(-1)^{\mathsf{a}\cdot(\mathsf{c}+\mathsf{s})}\right)\underline{\mathsf{c}}$$

$$= \left(\sum_{s\in S}(-1)^{\mathsf{a}\cdot\mathsf{s}}\right)\left(\sum_{\mathsf{c}\in\mathbb{Z}_2^n}(-1)^{\mathsf{a}\cdot\mathsf{c}}\underline{\mathsf{c}}\right)$$

$$= \chi_{V_S}(\mathsf{a})\mathcal{E}_\mathsf{a},$$

so that $\mathcal{E}_\mathsf{a}$ is an eigenvector corresponding to the eigenvalue $\chi_{V_S}(\mathsf{a})$.

We view $\mathcal{E}_\mathsf{a}$ as a column vector whose bth component is $(-1)^{\mathsf{a}\cdot\mathsf{b}}$. Taking the inner product of two such column vectors gives

$$\mathcal{E}_\mathsf{a} \cdot \mathcal{E}_{\mathsf{a}'} = \sum_{\mathsf{b}\in\mathbb{Z}_2^n}(-1)^{\mathsf{a}\cdot\mathsf{b}}(-1)^{\mathsf{a}'\cdot\mathsf{b}} = \sum_{\mathsf{b}\in\mathbb{Z}_2^n}(-1)^{(\mathsf{a}+\mathsf{a}')\cdot\mathsf{b}} \qquad (2.3)$$

$$= \begin{cases} 0 & \text{if } \mathsf{a} \neq \mathsf{a}' \text{ (equivalently, if } \mathsf{a}+\mathsf{a}' \neq \mathbf{0}), \\ 2^n & \text{if } \mathsf{a} = \mathsf{a}' \text{ (equivalently, if } \mathsf{a}+\mathsf{a}' = \mathbf{0}), \end{cases}$$

which is just a statement of the well-known orthogonality of the characters χ_{a} and $\chi_{\mathsf{a}'}$ for $\mathsf{a} \neq \mathsf{a}'$. Thus, the eigenvectors $\mathcal{E}_{\mathsf{a}}$ are orthogonal, hence linearly independent, and there are 2^n of them, so they determine a basis for $\mathbb{C}^{2^n}$. □

In the special case $\mathsf{S} = \{\varepsilon_i \mid i \in [1, n]\}$, we have $\mathsf{V_S} = \mathsf{V}$, and the representation graph is just the n-cube. In this case, Proposition 2.2 gives

Corollary 2.4. *Let* A *be the adjacency matrix of the n-cube. Then* A *has eigenvalues* $\chi_{\mathsf{V}}(\mathsf{a}) = \sum_{i=1}^{n} (-1)^{\mathsf{a} \cdot \varepsilon_i} = \sum_{i=1}^{n} (-1)^{\mathsf{a}_i} = n - 2h(\mathsf{a})$ *for* $\mathsf{a} = (\mathsf{a}_1, \ldots, \mathsf{a}_n) \in \mathbb{Z}_2^n$, *where* $h(\mathsf{a})$ *is the Hamming weight of* a, *and* $\mathcal{E}_{\mathsf{a}} = \sum_{\mathsf{b} \in \mathbb{Z}_2^n} (-1)^{\mathsf{a} \cdot \mathsf{b}} \underline{\mathsf{b}}$ *is an eigenvector for* A *corresponding to the eigenvalue* $\chi_{\mathsf{V}}(\mathsf{a})$. *Thus, the eigenvalues of* A *are* $n - 2h$ *for* $h = 0, 1, \ldots, n$ *and* $n - 2h$ *occurs with multiplicity* $\binom{n}{h}$, *and the eigenvectors* $\mathcal{E}_{\mathsf{a}}$, $\mathsf{a} \in \mathbb{Z}_2^n$, *form a basis for* $\mathbb{C}^{2^n}$.

Our next goal is to show the following (compare [S, Cor. 2.5]).

Corollary 2.5. *Let* $\mathsf{b}, \mathsf{c} \in \mathbb{Z}_2^n$ *and suppose* $h(\mathsf{b} + \mathsf{c}) = h$ *(i.e.* b *and* c *disagree in exactly h coordinates). Then the number of walks from* b *to* c *of k steps on the n-cube is given by*

$$(A^k)_{\mathsf{b},\mathsf{c}} = \frac{1}{2^n} \sum_{i=0}^{n} \sum_{j=0}^{h} (-1)^j \binom{h}{j} \binom{n-h}{i-j} (n - 2i)^k.$$

Proof. For $\mathsf{a} \in \mathbb{Z}_2^n$, we see from the calculation in (2.3) that $\mathcal{E}_{\mathsf{a}} \cdot \mathcal{E}_{\mathsf{a}} = 2^n$. Therefore, to get the matrix $\mathcal{E}$ in Proposition 2.1, we need to divide $\mathcal{E}_{\mathsf{a}}$ by $2^{n/2}$. Let $\mathcal{E}$ be the $(2^n \times 2^n)$-matrix whose ath column is $2^{-n/2} \mathcal{E}_{\mathsf{a}}$. Then by Proposition 2.1, we have for the adjacency matrix A of the n-cube,

$$(A^k)_{\mathsf{b},\mathsf{c}} = 2^{-n} \sum_{\mathsf{a} \in \mathbb{Z}_2^n} \mathcal{E}_{\mathsf{b},\mathsf{a}} \mathcal{E}_{\mathsf{c},\mathsf{a}} \lambda_{\mathsf{a}}^k$$

where

$$\mathcal{E}_{\mathsf{b},\mathsf{a}} \mathcal{E}_{\mathsf{c},\mathsf{a}} \lambda_{\mathsf{a}}^k = (-1)^{\mathsf{a} \cdot \mathsf{b}} (-1)^{\mathsf{a} \cdot \mathsf{c}} \left(\sum_{i=1}^{n} (-1)^{\mathsf{a}_i} \right)^k = (-1)^{\mathsf{a} \cdot (\mathsf{b} + \mathsf{c})} (n - 2h(\mathsf{a}))^k,$$

and $h(\mathsf{a})$ is the Hamming weight of a.

We count the number of $\mathsf{a} \in \mathbb{Z}_2^n$ with Hamming weight $h(\mathsf{a}) = i$ and with a having j ones in common with $\mathsf{b} + \mathsf{c}$ for $j = 0, 1, \ldots, h = h(\mathsf{b} + \mathsf{c})$. We can choose j ones in $\mathsf{b} + \mathsf{c}$ that agree with j ones in a in $\binom{h}{j}$ ways. The remaining $i - j$ ones in a can be placed in the remaining $n - h$ positions of a in $\binom{n-h}{i-j}$ ways. Since $\mathsf{a} \cdot (\mathsf{b} + \mathsf{c}) \equiv j \bmod 2$, we have the desired result. □

3 Consequences for the centralizer algebras $Z_k(\mathbb{Z}_2^n)$

Recall that in the n-cube case $\mathsf{G} = \mathbb{Z}_2^n$, $\mathsf{V} = \mathsf{X}^{\varepsilon_1} \oplus \cdots \oplus \mathsf{X}^{\varepsilon_n}$, and the irreducible modules for $\mathbb{Z}_2^n$ and for its centralizer algebras $Z_k(\mathbb{Z}_2^n) = \mathsf{End}_{\mathbb{Z}_2^n}(\mathsf{V}^{\otimes k})$ are labeled by elements $\mathsf{a} \in \mathbb{Z}_2^n$. Then Proposition 2.2 and Corollary 2.5 imply

Corollary 3.1.

