CYCLIC INCLUSION-EXCLUSION

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ABSTRACT. Following the lead of Stanley and Gessel, we consider a morphism which associates to an acyclic directed graph (or a poset) a quasi-symmetric function. The latter is naturally defined as multivariate generating series of non-decreasing functions on the graph.

We describe the kernel of this morphism, using a simple combinatorial operation that we call *cyclic inclusion-exclusion*. Our result also holds for the natural noncommutative analog and for the commutative and noncommutative restrictions to bipartite graphs.

An application to the theory of Kerov character polynomials is given.

1. INTRODUCTION

Given a poset $P = (V, <_P)$ or an acyclic directed graph $G = (V, E_G)$, it is natural to consider the following multivariate generating function

$$\Gamma_{P/G}(x_1, x_2, \cdots) = \sum_{\substack{f:V\to\mathbb{N} \\ f \text{ non-decreasing}}} \prod_{v\in V} x_{f(v)} \tag{1}$$

where $\mathbb{N}$ is the set of positive integers and *non-decreasing* means that $i <_P j$ (respectively $(i,j) \in E$) implies $f(i) \leq_G f(j)$. An example is given in Section 2.4.

This is a quite classical object in the algebraic combinatorics literature: using the terminology of the seminal book of Stanley [18], the non-decreasing functions on posets correspond to P-partitions when P has a *natural labelling* (up to reversing the order of P). The generating function Γ_P has then been considered by Gessel [10], see also Stanley's textbook [19, Section 7.19]. While not symmetric in the variables $x_1, x_2, \cdots$, this function exhibits some weaker symmetry property and belongs to the now well-studied algebra of *quasi-symmetric functions*[1].

Although posets are more common objects in the literature, the results of this paper are better formulated in terms of acyclic directed graphs. Obviously the map $\Gamma : G \to \Gamma_G$ defined by (1) can be extended by linearity to the vector space of formal linear combination of acyclic graphs, that we call here the *graph algebra*. A hint of the relevance of this map is the following: there are some natural Hopf

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[1] In fact, the terminology *quasi-symmetric function* was introduced in [10], precisely to study Γ_P.

algebra structures on the graph algebras and on quasi-symmetric functions, which turns the map Γ into a Hopf algebra morphism, see Section 2.5. However, we shall only focus here on the linear structure.

The main result of the present paper is a combinatorial description of the kernel of the application Γ from the graph algebra to quasi-symmetric functions (Theorem 2). This description relies on a simple combinatorial operation, that we call *cyclic inclusion-exclusion* (the definition and an example are given in Section 3.1). Before giving some background on this operation, let us mention that this description of the kernel of Γ is quite robust. Indeed, we shall prove that cyclic inclusion-exclusion also describes the kernel of some variants of Γ, namely:

- working with labeled (acyclic directed) graphs, it is natural to associate to them a multivariate generating series in *noncommuting variables* that lives in the algebra of *word quasi-symmetric functions* [15] (this algebra is also sometimes called *quasi-symmetric functions in noncommuting variables*, see [3]); we give a description of the kernel of this application (denoted Γ^{nc}) in Theorem 1.
- We also consider restrictions of the linear maps Γ and Γ^{nc} to bipartite graphs[2]. Analogs of Theorems 1 and 2 in the bipartite setting are given in Theorems 3 and 4.

  Note that, in the bipartite case, acyclic graphs and posets are the same objects. We explain below our motivation to consider such a restriction.

In all these cases, a byproduct of our proof is the surjectivity of the morphism Γ (respectively Γ^{nc} and their restriction to bipartite graphs). The surjectivity in the commutative non restricted case was observed by Stanley [20, Note p7], answering a question of Billera and Reiner.

Our proofs use a combination of basic linear algebra, graph combinatorics and (word) quasi-symmetric function manipulations. In the noncommutative/labeled case, we first exhibit a family of graphs so that their images form a $\mathbb{Z}$-basis of word quasi-symmetric functions. Then, we show that these graphs span the quotient of the graph algebra by cyclic inclusion-exclusion relations. With an easy linear algebra argument, this concludes the proof.

The commutative/unlabeled case can be obtained as a corollary of the noncommutative/labeled case. On the contrary, restrictions to bipartite graphs must be considered separately from the non-restricted setting (see Remark 5.1). The general structure of the proof is the same in the bipartite setting, although the arguments themselves are quite different.

Along the way, this gives natural bases of the word quasi-symmetric function ring: in particular, we find natural analogs of Gessel fundamental basis [10] and of two bases considered respectively by R. Stanley [20] and K. Luoto [13]. The analog of Luoto basis has been considered recently by the author and several coauthors

[2] A directed graph B is called *bipartite* if its vertex set can be split as $V \sqcup W$, so that for each edge $(v, w) \in E$, then v lies in V and w in W.

in [2]. It could be of interest for future work on the subject, as it has the nice property that any function $\Gamma^{\text{nc}}(B)$, where B is a bipartite graph, can be written as a multiplicity free sum of basis elements (see Proposition 5.8).

Let us now say a word about the cyclic inclusion-exclusion operation and how it has proved useful so far.

It has been introduced by the author (but not under this name) in the article [8] in the proof of a conjecture of Kerov on irreducible character values of the symmetric group. In fact, in this work, a two-alphabet variant of Γ_B is considered for bipartite graphs B. We explain in Section 6 how Theorem 4 can be used to simplify and generalize the proof of the former conjecture of Kerov.

Remarkably, this operation of cyclic inclusion-exclusion has also been fruitful in a quite different context in [5]: the purpose of this paper was to study some rational functions considered by Greene [11]. These functions are indexed by posets and defined as sums over linear extensions of the indexing poset : as such, they automatically verify cyclic inclusion-exclusion relations. This gives an efficient way to compute these rational functions and a powerful tool to study them; see [5].

The paper is organized as follows: Section 2 introduces some standard definitions and notations. In Section 3, cyclic inclusion-exclusion is defined and it is proved that this combinatorial construction gives some elements in the kernel of Γ^{nc}. Section 4 deals with the non-restricted setting and contains the proof of our main theorem in this case: the kernel of Γ and Γ^{nc} are spanned by the cyclic inclusion-exclusion relations (Theorems 1 and 2). The analogous results for the restrictions to bipartite graphs (Theorems 3 and 4) are established in Section 5. Finally, Section 6 describes the application of Theorem 4 to the theory of Kerov character polynomials.

2. Preliminaries

2.1. **Labelled and unlabeled graphs.**

Definition 2.1. A *labeled (directed) graph* G is a pair (V, E) where V is a finite set and E a subset of $V \times V$.

A *directed cycle* is a list $(v_1, \dots, v_k)$ of vertices of G such that (v_1, v_2), (v_2, v_3), $\cdots$, (v_{k-1}, v_k) and (v_k, v_1) are edges of G.

A graph without directed cycles is called *acyclic*.

For a non-negative integer n, we denote $[n]$ the set of positive integers smaller or equal to n. In this paper, we only consider graphs with vertex set $V = [n]$, for some integer n.

Denote S_n the group of permutations of n, that is of bijections from $[n]$ to $[n]$. If σ is a permutation of n and G a graph with vertex set $[n]$, then we consider the graph $\sigma(G)$ with vertex set $[n]$ and edge set

$$\sigma(E) = \big\{\{\sigma(v_1), \sigma(v_2)\} \text{ with } \{v_1, v_2\} \in E\big\}.$$

Definition 2.2. An *unlabeled (directed) graph* $\overline{G}$ is an equivalence class of labeled directed graphs under the relation

$$G \sim G' \iff \exists \sigma \in S_n \text{ s.t. } G = \sigma(G').$$

As this relation preserves acyclicity of graphs, there is a natural notion of unlabeled acyclic graphs. Namely an unlabeled graph is *acyclic* if at least one (or equivalently all) labeled graph(s) in the class is(are) acyclic.

