THE POLYTOPE OF TESLER MATRICES

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ABSTRACT. We introduce the Tesler polytope $\mathsf{Tes}_n(\mathbf{a})$, whose integer points are the Tesler matrices of size n with hook sums $a_1, a_2, \ldots, a_n \in \mathbb{Z}_{\geq 0}$. We show that $\mathsf{Tes}_n(\mathbf{a})$ is a flow polytope and therefore the number of Tesler matrices is counted by the type A_n Kostant partition function evaluated at $(a_1, a_2, \ldots, a_n, -\sum_{i=1}^{n} a_i)$. We describe the faces of this polytope in terms of "Tesler tableaux" and characterize when the polytope is simple. We prove that the h-vector of $\mathsf{Tes}_n(\mathbf{a})$ when all $a_i > 0$ is given by the Mahonian numbers and calculate the volume of $\mathsf{Tes}_n(1, 1, \ldots, 1)$ to be a product of consecutive Catalan numbers multiplied by the number of standard Young tableaux of staircase shape.

1. INTRODUCTION

Tesler matrices have played a major role in the works [2][13][14][15][19][27] in the context of diagonal harmonics. We examine them from a different perspective in this paper: we study the polytope, which we call the Tesler polytope, consisting of upper triangular matrices with nonnegative real entries with the same restriction as Tesler matrices on the hook sums: sum of the elements of a row minus the sum of the elements of a column. Then the integer points of this polytope are all Tesler matrices of given hook sums. We show that these polytopes are flow polytopes and are faces of transportation polytopes. We characterize the Tesler polytopes with nonnegative hook sums that are simple and we calculate their h-vectors. If the hook sums are all 1 the volume is the product of consecutive Catalan numbers multiplied by the number of standard Young tableaux of staircase shape. This result raises the question of the Tesler polytope's connection to the Chan-Robbins-Yuen polytope, a flow polytope whose volume is the product of consecutive Catalan numbers.

We now proceed to give the necessary definitions and state our main results. This section is broken down into three subsections for ease of reading: introduction to Tesler matrices and polytopes, introduction to flow polytopes and transportation polytopes, and our main results regarding Tesler polytopes. Section 2 and Section 3 are independent of each other, the first one is about the face structure and the other is about the volume of Tesler polytopes. Finally, in Section 4 we discuss some final remarks and questions.

1.1. **Tesler matrices and polytopes.** Let $\mathbb{U}_n(\mathbb{R}_{\geq 0})$ be the set of $n \times n$ upper triangular matrices with nonnegative real entries. The k^{th} **hook sum** of a matrix $(x_{i,j})$ in $\mathbb{U}_n(\mathbb{R}_{\geq 0})$ is the sum of all the elements of the k^{th} row minus the sum of

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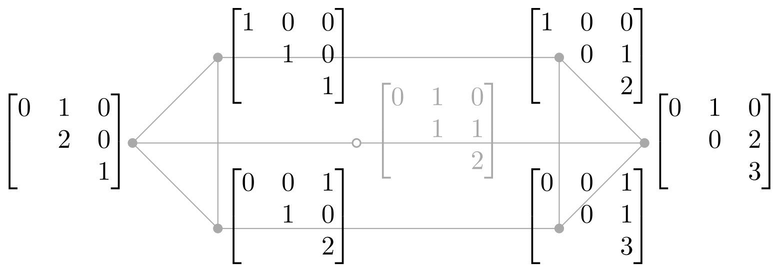


FIGURE 1. The seven 3×3 Tesler matrices with hook sums $(1,1,1)$. Six of them are vertices of the graph (depicted in gray) of the Tesler polytope $\mathsf{Tes}_n(1,1,1)$.

the elements in the k^{th} column excluding the term in the diagonal:

$$x_{k,k} + x_{k,k+1} + \cdots + x_{k,n} - (x_{1,k} + x_{2,k} + \cdots + x_{k-1,k})$$

Given a length n vector $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$ of nonnegative integers, the **Tesler polytope** $\mathsf{Tes}_n(\mathbf{a})$ with hook sums $\mathbf{a}$ is the set of matrices in $\mathbb{U}_n(\mathbb{R}_{\geq 0})$ where the k^{th} hook sum equals a_k, for $k = 1, \ldots, n$:

$$\mathsf{Tes}_n(\mathbf{a}) = \{(x_{i,j}) \in \mathbb{U}_n(\mathbb{R}_{\geq 0}) \,:\, x_{k,k} + \sum_{j=k+1}^{n} x_{k,j} - \sum_{i=1}^{k-1} x_{i,k} = a_k, 1 \leq k \leq n\}.$$

The lattice points of $\mathsf{Tes}_n(\mathbf{a})$ are called **Tesler matrices** with hook sums $\mathbf{a}$. These are $n \times n$ upper triangular matrices $B = (b_{i,j})$ with nonnegative integer entries such that for $k = 1, \ldots, n$, $b_{k,k} + \sum_{j=k+1}^{n} b_{k,j} - \sum_{i=1}^{k-1} b_{i,k} = a_k$. The set and number of such matrices are denoted by $\mathcal{T}_n(\mathbf{a})$ and $T_n(\mathbf{a})$ respectively. See Figure 1 for an example of the seven Tesler matrices in $\mathcal{T}_3(1,1,1)$.

Tesler matrices appeared recently in Haglund's study of diagonal harmonics [15] and their combinatorics and further properties were explored in [2][13][22][27]. The flavor of the results obtained for Tesler matrices in connection with diagonal harmonics is illustrated by the following example. Let $\mathcal{H}(DH_n, q, t)$ denote the bigraded Hilbert series of the space of diagonal harmonics DH_n. For more details regarding this polynomial in $\mathbb{N}[q,t]$ we refer the reader to [16, Ch. 5].

Example 1.1. When $\mathbf{a} = \mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{Z}^n$, Haglund [15] showed that

$$\mathcal{H}(DH_n, q, t) = \sum_{A \in \mathcal{T}_n(1,1,\ldots,1)} wt(A), \tag{1.1}$$

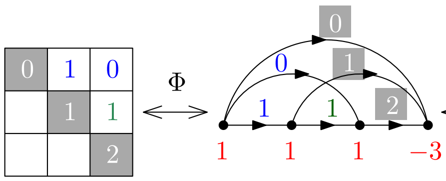
where

$$wt(A) = \frac{1}{(-M)^n} \prod_{i,j \,:\, a_{ij} > 0} (-M)[a_{ij}]_{q,t}, \quad M = (1-q)(1-t), \quad [b]_{q,t} = \frac{q^b - t^b}{q - t}. \tag{1.2}$$

The starting point for our investigation is the observation stated in the next lemma.

Lemma 1.2. *The Tesler polytope* $\mathsf{Tes}_n(\mathbf{a})$ *is a flow polytope* $\mathsf{Flow}_n(\mathbf{a})$,

$$\mathsf{Tes}_n(\mathbf{a}) \cong \mathsf{Flow}_n(\mathbf{a}). \tag{1.3}$$



$$(1,1,1,-3) = 1(\mathsf{e}_1-\mathsf{e}_2)+1(\mathsf{e}_1-\mathsf{e}_4)+1(\mathsf{e}_2-\mathsf{e}_3)+1(\mathsf{e}_2-\mathsf{e}_4)+2(\mathsf{e}_3-\mathsf{e}_4)$$

FIGURE 2. Correspondence between a 3×3 Tesler matrix with hook sums $(1,1,1)$, an integer flow in the complete graph k_4 and a vector partition of $(1,1,1,-3)$ into $\mathsf{e}_i-\mathsf{e}_j$ $1\le i<j\le 4$.

We now define flow polytopes to make Lemma 1.2 clear. For an illustration of the correspondence of polytopes in Lemma 1.2 see Figure 2.

1.2. **Flow polytopes.** Given $\mathbf{a}=(a_1,a_2,\dots,a_n)$, let $\mathsf{Flow}_n(\mathbf{a})$ be the **flow polytope** of the complete graph k_{n+1} with netflow a_i on vertex i for $i=1,\dots,n$ and the netflow on vertex $n+1$ is $-\sum_{i=1}^n a_i$. This polytope is the set of functions $f: E\to\mathbb{R}_{\ge 0}$, called flows, from the edge set $E=\{(i,j) : 1\le i<j\le n+1\}$ of k_{n+1} to the set of nonnegative real numbers such that for $k=1,\dots,n$, $\sum_{j>s} f(k,j)-\sum_{i<k} f(i,k)=a_k$. This forces $\sum_{i=1}^n f(i,n+1)=\sum_{i=1}^n a_i$. We can write $\mathsf{Flow}_n(\mathbf{a})=\{\mathbf{x}\in\mathbb{R}_{\ge 0}^{\binom{n+1}{2}} | A_{k_{n+1}}\mathbf{x}=(\mathbf{a},-\sum_{i=1}^n a_i)^T\}$, where $A_{k_{n+1}}$ is the matrix with columns $\mathsf{e}_i-\mathsf{e}_j$ for each edge (i,j) of k_{n+1}, $1\le i<j\le n+1$. It is then evident that the vertices of $\mathsf{Flow}_n(\mathbf{a})$ are integral, since $A_{k_{n+1}}$ is unimodular.

Proof of Lemma 1.2. Let $\Phi:\mathsf{Tes}_n(\mathbf{a})\to\mathsf{Flow}_n(\mathbf{a})$ defined by $\Phi: X=(x_{i,j})\mapsto f_X$ where $f_X(i,j)=\begin{cases} x_{i,j} \text{ if } j\ne n+1\\ x_{i,i} \text{ if } j=n+1\end{cases}$. The map Φ is a linear transformation that simply permutes the coordinates of $\mathsf{Tes}_n(\mathbf{a})$. Therefore the determinant of Φ is ± 1 and it follows that Φ is a volume preserving bijection between the polytopes. □

The type A_n **Kostant partition function** $K_{A_n}(\mathbf{a}')$ is the number of ways of writing $\mathbf{a}' := (\mathbf{a},-\sum_{i=1}^n a_i)$ as an $\mathbb{N}$-combination of the type A_n positive roots $\mathsf{e}_i-\mathsf{e}_j$, $1\le i<j\le n+1$ without regard to order. Kostant partition functions are very useful in representation theory for calculations of weight multiplicities and tensor product multiplicities. The value $K_{A_n}(\mathbf{a}')$ is also the number of lattice points of the polytope $\mathsf{Flow}_n(\mathbf{a})$, i.e. integral flows in the complete graph k_{n+1} with netflow a_i on vertex i (see Figure 2 for an example). Thus the following lemma is immediate from Lemma 1.2.

Lemma 1.3. *The number of Tesler matrices with hook sums* $(a_1,a_2,\dots,a_n)$ *is given by the value of the Kostant partition function at* $(a_1,\dots,a_n,-\sum_{i=1}^n a_i)$,

$$T_n(\mathbf{a})=K_{A_n}(\mathbf{a}'). \tag{1.4}$$

In the next example we include a brief discussion of another flow polytope of the complete graph, namely, $\mathsf{Flow}_n(1,0,\dots,0)$.

Example 1.4. The polytope $\mathsf{Flow}_n(1,0,\dots,0)$ is known as the **Chan-Robbins-Yuen polytope**. It has dimension $\binom{n}{2}$ and 2^{n-1} vertices. Stanley-Postnikov (unpublished), and Baldoni-Vergne [4, 5] proved that the normalized volume of this polytope is given by a value of the Kostant partition function (see (3.2))

$$\mathrm{vol}\,\mathsf{Flow}_n(1,0,\ldots,0) = K_{A_{n-1}}(0,1,2,\ldots,n-2,-\tbinom{n-1}{2}).$$

Then Zeilberger [28] used a variant of the Morris constant term identity [23] to compute this value of the Kostant partiton function as the product of the first $n-2$ Catalan numbers, proving a conjecture of Chan, Robbins and Yuen [9, 10].

$$K_{A_{n-1}}(0,1,2,\ldots,n-2,-\tbinom{n-1}{2}) = \prod_{i=0}^{n-2} \frac{1}{i+1}\binom{2i}{i}. \tag{1.5}$$

A Tesler polytope or flow polytope is itself a face of a well known kind of polytope called a transportation polytope which we define next.

1.3. **Transportation polytopes.** Given a vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ of nonnegative integers, the **transportation polytope**[1] $\mathsf{Trans}_n(\mathbf{s})$ is the set of all $n \times n$ matrices $M = (m_{i,j})$ with nonnegative real entries whose i^{th} row and i^{th} column respectively sum to s_i, for $i = 1, \ldots, n$. When all the s_i equal one, the polytope $\mathsf{Trans}_n(1, 1, \ldots, 1)$ is better known as the **Birkhoff polytope**. Next we show that the flow polytope $\mathsf{Flow}_n(\mathbf{a})$ is isomorphic to a face of the transportation polytope $\mathsf{Trans}_n(a_1, a_1 + a_2, \ldots, \sum_{i=1}^n a_i)$; see Figure 3.