(i) *The dimension of the irreducible $Z_k(\mathbb{Z}_2^n)$-module Z_k^a labeled by $\mathsf{a} \in \mathbb{Z}_2^n$ is given by*

$$\mathsf{dim}\, \mathsf{Z}_k^\mathsf{a} = (\mathsf{A}^k)_{\mathbf{0},\mathsf{a}} = \frac{1}{2^n} \sum_{i=0}^{n} \sum_{j=0}^{h} (-1)^j \binom{h}{j} \binom{n-h}{i-j} (n-2i)^k,$$

where $h = h(\mathsf{a})$ is the Hamming weight of a. In particular, the irreducible $Z_k(\mathbb{Z}_2^n)$-modules labeled by a and b with $h(\mathsf{a}) = h(\mathsf{b})$ have the same dimension.

(ii) $\mathsf{dim}\, Z_k(\mathbb{Z}_2^n) = \dfrac{1}{2^n} \displaystyle\sum_{i=0}^{n} \binom{n}{i} (n-2i)^{2k}$

Remark 3.2. *Part* (ii) *is a special case of* (i)*, since we know by Schur-Weyl duality that*

$$\mathsf{dim}\, Z_k(\mathbb{Z}_2^n) = \mathsf{dim}\, \mathsf{Z}_{2k}^{\mathbf{0}}$$

$$= \frac{1}{2^n} \sum_{i=0}^{n} \sum_{j=0}^{h} (-1)^j \binom{h}{j} \binom{n-h}{i-j} (n-2i)^{2k}, \;\; where\; h = h(\mathbf{0}) = 0,$$

$$= \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} (n-2i)^{2k}.$$

Next, we construct an explicit basis for $Z_k(\mathbb{Z}_2^n)$. Let $\{x_i = x_{\varepsilon_i} \mid i \in [1,n]\}$ be the basis for $\mathsf{V} = \mathsf{X}^{\varepsilon_1} \oplus \cdots \oplus \mathsf{X}^{\varepsilon_n}$ such that $\mathsf{e}^\mathsf{a} x_i = (-1)^{\mathsf{a} \cdot \varepsilon_i} x_i = (-1)^{\mathsf{a}_i} x_i$. Then for $\beta = (\beta_1, \ldots, \beta_k) \in [1,n]^k$, set $x_\beta = x_{\beta_1} \otimes \cdots \otimes x_{\beta_k}$. The elements x_β, $\beta \in [1,n]^k$, form a basis for $\mathsf{V}^{\otimes k}$ with

$$\mathsf{e}^\mathsf{a} x_\beta = (-1)^{\mathsf{a} \cdot \left(\varepsilon_{\beta_1} + \cdots + \varepsilon_{\beta_k} \right)} x_\beta.$$

Suppose $\Phi \in \mathsf{End}(\mathsf{V}^{\otimes k})$, and for $\alpha \in [1,n]^k$ assume

$$\Phi x_\alpha = \sum_{\beta \in [1,n]^k} \Phi_\alpha^\beta x_\beta,$$

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where $\Phi_\alpha^\beta \in \mathbb{C}$ for $\alpha, \beta \in [1, n]^k$. Then for all $\mathsf{a} \in \mathbb{Z}_2^n$,

$$\mathsf{e}^{\mathsf{a}} \Phi x_\alpha = \sum_{\beta \in [1,n]^k} (-1)^{\mathsf{a} \cdot (\varepsilon_{\beta_1} + \cdots + \varepsilon_{\beta_k})} \Phi_\alpha^\beta x_\beta$$

$$\Phi \mathsf{e}^{\mathsf{a}} x_\alpha = (-1)^{\mathsf{a} \cdot (\varepsilon_{\alpha_1} + \cdots + \varepsilon_{\alpha_k})} \sum_{\beta \in [1,n]^k} \Phi_\alpha^\beta x_\beta.$$

Thus, in order for Φ to belong to $\mathsf{Z}_k(\mathbb{Z}_2^n)$ we must have $\varepsilon_{\alpha_1} + \cdots + \varepsilon_{\alpha_k} = \varepsilon_{\beta_1} + \cdots + \varepsilon_{\beta_k}$ for all $\alpha, \beta \in [1, n]^k$ such that $\Phi_\alpha^\beta \neq 0$. Let $\mathsf{E}_\alpha^\beta \in \mathsf{End}(\mathsf{V}^{\otimes k})$ be given by

$$\mathsf{E}_\alpha^\beta x_\gamma = \delta_{\alpha,\gamma} x_\beta \quad \text{for all } \gamma \in [1, n]^k, \tag{3.3}$$

where $\delta_{\alpha,\gamma}$ is the Kronecker delta. Since the E_α^β with $\varepsilon_{\alpha_1} + \cdots + \varepsilon_{\alpha_k} = \varepsilon_{\beta_1} + \cdots + \varepsilon_{\beta_k}$ clearly satisfy the requisite condition to belong to $\mathsf{Z}_k(\mathbb{Z}_2^n)$, and they span $\mathsf{Z}_k(\mathbb{Z}_2^n)$, we have the following

Theorem 3.4. *A basis for the centralizer algebra $\mathsf{Z}_k(\mathbb{Z}_2^n)$ is*

$$\{\mathsf{E}_\alpha^\beta \mid \alpha, \beta \in [1, n]^k, \ \varepsilon_{\alpha_1} + \cdots + \varepsilon_{\alpha_k} = \varepsilon_{\beta_1} + \cdots + \varepsilon_{\beta_k}\},$$

where E_α^β is as in (3.3).

4 Partition diagrams

Let $\mathcal{P}(k, n)$ denote the set partitions of $[1, 2k]$ into at most n parts (blocks). We view the elements of $\mathcal{P}(k, n)$ diagrammatically and identify set partitions with their diagrams. For example, the diagram below corresponds to the set partition $\{1, 4\}, \{2, 6, 8, 9\}, \{3, 10\}, \{5, 7\}$ in $\mathcal{P}(5, n)$ for any $n \geq 4$.

$$d = \tag{4.1}$$
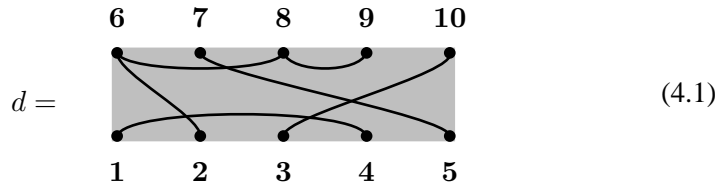
The way the edges are drawn is immaterial. What matters is that nodes in the same block are connected, and nodes in different blocks are not.

For $d \in \mathcal{P}(k, n)$, let

B_1 be the block of d containing 1;

B_2 be the block containing the smallest number not in B_1;

$\vdots$

B_j be the block containing the smallest number not in $\mathsf{B}_1 \cup \mathsf{B}_2 \cup \cdots \cup \mathsf{B}_{j-1}$.