We denote by $\mathscr{G}$ (respectively $\overline{\mathscr{G}}$) the vector space of linear combinations of labeled (respectively unlabeled) acyclic graphs. Then $\mathscr{G}$ (respectively $\overline{\mathscr{G}}$) is a graded vector space: the d-th homogeneous component $\mathscr{G}_d$ (respectively $\overline{\mathscr{G}}_d$) is by definition spanned by labeled (respectively unlabeled) graphs with vertex set $[d]$ (respectively with d vertices). The action of the symmetric group S_d on graphs with vertex set $[d]$ can be extended to $\mathscr{G}_d$. Then $\overline{\mathscr{G}}$ is the quotient of $\mathscr{G}$ by the vector space

$$\{x - \sigma(x), x \in \mathscr{G}_d, \sigma \in S_d \text{ for some } d \geq 1\}.$$

We denote this quotient map by φ_u (u stands for *unlabeling*).

2.2. **Quasi-symmetric functions.** As mentioned in Footnote 1, the ring of quasi-symmetric functions was introduced by I. Gessel [10] and may be seen as a generalization of the notion of symmetric functions. A modern introduction can be found in [19, Section 7.19] or [14, Section 3.3].

Let n be a nonnegative integer. A *composition* (or *integer composition*) of n is a sequence $I = (i_1, i_2, \dots, i_r)$ of positive integers, whose sum is equal to n. The notation $I \vDash n$ means that I is a composition of n and $\ell(I)$ denotes the number of parts of I. In numerical examples, it is customary to omit parentheses and commas. For example, 212 is a composition of 5.

Consider the algebra $\mathbb{C}[X]$ of polynomials[3] in a totally ordered alphabet of commutative variables $X = \{x_1, x_2, \dots\}$. Monomials $X^{\mathbf{v}} = x_1^{v_1} x_2^{v_2} \cdots$ correspond to sequences $\mathbf{v} = v_1, v_2, \dots$ with finitely many non-zero entries. For such a sequence, we denote by $\mathbf{v}_{\leftarrow}$ the list obtained by omitting the zero entries.

Definition 2.3. A polynomial $P \in \mathbb{C}[X]$ is said to be *quasi-symmetric* if and only if for any $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v}_{\leftarrow} = \mathbf{w}_{\leftarrow}$, the coefficients of $X^{\mathbf{v}}$ and $X^{\mathbf{w}}$ in P are equal.

One can easily prove that the set of quasi-symmetric polynomials is a subalgebra of $\mathbb{C}[X]$, called quasi-symmetric function ring and denoted $QSym$.

It should be clear that any symmetric polynomial is quasi-symmetric. The algebra $QSym$ of quasi-symmetric functions has a basis of monomial quasi-symmetric functions (M_I) indexed by compositions $I = (i_1, \dots, i_r)$, where

$$M_I = \sum_{k_1 < \dots < k_r} x_{k_1}^{i_1} \cdots x_{k_r}^{i_r}. \tag{2}$$

[3] Throughout the paper, we call "polynomial in infinitely many variables" an element of the inverse limit of the inverse system of graded algebras $(\mathbb{C}[x_1, \dots, x_n])_{n \geq 0}$ (the projection from $\mathbb{C}[x_1, \dots, x_n, x_{n+1}]$ to $\mathbb{C}[x_1, \dots, x_n]$ sends x_{n+1} to 0). In particular, it can have infinitely many monomials, but must have a bounded degree.

In particular, the dimension of the homogeneous space $QSym_n$ of degree n of $QSym$ is the number of compositions of n, that is 2^{n-1} for $n \geq 1$.

Example 2.4. $M_{212} = \sum_{k<l<m} x_k^2 x_l x_m^2$.

2.3. **Word quasi-symmetric functions.** The natural noncommutative analog of $QSym$ is the algebra of *word quasi symmetric functions*, denoted by $\mathbf{WQSym}$. We recall here its construction, following the presentation of Bergeron and Zabrocki [3, Section 5.2]. An equivalent, but slightly different presentation, using packed words instead of set compositions, can be found in a paper of Novelli and Thibon [15, Section 2.1].

Consider a totally ordered alphabet of noncommuting variables $\{a_1, a_2, \dots\}$. Monomials in these variables are canonically indexed by finite words w on the alphabet $\mathbb{N}$ as follows

$$\boldsymbol{a}_w = a_{w_1}\, a_{w_2} \dots a_{w_{|w|}}.$$

The *evaluation* $\mathrm{eval}(w)$ of a word w is the integer sequence $v = (v_1, v_2, \dots)$, where v_i is the number of letters i in w. Then the commutative image of $\boldsymbol{a}_w$ is $\boldsymbol{X}^{\mathrm{eval}(w)}$.

In the noncommutative framework, set compositions[4] play the role of compositions. A *set composition* of n is an *(ordered)* list $\mathbf{I} = (I_1, \dots, I_p)$ of pairwise disjoint non-empty subsets of $\{1, \dots, n\}$, whose union is $\{1, \dots, n\}$. In numerical example, we sort integers inside a part and use a vertical line to separate the parts. For example, the set composition $(\{1,5\}, \{3,4,6\}, \{2\})$ is denoted $15|346|2$.

To a word w on the (ordered) alphabet $\mathbb{N}$ of length ℓ, we associate the set composition $\mathbf{I} = \Delta(w)$ such that $j \in I_{|\{w_r : w_r \leq w_i\}|}$ (for every j in $[\ell]$). For example $\Delta(275525) = 15|346|2$.

Definition 2.5. A polynomial[5] in noncommuting variables $a_1, a_2, \dots$ is a *word quasi symmetric function* if and only if a_v and a_w are equal as soon as $\Delta(v)$ and $\Delta(w)$ coincide.

One can easily prove that the set $\mathbf{WQSym}$ of word quasi symmetric functions is an algebra. A linear basis of $\mathbf{WQSym}$ is given as follows:

$$\mathbf{M_I} = \sum_{\substack{w \text{ s.t.} \\ \Delta(w) = \mathbf{I}}} \boldsymbol{a}_w.$$

Clearly, if we only remember the sizes of the sets in a set composition $\mathbf{I}$, we get an integer composition that we denote $\varphi_c(\mathbf{I})$ (c stands for commuting). For example, $\varphi_c(15|346|2) = 231$. With this notation, the commutative image of $\mathbf{M_I}$ is $M_{\varphi_c(\mathbf{I})}$. Therefore, sending the variables $a_1, a_2, \dots$ to their commutative analogs $x_1, x_2, \dots$ defines a surjective projection from $\mathbf{WQSym}$ to $QSym$, that we abusively also denote φ_c. This projection can be alternatively realized as follows: the symmetric group S_n acts on the homogeneous component $\mathbf{WQSym}_n$

[4] Set compositions are also called sometimes *ordered set partitions*.

[5] As in the commutative setting, polynomials in infinitely may variables should be formally defined as inverse limit of a sequence of polynomials in finitely many variables.

of $\mathbf{WQSym}$ of degree n by permuting factors in every monomial. Then $QSym$ is the quotient of $\mathbf{WQSym}$ by the ideal spanned linearly by

$$\{x - \sigma(x), x \in \mathbf{WQSym}_d, \sigma \in S_d \text{ for some } d \geq 1\}.$$

To finish, let us mention that the ordered Bell numbers [17, A000670] count set compositions of $[n]$, and thus give the dimension of the homogeneous subspace of degree n of $\mathbf{WQSym}$.