Proposition 1.5. *For $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$ with $a_1 > 0$ we have that*

$$\mathsf{Tes}_n(\mathbf{a}) \cong \{(m_{i,j}) \in \mathsf{Trans}_n(a_1, a_1 + a_2, \ldots, \sum_{i=1}^n a_i) \,:\, m_{i,j} = 0 \text{ if } i - j \geq 2\}. \tag{1.6}$$

For example, the Chan-Robbins Yuen polytope $\mathsf{Tes}_n(1, 0, \ldots, 0)$ is isomorphic to a face of the Birkhoff polytope $\mathsf{Trans}_n(1, 1, \ldots, 1)$ [4, Lemma 18] and the Tesler polytope $\mathsf{Tes}_n(1, 1, \ldots, 1)$ is isomorphic to a face of the transportation polytope $\mathsf{Trans}_n(1, 2, \ldots, n)$. To prove the proposition we need the following characterization of the facets of transportation polytopes [21, Theorem 2] by Klee and Witzgall.

Lemma 1.6. [21] *Let $\mathbf{s} = (s_1, s_2, \ldots, s_n)$. The facets of $\mathsf{Trans}_n(\mathbf{s})$ are of the form $F_{i,j}(\mathbf{s}) := \{M \in \mathsf{Trans}_n(\mathbf{s}) \,:\, m_{i,j} = 0\}$ provided $s_i + s_j < \sum_{i=1}^n s_i$.*

Proof of Proposition 1.5. Fix $\mathbf{s} = (a_1, a_1 + a_2, \ldots, \sum_{i=1}^n a_i)$ and let F_n denote the set on the right-hand-side of (1.6). We claim that F_n is a face of $\mathsf{Trans}_n(\mathbf{s})$. If $n = 1, 2$ then $\mathsf{F}_n = \mathsf{Trans}_n(\mathbf{a})$ so the claim follows. For $n \geq 3$ we have that $\mathsf{F}_n = \bigcap_{i-j\geq 2} F_{i,j}(\mathbf{s})$. Since each $s_i \geq a_1 > 0$ then $s_i + s_j < \sum_{i=1}^n s_i$ and by Lemma 1.6 each $F_{i,j}(\mathbf{s})$ is a facet of $\mathsf{Trans}_n(\mathbf{s})$. Thus F_n is a face of this transportation polytope settling the claim.

Next, we build an isomorphism between $\mathsf{Flow}_n(\mathbf{a})$ and F_n. Then the result will follow by Lemma 1.2. Let $\Psi : \mathsf{Flow}_n(\mathbf{a}) \to \mathsf{F}_n$ be defined by $\Psi : f \mapsto (m_{i,j})$ where $m_{i,j} = \begin{cases} f(i, j+1) & \text{if } 1 \leq i \leq j \leq n \\ \sum_{t=1}^j a_t - \sum_{t=1}^j f(t, j+1) & \text{if } j = i-1, \\ 0 & \text{if } i - j \geq 2. \end{cases}$. See Figure 3 for an example.

We check that $(m_{ij}) = \Psi(f)$ is in $\mathsf{Trans}_n(\mathbf{s})$. We have that $m_{j+1,j} = \sum_{k=1}^j a_k - \sum_{k=1}^j f(k, j+1) \geq 0$ since $\sum_{k=1}^j f(k, j+1)$ is at most the total flow introduced

[1] In the literature transportation polytopes are more general [21]. The matrices can be rectangular and the i^{th} row sum and the i^{th} column sum can differ.

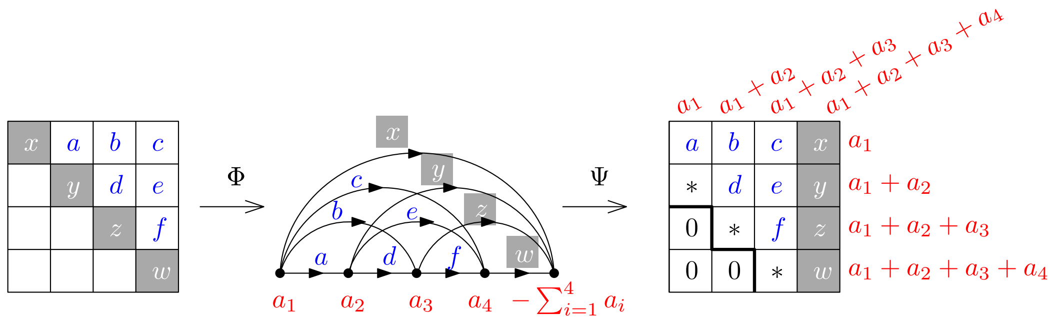


FIGURE 3. Correspondence among: a point X in the Tesler polytope $\mathsf{Tes}_4(\mathbf{a})$, a flow f on the complete graph k_5 with netflow $\mathbf{a}'$, and a point $M = (m_{ij})$ of the transportation polytope $\mathsf{Trans}_4(a_1, a_1 + a_2, \ldots, \sum_{i=1}^4 a_i)$ with $m_{31} = m_{41} = m_{42} = 0$ (the entries marked $*$ are determined by the others).

at vertices $1, 2, \ldots, j$, which is $\sum_{k=1}^{j} a_k$. Therefore all the entries of $\Psi(f)$ are nonnegative. By construction, for $k = 1, \ldots, n-1$ the sum of the k^{th} column is $\sum_{i=1}^{k} a_i$. The sum of the n^{th} column equals the netflow on vertex $n+1$ of the complete graph, $\sum_{i=1}^{n} m_{i,n} = \sum_{i=1}^{n} f(i, n+1) = \sum_{i=1}^{n} a_i$. For the rows, the netflow on vertex k, for $k = 1, \ldots, n$, is a_k which implies that $\sum_{j=k}^{n} f(k, j+1) - \sum_{i=1}^{k-1} f(i,k) = a_k$. Thus the k^{th} row sum equals

$$m_{k,k-1} + \sum_{j=k}^{n} m_{k,j} = \left(\sum_{i=1}^{k-1} a_i - \sum_{i=1}^{k-1} f(i,k)\right) + \sum_{j=k}^{n} f(k, j+1)$$
$$= \sum_{i=1}^{k-1} a_i + \left(\sum_{j=k}^{n} f(k,j+1) - \sum_{i=1}^{k-1} f(i,k)\right) = \sum_{i=1}^{k-1} a_i + a_k.$$

Therefore $\Psi(f)$ is in $\mathsf{Trans}_n(\mathbf{s})$. By construction $m_{i,j} = 0$ if $i - j \geq 2$, so $\Psi(f)$ is also in F_n and so Ψ is well defined. Finally, we leave to the reader to check that Ψ is a bijection with inverse $\Psi^{-1} : \mathsf{F}_n \to \mathsf{Flow}_n(\mathbf{a})$, $(m_{ij}) \mapsto f$ where $f(i,j) = m_{i,j-1}$ for $1 \leq i < j \leq n+1$. □

1.4. **The study of $\mathsf{Tes}_n(\mathbf{a})$.** Examples 1.1 and 1.4 served as our inspiration for studying the Tesler polytope $\mathsf{Tes}_n(\mathbf{a}) \cong \mathsf{Flow}_n(\mathbf{a})$. In Section 2 we prove that for any vector $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ of nonnegative integers, the polytope $\mathsf{Tes}_n(\mathbf{a})$ has dimension $\binom{n}{2}$ and at most $n!$ vertices, all of which are integral. When $\mathbf{a} \in (\mathbb{Z}_{>0})^n$ consists entirely of positive entries, we prove that $\mathsf{Tes}_n(\mathbf{a})$ has exactly $n!$ vertices. In this case, these vertices are the **permutation Tesler matrices** of order n, which are the $n \times n$ Tesler matrices with at most one nonzero entry in each row.

Recall that if P is a d-dimensional polytope, the **f-vector** $f(P) = (f_0, f_1, \ldots, f_d)$ of P is given by letting f_i equal the number of faces of P of dimension i. The **f-polynomial** of P is the corresponding generating function $\sum_{i=0}^{d} f_i x^i$. A polytope P is **simple** if each of its vertices is incident to $\dim(P)$ edges. If P is a simple polytope, the **h-polynomial** of P is the polynomial $\sum_{i=0}^{d} h_i x^i$ which is related to the f-polynomial of P by the equation $\sum_{i=0}^{d} f_i(x-1)^i = \sum_{i=0}^{d} h_i x^i$. The coefficient sequence $(h_0, h_1, \dots, h_d)$ of the h-polynomial of P is called the **h-vector** of P.

In Section 2 we characterize the vectors $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ for which the Tesler polytope $\mathsf{Tes}_n(\mathbf{a})$ is simple (Theorem 2.7). In particular, we show that $\mathsf{Tes}_n(\mathbf{a})$ is simple whenever $\mathbf{a} \in (\mathbb{Z}_{>0})^n$. In this case, the sum of its h-vector entries is given by $\sum_{i=0}^{\binom{n}{2}} h_i = f_0$. Since $\mathsf{Tes}_n(\mathbf{a})$ for $\mathbf{a} \in (\mathbb{Z}_{>0})^n$ has $n!$ vertices, this implies that $\sum_{i=0}^{\binom{n}{2}} h_i = n!$. One might expect that the h-polynomial $\sum_{i=0}^{\binom{n}{2}} h_i x^i$ of $\mathsf{Tes}_n(\mathbf{a})$ is the generating function of some interesting statistic on permutations. Indeed, we show in Section 2 that the h-polynomial of the Tesler polytope is the generating function for Coxeter length.

Theorem 1.7. *(Theorem 2.7, Corollary 2.9) Let $\mathbf{a} \in (\mathbb{Z}_{>0})^n$ be a vector of positive integers. The polytope $\mathsf{Tes}_n(\mathbf{a})$ is a simple polytope and its h-vector is given by the Mahonian numbers, that is, h_i is the number of permutations of $\{1, 2, \dots, n\}$ with i inversions. We have*

$$\sum_{i=0}^{\binom{n}{2}} f_i(x-1)^i = \sum_{i=0}^{\binom{n}{2}} h_i x^i = [n]!_x,$$

where $[n]!_x = \prod_{i=1}^{n}(1 + x + x^2 + \cdots + x^{i-1})$ and the f_i are the f-vector entries of $\mathsf{Tes}_n(\mathbf{1})$.

Just as $\mathsf{Tes}_n(1, 0, \dots, 0)$, i.e. the Chan-Robbins-Yuen polytope, $\mathsf{Flow}_n(1, 0, \dots, 0)$, has a product formula for its normalized volume involving Catalan numbers, so does the Tesler polytope $\mathsf{Tes}_n(\mathbf{1}) := \mathsf{Tes}_n(1, 1, \dots, 1)$. The following result is proven in Section 3 using a new iterated constant term identity (Lemma 3.5).

Theorem 1.8. *(Corollary 3.6) The normalized volume of the Tesler polytope $\mathsf{Tes}_n(\mathbf{1})$, or equivalently of the flow polytope $\mathsf{Flow}_n(1, 1, \dots, 1)$ equals*

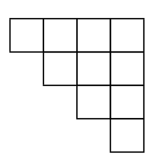
$$\begin{aligned} \operatorname{vol} \mathsf{Tes}_n(\mathbf{1}) = \operatorname{vol} \mathsf{Flow}_n(1, 1, \dots, 1) &= \frac{\binom{n}{2}! \cdot 2^{\binom{n}{2}}}{\prod_{i=1}^{n} i!} \\ &= f^{(n-1, n-2, \dots, 1)} \cdot \prod_{i=0}^{n-1} Cat(i), \end{aligned} \tag{1.7}$$

where $Cat(i) = \frac{1}{i+1}\binom{2i}{i}$ is the i^{th} Catalan number and $f^{(n-1,n-2,\dots,1)}$ is the number of Standard Young Tableaux of staircase shape $(n-1, n-2, \dots, 1)$.

2. The face structure of $\mathsf{Tes}_n(\mathbf{a})$

Let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$. The aim of this section is to describe the face poset of $\mathsf{Tes}_n(\mathbf{a})$. It will turn out that the combinatorial isomorphism type of $\mathsf{Tes}_n(\mathbf{a})$ only depends on the positions of the zeros in the integer vector $\mathbf{a}$.

Let rstc_n denote the reverse staircase of size n; the Ferrers diagram of rstc_4 is shown below.



We use the "matrix coordinates" $\{(i,j) : 1 \leq i \leq j \leq n\}$ to describe the cells of rstc_n. An **a-Tesler tableau** T is a $0,1$-filling of rstc_n which satisfies the following three conditions:

(1) for $1 \leq i \leq n$, if $a_i > 0$, there is at least one 1 in row i of T,
(2) for $1 \leq i < j \leq n$, if $T(i,j) = 1$, then there is at least one 1 in row j of T, and
(3) for $1 \leq j \leq n$, if $a_j = 0$ and $T(i,j) = 0$ for all $1 \leq i < j$, then $T(j,k) = 0$ for all $j \leq k \leq n$.