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In the example above, we have ordered the blocks in this fashion, so that $\mathsf{B}_1 = \{1,4\}$, $\mathsf{B}_2 = \{2,6,8,9\}$, $\mathsf{B}_3 = \{3,10\}$, and $\mathsf{B}_4 = \{5,7\}$.

For $\ell \in [1,2k]$, set

$$\zeta_\ell = j \ \text{ if } \ell \in \mathsf{B}_j, \text{ and let} \tag{4.2}$$

$$\zeta_d = (\zeta_1,\ldots,\zeta_k) \ \text{ and } \zeta'_d = (\zeta_{k+1},\ldots,\zeta_{2k})$$

In our running example,
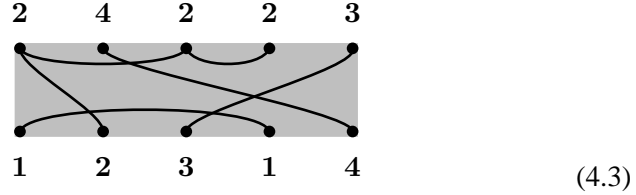
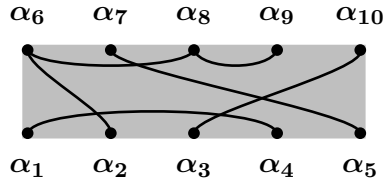so that $\zeta_d = (1,2,3,1,4)$ and $\zeta'_d = (2,4,2,2,3)$.

4.1 Definition of T_d

As in the previous section, assume $\alpha = (\alpha_1,\ldots,\alpha_k) \in [1,n]^k$ and let $x_\alpha = x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_k} \in \mathsf{V}^{\otimes k}$, where $x_i = x_{\varepsilon_i}$, $i = 1,\ldots,n$. We can regard V as the n-dimensional permutation module for the symmetric group S_n, with $\sigma\, x_j = x_{\sigma(j)}$ for all j. This extends to a diagonal action of S_n on $\mathsf{V}^{\otimes k}$ with $\sigma\, x_\alpha = x_{\sigma(\alpha)} = x_{\sigma(\alpha_1)} \otimes \cdots \otimes x_{\sigma(\alpha_k)}$.

Suppose $T \in \mathsf{End}(\mathsf{V}^{\otimes k})$ and $T = \sum_{\alpha,\beta \in [1,n]^k} T_\alpha^\beta\, \mathsf{E}_\alpha^\beta$, where the transformations E_α^β are given by (3.3) and the $T_\alpha^\beta \in \mathbb{C}$. Then,

$$T \in \mathsf{End}_{\mathsf{S}_n}(\mathsf{V}^{\otimes k}) \iff \sigma T = T\sigma \ \text{ for all } \sigma \in \mathsf{S}_n$$

$$\iff \sum_{\beta \in [1,n]^k} T_\alpha^\beta x_{\sigma(\beta)} = \sum_{\beta \in [1,n]^k} T_{\sigma(\alpha)}^\beta x_\beta \ \text{ for all } \alpha \in [1,n]^k$$

$$\iff T_\alpha^\beta = T_{\sigma(\alpha)}^{\sigma(\beta)} \ \text{ for all } \alpha,\beta \in [1,n]^k, \sigma \in \mathsf{S}_n. \tag{4.4}$$

Now let α, α' be two k-tuples in $[1,n]^k$, but assume $\alpha' = (\alpha_{k+1},\ldots,\alpha_{2k})$ so that the indices on the components of α' run from $k+1$ to $2k$ rather than from 1 to k. Thus, we can think of α as giving labels for the bottom row of a partition diagram d and α' as giving labels for the top row of d, as pictured below.

Let

$$T_d = \sum_{\alpha, \alpha' \in [1,n]^k} (T_d)_\alpha^{\alpha'} \, \mathsf{E}_\alpha^{\alpha'} \in \mathsf{End}(\mathsf{V}^{\otimes k}), \tag{4.5}$$

where

$$(T_d)_\alpha^{\alpha'} = \begin{cases} 1 & \text{if } \alpha_i = \alpha_j \text{ iff } i, j \text{ are in the same block in } d \text{ for } i, j \in [1, 2k], \\ 0 & \text{otherwise.} \end{cases}$$

In the example above,

$$(T_d)_\alpha^{\alpha'} = \begin{cases} 1 & \text{iff } \alpha_1 = \alpha_4; \; \alpha_2 = \alpha_6 = \alpha_8 = \alpha_9; \; \alpha_3 = \alpha_{10}; \; \alpha_5 = \alpha_7; \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $T_d \in \mathsf{Z}_k(\mathrm{S}_n) = \mathsf{End}_{\mathrm{S}_n}(\mathsf{V}^{\otimes k})$, as $(T_d)_\alpha^{\alpha'} = 1$ (resp. 0) exactly when $(T_d)_{\sigma(\alpha)}^{\sigma(\alpha')} = 1$ (resp. 0) for all $\sigma \in \mathrm{S}_n$, so that the condition in (4.4) is satisfied. In fact, the transformations T_d as d ranges over the diagrams in $\mathcal{P}(k, n)$ give a basis for the centralizer algebra $\mathsf{Z}_k(\mathrm{S}_n)$ (see for example, [HR2]).

For (α, α') and (β, β'), write $(\alpha, \alpha') \sim_{\mathrm{S}_n} (\beta, \beta')$ if $\alpha = \sigma(\beta)$ and $\alpha' = \sigma(\beta')$ for some $\sigma \in \mathrm{S}_n$. Then

$$T_d = \sum_{(\alpha, \alpha') \sim_{\mathrm{S}_n} (\zeta_d, \zeta_d')} \mathsf{E}_\alpha^{\alpha'} \tag{4.6}$$

and

$$\dim \mathsf{Z}_k(\mathrm{S}_n) = |\mathcal{P}(k, n)| = \sum_{j=1}^n \left\{ {2k \atop j} \right\} \tag{4.7}$$

where $\left\{ {2k \atop j} \right\}$ is the Stirling number of the 2nd kind, which counts the number of ways to partition $2k$ objects into j nonempty blocks. The Stirling number $\left\{ {2k \atop j} \right\} = 0$ whenever $j > 2k$.

Next we describe $\mathsf{Z}_k\big(\mathsf{G}(2, 1, n)\big) = \mathsf{End}_{\mathsf{G}(2,1,n)}(\mathsf{V}^{\otimes k})$, where $\mathsf{G}(2, 1, n) = \mathbb{Z}_2 \wr \mathrm{S}_n$ (the Weyl group of type B_n). Note that $\mathsf{G}(2, 1, n)$ acts on V so that relative to the basis $\{x_i \mid i \in [1, n]\}$, each element of $\mathsf{G}(2, 1, n)$ acts by a permutation matrix with entries ± 1. The inclusion $\mathrm{S}_n \subset \mathsf{G}(2, 1, n)$ implies the reverse inclusion of centralizer algebras, $\mathsf{Z}_k\big(\mathsf{G}(2, 1, n)\big) \subset \mathsf{Z}_k(\mathrm{S}_n)$. In [T], Tanabe investigated the centralizer algebras of the complex reflection groups $\mathsf{G}(m, p, n)$ acting on $\mathsf{V}^{\otimes k}$. Applying Tanabe's results to the particular case of $\mathsf{G}(2, 1, n)$, we have