Example 2.6. Consider the set composition $\boldsymbol{I} = 25|4|13$. Its evaluation is the integer composition 212. Then the associate basis element of $\mathbf{WQSym}$ is

$$\mathbf{M_I} = \sum_{k<l<m} a_m\, a_k\, a_m\, a_l\, a_k.$$

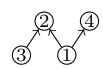
It is easy to check that its commutative image is M_{212} (given in Example 2.4), as claimed.

2.4. **Gessel's morphism.**

Definition 2.7. Let G be a graph on vertex set $[n]$. A function $f : [n] \to \mathbb{N}$ is called G *non-decreasing* if, for any edge (i,j) in E, one has $f(i) \leq f(j)$.

For a labeled graph G, we define $\Gamma^{\text{nc}}(G)$ as

$$\Gamma^{\text{nc}}(G) := \sum_{\substack{f:[n]\to\mathbb{N} \\ f \ \ G \text{ non-decreasing}}} a_{f(1)} \dots a_{f(n)}.$$

Example 2.8. Consider the graph $G =$ 

, then

$$\Gamma^{\text{nc}}(G) = \sum_{\substack{k_1,k_2,k_3,k_4 \\ k_3 \leq k_2,\ k_1 \leq k_2,\ k_1 \leq k_4}} a_{k_1} a_{k_2} a_{k_3} a_{k_4}.$$

It is clear that $\Gamma^{\text{nc}}(G)$ is a word quasi-symmetric function. Therefore, Γ^{nc} extends as a linear application from $\mathscr{G}$ to $\mathbf{WQSym}$.

The image $\varphi_c(\Gamma^{\text{nc}}(G))$ of $\Gamma^{\text{nc}}(G)$ in $QSym$ does not change if we replace G by an isomorphic labeled graph $G' = \sigma(G)$. Thus the morphism

$$\varphi_c \circ \Gamma^{\text{nc}} : \mathscr{G} \to QSym$$

factorizes through the quotient $\overline{\mathscr{G}}$ and defines a morphism $\overline{\mathscr{G}} \to QSym$. We recover of course the morphism Γ defined by Eq. (1) in the introduction and studied by Gessel in [10].

In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{\Gamma^{\text{nc}}} & \mathbf{WQSym} \\ {\scriptstyle \varphi_u}\downarrow & & \downarrow{\scriptstyle \varphi_c} \\ \overline{\mathscr{G}} & \xrightarrow[\Gamma]{} & QSym \end{array}.$$

2.5. **Hopf algebra structures.** In this Section, we mention known Hopf algebra structures of the spaces $\mathscr{G}$, $\overline{G}$, $QSym$ and $\mathbf{WQSym}$ which turns the morphisms described above into Hopf algebra morphisms.

As we focus in this paper on linear structures, this material won't be used and is only presented as additional motivation. This explains the lack of details and examples in this Section.

The space $\mathscr{G}$ has a Hopf algebra structure with the following product and coproduct:

- The product of G and G' is $G \sqcup (G')^{\uparrow |G|}$, where $(G')^{\uparrow |G|}$ means that we have shifted all vertex labels in G' by the number $|G|$ of vertices of G, so that the disjoint union is a graph with vertex set $[|G| + |G'|]$.
- The coproduct of a graph G with vertex set $[n]$ is given by

$$\Delta(G) = \sum_I \operatorname{std}(G[I]) \times \operatorname{std}\left(G\big[[n]\backslash I\big]\right),$$

  where the sum runs over subsets I of $[n]$ such that there is no edges going from $[n]\backslash I$ to I. Here, $G[I]$ and $G\big[[n]\backslash I\big]$ denote the graphs induced by G on I and $[n]\backslash I$ and $\operatorname{std}(H)$ consists in relabelling vertices of H in an order-preserving way so that the result has vertex set $[m]$ for some integer m.

These operations are compatible with the action of symmetric groups (in some sense that has to be precised) and thus are also naturally defined on the quotient $\overline{\mathscr{G}}$.

The spaces $QSym$ and $\mathbf{WQSym}$ have natural algebra structures inherited from the polynomial algebras, in which they live. It is also possible to define these products on the basis by some combinatorial operations on integer compositions and set compositions.

The coproducts of $QSym$ and $\mathbf{WQSym}$ are given on the bases by the formulas:

$$\Delta(M_I) = \sum_{k=0}^{\ell(I)} M_{(i_1,\dots,i_k)} \otimes M_{(i_{k+1},\dots,i_{\ell(I)})};$$

$$\Delta(\mathbf{M_I}) = \sum_{k=0}^{\ell(\mathbf{I})} \mathbf{M}_{(I_1,\dots,I_k)} \otimes \mathbf{M}_{(I_{k+1},\dots,I_{\ell(\mathbf{I})})}.$$

It is not difficult to check that these multiplication and comultiplication structures are compatible with all morphisms from the previous Section.

Remark 2.9. A detailed description of the Hopf algebra structure of $QSym$ can be found for example in [14, Section 3.3]. For $\mathbf{WQSym}$, we refer to [15, Section 2.1].

The Hopf algebra structure presented here for acyclic graphs is similar to the one considered on posets by Aguiar and Mahajan in [1, Section 13.1] with the formalism of Hopf monoids. It should be stressed that this Hopf algebra structure is different from the so-called *incidence Hopf algebra*, another Hopf algebra on posets considered in the litterature, see *e.g.* [7] and references therein.

3. CYCLIC INCLUSION-EXCLUSION

3.1. **Definition and example.** Let G be a directed graph. Consider G as a non directed graph and assume that it contains a cycle C.

Formally, such a cycle C is a list $C = (x_1, \dots, x_k)$ such that, for $1 \leq i \leq k$,

- either (x_i, x_{i+1}) is an edge of G;
- or (x_{i+1}, x_i) is an edge of G,

where, by convention, $x_{k+1} := x_1$. In the first case, we say that (x_i, x_{i+1}) is in a set C^+. In the second case, we say that (x_i, x_{i+1}) is in C^-.

Another description of the sets C^+ and C^- is the following. Edges of C have two orientations:

- their orientation in the cycle C;
- and their orientation as edges of G.

We denote C^+ (respectively C^-) the set of edges of C, for which these two orientations coincide (respectively do not coincide).

Finally, we define the following element of the graph algebra $\mathscr{G}$:

$$\mathrm{CIE}_{G,C} = \sum_{D \subseteq C^+} (-1)^{|D|} G \setminus D,$$

where $G \setminus D$ is the (directed acyclic) graph obtained from G by erasing the edges in D (and keeping the same set of vertices).

Example 3.1. Consider the graph G_{ex} from Fig. 1. The non-oriented version of G_{ex} contains several cycles, among them $C_{\text{ex}} = (6, 2, 3, 5, 1)$. This cycle is represented as a subgraph of G_{ex} in Fig. 1 with the two orientations described above. Then the set C_{ex}^+ is equal to $\{(6, 2), (2, 3), (3, 5)\}$ and $\mathrm{CIE}_{G_{\text{ex}}, C_{\text{ex}}}$ is given in Fig. 1.