For example, if $n = 4$ and $\mathbf{a} = (7,0,3,0)$, then three **a**-Tesler tableaux are shown below. We write the entries of **a** in a column to the left of a given **a**-Tesler tableau.

7	0	1	1	1	7	1	0	1	0	7	1	1	1	0
0		0	0	1	0		0	0	0	0		1	1	0
3			1	1	3			0	1	3			1	0
0				1	0				1	0				0

The **dimension** $\dim(T)$ of an **a**-Tesler tableau T is $\sum_{i=1}^{n}(r_i - 1)$, where

$$r_i = \begin{cases} \text{the number of 1's in row } i \text{ of } T & \text{if row } i \text{ of } T \text{ is nonzero,} \\ 1 & \text{if row } i \text{ of } T \text{ is zero.} \end{cases}$$

From left to right, the dimensions of the tableaux shown above are $3, 1$, and 3.

Given two **a**-Tesler tableaux T_1 and T_2, we write $T_1 \leq T_2$ to mean that for all $1 \leq i \leq j \leq n$ we have $T_1(i,j) \leq T_2(i,j)$. Moreover, we define a $0,1$-filling $\max(T_1, T_2)$ of rstc_n by $\max(T_1,T_2)(i,j) = \max(T_1(i,j), T_2(i,j))$.

We start with two lemmas on **a**-Tesler tableaux. Our first lemma states that any two zero-dimensional **a**-Tesler tableaux are componentwise incomparable.

Lemma 2.1. *Let* $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ *and let* T_1 *and* T_2 *be two* **a**-*Tesler tableaux with* $\dim(T_1) = \dim(T_2) = 0$. *If* $T_1 \leq T_2$, *then* $T_1 = T_2$.

Proof. Since $\dim(T_1) = \dim(T_2) = 0$, for all $1 \leq i \leq n$ we have that row i of either T_1 or T_2 consists entirely of 0's, with the possible exception of a single 1. Since $T_1 \leq T_2$, it is enough to show that if row i of T_2 contains a 1, then row i of T_1 also contains a 1. To prove this, we induct on i. If $i = 1$, then row 1 of T_2 contains a 1 if and only if $a_1 > 0$, in which case row 1 of T_1 contains a 1. If $i > 1$, suppose that row i of T_2 contains a 1. Then either $a_i > 0$ (in which case row i of T_1 also contains a 1) or $a_i = 0$ and there exists $i' < i$ such that $T_2(i', i) = 1$. But in the latter case we have that row i' of T_1 contains a 1 by induction. This combined with the condition $T_1 \leq T_2$ and the fact that T_1 and T_2 contain a unique 1 in row i' forces $T_1(i', i) = 1$. Therefore, row i of T_1 contains a 1. We conclude that $T_1 = T_2$. $\square$

Our next lemma states that the operation of componentwise maximum preserves the property of being an **a**-Tesler tableau.

Lemma 2.2. *Let* $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ *and let* T_1 *and* T_2 *be two* **a**-*Tesler tableaux. Then* $T := \max(T_1, T_2)$ *is also an* **a**-*Tesler tableau.*

Proof. If $a_i > 0$ for some $1 \leq i \leq n$, then row i of T is nonzero because row i of T_1 is nonzero. If $1 \leq i < j \leq n$ and $T(i,j) = 1$, then either $T_1(i,j) = 1$ or $T_2(i,j) = 1$. In turn, row j of either T_1 or T_2 is nonzero, forcing row j of T to be nonzero. Finally, if $1 \leq j \leq n$, $a_j = 0$, and $T(i,j) = 0$ for all $1 \leq i < j$, then

$T_1(i,j) = T_2(i,j) = 0$ for all $1 \leq i < j$. This means that row j of T_1 and T_2 is zero, so row j of T is also zero. □

The analogue of Lemma 2.2 for $\min(T_1, T_2)$ is false; the componentwise minimum of two $\mathbf{a}$-Tesler tableaux is not in general an $\mathbf{a}$-Tesler tableau. Faces of the Tesler polytope $\mathsf{Tes}_n(\mathbf{a})$ and $\mathbf{a}$-Tesler tableaux are related by taking supports.

Lemma 2.3. *Let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ and let F be a face of the Tesler polytope $\mathsf{Tes}_n(\mathbf{a})$. Define a function $T : \mathrm{rstc}_n \to \{0,1\}$ by $T(i,j) = 0$ if the coordinate equality $x_{i,j} = 0$ is satisfied on the face F and $T(i,j) = 1$ otherwise. Then T is an $\mathbf{a}$-Tesler tableau.*

Proof. If $a_i > 0$ for some $1 \leq i \leq n$, we have $x_{i,i} + x_{i,i+1} + \cdots + x_{i,n} \geq a_i$ on the face F, so that row i of T is nonzero. Suppose $T(i,j) = 1$ for some $1 \leq i < j \leq n$. Then $x_{i,j} > 0$ holds for some point in F, so that $x_{j,j} + x_{j,j+1} + \cdots + x_{j,n} \geq x_{i,j} > 0$ at that point. In particular, row j of T is nonzero. Finally, suppose that $a_j = 0$ and for all $1 \leq i < j$ we have $T(i,j) = 0$. Then on the face F we have $x_{j,j} + x_{j,j+1} + \cdots + x_{j,n} = 0$, forcing $x_{j,j} = x_{j,j+1} = \cdots = x_{j,n} = 0$ on F. This means that row j of T is zero. □

Lemma 2.3 shows that every face F of $\mathsf{Tes}_n(\mathbf{a})$ gives rise to an $\mathbf{a}$-Tesler tableaux T. We denote by $\phi : F \mapsto T$ the corresponding map from faces of $\mathsf{Tes}_n(\mathbf{a})$ to $\mathbf{a}$-Tesler tableaux; we will see that ϕ is a bijection. We begin by showing that ϕ bijects vertices of $\mathsf{Tes}_n(\mathbf{a})$ with zero-dimensional $\mathbf{a}$-Tesler tableaux.

Lemma 2.4. *Let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$. The map ϕ bijects the vertices of $\mathsf{Tes}_n(\mathbf{a})$ with zero-dimensional $\mathbf{a}$-Tesler tableaux.*

Proof. Let T be an $\mathbf{a}$-Tesler tableau with $\dim(T) = 0$. Then T contains at most a single 1 in every row. There exists a unique point $B_T \in \mathsf{Tes}_n(\mathbf{a})$ such that the support of the matrix B_T equals the set of nonzero entries of T. (Indeed, the vector $\mathbf{a}$ can be used to construct the matrix B_T row by row, from top to bottom.) By Lemma 2.1, we have that $B_{T_1} \neq B_{T_2}$ for distinct zero-dimensional $\mathbf{a}$-Tesler tableaux T_1 and T_2. We argue that the set

$$\{B_T \,:\, T \text{ an } \mathbf{a}\text{-Tesler tableau with } \dim(T) = 0\}$$

is precisely the set of vertices of $\mathsf{Tes}_n(\mathbf{a})$. Since this implies that $\phi(B_T) = T$, the lemma will follow.

To begin, we argue that $\mathsf{Tes}_n(\mathbf{a}) = \mathrm{conv}\{B_T \,:\, \dim(T) = 0\}$. To facilitate this inductive argument, given any matrix $B = (b_{i,j}) \in \mathsf{Tes}_n(\mathbf{a})$, define the *dimension* $\dim(B)$ to be $\dim(T)$, where T is the $\mathbf{a}$-Tesler tableau whose entries are

$$T(i,j) = \begin{cases} 0 & b_{i,j} = 0 \\ 1 & b_{i,j} \neq 0. \end{cases}$$

Fix a matrix $B \in \mathsf{Tes}_n(\mathbf{a})$. We want to show that $B \in \mathrm{conv}\{B_T \,:\, \dim(T) = 0\}$. We induct on $\dim(B)$. If $\dim(B) = 0$, then $B = B_T$ for some $\mathbf{a}$-Tesler tableau T with $\dim(T) = 0$ and the result follows, so assume $\dim(B) > 0$. Since $\dim(B) > 0$, at least one row of B has more than one positive entry. Let $1 \leq i_0 \leq n-1$ be maximal such that row i_0 of B has more than one positive entry.

For any $i_0 < j \leq n$ with $b_{i_0,j} > 0$, we define a subset $P_j = \{(p_1,q_1),(p_2,q_2),\ldots\}$ of the matrix coordinates of B (called the *positive path at* j) as follows. Let $(p_1,q_1) = (i_0,j)$. Given $(p_r,q_r) \in P_j$ with $p_r < n$, we define (p_{r+1},q_{r+1}) by letting $p_{r+1} = q_r$ and letting q_{r+1} be the column of the unique nonzero entry in row q_r of

B. We also set $P_{i_0} = \{(i_0, j_0)\}$. For example, if $\mathbf{a} = (4,3,1,1,1,2)$ and B is the point in $\mathsf{Tes}_6(\mathbf{a})$ shown below, we have $i_0 = 2$ and $P_3 = \{(2,3),(3,5),(5,5)\}, P_4 = \{(2,4),(4,4)\}$, and $P_6 = \{(2,6),(6,6)\}$. In general, for any distinct j, j' we have $P_j \cap P_{j'} = \emptyset$.

$$B = \begin{bmatrix} 0 & 2 & 0 & 1 & 1 & 0 \\ & 0 & 2 & 2 & 0 & 1 \\ & & 0 & 0 & 3 & 0 \\ & & & 4 & 0 & 0 \\ & & & & 5 & 0 \\ & & & & & 3 \end{bmatrix}$$

Let $i_0 \leq j_0 < j_1 \leq n$ be such that $c := b_{i_0,j_0}$ and $d := b_{i_0,j_1}$ are positive. We define two new upper triangular $n \times n$ matrices $B' = (b'_{i,j})$ and $B'' = (b''_{i,j})$ by the rules

$$b'_{i,j} = \begin{cases} b_{i,j} + d & (i,j) \in P_{j_0} \\ b_{i,j} - d & (i,j) \in P_{j_1} \\ b_{i,j} & \text{otherwise} \end{cases} \tag{2.1}$$

and

$$b''_{i,j} = \begin{cases} b_{i,j} - c & (i,j) \in P_{j_0} \\ b_{i,j} + c & (i,j) \in P_{j_1} \\ b_{i,j} & \text{otherwise.} \end{cases} \tag{2.2}$$

For example, if B is as above with $i_0 = 2$, if we make the choices $j_0 = 3$ and $j_1 = 6$ the matrices B' and B'' are as follows.

$$B' = \begin{bmatrix} 0 & 2 & 0 & 1 & 1 & 0 \\ & 0 & 3 & 2 & 0 & 0 \\ & & 0 & 0 & 4 & 0 \\ & & & 4 & 0 & 0 \\ & & & & 6 & 0 \\ & & & & & 2 \end{bmatrix} \qquad B'' = \begin{bmatrix} 0 & 2 & 0 & 1 & 1 & 0 \\ & 0 & 0 & 2 & 0 & 3 \\ & & 0 & 0 & 1 & 0 \\ & & & 4 & 0 & 0 \\ & & & & 3 & 0 \\ & & & & & 5 \end{bmatrix}$$

It is straightforward to verify that both B' and B'' lie in $\mathsf{Tes}_n(\mathbf{a})$. Since B' and B'' have one fewer positive entry than B in row i_0, we have $\dim(B') < \dim(B)$ and $\dim(B'') < \dim(B)$, so that inductively $B' \in \text{conv}\{B_T : \dim(T) = 0\}$ and $B'' \in \text{conv}\{B_T : \dim(T) = 0\}$. Since $B = \frac{1}{c+d}(cB' + dB'')$, we conclude that $B \in \text{conv}\{B_T : \dim(T) = 0\}$.

Since $\mathsf{Tes}_n(\mathbf{a}) = \text{conv}\{B_T : \dim(T) = 0\}$, every vertex of $\mathsf{Tes}_n(\mathbf{a})$ is of the form B_T for some $\mathbf{a}$-Tesler tableau T with $\dim(T) = 0$. We argue that every matrix B_T is actually a vertex of $\mathsf{Tes}_n(\mathbf{a})$. For otherwise, there would exist some $\mathbf{a}$-Tesler tableau T with $\dim(T) = 0$ such that

$$B_T = \sum_{\substack{\dim(T')=0 \\ T' \neq T}} c_{T'} B_{T'},$$

for some $c_{T'} \geq 0$ with $\sum c_{T'} = 1$. But this is impossible by Lemma 2.1. We conclude that B_T is a vertex of $\mathsf{Tes}_n(\mathbf{a})$. □

We are ready to characterize the face poset of $\mathsf{Tes}_n(\mathbf{a})$.