Proposition 4.8. *Let* $\mathcal{P}_{\mathsf{even}}(k, n)$ *be the set of partitions of* $[1, 2k]$ *into blocks of even size such that there are at most* n *blocks. Then* $\{T_d \mid d \in \mathcal{P}_{\mathsf{even}}(k, n)\}$ *is a basis for* $\mathsf{Z}_k\big(\mathsf{G}(2, 1, n)\big)$, *where* $T_d \in \mathsf{End}(\mathsf{V}^{\otimes k})$ *is as in* (4.5) *(or equivalently, as in* (4.6)*).*

Proof. By [T, Lem. 2.1], a basis for $\mathsf{Z}_k\big(\mathsf{G}(2,1,n)\big)$ consists of the transformations T_d such that $d \in \mathcal{P}(k,n)$ and

$$(\#j \text{ in } \zeta_d) \equiv (\#j \text{ in } \zeta'_d) \mod 2 \text{ for each } j \in [1,n]. \tag{4.9}$$

This condition is equivalent to saying that the blocks of d are of even size. $\qquad\square$

For the example in (4.3), $\zeta_d = (1,2,3,1,4)$ and $\zeta'_d = (2,4,2,2,3)$, so that $(\#1 \text{ in } \zeta_d) = 2 \equiv 0 = (\#1 \text{ in } \zeta'_d)$; $(\#2 \text{ in } \zeta_d) = 1 \equiv 3 = (\#2 \text{ in } \zeta'_d)$; $(\#3 \text{ in } \zeta_d) = 1 \equiv 1 = (\#3 \text{ in } \zeta'_d)$; and $(\#4 \text{ in } \zeta_d) = 1 \equiv 1 = (\#4 \text{ in } \zeta'_d)$; and $(\#j \text{ in } \zeta_d) = 0 \equiv 0 = (\#j \text{ in } \zeta'_d)$ for all $j \in [5,n]$. There are 4 blocks in d, and they have sizes $2,4,2,2$. Thus, d satisfies condition (4.9), and T_d is a basis element of $\mathsf{Z}_5\big(\mathsf{G}(2,1,n)\big)$.

4.2 $\dim \mathsf{Z}_k\big(\mathsf{G}(2,1,n)\big)$ and $\dim \mathsf{Z}_k(\mathbb{Z}_2^n)$

Let $\mathrm{T}(k,r)$ be the number of partitions of a set of size $2k$ into r nonempty blocks of even size. In particular, $\mathrm{T}(k,r) = 0$ if $r > k$, and $\mathrm{T}(k,1) = 1$. These numbers correspond to sequence A156289 in the Online Encyclopedia of Integer Sequences [OEIS], and are known to satisfy

$$\mathrm{T}(k,r) = \frac{1}{r!\, 2^{r-1}} \sum_{j=1}^{r} (-1)^{r-j} \binom{2r}{r-j} j^{2k} \tag{4.10}$$

In particular, $\mathrm{T}(4,2) = \frac{1}{4}\left((-1)^{2-1}\binom{4}{1}1^8 + (-1)^0\binom{4}{0}2^8\right) = \frac{1}{4}(256-4) = 63$. Each such set partition determines an integer solution to

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = k, \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0; \tag{4.11}$$

hence, a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of k into r nonzero parts. Let ℓ_j^λ be the multiplicity of j in the partition λ. Then an alternate expression for $\mathrm{T}(k,r)$ is

$$\mathrm{T}(k,r) = \sum \frac{1}{\ell_1^\lambda!\,\ell_2^\lambda!\,\cdots}\binom{2k}{2\lambda_1\ 2\lambda_2\ \cdots\ 2\lambda_r} \quad \text{(multinomial notation)}$$

$$= \sum \frac{1}{\ell_1^\lambda!\,\ell_2^\lambda!\,\cdots}\binom{2k}{2\lambda_1}\binom{2k-2\lambda_1}{2\lambda_2}\cdots\binom{2\lambda_r}{2\lambda_r}, \tag{4.12}$$

where the sum is over all $\lambda = (\lambda_1, \ldots, \lambda_r)$ satisfying (4.11). For example, when $k = 4$ and $r = 2$, there are two solutions $3+1 = 4$, $2+2 = 4$ to (4.11) and

$$\mathrm{T}(4,2) = \binom{8}{6}\binom{2}{2} + \frac{1}{2!}\binom{8}{4}\binom{4}{4} = 28 + 35 = 63.$$

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The next result is an immediate consequence of Proposition 4.8.

Proposition 4.13. $\dim Z_k\big(G(2,1,n)\big) = \sum_{r=1}^{n} T(k,r)$.

Recall that the transformations E_α^β, where $\alpha, \beta \in [1,n]^k$ and $\sum_{i=1}^{k} \varepsilon_{\alpha_i} = \sum_{i=1}^{k} \varepsilon_{\beta_i}$, form a basis for $Z_k(\mathbb{Z}_2^n)$. Note that the condition $\sum_{i=1}^{k} \varepsilon_{\alpha_i} = \sum_{i=1}^{k} \varepsilon_{\beta_i}$ says that $(\#j \text{ in } \alpha) \equiv (\#j \text{ in } \beta) \mod 2$ for each $j \in [1,n]$. For example, if $\alpha = (4,3,2,4,1)$ and $\beta = (3,1,3,3,2)$, then this condition is satisfied. Moreover, if we label the nodes of a diagram with the components of α on the bottom, and the components of β on top and connect nodes that have the same label, we obtain a diagram d satisfying (4.9), since there is a $\sigma \in S_n$ such that $\sigma(\alpha) = \zeta_d$ and $\sigma(\beta) = \zeta_d'$ (in fact, in this example $\sigma = (1\,4)(2\,3)$ will do the job).
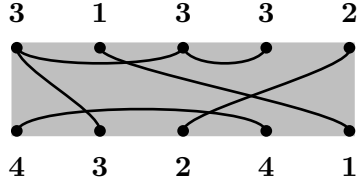
Therefore, E_α^β is one of the summands of T_d. We note that E_α^β doesn't commute with $G(2,1,n)$. But it does commute with its normal subgroup $\mathbb{Z}_2^n$.

Example 4.14. *Suppose $n = 3$ and $k = 2$. The dimension of $Z_2(S_3)$ is $|\mathcal{P}(2,3)| = \left\{{4 \atop 1}\right\} + \left\{{4 \atop 2}\right\} + \left\{{4 \atop 3}\right\} = 14$ where the summands are Stirling numbers of the 2nd kind. There are only 4 diagrams $d \in \mathcal{P}(2,3)$ that have ≤ 3 blocks of even size; namely, the ones pictured below, where we have indicated ζ_d and ζ_d' on each diagram.*
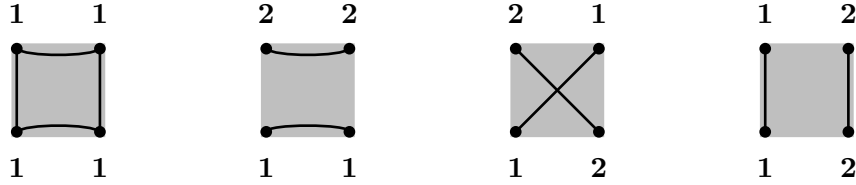
Assuming these diagrams are numbered $d_1, \ldots, d_4$ from left to right, we have