3.2. **Cyclic inclusion-exclusion relations.**

Proposition 3.2. *For any graph G and cycle C of G, one has:*

$$\Gamma^{nc}(\mathrm{CIE}_{G,C}) = 0.$$

Proof. Let n be the size of G. Using the definitions of the morphism Γ^{nc} and of the element $\mathrm{CIE}_{G,C}$, one has:

$$\Gamma^{\text{nc}}(\mathrm{CIE}_{G,C}) = \sum_{D \subseteq C^+} (-1)^{|D|} \left[\sum_{\substack{f:[n]\to\mathbb{N} \\ f\ (G\setminus D) \text{ non-decreasing}}} a_{f(1)} \cdots a_{f(n)} \right]$$
$$= \sum_{f:[n]\to\mathbb{N}} \big(a_{f(1)} \cdots a_{f(n)}\big) \left(\sum_{D \subseteq C^+} (-1)^{|D|} \big[f\ (G \setminus D) \text{ non-decreasing}\big] \right),$$

where $[\text{condition}]$ is 1 if the condition is fulfilled and 0 else. The idea of the proof is to show that for any function $f : [n] \to \mathbb{N}$, its contribution

$$\sum_{D \subseteq C^+} (-1)^{|D|} \big[f\ (G \setminus D) \text{ non-decreasing}\big] \tag{3}$$

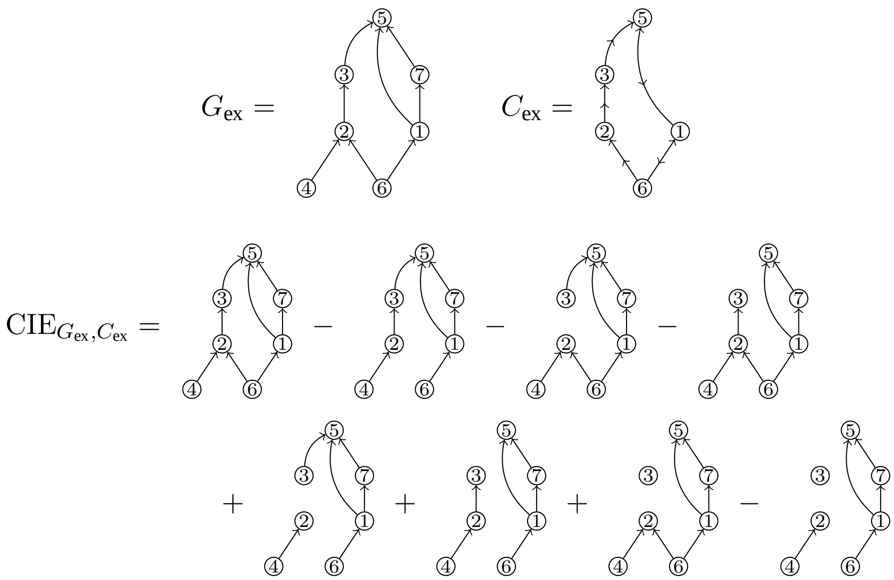


FIGURE 1. Graph G_{ex}, cycle C_{ex} and the graph algebra element $\text{CIE}_{G_{\text{ex}},C_{\text{ex}}}$ from Example 3.1.

is zero.

If f is not a $G\backslash C^+$ non-decreasing function, then each summand of (3) is zero and the conclusion holds trivially in this case.

Let $f : [n] \to \mathbb{N}$ be a $G\backslash C^+$ non-decreasing function, define

$$D_f = \big\{(x, y) \in C^+ \text{ s.t. } f(x) > f(y)\big\} \subseteq C^+.$$

It is straightforward that D_f fulfills the following property:

$$\forall D \subseteq C^+, \quad f \text{ is } G \setminus D \text{ non-decreasing} \iff D_f \subseteq D. \tag{4}$$

Hence Eq. (3) can be rewritten as:

$$\sum_{D_f \subseteq D \subseteq C^+} (-1)^{|D|}.$$

which is equal to zero if and only if $D_f \neq C^+$. Therefore, to end the proof of the proposition, it is enough to show that, for any $G\backslash C^+$ non-decreasing function, D_f is strictly included in C^+.

We proceed by contradiction. Suppose that we can find a $G\backslash C^+$ non-decreasing function f for which $D_f = C^+$. This means that, for each (x, y) in C^+, one has $f(x) > f(y)$.

Besides, since f is a $G\backslash C^+$ non-decreasing function, one has $f(x) \leq f(y)$ for any edge (x, y) of G which is not in C^+, so in particular for any couple (y, x) in C^-.

Recall now that C is a cycle in the undirected version of G. Formally, C is a list $(x_1, \dots, x_k)$ such that, for $1 \leq i \leq \ell$, (by convention, $x_{k+1} = x_1$)

- either (x_i, x_{i+1}) is an edge of G and $(x_i, x_{i+1}) \in C^+$;
- or (x_{i+1}, x_i) is an edge of G and $(x_i, x_{i+1}) \in C^-$.

Using the remarks above, we can conclude in both cases that $f(x_i) \geq f(x_{i+1})$. Bringing everything together,

$$f(x_1) \geq f(x_2) \geq \cdots \geq f(x_\ell - 1) \geq f(x_\ell) \geq f(x_1).$$

As C^+ can not be empty (otherwise, $(x_k, \dots, x_1)$ would be a directed cycle), at least one of these inequalities should be strict. We have reached a contradiction and D_f must be strictly included in C^+. □

Proposition 3.2 gives some relations between the word quasi-symmetric functions $\Gamma^{\mathrm{nc}}(G)$. We call these relations *cyclic inclusion-exclusion relations* (CIE relations for short). Formally, the elements $(\mathrm{CIE}_{G,C})$ span linearly a subspace, that we shall denote $\mathscr{C}$, which is included in the kernel of Γ^{nc}.

We shall prove in the next Section that any relation among the $\Gamma^{\mathrm{nc}}(G)$ can be deduced from CIE relations. In other terms, the space $\mathscr{C}$ is exactly the kernel of Γ^{nc}. We will also prove that analog results hold for some quotients/restrictions of Γ^{nc}.

Special case 3.3. We describe here the special case where $|C^+| = 1$. If $e = (v_1, v_2)$ is the element of C^+, this means that the graph G contains another path[6] from v_1 to v_2. Informally, e can be obtained from other edges of G by *transitivity*.

In this case, the inclusion-exclusion relation yields $\Gamma^{\mathrm{nc}}(G) = \Gamma^{\mathrm{nc}}(G \backslash \{e\})$. This is indeed true, as non-decreasing functions on both graphs are the same.

Remark 3.4. A weaker form of Proposition 3.2 (in the commutative setting) has been established in [5, Theorem 4.1] and widely used to extend some rational identity due to Greene [11]. The structure of the proof is exactly the same.

4. THE KERNEL IN THE NON-RESTRICTED CASE

4.1. The graphs $G_{\mathbf{I}}$.

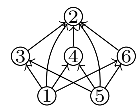
Definition 4.1. Let $\mathbf{I} = (I_1, \dots, I_r)$ be a set composition of $[n]$. We consider the directed graph $G_{\mathbf{I}}$ with vertex set $[n]$ and edge set

$$\bigsqcup_{j<k} I_j \times I_k.$$

In other terms, there is an edge between x and y if the index of the set of $\mathbf{I}$ containing x is smaller than the one of the set containing y.

[6] A path form x to y is a list $(v_0, v_1, \dots, v_k)$ with $v_0 = x$ and $v_k = y$ such that for every i in $[k]$, the pair (v_{i-1}, v_i) is an edge of G.

Example 4.2. Take $\mathbf{I}_{\text{ex}} = 15|346|2$. Then $G_{\mathbf{I}_{\text{ex}}}$ and the associated word quasi symmetric function are

(5) $G_{\mathbf{I}_{\text{ex}}} =$ 

$; \quad \Gamma^{\text{nc}}(G_{\mathbf{I}_{\text{ex}}}) = \sum_{\substack{k_1,\dots,k_6 \\ \max(k_1,k_5)\leq\min(k_3,k_4,k_6) \\ \max(k_3,k_4,k_6)\leq k_2}} a_{k_1}\cdots a_{k_6}.$

4.2. **A $\mathbb{Z}$-basis of $\mathbf{WQSym}$.** The purpose of this Section is to prove that $\Gamma^{\text{nc}}(G_{\mathbf{I}})$ is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$. The proof requires to consider two additional bases of $\mathbf{WQSym}$ and prove that three change of basis matrices are unitriangular (with respect to different orders of the basis elements).

As in [21, Section 3.1], it will be convenient to work with descent-starred permutations instead of set compositions.