Theorem 2.5. *Let* $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$*. The support map* $\phi : F \mapsto T$ *gives an isomorphism from the face poset of* $\mathsf{Tes}_n(\mathbf{a})$ *to the set of* $\mathbf{a}$*-Tesler tableaux, partially ordered by* $\leq$*. For any face* F*, we have that* $\dim(F) = \dim(\phi(F))$*.*

Proof. For any $\mathbf{a}$-Tesler tableau T, define a face $F(T) \subseteq \mathsf{Tes}_n(\mathbf{a})$ by letting $F(T)$ be the intersection of the hyperplanes $\{x_{i,j} = 0 : T(i,j) = 0\}$ within the ambient affine subspace

$$\bigcap_{i=1}^{n} \{x_{i,i} + x_{i,i+1} + \cdots + x_{i,n} = a_i + x_{1,i} + \cdots + x_{i-1,i}\}$$

of $\{(x_{i,j}) : x_{i,j} \in \mathbb{R}, 1 \leq i \leq j \leq n\}$. It is evident that $\dim(F(T)) = \dim(T)$ and that $\phi(F(T)) = T$. Moreover, we have that $T_1 \leq T_2$ if and only if $F(T_1) \subseteq F(T_2)$. It therefore suffices to show that every face of $\mathsf{Tes}_n(\mathbf{a})$ is of the form $F(T)$ for some $\mathbf{a}$-Tesler tableau T.

Let F be a face of $\mathsf{Tes}_n(\mathbf{a})$. By Lemma 2.4, there exist zero-dimensional $\mathbf{a}$-Tesler tableaux $T_1, \ldots, T_k$ such that $B_{T_1}, \ldots, B_{T_k}$ are the vertices of F. Let $T = \max(T_1, \ldots, T_k)$. By Lemma 2.2 we have that T is an $\mathbf{a}$-Tesler tableau. It is clear that $F \subseteq F(T)$. We argue that $F(T) \subseteq F$. To see this, suppose that $1 \leq i \leq j \leq n$ and the defining hyperplane $x_{i,j} = 0$ of $\mathsf{Tes}_n(\mathbf{a})$ contains F. Then in particular we have that $x_{i,j} = 0$ contains $B_{T_1}, \ldots, B_{T_k}$, so that $T_1(i,j) = \cdots = T_k(i,j) = 0$. This means that $T(i,j) = 0$, so that $x_{i,j} = 0$ contains $F(T)$. We conclude that $F = F(T)$. □

Given any vector $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$, we let $\epsilon(\mathbf{a}) \in \{0, +\}^n$ be the associated **signature**; for example, $\epsilon(7, 0, 3, 0) = (+, 0, +, 0)$. Theorem 2.5 implies that the combinatorial isomorphism type of $\mathsf{Tes}_n(\mathbf{a})$ depends only on the signature $\epsilon(\mathbf{a})$.

As a first application of Theorem 2.5, we determine the dimension of $\mathsf{Tes}_n(\mathbf{a})$ and give an upper bound on the number of its vertices. When $\mathbf{a} \in \mathbb{Z}^n_{>0}$ the result about the dimensionality also follows from [4]. Observe that if $a_1 = 0$, the first rows of the matrices in $\mathsf{Tes}_n(\mathbf{a})$ vanish and we have the identification $\mathsf{Tes}_n(\mathbf{a}) = \mathsf{Tes}_{n-1}(a_2, a_3, \ldots, a_n)$. We may therefore restrict to the case where $a_1 > 0$.

Corollary 2.6. *Let* $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$ *and assume* $a_1 > 0$*. The polytope* $\mathsf{Tes}_n(\mathbf{a})$ *has dimension* $\binom{n}{2}$ *and at most* $n!$ *vertices. Moreover, the polytope* $\mathsf{Tes}_n(\mathbf{a})$ *has exactly* $n!$ *vertices if and only if* $a_2, a_3, \ldots, a_{n-1} > 0$*.*

Proof. The claim about dimension follows from the fact that the mapping $T(i,j) = 1$ for $1 \leq i \leq j \leq n$ is an $\mathbf{a}$-Tesler tableau of dimension $\binom{n}{2}$ (since $a_1 > 0$).

Recall that a **file rook** is a rook which can attack horizontally, but not vertically (see for example [7, Definition 1]). There is an injective mapping from the set of zero-dimensional $\mathbf{a}$-Tesler tableaux to the set of maximal file rook placements on rstc_n by placing a file rook in the position of every 1 in T, together with a file rook on the main diagonal of any zero row of T. Since there are $n!$ maximal file rook placements on rstc_n, by Theorem 2.5 we have that $\mathsf{Tes}_n(\mathbf{a})$ has at most $n!$ vertices.

If $a_2, a_3, \ldots, a_{n-1} > 0$, then a zero-dimensional $\mathbf{a}$-Tesler tableau T contains a unique 1 in every row, with the possible exception of row n (which consists of a single cell). Thus, every maximal file rook placement on rstc_n arises from a zero-dimensional $\mathbf{a}$-Tesler tableau. It follows that $\mathsf{Tes}_n(\mathbf{a})$ has $n!$ vertices. On the other hand, if $a_i = 0$ for some $1 < i < n$, then for any zero-dimensional $\mathbf{a}$-Tesler tableau T we have that $T(j,k) = 0$ for all $j < k$ implies $T(i,i) = 0$. In terms of the corresponding file rook placements, this means that if the file rooks in every row other than i are on the main diagonal, then the file rook in row i is also on the main diagonal. In particular, the mapping from zero-dimensional $\mathbf{a}$-Tesler tableaux to maximal file rook placements on rstc_n is not surjective and the polytope $\mathsf{Tes}_n(\mathbf{a})$ has $< n!$ vertices. □

Theorem 2.5 can also be used to characterize when $\mathsf{Tes}_n(\mathbf{a})$ is a simple polytope.

Theorem 2.7. *Let $\mathbf{a} = (a_1, \ldots, a_n) \in (\mathbb{Z}_{\geq 0})^n$ and let $\epsilon(\mathbf{a}) = (\epsilon_1, \ldots, \epsilon_n) \in \{0, +\}^n$ be the associated signature. Assume that $\epsilon_1 = +$. The polytope $\mathsf{Tes}_n(\mathbf{a})$ is a simple polytope if and only if $n \leq 3$ or $\epsilon(\mathbf{a})$ is one of $+^n, +^{n-1}0, +0+^{n-2}$ or $+0+^{n-3}0$.*

Proof. When $n = 1$ the polytope $\mathsf{Tes}_1(\mathbf{a})$ is a single point. When $n = 2$ the polytope $\mathsf{Tes}_2(\mathbf{a})$ is an interval. When $n = 3$ the polytope $\mathsf{Tes}_3(\mathbf{a})$ is a 3-simplex Δ_3 if $\epsilon_2 = 0$ and the triangular prism $\Delta_1 \times \Delta_2$ if $\epsilon_2 = +$. In either case, we have that $\mathsf{Tes}_3(\mathbf{a})$ is simple.

In general, the vertices of $\mathsf{Tes}_n(\mathbf{a})$ correspond to zero-dimensional $\mathbf{a}$-Tesler tableaux T. We may therefore speak of "adjacent" zero-dimensional $\mathbf{a}$-Tesler tableaux T_1 and T_2 to mean that the corresponding vertices B_{T_1} and B_{T_2} are connected by an edge of $\mathsf{Tes}_n(\mathbf{a})$. Given two distinct $\mathbf{a}$-Tesler tableau T_1, T_2 with $\dim(T_1) = \dim(T_2) = 0$, by Theorem 2.5 we know that T_1 and T_2 are adjacent if and only if for all $1 \leq i \leq n$, row i of T_2 can be obtained from row i of T_1 by

(1) leaving row i of T_1 unchanged,
(2) changing the unique 1 in row i of T_1 to a 0,
(3) changing a single 0 in row i to T_1 to a 1 (if row i of T_1 is a zero row), or
(4) moving the unique 1 in row i of T_1 to a different position in row i.

Moreover, the Operation (4) must take place in precisely one row of T_1.

Given a fixed $\mathbf{a}$-Tesler tableau T with $\dim(T) = 0$, we can replace the 0's in T with entries in the set $\{\textcircled{i} : i \in \mathbb{Z}_{\geq 0}\}$ to keep track of some of the adjacent zero-dimensional $\mathbf{a}$-Tesler tableaux. In particular, we define a new filling T° of rstc_n using the alphabet $\{1, \textcircled{0}, \textcircled{1}, \textcircled{2}, \ldots\}$ as follows.

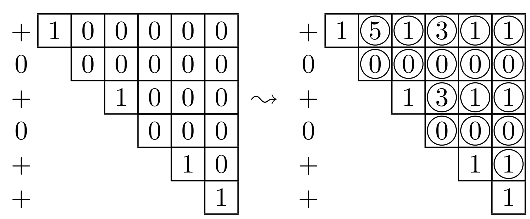
- If $T(i,j) = 1$, set $T^\circ(i,j) = 1$.
- If $T(i,j) = 0$ and row i of T is zero, then set $T^\circ(i,j) = \textcircled{0}$.
- If $T(i,j) = 0$, row i of T is nonzero, and row j of T is nonzero, then set $T^\circ(i,j) = \textcircled{1}$.
- If $T(i,j) = 0$, row i of T is nonzero, and row j of T is zero, then set $T^\circ(i,j) = \textcircled{j'}$, where $j' = n - j + 1$ is the number of boxes in row j.

Observe that in the first case we necessarily have $\epsilon_i = 0$ and in the third case we necessarily have $\epsilon_j = 0$. For example, suppose $n = 5$ and $(\epsilon_1, \ldots, \epsilon_5) = (+, 0, 0, 0, +)$. Applying the above rules to the zero-dimensional $\mathbf{a}$-Tesler tableau T shown below yields the given T°.

$$T = \begin{array}{c|c|c|c|c|c|} + & 0 & 0 & 0 & 1 & 0 \\ 0 & & 0 & 0 & 0 & 0 \\ 0 & & & 0 & 0 & 0 \\ 0 & & & & 0 & 1 \\ + & & & & & 1 \end{array} \rightsquigarrow \begin{array}{c|c|c|c|c|c|} + & \textcircled{1} & \textcircled{4} & \textcircled{3} & 1 & \textcircled{1} \\ 0 & & \textcircled{0} & \textcircled{0} & \textcircled{0} & \textcircled{0} \\ 0 & & & \textcircled{0} & \textcircled{0} & \textcircled{0} \\ 0 & & & & \textcircled{1} & 1 \\ + & & & & & 1 \end{array} = T^\circ$$

For any $\mathbf{a}$-Tesler tableau T with $\dim(T) = 0$, we claim that the number of adjacent zero-dimensional $\mathbf{a}$-Tesler tableaux is at least the sum of the circled entries in the associated tableau T°. For example, the number of adjacent tableaux in the case shown above is $\geq 1+4+3+1+1 = 10$. To see this, observe that for any adjacent zero-dimensional **a**-Tesler tableau T', there is precisely one row i such that both T and T' contain a 1 in row i, but this 1 is in a different position (corresponding to Operation (4) above). We can view T' as being obtained from T by moving this 1 in row i, and then possibly changing entries in lower rows (corresponding to Operations (2) and (3) above). If this 1 is moved to a position (i, j) such that row j of T is zero, then one of the $j' = n - j + 1$ 0's in row j of T' must be changed to a 1. In the example above, if the 1 in position $(1, 4)$ is moved to $(1, 2)$, then one of the four 0's in positions $(2, 2), (2, 3), (2, 4)$, and $(2, 5)$ must be changed to a 1, which corresponds to the circled 4 in position $(1, 2)$ of T°. We emphasize that this lower bound on the number of adjacent tableaux is not tight in general; for example, if we move the 1 in row 1 in the above tableau from $(1, 4)$ to $(1, 2)$ and change the 0 in position $(2, 3)$ to a 1, then we must change one of the three 0's in row 3 to a 1, leading to more options for adjacent tableaux. In particular, the number of adjacent tableaux to the tableau T shown above is $> 10 = \binom{5}{2} = \dim(\mathsf{Tes}_5(+, 0, +, 0, +, +))$ and the polytope $\mathsf{Tes}_5(+, 0, +, 0, +, +)$ is not simple.

Suppose that $n > 3$ and there exist indices $1 < i < j < n$ such that $\epsilon_i = +$ and $\epsilon_j = 0$. We argue that $\mathsf{Tes}_n(\mathbf{a})$ is not simple by exhibiting an **a**-Tesler tableau T such that T has $> \binom{n}{2} = \dim(\mathsf{Tes}_n(\mathbf{a}))$ adjacent zero-dimensional **a**-Tesler tableaux. Indeed, let T be the "diagonal" **a**-Tesler tableau defined by $T(k, \ell) = 0$ whenever $1 \leq k < \ell \leq n$, $T(i, i) = 1$ if $\epsilon_i = +$, and $T(i, i) = 0$ if $\epsilon_i = 0$. Perform the above circling procedure to T to get the tableau T°; the example $\epsilon = (+, 0, +, 0, +, +)$ is shown below.