$$
\begin{aligned}
T_{d_1} &= E_{11}^{11} + E_{22}^{22} + E_{33}^{33} \\
T_{d_2} &= E_{11}^{22} + E_{22}^{33} + E_{33}^{11} + E_{11}^{33} + E_{33}^{22} + E_{11}^{22} \\
T_{d_3} &= E_{12}^{21} + E_{23}^{32} + E_{31}^{13} + E_{13}^{31} + E_{32}^{23} + E_{21}^{12} \\
T_{d_4} &= E_{12}^{12} + E_{23}^{23} + E_{31}^{31} + E_{13}^{13} + E_{32}^{32} + E_{21}^{21},
\end{aligned}
$$

and $\{T_{d_j} \mid j \in [1,4]\}$ is a basis for $\mathsf{Z}_2\big(\mathsf{G}(2,1,3)\big)$. *Note there are a total of 21 summands* E_α^β *in these expressions. According to Corollary 3.1,*

$$\dim \mathsf{Z}_2(\mathbb{Z}_2^3) = \frac{1}{2^3} \sum_{i=0}^{3} \binom{3}{i}(3-2i)^4 = \frac{1}{8}\left(3^4 + 3 + 3 + 3^4\right) = 21.$$

Recall that a basis element E_α^β for $\mathsf{Z}_k(\mathbb{Z}_2^n)$ corresponds to a partition diagram d with $2k$ nodes obtained by labeling the nodes of d with the components of α on the bottom, and the components of β on top. Nodes having the same label are connected by an edge. The blocks have even size, and there are r blocks for some $r \le n$. Labeling the blocks with different numbers amounts to coloring the blocks of d with different colors chosen from n colors. Therefore, there are $\mathrm{T}(k,r)\dfrac{n\,!}{(n-r)!}$ basis elements E_α^β corresponding to diagrams with r blocks. Combining this with Schur-Weyl duality gives

Proposition 4.15.

$$\dim \mathsf{Z}_k(\mathbb{Z}_2^n) = \sum_{r=1}^{n} \mathrm{T}(k,r)\frac{n\,!}{(n-r)!} \tag{4.16}$$

$$= \textit{the number of walks of } 2k \textit{ steps}$$

$$\textit{from } \mathbf{0} \textit{ to } \mathbf{0} \textit{ on the } n\textit{-cube}.$$

Example 4.17.

$$\dim \mathsf{Z}_2(\mathbb{Z}_2^3) = \mathrm{T}(2,1)\frac{3!}{2!} + \mathrm{T}(2,2)\frac{3!}{1!}$$

$$= 1 \cdot 3 + 3 \cdot 6 = 21.$$

For $i \in [1,n]$, let $\mathsf{p}_i : \mathsf{V} \to \mathsf{X}^{\varepsilon_i}$ be the projection map onto the ith summand. For $\alpha = (\alpha_1, \ldots, \alpha_k) \in [1,n]^k$, set

$$\mathsf{p}_\alpha = \mathsf{p}_{\alpha_1} \otimes \mathsf{p}_{\alpha_2} \otimes \cdots \otimes \mathsf{p}_{\alpha_k} \in \mathsf{End}(\mathsf{V}^{\otimes k}).$$

Then for $\beta \in [1,n]^k$, $\mathsf{p}_\alpha(x_\beta) = \prod_{j=1}^{k} \delta_{\alpha_j,\beta_j} x_\alpha = \delta_{\alpha,\beta} x_\alpha$. Moreover,

$$\mathsf{p}_\beta T_d \mathsf{p}_\alpha = \mathsf{E}_\alpha^\beta.$$

Combining the results of this section, we have

Proposition 4.18. *The elements* T_d *such that* $d \in \mathcal{P}_{\mathsf{even}}(k,n)$ *together with the projections* p_α, $\alpha \in [1,n]^k$, *generate the centralizer algebra* $\mathsf{Z}_k(\mathbb{Z}_2^n)$.

4.3 The Bratteli diagram and a bijection

The *Bratteli diagram* $\mathcal{B}_\mathsf{V}(\mathbb{Z}_2^n)$ associated to the group $\mathbb{Z}_2^n$ and the module V is the infinite graph with vertices labeled by the elements of $\mathsf{a} \in \mathbb{Z}_2^n$ on level k that can be reached by a walk of k steps on the representation graph $\mathcal{R}_\mathsf{V}(\mathbb{Z}_2^n)$. Such a walk corresponds to a sequence $(\mathsf{a}^0, \mathsf{a}^1, \ldots, \mathsf{a}^k)$ starting at $\mathsf{a}^0 = \mathbf{0} = (0, \ldots, 0)$, such that $\mathsf{a}^j \in \mathbb{Z}_2^n$ for each $1 \le j \le k$, and $\mathsf{a}^j = \mathsf{a}^{j-1} + \varepsilon_i$ for some $i \in [1, n]$. Thus, a^j is connected to a^{j-1} by an edge in $\mathcal{R}_\mathsf{V}(\mathbb{Z}_2^n)$. This corresponds to a unique path on $\mathcal{B}_\mathsf{V}(\mathbb{Z}_2^n)$ starting at $\mathbf{0}$ on the top and going to a at level k. The subscript on node a at level k in $\mathcal{B}_\mathsf{V}(\mathbb{Z}_2^n)$ indicates the number m_k^a of such paths (hence, the number of walks on $\mathcal{R}_\mathsf{V}(\mathbb{Z}_2^n)$ of k steps from $\mathbf{0}$ to a). This can be easily computed by summing, in a Pascal triangle fashion, the subscripts of the vertices at level $k - 1$ that are connected to a. This is the multiplicity of the irreducible $\mathbb{Z}_2^n$-module X^a in $\mathsf{V}^{\otimes k}$, which is also the dimension of the irreducible $\mathsf{Z}_k(\mathbb{Z}_2^n)$-module Z_k^a by Schur-Weyl duality. The sum of the squares of those dimensions at level k is the number on the right, which is the dimension of the centralizer algebra $\mathsf{Z}_k(\mathbb{Z}_2^n)$. Levels $0,1,\ldots,6$ of the Bratteli diagram for $n = 3$ are displayed below, where to simplify the notation we have omitted the commas in writing the elements a of $\mathbb{Z}_2^n$.
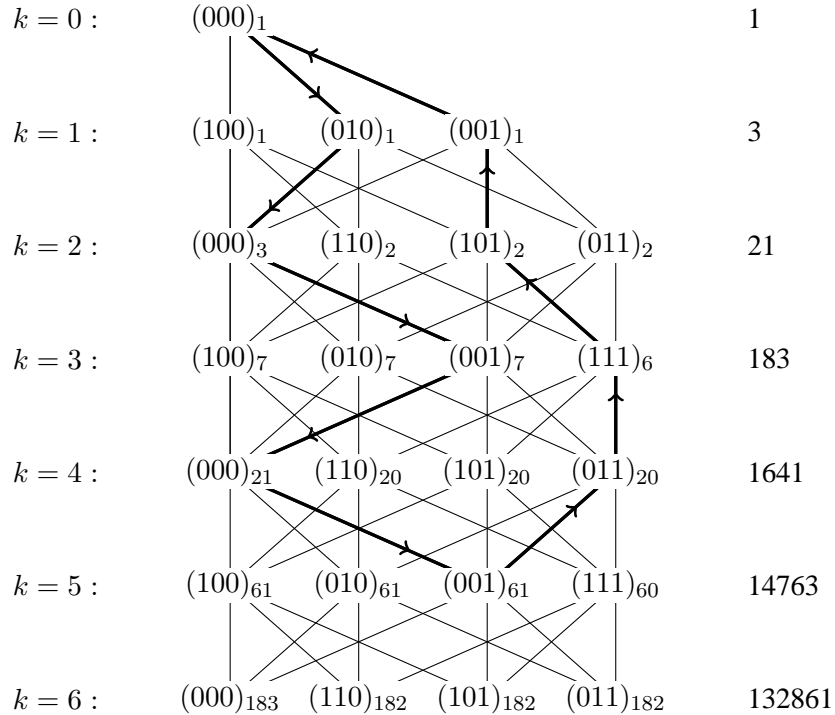
A pair (ϱ_1, ϱ_2) of paths $\varrho_1 = (\mathsf{a}^0, \mathsf{a}^1, \ldots, \mathsf{a}^k)$, $\varrho_2 = (\mathsf{b}^0, \mathsf{b}^1, \ldots, \mathsf{b}^k)$ starting from $\mathsf{a}^0 = \mathsf{b}^0 = \mathbf{0} = (0, \ldots, 0)$ at the top and going to $\mathsf{a}^k = \mathsf{b}^k = \mathsf{a} \in \mathbb{Z}_2^n$ at level k in $\mathcal{B}_{\mathsf{V}}(\mathbb{Z}_2^n)$ determines a closed path from $\mathbf{0}$ to $\mathbf{0}$ by reversing the second path and concatenating the two paths. This is illustrated by the darkened edges in the diagram above. Such a pair determines two k-tuples $\alpha = (\alpha_1, \ldots, \alpha_k)$, and $\beta = (\beta_1, \ldots, \beta_k)$ in $[1, n]^k$, such that $\mathsf{a}^j = \mathsf{a}^{j-1} + \varepsilon_{\alpha_j}$ for $j = 1, \ldots, k$, and $\mathsf{b}^{j-1} = \mathsf{b}^j + \varepsilon_{\beta_{k+1-j}}$ for $j = k, \ldots, 1$. Then since $\varepsilon_{\alpha_1} + \cdots + \varepsilon_{\alpha_k} + \varepsilon_{\beta_1} + \cdots + \varepsilon_{\beta_k} = \mathbf{0}$, the condition in Theorem 3.4 is satisfied, and α and β determine a labeled partition diagram $d \in \mathcal{P}_{\mathsf{even}}(k, n)$, in which nodes with the same label are connected by an edge.