Definition 4.3. We call *descent-starred permutation* a couple (σ, D) such that D is a subset of the descent set $\{i, \sigma(i) > \sigma(i+1)\}$ of σ.
The descents in D are termed *starred*.

In numerical example, we represent a descent-starred permutation (σ, D) by the word notation of σ in which the elements of index in D are followed by a star. For example the descent-starred permutation $(3142, \{3\})$ will be denoted $314_\star2$.

Lemma 4.4. *Descent-starred permutations of* n *are in bijection with set compositions of* $[n]$.

Proof. From the numerical notation of a set composition $\mathbf{I}$, we sort each part in decreasing order and remove vertical bars to get the word notation of σ. Then mark with a star the descents inside the same part of $\mathbf{I}$. This is clearly a bijection. □

For example, the descent-starred permutation associated to $15|346|2$ is $5_\star16_\star4_\star32$.
Let us define three families of word quasi-symmetric functions indexed by descent-starred permutations $\mathbf{M}_{(\sigma,D)}$, $\mathbf{L}_{(\sigma,D)}$ and $\mathbf{F}_{(\sigma,D)}$. All of them are defined as a sum

$$\sum a_{k_1}\cdots a_{k_n}$$

over lists $\mathbf{k} = (k_1, \dots, k_n)$ of positive integers with conditions given in the following table (for integers x in $[n-1]$):

	$\mathbf{M}_{(\sigma,D)}$	$\mathbf{L}_{(\sigma,D)}$	$\mathbf{F}_{(\sigma,D)}$
$x \in D$	$k_{\sigma(x)} = k_{\sigma(x+1)}$	$k_{\sigma(x)} = k_{\sigma(x+1)}$	$k_{\sigma(x)} < k_{\sigma(x+1)}$
$x \notin D$	$k_{\sigma(x)} < k_{\sigma(x+1)}$	$k_{\sigma(x)} \leq k_{\sigma(x+1)}$	$k_{\sigma(x)} \leq k_{\sigma(x+1)}$

In the definitions of $\mathbf{M}_{(\sigma,D)}$ and $\mathbf{L}_{(\sigma,D)}$, we require that $k_{\sigma(x)} = k_{\sigma(x+1)}$ for $x \in D$, which implies that the function $x \mapsto k_x$ should be constant on the parts of the associated set composition $\mathbf{I}$. Moreover, in $\mathbf{M}_{(\sigma,D)}$, together the strict inequalities for $x \notin D$, this is equivalent to $\Delta(\mathbf{k}) = \mathbf{I}$, so that we have $\mathbf{M}_{(\sigma,D)} = M_{\mathbf{I}}$.

Remark 4.5. The commutative projection of $\mathbf{F}_{(\sigma,D)}$ is F_J, where F is the so-called *fundamental basis* of $QSym$ and J the (integer) composition associated with the set D (we use here the terminology of [19, Section 7.19]).

Lemma 4.6. *The families* $(\mathbf{L}_{(\sigma,D)})$ *and* $(\mathbf{F}_{(\sigma,D)})$*, indexed by descent-starred permutations, are* $\mathbb{Z}$*-basis of* $\mathbf{WQSym}$.

Proof. We start by recalling some classical terminology: we say that a set-partition $\mathbf{I}$ is finer than $\mathbf{J}$ and denote $\mathbf{I} \lhd \mathbf{J}$ if $\mathbf{J}$ can be obtained from $\mathbf{I}$ by removing vertical lines and reordering the blocks: for example, $15|346|2$ is finer than $13456|2$ and than $15|2346$.

Let (σ, D) be a descent-starred permutation and $\mathbf{I}$ the associated set composition. Using the remark above, the definition of $\mathbf{L}_{(\sigma,D)}$ (that we will also denote $\mathbf{L}_\mathbf{I}$) can be rewritten as

$$\mathbf{L}_\mathbf{I} = \sum a_{k_1} \cdots a_{k_n},$$

where the sum runs over lists $(k_1, \dots, k_n)$ that are constant on the parts of I and such that the value of k on I_m is at most the one on I_{m+1} (for each m in $[\ell(I)-1]$). If we cut the sum depending on which indices i_ℓ are equal, we obtain[7]

$$\mathbf{L}_\mathbf{I} = \sum_{J \rhd I} \mathbf{M}_\mathbf{J}.$$

This implies that $(\mathbf{L}_\mathbf{I})$ is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$ as its matrix in the basis $(\mathbf{M}_\mathbf{I})$ is unitriangular.

Consider now the family $\mathbf{F}_{(\sigma,D)}$. We first rewrite the definitions of $\mathbf{F}_{(\sigma,D)}$ and $\mathbf{L}_{(\sigma,D)}$ as follows:

$$\mathbf{F}_{(\sigma,D)} = \sum a_{k_1} \cdots a_{k_n} \prod_{x \in D} \left(1 - \delta_{k_{\sigma(x)}, k_{\sigma(x+1)}}\right), \tag{6}$$

$$\mathbf{L}_{(\sigma,D)} = \sum a_{k_1} \cdots a_{k_n} \prod_{x \in D} \left(\delta_{k_{\sigma(x)}, k_{\sigma(x+1)}}\right), \tag{7}$$

where both sums run over lists $(k_1, \cdots, k_n)$ that satisfy $k_{\sigma(1)} \leq \cdots \leq k_{\sigma(n)}$ and $\delta_{i,j}$ is the usual Kronecker symbol. Expanding the product in (6), we get

$$\mathbf{F}_{(\sigma,D)} = \sum_{D' \subseteq D} (-1)^{|D'|} \mathbf{L}_{(\sigma,D')}.$$

Hence the matrix of the family $\mathbf{F}_{(\sigma,D)}$ in the basis $\mathbf{L}_{(\sigma,D)}$ is unitriangular with respect to the following order[8]:

$$(\sigma', D') \leq_1 (\sigma, D) \Rightarrow \begin{cases} \sigma = \sigma' \\ D' \subseteq D \end{cases}$$

This proves that $(\mathbf{F}_{(\sigma,D)})$ is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$. □

[7] See [10, Eq. (2)] for the commutative analog of this statement.

[8] This order is isomorphic to the order on set compositions denoted $\leq_\star$ in [3, Section 6].

We now explain how $\Gamma^{\text{nc}}(G_{\mathbf{I}})$ writes on the $\mathbf{F}$ basis. If $\mathbf{I} = (I_1, \dots, I_r)$ is a set composition, we consider the following set $\text{MP}(\mathbf{I})$ of descent-starred permutations:

- As a word $\sigma = w_1 \cdots w_r$, where w_m contains exactly once each element of I_m ;
- The descent in position x is starred if σ_x and σ_{x+1} are in the same part of $\mathbf{I}$. In other words, for each m, we mark the descents in w_m, but not the potential descent created by concatenating w_m and w_{m+1}.

For example, take $\mathbf{I}_{\text{ex}} = 15|346|2$, then $\text{MP}(\mathbf{I}_{\text{ex}})$ contains the following 12 descent-starred permutations:

$$153462,\ 5_\star13462,\ 154_\star362,\ 5_\star14_\star362,\ 156_\star4_\star32,\ 5_\star16_\star4_\star32,$$
$$1536_\star42,\ 5_\star136_\star42,\ 1546_\star32,\ 5_\star146_\star32,\ 156_\star342,\ 5_\star16_\star342$$

Proposition 4.7. *For any set composition* $\mathbf{I}$*, one ha:*

$$\Gamma^{nc}(G_{\mathbf{I}}) = \sum_{(\sigma,D)\in MP(\mathbf{I})} \mathbf{F}_{(\sigma,D)}.$$

Proof. Let f be a $G_{\mathbf{I}}$ non-decreasing function from $[n]$ to $\mathbb{N}$. For each part I_m in the set composition $\mathbf{I}$, let us consider the restriction f_m of f to I_m. Then there exists a unique word w_m containing exactly once each number in I_m such that

$$y \text{ appear before } z \text{ in } w_m \Leftrightarrow \begin{cases} f_m(y) \le f_m(z) & \text{if } y < z; \\ f_m(y) < f_m(z) & \text{if } y > z; \end{cases}$$

Indeed this word is obtained by ordering lexicographically the pair $((f_m(y), y))_{y\in I_m}$ and keeping only the second element of each pair[9].