We claim that the sum of the circled entries in row 1 of T°, plus the number of circled positive entries in the remaining rows of T°, equals $\binom{n}{2}$. Indeed, since $\epsilon_1 > 0$, we have the entry in position $(1, k)$ of T° is a positive circled number for $2 \leq k \leq n$. If $T^\circ(1, k) = \textcircled{1}$, then row k of T is nonzero, so that row k of T° consists of precisely one 1, together with $n - k$ $\textcircled{1}$'s. If $T^\circ(1, k) = \textcircled{k'}$ for some $k' > 1$, we must have that $k' = n - k + 1$, $\epsilon_k = 0$, and row k of T° consists entirely of $\textcircled{0}$'s. In either case, the circled entry in $T^\circ(1, k)$, plus the number of positive circled entries in row k of T°, is one plus the number of boxes in row k of T°. On the other hand, the entry in position (i, j) of T° is a circled number > 1 because $\epsilon_j = 0$ and $j < n$. This means that the sum of the circled entries is $> \binom{n}{2}$, the tableau T has $> \binom{n}{2}$ adjacent zero-dimensional tableaux, and the polytope $\mathsf{Tes}_n(\mathbf{a})$ is not simple.

Suppose that $n > 3$ and ϵ has the form $\epsilon = +0^i +^{n-i-1}$ for some $1 < i < n$. Let T be the "near-diagonal" zero-dimensional **a**-Tesler tableau defined by $T(1, 2) = T(2, 2) = 1$, $T(j, j) = 1$ for $i < j \leq n$, and $T(k, \ell) = 0$ otherwise. Perform the above circling procedure to T to get T°; the case $\epsilon = (+, 0, 0, 0, +, +)$ is shown below.

$$
\begin{array}{c|c|c|c|c|c|c|}
+ & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & & 1 & 0 & 0 & 0 & 0 \\
0 & & & 0 & 0 & 0 & 0 \\
0 & & & & 0 & 0 & 0 \\
+ & & & & & 1 & 0 \\
+ & & & & & & 1
\end{array}
\rightsquigarrow
\begin{array}{c|c|c|c|c|c|c|}
+ & \textcircled{1} & 1 & \textcircled{4} & \textcircled{3} & \textcircled{1} & \textcircled{1} \\
0 & & 1 & \textcircled{4} & \textcircled{3} & \textcircled{1} & \textcircled{1} \\
0 & & & 0 & \textcircled{0} & \textcircled{0} & \textcircled{0} \\
0 & & & & 0 & \textcircled{0} & \textcircled{0} \\
+ & & & & & 1 & \textcircled{1} \\
+ & & & & & & 1
\end{array}
$$

A similar argument as in the last paragraph shows that the sum of the circled entries in row 1 of T°, plus the number of positive circled entries in the remaining rows of T°, equals $\binom{n}{2}$. On the other hand, since $1 < i < n$ and $n > 3$, at least one of the circled entries in row 2 of T° is > 1. We conclude that the sum of all the circled entries is $> \binom{n}{2}$, so that $\mathsf{Tes}_n(\mathbf{a})$ is not simple.

If $\epsilon_n = +$, let $\mathbf{a}' = (a_1, a_2, \ldots, a_{n-1}, 0)$. We claim that the polytopes $\mathsf{Tes}_n(\mathbf{a})$ and $\mathsf{Tes}_n(\mathbf{a}')$ are affine isomorphic: $\mathsf{Tes}_n(\mathbf{a}) \cong \mathsf{Tes}_n(\mathbf{a}')$. Indeed, an isomorphism $B \mapsto B'$ is obtained by subtracting a_n from the (n,n)-entry of any matrix $B \in \mathsf{Tes}_n(\mathbf{a})$. By this fact and the last two paragraphs, the polytope $\mathsf{Tes}_n(\mathbf{a})$ is not simple unless $\epsilon(\mathbf{a})$ has one of the four forms given in the statement of the theorem. Also by this fact, to complete the proof we need only show that $\mathsf{Tes}_n(\mathbf{a})$ is simple when $\epsilon(\mathbf{a})$ has one of the two forms $+^n$ or $+0+^{n-2}$.

If $\epsilon(\mathbf{a}) = +^n$, then any zero-dimensional $\mathbf{a}$-Tesler tableau has a unique 1 in every row. Given an $\mathbf{a}$-Tesler tableau T with $\dim(T) = 0$, the tableaux adjacent to T can be obtained by moving a single 1 to a different position in its row. There are $(n-1) + (n-2) + \cdots + 1 = \binom{n}{2} = \dim(\mathsf{Tes}_n(\mathbf{a}))$ ways to do this, so the polytope $\mathsf{Tes}_n(\mathbf{a})$ is simple.

If $\epsilon(\mathbf{a}) = +0+^{n-2}$, then any zero-dimensional $\mathbf{a}$-Tesler tableau T has a unique 1 in every row, with the possible exception of row 2. In particular, row 2 of T contains a 1 if and only if the 1 in row 1 of T is in position $(1,2)$. In either case, we see that T is adjacent to precisely $\binom{n}{2}$ tableaux, so that $\mathsf{Tes}_n(\mathbf{a})$ is simple. $\square$

We now focus on the case of greatest representation theoretic interest in the context of diagonal harmonics: where $\epsilon(\mathbf{a}) = +^n$, so that every entry of $\mathbf{a}$ is a positive integer. The combinatorial isomorphism type of $\mathsf{Tes}_n(\mathbf{a})$ is immediate from Theorem 2.5. We denote by Δ_d the d-dimensional simplex in $\mathbb{R}^{d+1}$ defined by $\Delta_d := \{(x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \,:\, x_1 + \cdots + x_{d+1} = 1, x_1 \geq 0, \ldots, x_{d+1} \geq 0\}$.

Corollary 2.8. *Let $\mathbf{a} \in (\mathbb{Z}_{>0})^n$ be a vector of positive integers. The face poset of the Tesler polytope $\mathsf{Tes}_n(\mathbf{a})$ is isomorphic to the face poset of the Cartesian product of simplices $\Delta_1 \times \Delta_2 \times \cdots \times \Delta_{n-1}$.*

Corollary 2.9. *Let $\mathbf{a} \in (\mathbb{Z}_{>0})^n$ be a vector of positive integers. The h-polynomial of the Tesler polytope $\mathsf{Tes}_n(\mathbf{a})$ is the Mahonian distribution*

$$\sum_{i=0}^{\binom{n}{2}} h_i x^i = [n]!_x = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1}).$$

Proof. We give two proofs of this result, one relying on Corollary 2.8 and one relying on generic linear forms.

First proof: Let P and Q be arbitrary simple polytopes and let $P \times Q$ be their Cartesian product. The polytope $P \times Q$ is simple and the h-polynomial of $P \times Q$ is the product of the h-polynomials of P and Q. To see this, observe that a typical i-dimensional face of $P \times Q$ is given by the product of an j-dimensional face of

P and a $i-j$-dimensional face of Q, for some $0 \leq j \leq i$. Therefore, the f-vectors $f(P) = (f_0(P), f_1(P), \dots)$ and $f(Q) = (f_0(Q), f_1(Q), \dots)$ are related to the f-vector of the product $f(P \times Q)$ by $f_i(P \times Q) = \sum_{j=0}^{i} f_i(P) f_{i-j}(Q)$. The h-polynomials are therefore related by:

$$\begin{aligned}
\sum_{i=0}^{\dim(P)+\dim(Q)} h_i(P \times Q) x^i &= \sum_{i=0}^{\dim(P)+\dim(Q)} f_i(P \times Q)(x-1)^i \\
&= \sum_{i=0}^{\dim(P)+\dim(Q)} \left(\sum_{j=0}^{i} f_j(P)(x-1)^j f_{i-j}(Q)(x-1)^{i-j} \right) \\
&= \left(\sum_{i=0}^{\dim(P)} f_i(P)(x-1)^i \right) \left(\sum_{j=0}^{\dim(Q)} f_j(Q)(x-1)^j \right),
\end{aligned}$$

which equals the product of the h-polynomials of P and Q. This multiplicative property of h-polynomials is surely well known, but the authors could not find a reference.

It remains to observe that the h-polynomial of the d-dimensional simplex Δ_d is given by $\sum_{i=0}^{d} h_i(\Delta_d) x^i = \sum_{i=0}^{d} \binom{d+1}{i+1}(x-1)^i = 1 + x + \cdots + x^d$, where we used the fact that Δ_d has $\binom{d+1}{i+1}$ faces of dimension i.

Second proof: Let λ be any generic linear form on the vector space spanned by $\mathsf{Tes}_n(\mathbf{1})$. Then λ induces an orientation on the 1-skeleton of $\mathsf{Tes}_n(\mathbf{a})$ by requiring that the value of λ increase along each oriented edge. It follows (see for example [29, §8.3]) that the h-vector entry $h_i(\mathsf{Tes}_n(\mathbf{a}))$ equals the number of vertices in this oriented 1-skeleton with outdegree i.

By Theorem 2.5, the vertices of $\mathsf{Tes}_n(\mathbf{a})$ are the permutation Tesler matrices of size n and the edges of $\mathsf{Tes}_n(\mathbf{a})$ emanating from a fixed vertex correspond to changing the support of the corresponding permutation Tesler matrix of the vertex in exactly two positions belonging the the same row. Let λ be any linear form such that moving from one to another permutation Tesler matrix by shifting the support to the right in a single row corresponds to an increase in λ. Then if the support of a permutation Tesler matrix is given by $\{(i, b_i) : 1 \leq i \leq n\}$, its outdegree in the orientation induced by λ is $\sum_{i=1}^{n}(n - b_i)$. The corresponding generating function for outdegree is $\sum_{i=0}^{\binom{n}{2}} h_i(\mathsf{Tes}_n(\mathbf{a}))x^i = \prod_{i=1}^{n} \left(\sum_{a_i=i}^{n} x^{n-b_i} \right) = [n]!_x$. □

Corollaries 2.8 and 2.9 are also true for Tesler polytopes $\mathsf{Tes}_n(\mathbf{a})$, where $\epsilon(a) = +^{n-1}0$. In light of Theorem 2.7, it is natural to ask for an analog to these results when $\epsilon(a)$ is of the form $+0+^{n-2}$ or $+0+^{n-3}0$. Such an analog is provided by the following corollary.

Corollary 2.10. *Let $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ and assume that $\epsilon(\mathbf{a})$ has one of the forms $+0+^{n-2}$ or $+0+^{n-3}0$. Let P be the quotient polytope $(\Delta_{n-2} \times \Delta_{n-1})/\sim$, where we declare $(p,q) \sim (p',q)$ whenever $q \in \Delta_{n-1}$ belongs to the facet of Δ_{n-1} defined by $x_2 = 0$ and $p, p' \in \Delta_{n-2}$.*

The face poset of the polytope $\mathsf{Tes}_n(\mathbf{a})$ is isomorphic to the face poset of the Cartesian product $\Delta_1 \times \Delta_2 \times \cdots \Delta_{n-3} \times P$. Moreover, we have that $\mathsf{Tes}_n(\mathbf{a})$ has $2(n-1)!$ vertices and h-polynomial $(1 + x^{n-1})[n-1]!_x$.

Proof. (Sketch.) The second row of any $\mathbf{a}$-Tesler tableau T is nonzero if and only if $T(1,2) = 1$. All other rows of any $\mathbf{a}$-Tesler tableau are nonzero. By Theorem 2.5, we get the claimed Cartesian product decomposition of $\mathsf{Tes}_n(\mathbf{a})$. The fact that $\mathsf{Tes}_n(\mathbf{a})$ has $2(n-1)!$ vertices arises from the fact that the quotient polytope P has $2(n-1)$ vertices. The fact that $\mathsf{Tes}_n(\mathbf{a})$ has h-polynomial $(1+x^{n-1})[n-1]!_x$ can be deduced from the multiplicative property of h-polynomials of the first proof of Corollary 2.9 and the fact that P has h-polynomial $(1+x^{n-1})[n-1]_x$. □

Remark 2.11. All of the results of this section are still true when one considers the "generalized" Tesler polytopes polytopes $\mathsf{Tes}_n(\mathbf{a})$ defined for real vectors $\mathbf{a}$; one simply replaces $(\mathbb{Z}_{\geq 0})^n$ and $(\mathbb{Z}_{>0})^n$ with $(\mathbb{R}_{\geq 0})^n$ and $(\mathbb{R}_{>0})^n$ throughout. The proofs are identical.

Remark 2.12. When $\mathbf{a} \in (\mathbb{Z}_{>0})^n$ is a vector of positive integers, Theorem 2.5 can be deduced from results of Hille [20]. In particular, if Q denotes the quiver on the vertex set $Q_0 = [n+1]$ with arrows $i \to j$ for all $1 \leq i < j \leq n+1$ and if $\theta : Q_0 \to \mathbb{R}$ denotes the weight function defined by $\theta(i) = a_i$ for $1 \leq i \leq n$ and $\theta(n+1) = -a_1 - \cdots - a_n$, then the Tesler polytope $\mathsf{Tes}_n(\mathbf{a})$ is precisely the polytope $\Delta(\theta)$ considered in [20, Theorem 2.2]. By the argument in the last paragraph of [20, Theorem 2.2] and [20, Proposition 2.3], the genericity condition on θ in the hypotheses of [20, Theorem 2.2] is equivalent to every entry of $\mathbf{a}$ being positive. The conclusion of [20, Theorem 2.2] is essentially the same as the special case of Theorem 2.5 when $\mathbf{a} \in (\mathbb{Z}_{>0})^n$. When some entries of $\mathbf{a}$ are zero, in the terminology of [20] the weight function θ lies on a wall, and the results of [20] do not apply to $\mathsf{Tes}_n(\mathbf{a})$.