4.19. *This process establishes a bijection between the pairs (ϱ_1, ϱ_2) of paths from $\mathbf{0}$ at the top of the Bratteli diagram to $\mathsf{a} \in \mathbb{Z}_2^n$ at level k and the basis elements E_α^β of $\mathsf{Z}_k(\mathbb{Z}_2^n)$ such that (a) $\alpha, \beta \in [1, n]^k$; (b) $(\#\alpha_i = j) \equiv (\#\beta_i = j) \bmod 2$ for all $j \in [1, n]$; and (c) $\sum_{i=1}^k \varepsilon_{\alpha_i} = \sum_{i=1}^k \varepsilon_{\beta_i} = \mathsf{a}$.*

In the example above, $\varepsilon_2, \varepsilon_2, \varepsilon_3, \varepsilon_3, \varepsilon_3$ have been added in succession to (000) to arrive at $\mathsf{a} = (001)$ so that $\alpha = (2, 2, 3, 3, 3)$; and $\varepsilon_2, \varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_3$ have been added in succession to $\mathsf{a} = (001)$ to return back (000) so that $\beta = (2, 1, 2, 1, 3)$. The resulting labeled partition diagram $d \in \mathcal{P}_{\mathsf{even}}(5, 3)$ is
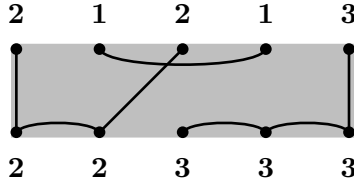
which is identified with the basis element $\mathsf{E}_{22333}^{21213}$ of $\mathsf{Z}_5(\mathbb{Z}_2^3)$.

Remark 4.20. *It is evident that the following hold:*

(i) *For $\mathsf{a} \in \mathbb{Z}_2^n$, a basis for the $\mathsf{Z}_k(\mathbb{Z}_2^n)$-module $\mathsf{Z}_k^{\mathsf{a}} \subseteq \mathsf{V}^{\otimes k}$ is*

$$\left\{ x_\alpha = x_{\alpha_1} \otimes \cdots \otimes x_{\alpha_k} \mid \alpha_j \in [1, n] \text{ for all } j \in [1, k], \text{ and } \sum_{j=1}^k \varepsilon_{\alpha_j} = \mathsf{a} \right\}.$$

(ii) *$e^{\mathsf{b}} \cdot x_\alpha = (-1)^{\mathsf{a} \cdot \mathsf{b}} x_\alpha$ for all $\mathsf{b} \in \mathbb{Z}_2^n$ and all x_α in (i) so that $\mathsf{Z}_k^{\mathsf{a}}$ is also a $\mathbb{Z}_2^n$-submodule of $\mathsf{V}^{\otimes k}$; it is the sum of all copies of the $\mathbb{Z}_2^n$-module X^{a} in $\mathsf{V}^{\otimes k}$.*

(iii) *$\mathsf{End}_{\mathbb{Z}_2^n}(\mathsf{Z}_k^{\mathsf{a}})$ has a basis consisting of the transformations $\mathsf{E}_\alpha^\beta \in \mathsf{End}(\mathsf{V}^{\otimes k})$ such that (a) $\alpha, \beta \in [1, n]^k$; (b) $(\#\alpha_i = j) \equiv (\#\beta_i = j) \bmod 2$ for all $j \in [1, n]$; and (c) $\sum_{i=1}^k \varepsilon_{\alpha_i} = \sum_{i=1}^k \varepsilon_{\beta_i} = \mathsf{a}$.*

4.4 Poincaré series and exponential generating functions

The assumptions of Theorem 1.2 hold for $G = \mathbb{Z}_2$ and $V = X^{\varepsilon_1} \oplus \cdots \oplus X^{\varepsilon_n}$, so by (1.3), the Poincaré series for the multiplicities m_k^a of the irreducible $\mathbb{Z}_2^n$-module labeled by $a \in \mathbb{Z}_2^n$ in $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ (hence, for the number m_k^a of walks of k steps from $\mathbf{0}$ to a on the n-cube) is given by

$$m^a(t) = \sum_{k \geq 0} m_k^a t^k = \frac{\det(M^a)}{\det(I - tA)} = \frac{\det(M^a)}{\prod_{c \in \mathbb{Z}_2^n} (1 - \chi_V(c)t)}, \tag{4.21}$$

where A is the adjacency matrix of the n-cube, and M^a is the matrix obtained from $I - tA$ by replacing column a with $\underline{\delta} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Here we demonstrate how to compute these series. Since $\chi_V = \sum_{i=1}^n \chi_{\varepsilon_i}$, the character values are

$$\chi_V(c) = \sum_{i=1}^n \chi_{\varepsilon_i}(c) = \sum_{i=1}^n (-1)^{c \cdot \varepsilon_i} = n - 2h(c),$$

where $h(c)$ is the Hamming weight of $c \in \mathbb{Z}_2^n$. Elements with the same Hamming weight have the same character value on V. Therefore the denominator in (4.21) is given by

$$\prod_{c \in \mathbb{Z}_2^n} (1 - \chi_V(c)t) = \prod_{h=0}^n \left(1 - (n - 2h)t\right)^{\binom{n}{h}} \tag{4.22}$$