We mark the descent in w_m and by concatenating all the words w_m (for $1 \le m \le r$), we get a descent-starred permutation (σ, D) in $\text{MP}(\mathbf{I})$. This descent-starred permutation is the only one in $\text{MP}(\mathbf{I})$ such that $a_{f(1)} \cdots a_{f(n)}$ appears in $\mathbf{F}_{(\sigma,D)}$, which explains the formula of the proposition. □

Example 4.8. Take $\mathbf{I}_{\text{ex}}$ as above, $\Gamma^{\text{nc}}(G_{\mathbf{I}_{\text{ex}}})$ is given by Eq. (5). The summation set can be split as follows:

- either $k_1 \le k_5$ or $k_5 < k_1$;
- besides, the integers k_3, k_4 and k_6 fulfill exactly one of the 6 following inequalities:

$$k_3 \le k_4 \le k_6, \quad k_4 < k_3 \le k_6, \quad k_3 \le k_6 < k_4,$$
$$k_4 \le k_6 < k_3, \quad k_6 < k_3 \le k_4, \quad k_6 < k_4 < k_3.$$

Combining both case distinctions yield 12 different cases, and $\Gamma^{\text{nc}}(G_{\mathbf{I}_{\text{ex}}})$ is a sum of 12 different terms which are the $\mathbf{F}$ functions indexed by the 12 descent-starred permutations in $\text{MP}(\mathbf{I})$ (which are listed above).

Corollary 4.9. *The family* $\big(\Gamma^{nc}(G_{\mathbf{I}})\big)$ *is a* $\mathbb{Z}$*-basis of* $\mathbf{WQSym}$.

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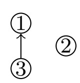
[9] Existence and uniqueness of the word w_m can also be seen as a special case of Stanley fundamental theorem on P-partitions [18, Theorem 6.2] (see also [12]), where the poset P has element set I_m and no relations.

Proof. If (σ, D) is the descent-starred permutation associated by Lemma 4.6 to a set composition $\mathbf{I}$ of n of length r, then the size of D is $n - r$. Besides, for each element $(\sigma', D') \in MP(\mathbf{I})$, the size of D' is smaller than $n - r$, unless $(\sigma', D') = (\sigma, D)$. Hence the proposition implies that the matrix of $\Gamma^{\mathrm{nc}}(G_{\mathbf{I}})$ in the basis $\mathbf{F}_{(\sigma,D)}$ is unitriangular with respect to the order

$$(\sigma', D') <_2 (\sigma, D) \Leftrightarrow |D'| < |D|$$

and $\Gamma^{\mathrm{nc}}(G_{\mathbf{I}})$ is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$. □

Remark 4.10. Stanley fundamental theorem on P-partitions [18, Theorem 6.2] (see also Knuth's paper [12]) implies that, if G is a naturally labeled graph (*i.e.* such that $(i, j) \in E$ implies $i \leq j$ as positive integer as positive integerss), then $\Gamma^{\mathrm{nc}}(G)$ has a non-negative expansion on the $\mathbf{F}_{(\sigma,D)}$ basis. Proposition 4.7 gives examples of non-necessarily naturally labeled graphs G, such that the $\mathbf{F}_{(\sigma,D)}$ expansion of $\Gamma^{\mathrm{nc}}(G)$ has non-negative coefficients. But, this is not the case for any graph G, as shown by the following example (we skip details in the computation):

$\Gamma^{\mathrm{nc}}\Bigg($  $\Bigg)$

$$= \mathbf{F}_{231} + \mathbf{F}_{3_\star 2_\star 1} + \mathbf{F}_{312} - \mathbf{L}_{3_\star 2_\star 1}$$
$$= \mathbf{F}_{231} + \mathbf{F}_{312} + \mathbf{F}_{3_\star 21} + \mathbf{F}_{32_\star 1} - \mathbf{F}_{321}.$$

Such negative signs do not occur in the commutative setting: indeed, any function $\Gamma(\overline{G})$ is a non-negative linear combination of fundamental quasi-symmetric functions, see [19, Corollary 7.19.5].

4.3. **A generating family for the quotient.** We will now show that $(G_{\mathbf{I}})$, where $\mathbf{I}$ runs over all set compositions, is a generating family in the quotient $\mathscr{G}/\mathscr{C}$. As explained in Section 4.4, together with the results of Section 4.2 and Remark 3.4, this implies that $\Gamma^{\mathrm{nc}} : \mathscr{G}/\mathscr{C} \to \mathbf{WQSym}$ is an isomorphism.

Here is the key combinatorial lemma in this section.

Lemma 4.11. *Let G be a unlabeled poset. Then either G is equal to some $G_{\mathbf{I}}$ or, in the quotient $\mathscr{G}/\mathscr{C}$, one can write G as a linear combination of graphs with the same set of vertices and more edges.*

Proof. Let G be an acyclic directed graph with vertex set $[n]$ and edge set E_G.

Throughout the proof, we denote $\sim$ the following symmetric relation: $x \sim y$ if, in G, there is no directed path (see Footnote 6 for the definition) from x to y, nor from y to x. When $x \sim y$, the graphs $G_{(x,y)}$ and $G_{(y,x)}$ obtained from G by adding respectively an edge from x to y or from y to x are still acyclic.

We distinguish three cases.

Case 1: G is not the graph of a transitive relation.

In other terms, there exist x, y and z such that

- there is an edge from x to y and from y to z in G;
- there is no edge from x to z.

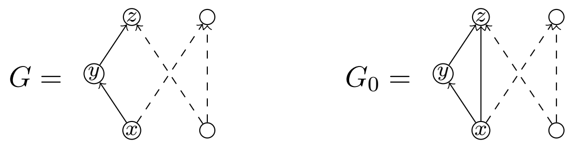


FIGURE 2. Graphs G and G_0 in the first case of the proof of Lemma 4.11.

We consider $G_0 = G_{(x,z)}$ the graph obtained from G by adding an edge between x and z. As a directed graph, G_0 is acyclic: otherwise, there would be a path from z to x in G and, together with (x, y) and (y, z), this path would be a directed cycle in G. But the non-oriented version of G_0 contains a cycle $C = (x, z, y)$. Using the notation of Section 3.1 (see also Footnote 6), one has $C^+ = \{(x, z)\}$ and the corresponding cyclic inclusion-exclusion element is

$$\mathrm{CIE}_{G_0,C} = G_0 - G.$$

Hence, in $\mathscr{G}/\mathscr{C}$, one has $G = G_0$ and the statement is true in this case.

This case is illustrated in Fig. 2 with examples of graphs G and G_0. Dashed edges are edges of G and G_0 that do not play a role in the proof.

Case 2: the relation $\sim$ is not an equivalence relation.
By assumption, there exist vertices x, y, z such that

- there is a path $(x, v_1, \cdots, v_k, z)$ from x to z in G;
- one has $x \sim y$ and $y \sim z$.

By definition of $\sim$, the graph $G_{(x,y)}$ is acyclic. Moreover, it does not contain a path from z to y. Indeed, as $y \sim z$ in G, such a path should use the edge (x, y) and thus be the concatenation of a path from z to x with the edge (x, y). But G does not contain a path from z to x (indeed, it contains a path from x to z and no directed cycles).