Remark 2.13. When $\mathbf{a} \in (\mathbb{Z}_{>0})^n$ is a vector of positive integers, the simplicity of $\mathsf{Tes}_n(\mathbf{a})$ guaranteed by Theorem 2.7 had been observed previously in the context of flow polytopes. The condition that every entry in $\mathbf{a}$ is positive is equivalent to $\mathbf{a}$ lying in the "nice chamber" defined by Baldoni and Vergne in [4, p. 458]. In [6, p. 798], Brion and Vergne observe that this condition on $\mathbf{a}$ implies the simplicity of $\mathsf{Tes}_n(\mathbf{a})$. The simplicity of $\mathsf{Tes}_n(\mathbf{a})$ in this case can also be derived from Hille's characterization of the face poset [20] using exactly the same argument as in the proof of Theorem 2.7.

3. Volume of the Tesler polytope $\mathsf{Tes}_n(\mathbf{1})$

The aim of this section is to prove Theorem 1.8 through a sequence of results. For ease of reading the section is broken down into several subsections. We start by stating previous results on volumes and Ehrhart polynomials of flow polytopes and then prove specific lemmas regarding $\mathsf{Tes}_n(\mathbf{1})$.

In this section we work in the field of *iterated formal Laurent series* with m variables as discussed by Haglund, Garsia and Xin in [13, §4]. We choose a total order of the variables: $x_1, x_2, \ldots, x_m$ to extract *iteratively* coefficients, constant coefficients, and residues of an element $f(\mathbf{x})$ in this field. We denote these respectively by

$$\mathrm{CT}_{x_m} \cdots \mathrm{CT}_{x_1} f, \quad [\mathbf{x}^{\mathbf{a}}] := [x_m^{a_m} \cdots x_1^{a_1}] f, \quad \mathrm{Res}_{x_m} \cdots \mathrm{Res}_{x_1} f.$$

For more on these iterative coefficient extractions see [26, §2].

3.1. **Generating function of $K_{A_n}(\mathbf{a}')$ and the Lidskii formulas.** Recall that by Lemmas 1.2 and 1.3 we have that the normalized volume $\mathrm{vol}\,\mathsf{Tes}_n(\mathbf{a})$ equals the normalized volume $\mathrm{vol}\,\mathsf{Flow}_n(\mathbf{a})$ and that the number $T_n(\mathbf{a})$ of Tesler matrices is given by the Kostant partition function $K_{A_n}(\mathbf{a}')$. By definition, the latter is given by the following iterated coefficient extraction.

$$K_{A_n}(\mathbf{a}') = [\mathbf{x}^{\mathbf{a}'}] \prod_{1\leq i<j\leq n+1} (1 - x_i x_j^{-1})^{-1}. \tag{3.1}$$

In addition, the Kostant partition function is invariant under reversing the order and sign of the netflow vector.

Proposition 3.1.

$$K_{A_n}(a_1, a_2, \ldots, a_n, -\sum_{i=1}^n a_i) = K_{A_n}(\sum_{i=1}^n a_i, -a_n, \ldots, -a_2, -a_1).$$

Proof. Reversing an (integer) flow on the complete graph k_n gives an involution between (integer) flows with netflow $(a_1, a_2, \ldots, a_n, -\sum_{i=1}^n a_i)$ and (integer) flows with netflow $(\sum_{i=1}^n a_i, -a_n, \ldots, -a_2, -a_1)$. □

Assume that $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ satisfies $a_i \geq 0$ for $i = 1, \ldots, n$. Then the **Lidskii formulas** [4, Proposition 34, Theorem 37] state that

(3.2)
$$\mathrm{vol}\,\mathsf{Flow}_n(\mathbf{a}) = \sum_{\mathbf{i}} \binom{\binom{n}{2}}{i_1, i_2, \ldots, i_n} a_1^{i_1} \cdots a_n^{i_n} \cdot K_{A_{n-1}}(i_1 - n + 1, i_2 - n + 2, \ldots, i_n),$$

and
(3.3)
$$K_{A_n}(\mathbf{a}') = \sum_{\mathbf{i}} \binom{a_1 + n - 1}{i_1}\binom{a_2 + n - 2}{i_2} \cdots \binom{a_n}{i_n} \cdot K_{A_{n-1}}(i_1 - n+1, i_2 - n+2, \ldots, i_n),$$

where both sums are over weak compositions $\mathbf{i} = (i_1, i_2, \ldots, i_n)$ of $\binom{n}{2}$ with n parts which we denote as $\mathbf{i} \models \binom{n}{2}$, $\ell(\mathbf{i}) = n$.

Example 3.2. The Tesler polytope $\mathsf{Tes}_3(1,1,1) \cong \mathsf{Flow}_3(1,1,1)$ has normalized volume 4 since by (3.2)

$$\mathrm{vol}\,\mathsf{Flow}_3(1,1,1) = \binom{3}{3,0,0} K_{A_2}(1,-1,0) + \binom{3}{2,1,0} K_{A_2}(0,0,0) + 0 = 1\cdot 1 + 3\cdot 1 = 4.$$

And this polytope has $T_3(1,1,1) = K_{A_3}(1,1,1,-3) = 7$ lattice points (the seven 3×3 Tesler matrices with hook sums $(1,1,1)$; see Figure 1). Indeed by (3.3)

$$K_{A_3}(1,1,1,-3) = \binom{1+2}{3}\binom{1+1}{0} K_{A_2}(1,-1,0) + \binom{1+2}{2}\binom{1+1}{1} K_{A_2}(0,0,0) = 7.$$

Example 3.3. [4] If one uses (3.2) on the Chan-Robbins-Yuen polytope $\mathsf{Tes}_n(\mathsf{e}_1)$ one obtains

$$\mathrm{vol}\,\mathsf{Tes}_n(1,0,\ldots,0) = K_{A_{n-1}}(-\binom{n-1}{2}, -n+2, \ldots, -1, 0),$$

since the only composition $\mathbf{i}$ that does not vanish is $i_1 = \binom{n}{2}, i_2 = 0, \ldots, i_n = 0$. By Proposition 3.1 this is equivalent to the first identity in Example 1.4.

3.2. **Volume of $\mathsf{Tes}_n(\mathbf{1})$ as a constant term.** In this short section we use (3.2) and the generating series (3.1) of Kostant partition functions to write the volume of $\mathsf{Tes}_n(\mathbf{1})$ as an iterated constant term of a formal Laurent series.

Lemma 3.4.

$$\operatorname{vol}\mathsf{Tes}_n(\mathbf{1}) = \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\ (x_1+\cdots+x_n)^{\binom{n}{2}}\prod_{1\le i<j\le n}(x_j-x_i)^{-1}, \tag{3.4}$$

where $\mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1} f$ *denotes the iterated constant term of* f.

Proof. By (3.2) and Proposition 3.1 we have that

$$\begin{aligned}\operatorname{vol}\mathsf{Tes}_n(\mathbf{1}) &= \sum_{\mathbf{i}\models\binom{n}{2},\ell(\mathbf{i})=n}\binom{\binom{n}{2}}{i_1,i_2,\ldots,i_n}\cdot K_{A_{n-1}}(i_1-n+1,i_2-n+2,\ldots,i_n)\\ &= \sum_{\mathbf{i}\models\binom{n}{2},\ell(\mathbf{i})=n}\binom{\binom{n}{2}}{i_1,i_2,\ldots,i_n}\cdot K_{A_{n-1}}(-i_n,1-i_{n-1},2-i_{n-2},\ldots,n-1-i_1).\end{aligned}$$

We use (3.1) to rewrite this as

$$\operatorname{vol}\mathsf{Tes}_n(\mathbf{1}) = \sum_{\mathbf{i}\models\binom{n}{2},\ell(\mathbf{i})=n}\binom{\binom{n}{2}}{i_1,i_2,\ldots,i_n}[\mathbf{x}^{\delta_\mathbf{n}-\mathbf{i}}]\prod_{1\le i<j\le n}(1-x_ix_j^{-1})^{-1},$$

where $\delta_n=(0,1,2\ldots,n-1)$. Since $[\mathbf{x}^\mathbf{a}]f=\mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\mathbf{x}^{-\mathbf{a}}f$ then

$$\operatorname{vol}\mathsf{Tes}_n(\mathbf{1}) = \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\sum_{\mathbf{i}\models\binom{n}{2},\ell(\mathbf{i})=n}\mathbf{x}^{\mathbf{i}-\delta_n}\binom{\binom{n}{2}}{i_1,i_2,\ldots,i_n}\prod_{1\le i<j\le n}(1-x_ix_j^{-1})^{-1}.$$

Using $\prod_{1\le i<j\le n}(1-x_ix_j^{-1})^{-1}=\mathbf{x}^{\delta_n}\prod_{1\le i<j\le n}(x_j-x_i)^{-1}$ we get

$$\operatorname{vol}\mathsf{Tes}_n(\mathbf{1}) = \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\prod_{1\le i<j\le n}(x_j-x_i)^{-1}\sum_{\mathbf{i}\models\binom{n}{2},\ell(\mathbf{i})=n}\binom{\binom{n}{2}}{i_1,i_2,\ldots,i_n}\mathbf{x}^{\mathbf{i}}.$$

An application of the multinomial theorem yields the desired result. □

3.3. **A Morris-type constant term identity.** Let $e_k=e_k(x_1,x_2,\ldots,x_n)$ denote the k^{th} elementary symmetric polynomial. In particular $e_1=x_1+x_2+\cdots+x_n$. For $n\ge 2$ and nonnegative integers a,c we define $L_n(a,c)$ to be the following iterated constant term:

$$L_n(a,c) := \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\ e_1^{(a-1)n+c\binom{n}{2}}\prod_{i=1}^{n}x_i^{-a+1}\prod_{1\le i<j\le n}(x_i-x_j)^{-c}. \tag{3.5}$$

Note that by Lemma 3.4 we have that

$$\operatorname{vol}\mathsf{Tes}_n(\mathbf{1}) = L_n(1,1). \tag{3.6}$$

Next we give a product formula for $L_n(a,c)$ that for $a=c=1$ yields (1.7). We postpone the proof to the next section.

Lemma 3.5. *For $n\ge 2$ and nonnegative integers a,c we have that*

$$L_n(a,c) = \big((a-1)n+c\tbinom{n}{2}\big)!\prod_{i=0}^{n-1}\frac{\Gamma(1+c/2)}{\Gamma(1+(i+1)c/2)\Gamma(a+ic/2)}, \tag{3.7}$$

where $\Gamma(\cdot)$ *is the Gamma function.*

Corollary 3.6.

$$L_n(1,1) = \frac{\binom{n}{2}! \cdot 2^{\binom{n}{2}}}{\prod_{i=1}^n i!}. \tag{3.8}$$

Proof. Set $a = 1$ and $c = 1$ in (3.7) and obtain

$$L_n(1,1) = \binom{n}{2}! \prod_{i=0}^{n-1} \frac{\Gamma(3/2)}{\Gamma(1+(i+1)/2)\Gamma(1+i/2)},$$

since $\Gamma(3/2) = \sqrt{\pi}/2$ and by the duplication formula of $\Gamma(\cdot)$ this becomes

$$L_n(1,1) = \binom{n}{2}! \prod_{i=0}^{n-1} \frac{2^i}{(i+1)!} = \frac{\binom{n}{2}! \cdot 2^{\binom{n}{2}}}{\prod_{i=1}^n i!},$$

as desired. □

Drew Armstrong (private communication) noted the resemblance of the product in the RHS (3.8) with the number of standard Young tableaux of staircase shape. Indeed, if we let $f^{(n-1,n-2,\ldots,1)}$ be the number of standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$ which by the hook-length formula equals

$$f^{(n-1,n-2,\ldots,1)} = \frac{\binom{n}{2}!}{\prod_{k=1}^{n-1}(2k-1)^{n-k}},$$

then one can show that $L_n(1,1)$ is divisible by this number. The ratio of these numbers is a product of consecutive Catalan numbers.