In particular, when $n = 3$,

$$\prod_{c \in \mathbb{Z}_2^3} (1 - \chi_V(c)t) = (1 - 3t)(1 - t)^3(1 + t)^3(1 + 3t) = (1 - 9t^2)(1 - t^2)^3.$$

Below we display the numerators for the various choices of $a \in \mathbb{Z}_2^3$ and the Poincaré series $m^a(t)$. For elements of $\mathbb{Z}_2^3$ with the same Hamming weight, the numerators are the same, as are the Poincaré series, so we list a single representative for each Hamming weight. On the right we indicate the corresponding OEIS label.

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$$\mathsf{m}^{(000)}(t) = \frac{(1-7t^2)(1-t^2)^2}{(1-9t^2)(1-t^2)^3} = 1 + 3t^2 + 21t^4 + 183t^6 + 1641t^8 + \cdots \quad \text{(A054879)}$$

$$\mathsf{m}^{(100)}(t) = \frac{t(1-3t^2)(1-t^2)^2}{(1-9t^2)(1-t^2)^3} = t + 7t^3 + 61t^5 + 547t^7 + \cdots \quad \text{(A066443)}$$

$$\mathsf{m}^{(110)}(t) = \frac{2t^2(1-t^2)^2}{(1-9t^2)(1-t^2)^3} = 2t^2 + 20t^4 + 182t^6 + 1640t^8 + \cdots \quad \text{(A125857)}$$

$$\mathsf{m}^{(111)}(t) = \frac{6t^3(1-t^2)^2}{(1-9t^2)(1-t^2)^3} = 6t^3 + 60t^5 + 546t^7 + \cdots \quad \text{(A054880)}$$

The numbers appearing as coefficients in these series are the subscripts of the element a in the Bratteli diagram, and the exponent of t indicates the level.

In the $n = 3$ example, the minimum polynomial of the adjacency matrix A is $p(t) = (t^2 - 1)(t^2 - 9) = t^4 - 10t^2 + 9$, so by (1.7), the multiplicities satisfy the recursion relation

$$\mathsf{m}^{\mathsf{a}}_{k+4} - 10\mathsf{m}^{\mathsf{a}}_{k+2} + 9\mathsf{m}^{\mathsf{a}}_k = 0$$

for all $k \geq 0$. For example, when $\mathsf{a} = (110)$,

$$\mathsf{m}^{(110)}_8 - 10\mathsf{m}^{(110)}_6 + 9\mathsf{m}^{(110)}_4 = 1640 - (10 \times 182) + (9 \times 20) = 0.$$

Next we apply Theorem 1.10 to determine information about the exponential generating function $\mathsf{g}^{\mathsf{a}}(t) = \sum_{k \geq 0} \mathsf{m}^{\mathsf{a}}_k \frac{t^k}{k!}$ for the multiplicities $\mathsf{m}^{\mathsf{a}}_k$ (i.e., for the number of walks of k steps from $\mathbf{0}$ to $\mathsf{a} \in \mathbb{Z}_2^n$ on the n-cube) for arbitrary n.

The eigenvalues of the adjacency matrix A of the n-cube are $n - 2h$, $h = 0, 1, \ldots, n$, and the vectors $\mathcal{E}_{\mathsf{a}}$ in Corollary 2.4 with $h(\mathsf{a}) = h$ are the eigenvectors corresponding to the eigenvalue $n - 2h$. Since the $\mathcal{E}_{\mathsf{a}}$, $\mathsf{a} \in \mathbb{Z}_2^n$ form a basis for $\mathbb{C}^{2^n}$, it follows that the minimal polynomial of A is

$$p(t) = \prod_{h=0}^n (t - n + 2h)$$
$$= \begin{cases} t(t^2 - 4)(t^2 - 16) \cdots (t^2 - n^2), & \text{if } n \text{ is even,} \\ (t^2 - 1)(t^2 - 9) \cdots (t^2 - n^2) & \text{if } n \text{ is odd.} \end{cases}$$

Suppose $p(t) = t^{n+1} + p_n t^n + \cdots + p_1 t + p_0 = \prod_{h=0}^n (t - n + 2h)$. Then we know by Theorem 1.10 that $\mathsf{g}^{\mathsf{a}}(t)$ satisfies the differential equation

$$y^{(n+1)} + p_n y^{(n)} + \cdots + p_1 y^{(1)} + p_0 y = 0, \quad (4.23)$$

and a general solution of (4.23) is a linear combination of the following exponential functions

$$e^{\pm nt}, e^{\pm(n-2)t}, \ldots, e^{\pm 2t}, 1 \qquad \text{if } n \text{ is even}$$

$$e^{\pm nt}, e^{\pm(n-2)t}, \ldots, e^{\pm t} \qquad \text{if } n \text{ is odd.}$$

Theorem 4.24. *Let* $g^a(t)$ *be the exponential generating function for the number* m_k^a *of walks on the* n-*cube of* k *steps starting at* $\mathbf{0}$ *and ending at* a *(equivalently, for the dimension of the irreducible modules* Z_k^a *for the centralizer algebra* $Z_k(\mathbb{Z}_2^n) = \mathrm{End}_{\mathbb{Z}_2^n}(V^{\otimes k})$ *for* $k \geq 0$*). Then,*

$$g^a(t) = \sum_{k \geq 0} m_k^a \frac{t^k}{k!} = (\cosh t)^{n-h} (\sinh t)^h$$

where $h = h(a)$*, the Hamming weight of* a*, and* $\cosh t$ *and* $\sinh t$ *are hyperbolic cosine and sine.*

Proof. As we have noted earlier, $m_k^a = m_k^b$ for all $k \geq 0$ whenever $h(a) = h(b)$, so it suffices to assume $a = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{Z}_2^n$, where there are $h = h(a)$ ones. The multiplicity m_k^a is the number of (ordered) k-tuples $\alpha = (\alpha_1, \ldots, \alpha_k) \in [1, n]^k$ such that

$$\varepsilon_{\alpha_1} + \cdots + \varepsilon_{\alpha_k} = a.$$

For each $j = 1, \ldots, h$, the number of α_i in α equal to j is odd, and for each $j = h+1, \ldots, n$, the number of α_i in α equal to j is even. Thus, the multiplicity is obtained by the following computation:

$$m_k^a = \sum_{r=1}^{n} \sum \frac{1}{\ell_1^\lambda! \, \ell_2^\lambda! \, \cdots} \binom{k}{\lambda_1} \binom{k-\lambda_1}{\lambda_2} \cdots \binom{\lambda_r}{\lambda_r} \times h! \times \frac{(n-h)!}{(n-r)!}, \quad (4.25)$$

where the second sum is over all partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ of k with r nonzero parts, such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = k, \quad \lambda_1 \geq \cdots \geq \lambda_h, \quad \lambda_{h+1} \geq \cdots \geq \lambda_r;$$

$\lambda_1, \ldots, \lambda_h$ are odd numbers; and $\lambda_{h+1}, \ldots, \lambda_r$ are even numbers. The factor $h!$ in (4.25) counts the number of ways to assign a number from $\{1, \ldots, h\}$ to each odd number λ_i, $i = 1, \ldots, h$. Similarly, the factor $(n-h)! / (n-r)! = (n-h) \times \cdots \times (n - h - (r - h - 1))$ counts the ways to assign a number from $\{h+1, \ldots, n\}$ to each even number λ_i, $i = h+1, \ldots, r$.