Therefore, the graph G_0 obtained from $G_{(x,y)}$ by adding an edge from y to z is an acyclic directed graph. However, its undirected version contains a cycle

$$C = (x, y, z, v_k, \cdots, v_1).$$

Using the notation of Section 3.1, for this cycle, one has $C^+ = \{(x, y), (y, z)\}$. Hence,

$$\mathrm{CIE}_{G_0,C} = G_0 - G_0 \setminus \{(x, y)\} - G_0 \setminus \{(y, z)\} + G_0 \setminus \{(x, y), (y, z)\}.$$

But $G_0 \setminus \{(x, y), (y, z)\}$ is G, so, in the quotient $\mathscr{G}/\mathscr{C}$, one has

$$G = -G_0 + G_0 \setminus \{(x, y)\} + G_0 \setminus \{(y, z)\}$$

and the statement is proved in this case.

This case is illustrated in Fig. 3 with examples of graphs G and G_0. Here, the dashed edge illustrates the fact that we do not know the length of the path P from x to z. Potential extra edges and vertices of G and G_0 have not been represented for more readability.

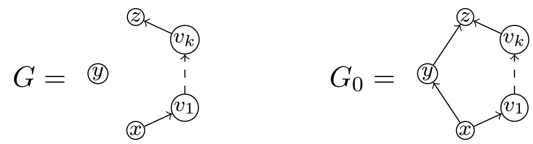


FIGURE 3. Graphs G and G_0 in the second case of the proof of Lemma 4.11.

Case 3: G is the graph of a transitive relation and the relation $\sim$ is an equivalence relation.

In this case, we will prove that G is necessarily equal to $G_{\mathbf{I}}$, for some set composition $\mathbf{I}$.

Let us start by a remark: in the graph of a transitive relation, the existence of a path from x to y implies the existence of an edge from x to y. Hence $x \nsim y$ means that there is either an edge from x to y or from y to x.

Denote $(V_j)_{j \in J}$ the partition of the vertex set of G into equivalence classes of $\sim$. Consider two such classes V_j and V_k. We will prove that either $V_j \times V_k$ or $V_k \times V_j$ is included in E_G.

Select arbitrarily a pair (v_0, w_0) in $V_j \times V_k$. As $v_0 \nsim w_0$, by eventually swapping v_0 and w_0 (and simultaneously j and k), we may assume that (v_0, w_0) is an edge of G.

Then, for any w in V_k, the pair (v_0, w) is also an edge of G. Indeed, if this is not the case, as $v_0 \nsim w$, this would imply that (w, v_0) is an edge of V. But, then by transitivity, (w, w_0) should be an edge of G, which is impossible as $w \sim w_0$.

The same argument proves that, for any v in V_j, the pair (v, w) must be an edge of G, which proves the inclusion of $V_j \times V_k$ in E_G.

As we may have swapped v_0 and w_0 at the beginning, we have in fact proved that for any pair (j, k) in J^2, either $V_j \times V_k$ or $V_k \times V_j$ is included in E_G. As G does not have any directed cycle, there exists a total order $<_J$ on J such that $V_j \times V_k$ is included in E_G if and only if $j <_J k$.

By definition of $\sim$, there is no edges with both extremities in the same V_j. Besides, there can not be an edge from V_k to V_j (with $j <_J k$), as this would create a directed cycle of length 2. Finally, the set of edges of G is exactly

$$\bigsqcup_{j <_J k} V_j \times V_k,$$

which means that $G = G_{\mathbf{I}}$ for $\mathbf{I} = (V_j)_{j \in J}$. $\square$

Let G be an acyclic directed graph. Iterating Lemma 4.11, one can write G as an integer linear combination of $G_{\mathbf{I}}$ in the quotient space $\mathscr{G}/\mathscr{C}$. In other terms, $G_{\mathbf{I}}$ is a generating family of the vector space $\mathscr{G}/\mathscr{C}$.

4.4. **First main result.** We are now ready to prove the following statement.

Theorem 1. *The space $\mathscr{C}$, spanned by cyclic inclusion-exclusion elements, is the kernel of the surjective morphism Γ^{nc} from $\mathscr{G}$ to $\mathbf{WQSym}$.*

Proof. Denote $\mathscr{K}$ the kernel of Γ^{nc}. By Proposition 3.2, it contains $\mathscr{C}$. On the one hand (Fig. 3), we know that $\mathscr{G}/\mathscr{C}$ is spanned by the family $(G_{\mathbf{I}})$. On the other hand (Corollary 4.9), the family $\Gamma^{\mathrm{nc}}(G_{\mathbf{I}})$ is a basis of $\mathbf{WQSym}$, which implies in particular that the $(G_{\mathbf{I}})$ are linearly independent in $\mathscr{G}/\mathscr{K}$ and hence in $\mathscr{G}/\mathscr{C}$.

Therefore $(G_{\mathbf{I}})$ is a basis of $\mathscr{G}/\mathscr{C}$ and Γ^{nc} is an isomorphism from $\mathscr{G}/\mathscr{C}$ to $\mathbf{WQSym}$ (it sends a basis on a basis), which concludes the proof. □

Remark 4.12. In fact, we have proved a stronger result: the subspace of $\mathscr{G}$ spanned by cyclic inclusion-exclusion associated to cycles C with $|C^+| = 1$ and $|C^+| = 2$ is the kernel of Γ (and hence coincides with $\mathscr{C}$).

4.5. **Unlabeled commutative framework and second main result.** Consider a unlabeled directed graph $\overline{G}$ and a cycle $\overline{C}$ of the undirected version of $\overline{G}$. As in Section 3.1, we can define $\overline{C}^+$ and an element

$$\overline{\mathrm{CIE}}_{\overline{G},\overline{C}} = \sum_{D \subseteq \overline{C}^+} (-1)^{|D|} \overline{G} \setminus D.$$

But $\overline{G}$ is the equivalence class of some graph G, whose undirected version contains a cycle C, which projects on $\overline{C}$. With this in mind, $\overline{\mathrm{CIE}}_{\overline{G},\overline{C}}$ is simply the image of $\mathrm{CIE}_{G,C}$ by the morphism $\varphi_u : \mathscr{G} \to \overline{\mathscr{G}}$.

Let us consider the subspace $\overline{\mathscr{C}}$ of $\overline{\mathscr{G}}$ spanned by cyclic inclusion-exclusion elements. Equivalently this is the image of $\mathscr{C}$ by the morphism φ_c.

Theorem 2. *The ideal $\overline{\mathscr{C}}$, spanned by inclusion-exclusion elements, is the kernel of the surjective morphism Γ from $\overline{\mathscr{G}}$ to $QSym$.*

Proof. This follows from Theorem 1, and the fact that the morphism Γ^{nc} is compatible with the action of S_n on homogeneous components described in Sections 2.1 and 2.3. Indeed, one can write

$$\begin{aligned} \overline{\mathscr{G}}/\langle \overline{\mathrm{CIE}}_{\overline{G},\overline{C}} \rangle &\simeq \big(\mathscr{G}/\langle x - \sigma.x \rangle\big)/\langle \overline{\mathrm{CIE}_{G,C}} \rangle \simeq \mathscr{G}/\langle x - \sigma.x, \mathrm{CIE}_{G,C} \rangle \\ &\simeq \big(\mathscr{G}/\langle \mathrm{CIE}_{G,C} \rangle\big)/\langle x - \sigma.x \rangle \simeq \mathbf{WQSym}/\langle x - \sigma.x \rangle \simeq QSym. \end{aligned}$$ □

Remark 4.13. The function $\Gamma\big(\overline{G_{\mathbf{I}}}\big)$ in $QSym$ depends only on the integer composition $I = \varphi_c(\mathbf{I})$. Therefore, from Section 4.2, we know that this family, indexed by integer compositions, is a $\mathbb{Z}$-basis of $QSym$. This family has appeared in a paper of Stanley [20, Note p7] which noticed that the change of basis matrix with the fundamental basis is unitriangular (commutative version of Proposition 4.7).