Proposition 3.7.

$$\frac{\binom{n}{2}! \cdot 2^{\binom{n}{2}}}{\prod_{i=1}^n i!} = f^{(n-1,n-2,\ldots,1)} \cdot \prod_{i=1}^{n-1} Cat(i). \tag{3.9}$$

Proof. The identity is easily verified using the formula for $f^{(n-1,n-2,\ldots,1)}$ and for $Cat(i) = \frac{1}{i+1}\binom{2i}{i}$. □

Remark 3.8. When we set $a = 1$ and $c = 2$ in (3.7) one can also show that

$$L_n(1,2) = \frac{(n(n-1))!}{n!(\prod_{i=1}^{n-1} i!)^2} = f^{(n-1)^n} \cdot \prod_{i=1}^{n-1} \left(\frac{i+1}{2} Cat(i)^2 \right), \tag{3.10}$$

where $f^{(n-1)^n}$ is the number of standard Young tableaux of rectangular shape $(n-1)^n$ which equals $(n(n-1))! \prod_{k=0}^{n-1} k! \,/\, \prod_{k=0}^{n-1}(n+k-1)!$. We were unable to find similar identities relating $L_n(1,c)$, $c \geq 3$ with the number of SYT of shape λ.

Remark 3.9. A similar iterated constant term identity to (3.7) is Zeilberger's variation of the Morris constant term identity [28] used to prove (1.5). We state the version in [26, §3.5]: for $n \geq 2$ and nonnegative integers a, b, c let

$$M_n(a,b,c) := \mathrm{CT}_{x_n} \cdots \mathrm{CT}_{x_1} \prod_{i=1}^n x_i^{-a+1}(1-x_i)^{-b} \prod_{1 \leq i < j \leq n} (x_i - x_j)^{-c} \tag{3.11}$$

then

$$M_n(a,b,c) = \prod_{j=0}^{n-1} \frac{\Gamma(1+c/2)\Gamma(a+b-1+(n+j-1)c/2)}{\Gamma(1+(j+1)c/2)\Gamma(a+jc/2)\Gamma(b+jc/2)},$$

and in particular

$$M_n(1,1,1) = \prod_{i=0}^{n-1} \frac{1}{i+1}\binom{2i}{i}.$$

Moreover, let $h_k(x_1,\ldots,x_n)$ denote the k^{th} complete symmetric polynomial in the variables $x_1,\ldots,x_n$. Since $\prod_{i=1}^n(1-x_i)^{-1} = \sum_{k\geq 0} h_k(x_1,\ldots,x_n)$ then by linearity of $\mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}$ and degree considerations, $M_n(a,1,c)$ can be expressed as a sum of iterated constant term extractions all except one are zero. Thus

(3.12)

$$M_n(a,1,c) = \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\ h_{((a-1)n+c\binom{n}{2})}(x_1,\ldots,x_n)\prod_{i=1}^n x_i^{-a+1}\prod_{1\leq i<j\leq n}(x_i-x_j)^{-c}.$$

This alternate description of $M_n(a,1,c)$ resembles the original definition of $L_n(a,c)$ in (3.5). Conversely, one can show using $(1-e_1)^{-1} = \sum_{k\geq 0} e_1^k$, linearity, and degree considerations that $L_n(a,c)$ equals the following iterated constant term

$$L_n(a,c) = \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\ (1-e_1)^{-1}\prod_{i=1}^n x_i^{-a+1}\prod_{1\leq i<j\leq n}(x_i-x_j)^{-c}, \tag{3.13}$$

which resembles the original description of $M_n(a,1,c)$.

3.4. **Proof of Lemma 3.5 via Baldoni-Vergne recurrence approach.** To prove Lemma 3.5 we follow Xin's [26, §3.5] simplified recursion approach of the proof by Baldoni-Vergne [5] of the Morris identity (3.11).

Outline of the proof: First, for nonnegative integers $n\geq 2, a, c$ and $\ell = 0,\ldots,n$ we introduce the constants

$$C_n(\ell,a,c) := \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\ \frac{P_\ell\cdot e_1(x_1,\ldots,x_n)^{(a-1)n+c\binom{n}{2}-\ell}}{\prod_{i=1}^n x_i^{a-1}\prod_{i=1}^n(x_i-x_j)^c},$$

where $P_\ell = \ell!(n-\ell)!e_\ell(x_1,\ldots,x_n)$. Note that $C_n(0,a,c) = n!L_n(a,c)$. Second, we show that $C_n(\ell,a,c)$ satisfy certain linear relations (Proposition 3.10). Third, we show that these relations uniquely determine the constants $C_n(\ell,a,c)$ (Proposition 3.11). Lastly, in Proposition 3.12 we define $C'_n(\ell,a,c)$ as certain products of Gamma functions such that $C'_n(0,a,c)/n!$ coincides with the expression on the right-hand-side of (3.7). We then show that $C'_n(\ell,a,c)$ satisfy the same relations as $C_n(\ell,a,c)$ and since these relations determine uniquely the constants then $C'_n(\ell,a,c) = C_n(\ell,a,c)$. This completes the proof of the Lemma.

The $C_n(\ell,a,c)$ satisfy the following relations.

Proposition 3.10. *Let $C_n(\ell,a,c)$ be defined as above then for $1\le \ell\le n$ we have:*

$$\frac{C_n(\ell,a,c)}{C_n(\ell-1,a,c)}=\frac{a-1+c(n-\ell)/2}{(a-1)n+c\binom{n}{2}-\ell+1}, \tag{3.14}$$

$$C_n(n,a,c)=C_n(0,a-1,c), \tag{3.15}$$

$$C_n(n-1,1,c)=C_{n-1}(0,c,c), \qquad (\textit{if } n>1) \tag{3.16}$$

$$C_n(0,1,0)=n!, \tag{3.17}$$

$$C_n(\ell,0,c)=0. \tag{3.18}$$

Proof. The relations (3.15)-(3.18) follow from the same proof as in [26, Theorem 3.5.2] $C_n(\ell,a,c)$.

We now prove (3.14). Let $U_\ell=e_1^{(a-1)n+c\binom{n}{2}-\ell}/(\prod_{i=1}^n x_i^a\prod_{i=1}^n(x_i-x_j)^c)$, since $\mathrm{CT}_y\, g(y)=\mathrm{Res}_y\, yg(y)$ then

$$C_n(\ell,a,c)=\mathrm{Res}_{x_n}\cdots\mathrm{Res}_{x_1}\; P_\ell U_\ell, \tag{3.19}$$

Next we calculate the following derivative with respect to x_1.

$$\frac{\partial}{\partial x_1}e_1\cdot x_1x_2\cdots x_\ell U_\ell=\left((a-1)n+c\tbinom{n}{2}-\ell+1\right)x_1\cdots x_\ell U_\ell+(1-a)x_2\cdots x_\ell U_{\ell-1}+ \\ -c\cdot x_1\cdots x_\ell\sum_{j=2}^n\frac{U_{\ell-1}}{x_1-x_j}. \tag{3.20}$$

If c is odd then U_ℓ is anti-symmetric. If we anti-symmetrize (3.20) over the symmetric group $\mathfrak{S}_n$, we get

$$\sum_{w\in\mathfrak{S}_n}(-1)^{inv(w)}w\cdot\left(\frac{\partial}{\partial x_1}e_1\cdot x_1x_2\cdots x_\ell U_\ell\right)=$$

$$\left((a-1)n+c\tbinom{n}{2}-\ell+1\right)P_\ell U_\ell+(1-a)P_{\ell-1}U_{\ell-1}-c\sum_{w\in\mathfrak{S}_n}w\cdot x_1\cdots x_\ell\sum_{j=2}^n\frac{U_{\ell-1}}{x_1-x_j}$$

One can check that

$$2\sum_{w\in\mathfrak{S}_n}w\cdot x_1\cdots x_\ell\sum_{j=2}^n\frac{1}{x_1-x_j}=(n-\ell)P_{\ell-1}.$$

So putting everything together for c odd we obtain

$$\sum_{w\in\mathfrak{S}_n}(-1)^{inv(w)}w\cdot\left(\frac{\partial}{\partial x_1}e_1\cdot x_1x_2\cdots x_\ell U_\ell\right)= \\ \left((a-1)n+c\tbinom{n}{2}-\ell+1\right)P_\ell U_\ell-(a-1+c(n-\ell)/2)P_{\ell-1}U_{\ell-1}. \tag{3.21}$$

Next, if c is even, U_ℓ is symmetric. If we symmetrize (3.20) over $\mathfrak{S}_n$ and do similar simplifications as in the previous case we get

$$\sum_{w\in\mathfrak{S}_n}w\cdot\left(\frac{\partial}{\partial x_1}e_1x_1x_2\cdots x_\ell U_\ell\right)= \\ \left((a-1)n+c\tbinom{n}{2}-\ell+1\right)P_\ell U_\ell-(a-1+c(n-\ell)/2)P_{\ell-1}U_{\ell-1}. \tag{3.22}$$

Finally, we take the iterated residue $\mathrm{Res}_{x_n}\cdots\mathrm{Res}_{x_1}$ of (3.21) and (3.22). Since the left-hand-side of these two equations consist of sums of derivatives with respect to $x_1,\dots,x_n$, then their iterated residues $\mathrm{Res}_{\mathbf{x}}$ are zero [5, Remark 3(c), p. 15]. This combined with (3.19) yields

$$0=\left((a-1)n+c\binom{n}{2}-\ell+1\right)C_n(\ell,a,c)-(a-1+c(n-\ell)/2)C_n(\ell-1,a,c),$$

which proves (3.14) for c even or odd. □

We now show that the recurrences (3.14)-(3.18) determine entirely the constants $C_n(\ell,a,c)$ (same algorithm as in [5, p. 10]).

Proposition 3.11. [5, p. 10] *The recurrences* (3.14)-(3.18) *determine uniquely the constants* $C_n(\ell,a,c)$.

Proof. We give an algorithm to compute the constants $C_n(\ell,a,c)$ recursively using (3.14)-(3.18). The algorithm has the following three cases:

Case 1. If $c=0$ and $a>1$ we use (3.14) repeatedly to increase ℓ up to n. We can use this recursion since $a-1+c(n-\ell)=a-1>0$. If $\ell=n$ then we can apply (3.15) and go from $C_n(n,a,0)$ to $C_n(0,a-1,0)$:

$$C_n(\ell,a,0)\xrightarrow{(3.14)}C_n(\ell+1,a,0)\overset{(3.14)^*}{\dashrightarrow}C_n(n,a,0)\xrightarrow{(3.15)}C_n(0,a-1,0).$$

Thus computing $C_n(\ell,a,0)$ reduces to finding $C_n(0,1,0)$ which equals $n!$ by (3.17).

Case 2. If $c>0$ and $a>1$ we use (3.14) repeatedly to increase ℓ up to n. We can use this recursion since $a-1+c(n-\ell)=a-1>0$. If $\ell=n$ then we apply (3.15) and go from $C_n(n,a,c)$ to $C_n(0,a-1,c)$:

$$C_n(\ell,a,c)\xrightarrow{(3.14)}C_n(\ell+1,a,c)\overset{(3.14)^*}{\dashrightarrow}C_n(n,a,c)\xrightarrow{(3.15)}C_n(0,a-1,c).$$

Thus computing $C_n(\ell,a,c)$ reduces to finding $C_n(0,1,c)$.

Case 3. To compute $C_n(0,1,c)$ with $c>0$, we use (3.14) repeatedly to increase ℓ from 0 up to $n-1$. Then we can apply (3.16) and go from $C_n(n-1,1,c)$ to $C_{n-1}(0,c,c)$:

$$C_n(0,1,c)\xrightarrow{(3.14)}C_n(1,1,c)\overset{(3.14)^*}{\dashrightarrow}C_n(n-1,1,c)\xrightarrow{(3.16)}C_{n-1}(0,c,c).$$

Thus by iterating this reduction with Case 2 we see that computing $C_n(0,1,c)$ reduces to finding $C_1(\ell,a,c)$. Having $n=1$ guarantees there is no term

$$\prod_{1\le i<j\le n}(x_i-x_j)^{-c}.$$

So $C_1(\ell,a,c)=C_1(\ell,a,0)$ which we can compute with Case 1. □

Next we give an explicit product formula for $C_n(\ell,a,c)$. We prove this by showing that the formula satisfies relations (3.14)-(3.18) which by Proposition 3.11 determine uniquely $C_n(\ell,a,c)$.

Proposition 3.12. *If $c>0$ or if $a>1$ then for $1\le\ell\le n$ then*

$$C_n(\ell,a,c)=C_n(0,a,c)\prod_{j=1}^{\ell}\frac{a-1+(n-j)c/2}{(a-1)n+c\binom{n}{2}-j+1}. \tag{3.23}$$

if $a \geq 1$ *then*

$$C_n(0,a,c) = n! \cdot \Gamma\big(1+(a-1)n+c\tbinom{n}{2}\big) \prod_{i=0}^{n-1} \frac{\Gamma(1+c/2)}{\Gamma(1+(i+1)c/2)\Gamma(a+ic/2)}. \tag{3.24}$$

Proof. By Proposition 3.11 it suffices to check that the formulas for $C_n(\ell,a,c)$ and $C_n(0,a,c)$ in (3.23), (3.24) satisfy the relations (3.14)-(3.18).

Let $C'_n(\ell,a,c)$ and $C'_n(0,a,c)$ be the formulas in the right-hand-side of (3.23) and (3.24) respectively.