Therefore,

$$\frac{1}{k\,!}\mathsf{m}_k^{\mathsf{a}} = \sum_{r=1}^{n}\sum \frac{1}{\ell_1^{\lambda_1}!\,\ell_2^{\lambda_2}!\,\ldots}\,\frac{1}{\lambda_1!\,\lambda_2!}\,\cdots\,\frac{1}{\lambda_r!}\times h\,!\times\frac{(n-h)!}{(n-r)!}\,. \qquad (4.26)$$

For a fixed value of r, the expression in the inner sum of (4.26) can be gotten by summing the coefficients of products of $r-h$ different factors from $\left(\cosh t\right)^{n-h} = \left(\sum_{j=0}^{\infty}\frac{t^{2j}}{(2j)!}\right)^{n-h}$ and h different factors from $\left(\sinh t\right)^{h} = \left(\sum_{j=0}^{\infty}\frac{t^{2j+1}}{(2j+1)!}\right)^{h}$ such that the total power of t of those r factors is k. Hence, $\dfrac{1}{k!}\mathsf{m}_k^{\mathsf{a}}$ equals the coefficient of t^k in

$$\left(\cosh t\right)^{n-h}\left(\sinh t\right)^{h} = \left(\sum_{j=0}^{\infty}\frac{t^{2j}}{(2j)!}\right)^{n-h}\left(\sum_{j=0}^{\infty}\frac{t^{2j+1}}{(2j+1)!}\right)^{h}.$$

$\square$

In the special case that $\mathsf{a} = \mathbf{0}$, Theorem 4.24 implies the following

Corollary 4.27. *Let* $\mathrm{g}^{\mathbf{0}}(t) = \displaystyle\sum_{k\geq 0}\mathsf{m}_{2k}^{\mathbf{0}}\frac{t^{2k}}{(2k)!}$ *be the exponential generating function for the number* $\mathsf{m}_{2k}^{\mathbf{0}}$ *of walks on the n-cube of $2k$ steps starting and ending at* $\mathbf{0}$ *(equivalently, for the dimension* $\mathsf{m}_{2k}^{\mathbf{0}}$ *of the space of $\mathbb{Z}_2^n$-invariants in $V^{\otimes(2k)}$; equivalently, for the dimension of the centralizer algebra $\mathsf{Z}_k(\mathbb{Z}_2^n)$ for $k \geq 0$). Then,*

$$\mathrm{g}^{\mathbf{0}}(t) = \left(\cosh t\right)^{n} = \left(\frac{\mathrm{e}^t + \mathrm{e}^{-t}}{2}\right)^{n} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\mathrm{e}^{(n-2i)t} \qquad (4.28)$$

Remark 4.29. *Corollary 4.27 implies*

$$\mathrm{g}^{\mathbf{0}}(t) = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\mathrm{e}^{(n-2i)t} = \frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(\sum_{r\geq 0}\frac{(n-2i)^r t^r}{r\,!}\right)$$

$$= \sum_{r\geq 0}\left(\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}(n-2i)^r\right)\frac{t^r}{r\,!},$$

so that the number of walks of r steps from $\mathbf{0}$ to $\mathbf{0}$ is $\mathsf{m}_r^{\mathbf{0}} = \dfrac{1}{2^n}\displaystyle\sum_{i=0}^{n}\binom{n}{i}(n-2i)^r$, *which is 0 unless $r = 2k$ for some $k \geq 0$. Compare Corollary 2.5 with $\mathsf{b} = \mathsf{c} = \mathbf{0}$.*

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Remark 4.30. *One may define* $(\cosh t)^x$ *for any* $x \in \mathbb{C}$, *in particular, for* $x = -1$. *In that case*

$$(\cosh t)^{-1} = \sum_{j \geq 0} \mathrm{E}_j \frac{t^j}{j!},$$

where E_j *is the jth Euler number. Generalized Euler numbers arising from the series expansion of* $(\cosh t)^{-x} = (\operatorname{sech} t)^x$ *have been studied and shown to have connections with Stirling numbers of the first and second kind (see for example,* *[L], [KJR]). The* m_{2k}^0 *in Corollary 4.27 are examples of such generalized Euler numbers.*

5   Other directions

We conclude with a few remarks on some other directions which have been investigated that are related to walks on the n-cube.

Remark 5.1. *Let* R, L *be the raising and lowering transformations on* $\operatorname{span}_{\mathbb{C}}\{\underline{\mathsf{b}} \mid \mathsf{b} \in \mathbb{Z}_2^n\}$ *defined by*

$$\mathrm{R}(\underline{\mathsf{b}}) = \sum_{i, h(\mathsf{b}+\varepsilon_i) > h(\mathsf{b})} \underline{\mathsf{b} + \varepsilon_i} \qquad \mathrm{L}(\underline{\mathsf{b}}) = \sum_{i, h(\mathsf{b}+\varepsilon_i) < h(\mathsf{b})} \underline{\mathsf{b} + \varepsilon_i},$$

and let $\mathrm{A}^*(\underline{\mathsf{b}}) = \big(n - 2h(\mathsf{b})\big)\underline{\mathsf{b}}$. *Then* $\mathrm{R} + \mathrm{L} = \mathrm{A}$ *(the adjacency matrix of the* *n-cube), and* $\mathrm{R}, \mathrm{L}, \mathrm{A}^*$ *determine a canonical basis for a copy of* $\mathfrak{sl}_2$ *such that* $[\mathrm{L}, \mathrm{R}] = \mathrm{A}^*$, $[\mathrm{A}^*, \mathrm{L}] = 2\mathrm{L}$ *and* $[\mathrm{A}^*, \mathrm{R}] = -2\mathrm{R}$. *The transformations* A, A^* *form a tridiagonal pair and generate the Terwilliger algebra of the n-cube. The details can be found in [G].*

Remark 5.2. *In [DG2], Diaconis and Graham considered the Markov chain arising from the affine walk on the n-cube given by* $X_r = C X_{r-1} + \epsilon_r$, *with* $X_r \in \mathbb{Z}_2^n$, *C an invertible matrix with entries in* $\mathbb{Z}_2$, *and* ϵ_r *a random vector in* $\mathbb{Z}_2^n$ *of disturbance terms. Their analysis of such walks relies on codes made from binomial coefficients* $\mod 2$. *When C is the lower triangular matrix of a single Jordan block corresponding to the eigenvalue 1, and the random vector is nonzero, the distribution of* X_r *tends to the uniform distribution on* $\mathbb{Z}_2^n$. *Without the random vector,* *X_r is a deterministic walk that returns to the starting point in n steps. (See also [DGM] for more on random walks on the n-cube.)*

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