Relation (3.14) is apparent from the definition of $C'_n(\ell,a,c)$.

Next we check that $C'_n(\ell,a,c)$ satisfies (3.15). Using $\Gamma(t+1)=t\Gamma(t)$ repeatedly we obtain:

$$\begin{aligned}
\frac{C'_n(n-1,a,c)}{C'_n(0,a-1,c)} &= \\
&= \frac{\Gamma(1+(a-1)n+c\binom{n}{2})}{\Gamma(1+(a-2)n+c\binom{n}{2})} \prod_{j=1}^{n} \frac{a-1+(n-j)c/2}{(a-1)n+c\binom{n}{2}-j+1} \prod_{i=0}^{n} \frac{\Gamma(a-1+ic/2)}{\Gamma(a+ic/2)} \\
&= \prod_{j=1}^{n} ((a-1)n+c\tbinom{n}{2}-j+1) \prod_{j=1}^{n} \frac{a-1+(n-j)c/2}{(a-1)n+c\binom{n}{2}-j+1} \prod_{i=0}^{n} \frac{1}{a-1+ic/2} \\
&= 1,
\end{aligned}$$

as desired.

Next we verify (3.16). Again, using $\Gamma(t+1)=t\Gamma(t)$ repeatedly we obtain:

$$\begin{aligned}
\frac{C'_n(n-1,1,c)}{C'_{n-1}(0,c,c)} &= \\
&= \frac{\prod_{j=1}^{n-1}(n-j)c/2}{\prod_{j=1}^{n-1} c\binom{n}{2}-j+1} \frac{n\Gamma(1+c\binom{n}{2})}{\Gamma(1+c\binom{n}{2}-(n-1))} \times \\
&\quad \times \frac{\Gamma(1+c/2)}{\Gamma(1+(n-1)c/2)\Gamma(1+nc/2)} \frac{\prod_{i=0}^{n-2}\Gamma(c(i+2)/2)}{\prod_{i=0}^{n-2}\Gamma(1+ic/2)} \\
&= \frac{\prod_{j=1}^{n-1}(n-j)c/2}{\prod_{j=1}^{n-1} c\binom{n}{2}-j+1} \frac{n\prod_{j=1}^{n-1} c\binom{n}{2}-j+1}{1} \prod_{j=2}^{n} \frac{\Gamma(jc/2)}{\Gamma(1+jc/2)} \\
&= n\prod_{j=1}^{n-1}(n-j)c/2 \prod_{j=2}^{n} \frac{1}{jc/2} = 1,
\end{aligned}$$

as desired.

Finally, it is trivial to check that $C'_n(\ell,a,c)$ satisfy (3.17) and (3.18). Thus since $C'_n(\ell,a,c)$ satisfy relations (3.14)-(3.18) and by Proposition 3.11 these relations uniquely determine the constants $C_n(\ell,a,c)$ then $C'_n(\ell,a,c)=C_n(\ell,a,c)$. □

To conclude, since $C_n(0,a,c) = n!\cdot L_n(a,c)$ then Lemma 3.5 follows from (3.24) in Proposition 3.12. By Corollary 3.6 and Proposition 3.7 $L_n(1,1)$ yields the desired formula for the volume of $\mathsf{Tes}_n(\mathbf{1})$ which completes the proof of Theorem 1.8.

4. Final remarks

4.1. **Diagonal harmonics and polytopes.** Example 1.1 states Haglund's result from [15] showing that the bigraded Hilbert series of the space DH_n is given by a weighted sum over Tesler matrices in $\mathcal{T}_n(1,1,\dots,1)$. The space DH_n has dimension $(n+1)^{n-1}$, the number of parking functions of size n. A conjecture of Haglund and Loehr [18], settled by Carlsson and Mellit [8] with their proof of the more general *shuffle conjecture* [17], expresses the LHS as

$$\mathcal{H}(DH_n, q, t) = \sum_{\pi} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)}, \tag{4.1}$$

where the sum is over parking functions π. For definitions of the statistics dinv and area see [16]. By definition $\mathcal{H}(DH_n, q, t)$ is a polynomial in $\mathbb{N}[q,t]$ and symmetric in q and t. The right-hand sides of (4.1) and (1.1) give different combinatorial models for this Hilbert series where the (q,t positivity, q,t symmetry) are (trivial, non-trivial) and (non-trivial, trivial) respectively. It remains open to prove directly the equality of these models:

$$\sum_{\pi} q^{\mathrm{dinv}(\pi)} t^{\mathrm{area}(\pi)} = \sum_{A \in \mathcal{T}_n(1,1,\dots,1)} wt(A), \tag{4.2}$$

for $wt(A)$ as defined in (1.2). Levande [22] verified this identity for $(q,0)$ and $(1,t)$. In particular, when $q=1, t=1$, $wt(A)\mid_{q=1,t=1} = 0$ for any $n \times n$ Tesler matrix A with more than n nonzero entries and the matrices that survive are the permutation Tesler matrices each with n nonzero entries. Thus (4.2) at $q=1, t=1$ becomes

$$(n+1)^{n-1} = \sum_{A} \prod_{i,j\,:\,a_{ij}>0} a_{ij}$$

where the sum is over the $n!$ permutation Tesler matrices in $\mathcal{T}_n(\mathbf{1})$; the vertices of polytope $\mathsf{Tes}_n(\mathbf{a})$. This curious identity was proved combinatorially in [2, §5] extending a function from Levande [22] from Tesler matrices to permutations.

Analogously, an important subspace of the space DH_n is the alternant DH_n^{ε} that has dimension $Cat(n) = \frac{1}{n+1}\binom{2n}{n}$. The bigraded Hilbert series of DH_n^{ε} has the following combinatorial model by Garsia and Haglund [11, 12]

$$\mathcal{H}(DH_n^{\varepsilon}, q, t) = \sum_{P} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}, \tag{4.3}$$

where the sum is over *Dyck paths* P of size n, see [16, §3] for the definition of bounce. Gorsky and Negut [14] also expressed this Hilbert series as a weighted sum over Tesler matrices:

$$\mathcal{H}(DH_n^{\varepsilon}, q, t) = \sum_{A \in \mathcal{T}_n(1,1,\dots,1)} wt'(A), \tag{4.4}$$

where

$$wt'(A) = \prod_{a_{ii+1}>0} ([a_{ii+1}+1]_{q,t} - [a_{ii+1}]_{q,t}) \prod_{j>i+1\,:\,a_{ij}>0} (-M)[a_{ij}]_{q,t},$$

for $M = (1-q)(1-t)$ and $[b]_{q,t} = (q^b - t^b)/(q-t)$ as in (1.2). When we set $q=1, t=1$ in (4.4), by the definition of $wt'(A)$ only the Tesler matrices A with support in the diagonals a_{ii} and a_{ii+1} survive each with weight 1. So (4.4) becomes

$$\frac{1}{n+1}\binom{2n}{n} = \#\{A \in \mathcal{T}_n(1,1,\dots,1) \,:\, a_{ij} = 0, j > i+1\}. \tag{4.5}$$

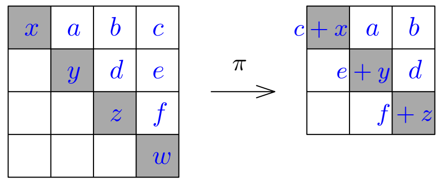


FIGURE 4. Illustration of the projection π used in the proof of Proposition 4.1.

This identity can be proved in the context of flow polytopes. Namely, translating from flow polytopes (see Lemma 1.2) Baldoni-Vergne [4] noticed that the polytope

$$\{(m_{i,j}) \in \mathsf{Tes}_n(\mathbf{a}) \,:\, m_{i,j} = 0, j > i+1\}$$

is the *Pitman-Stanley* polytope [24] and when $\mathbf{a} = \mathbf{1}$, this polytope has $\frac{1}{n+1}\binom{2n}{n}$ lattice points, explaining (4.5), and volume $(n+1)^{n-1}$ (see [24, §1, §5]).

4.2. **Enumeration of Tesler matrices.** There is no known explicit formula for the number $T_n(\mathbf{1})$ of Tesler matrices of size n. More than 20 terms of the sequence $\{T_n(\mathbf{1})\}_{n=1}$ have been computed in the OEIS [25, A008608]:

1, 2, 7, 40, 357, 4820, 96030, 2766572, 113300265, 6499477726, 515564231770, 55908184737696, ...

Regarding asymptotic of this sequence we give some preliminary lower and upper bounds that follows from a recursive construction by Drew Armstrong [1].

Proposition 4.1. $n! \leq T_n(\mathbf{1}) \leq 2^{\binom{n}{2}}$.

Proof. Let $\pi : \mathcal{T}_n(a_1,\ldots,a_{n-1},a_n) \to \mathcal{T}_{n-1}(a_1,\ldots,a_{n-1})$ defined by $\pi : (a_{i,j}) \mapsto (b_{i,j})$ where $b_{i,j} = \begin{cases} a_{i,i} + a_{i,n} & \text{if } i = j, \\ a_{i,j} & \text{if } i \neq j \end{cases}$. See Figure 4 for an example of π. The map π is surjective and for each $B \in \mathcal{T}_{n-1}(a_1,\ldots,a_{n-1})$, the size of the preimage is $\pi^{-1}(B) = \prod_{i=1}^{n-1}(1 + b_{i,i})$. Thus

$$T_n(a_1,\ldots,a_n) = \sum_{B \in \mathcal{T}(a_1,\ldots,a_{n-1})} \prod_{i=1}^{n-1}(1 + b_{i,i}). \tag{4.6}$$

For the case $\mathbf{a} = \mathbf{1}$ one can show that if $B \in \mathcal{T}_{n-1}(\mathbf{1})$ then $n \leq \pi^{-1}(B) \leq 2^{n-1}$. Using these bounds for $\pi^{-1}(B)$ in (4.6) yields

$$n \cdot T_{n-1}(\mathbf{1}) \leq T_n(\mathbf{1}) \leq 2^{n-1} \cdot T_{n-1}(\mathbf{1}).$$

Iterating these bounds give the desired result.

An alternative proof of the lower bound is as follows: the matrices in $\mathcal{T}_n(\mathbf{1})$ include the $n!$ permutation Tesler matrices of size n. □

4.3. **Combinatorial proof volume of CRY and Tesler polytopes.** The product formulas (1.5) and (1.7) for the volumes of the CRY and the Tesler polytopes involving Catalan numbers and number of SYT suggest a combinatorial proof that has been elusive since Zeilberger's proof of (1.5). The current proofs of the formulas use the Lidskii formula (3.2) for the volume of flow polytopes to translate the problem to evaluations of Kostant partition functions via constant term identities.

It is also not clear why the volume of the CRY polytope divides the volume of the Tesler polytope in terms of operations on polytopes. Curiously, using constant term identities it is possible to express the volume of the Tesler polytope as a nonnegative sum of terms two of which are $f^{(n-1,n-2,\ldots,1)}$ and $\prod_{i=0}^{n-1} Cat(i)$. Namely, by (3.4) the volume of the Tesler polytope $\mathsf{Tes}_n(\mathbf{1})$ is the constant term of $(e_1(x_1,\ldots,x_n))^{\binom{n}{2}} \prod_{1\leq i<j\leq n}(x_i-x_j)^{-1}$. Since $e_1^{\binom{n}{2}} = \sum_{\lambda\vdash\binom{n}{2}} f^{\lambda} s_{\lambda}$ where s_λ is the Schur function of λ, then by linearity of $\mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}$

$$f^{(n-1,n-2,\ldots,1)}\prod_{i=0}^{n-1} Cat(i) = \sum_{\lambda\vdash\binom{n}{2}} f^{\lambda}\,\mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\, s_{\lambda}(x_1,\ldots,x_n)\prod_{1\leq i<j\leq n}(x_i-x_j)^{-1}.$$

First, when $\lambda=(n-1,n-2,\ldots,1)$ then we get $f^{(n-1,n-2,\ldots,1)}$ and by degree considerations and (3.1) one can show that

$$\mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\, s_{(n-1,n-2,\ldots,1)}\prod_{1\leq i<j\leq n}(x_i-x_j)^{-1} = \mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\, x_2x_3^2\cdots x_n^{n-1}\prod_{1\leq i<j\leq n}(x_i-x_j)^{-1} = K_{A_{n-1}}(\mathbf{0}) = 1.$$

Second, when $\lambda=(\binom{n}{2})$ then $f^{(\binom{n}{2})}=1$ and by the version (3.12) of the Morris identity we get

$$\mathrm{CT}_{x_n}\cdots\mathrm{CT}_{x_1}\, s_{(\binom{n}{2})}\prod_{1\leq i<j\leq n}(x_i-x_j)^{-1} = M_n(1,1,1) = \prod_{i=0}^{n-1} Cat(i).$$

This of course this still leaves the question of why the nonnegative sum ends up being the product $f^{(n-1,n-2,\ldots,1)}\cdot\prod_{i=0}^{n-1} Cat(i)$ unanswered.

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