Remark 4.14. A direct proof of Theorem 2 along the same lines as the proof of Theorem 1 is of course possible.

5. The kernel in the bipartite case

The purpose of this Section is to show that the kernel of Γ and Γ^{nc} restricted to bipartite graphs is also generated by cyclic-inclusion relations.

5.1. **Preliminaries for the bipartite setting.** Recall that a directed graph is called *bipartite* if its vertex set can be split in $V \sqcup W$, such that if $(v, w) \in E$, then v lies in V and w in W (in other words, the edge set is included in $V \times W$). Note that this bipartition is not unique as isolated vertices can be either in V or W, but this is the only degree of freedom.

The subalgebra of the graph algebra $\mathscr{G}$ spanned by bipartite graphs will be denoted $\mathscr{G}_b$. If B is a bipartite graph and C a cycle in the undirected version of B, then the cyclic inclusion-exclusion element $\mathrm{CIE}_{B,C}$ lies in $\mathscr{G}_b$. We denote $\mathscr{C}_b$ the subspace of $\mathscr{G}_b$ spanned by these elements.

Finally, we consider the restriction of Γ^{nc} to $\mathscr{G}_b$, that we denote Γ^{nc}_b. Clearly, from Proposition 3.2, the space $\mathscr{C}_b$ is included in the kernel of Γ^{nc}_b.

Remark 5.1. The kernel of Γ^{nc}_b is, from Theorem 1, equal to $\mathscr{C} \cap \mathscr{G}_b$. But, even if $\mathscr{C}$ is by definition generated by cyclic inclusion-exclusion elements, we do not know *a priori* whether this intersection is spanned by the cyclic inclusion-exclusion elements that lie in it.

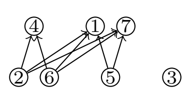
5.2. **The bipartite graphs $B_{(\mathbf{I},\mathbf{J})}$.** Consider a set composition of $[n]$. In the following, it will be convenient to distinguish odd and even-indexed parts of the composition. Therefore we denote I_1 its first part, J_1 its second part, I_2 its third and so on until J_r which is eventually empty if the number of parts of the set composition is odd. In this context, a set composition is denoted $(\mathbf{I}, \mathbf{J})$ and r is called its *semi-length*. We draw the attention of the reader on the fact that, from this viewpoint, a pair $(\mathbf{I}, \mathbf{J})$ is a *single* set composition and not a pair of set compositions.

Definition 5.2. Let $(\mathbf{I}, \mathbf{J})$ be a set composition of $[n]$. We consider the bipartite directed graph $B_{(\mathbf{I},\mathbf{J})}$ with vertex set $[n]$ and edge set

$$\bigsqcup_{h<k} I_h \times J_k.$$

Example 5.3. Consider the set composition $26|4|5|17|3$. With the notations of this section, it writes as $(\mathbf{I}_{\mathrm{ex}}, \mathbf{J}_{\mathrm{ex}}) = (26|5|3, 4|17|)$ (in this case , J_3 is empty, which explains the vertical bar at the end of the numerical notation of $\mathbf{J}_{\mathrm{ex}}$). The graph $B_{(\mathbf{I}_{\mathrm{ex}},\mathbf{J}_{\mathrm{ex}})}$ and the associated word quasi symmetric function are

(8)

$B_{(\mathbf{I}_{\mathrm{ex}},\mathbf{J}_{\mathrm{ex}})} =$ 

$; \quad \Gamma^{\mathrm{nc}}(B_{(\mathbf{I}_{\mathrm{ex}},\mathbf{J}_{\mathrm{ex}})}) = \sum_{\substack{k_1,\dots,k_7 \\ \max(k_2,k_6) \le \min(k_1,k_4,k_7) \\ k_5 \le \min(k_1,k_7)}} a_{k_1} \cdots a_{k_7}.$

5.3. **A combinatorial lemma.** If $V \sqcup W = [n]$ is a bipartition of $[n]$, we denote $K_{V,W}$ the complete directed bipartite graph between V and W, that is the graph with vertex set $[n]$ and edge set $V \times W$. Let D be a subset of $V \times W$. Then we consider the directed graph K^D obtained from $K_{V,W}$ by turning the edges in D around (in general, K^D is not a directed bipartite graph).

For example, consider $V = \{1, 2, 4, 6\}$ and $W = \{3, 5\}$. The corresponding complete bipartite graph is the left-most graph in Fig. 4. We now choose a subset

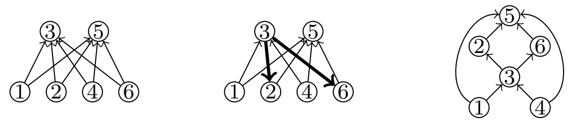


FIGURE 4. A complete bipartite graph (left-most graph), the graph obtained after turning some edges around (middle graph) and a graph from the family $H_{(\mathbf{I},\mathbf{J})}$ (right-most graph). Note that the last two are identical.

of $V \times W$, *e.g.* $D = \{(2,3),(6,3)\}$. The corresponding graph K^D is drawn in the middle of Fig. 4.

We are also interested in the following family of graphs. If $(\mathbf{I},\mathbf{J})$ is a set composition of $[n]$, we define $H_{(\mathbf{I},\mathbf{J})}$ as the graph with vertex set $[n]$ and edge set

$$\bigsqcup_{m \leq m'} (I_m \times J_{m'}) \sqcup \bigsqcup_{m < m'} (J_m \times I_{m'}).$$

As an example, let us choose $\mathbf{I} = 14|26$ and $\mathbf{J} = 3|5$, that is $\mathbf{K} = 14|3|26|5$. The corresponding graph $H_{(\mathbf{I},\mathbf{J})}$ is the right-most graph of Fig. 4. The examples have been chosen so that $H_{(\mathbf{I},\mathbf{J})}$ and K^D are the same graph. We will now see that the family $H_{(\mathbf{I},\mathbf{J})}$ roughly corresponds to the family of *acyclic* graphs among the K^D.

The following lemma will be useful in the next Section.

Lemma 5.4. *Let V, W and D as above. Assume that each vertex in W is the extremity of at least one edge not in D. Then, either K^D contains a directed cycle, or there exists a set composition $(\mathbf{I},\mathbf{J})$ with $\bigsqcup_{1 \leq k \leq r} I_k = V$ and $\bigsqcup_{1 \leq k \leq r} J_k = W$ such that $K^D = H_{(\mathbf{I},\mathbf{J})}$.*

Moreover, each such set composition $(\mathbf{I},\mathbf{J})$ corresponds to exactly one set D such that K^D is acyclic.

Proof. Assume K^D is acyclic. Denote I_1 the subset of V of elements x such that

$$\{(x,y), y \in W\} \cap D = \emptyset,$$

i.e. none of the edges starting x have been turned around.

We will prove by contradiction that I_1 is non empty. Assume $I_1 = \emptyset$. Then K^D is a directed graph, where all vertices have at least one incoming edge (vertices in W have at least one incoming edge because of our hypothesis and vertices in V have an incoming edge because I_1 is empty). Such a graph necessarily contains a directed cycle (start from an arbitrary vertex and follow backwards incoming edges until you encounter twice the same vertex, which will happen eventually; you have found a directed cycle).

Thus I_1 is non-empty and, by construction, $I_1 \times W$ is included in the edge set $E(K^D)$ of K^D.

Consider now the set J_1 of elements y such that

$$\{(x,y), x \in V \setminus I_1\} \subseteq D,$$

i.e. all edges going to y, except those starting from an element of I_1, have been turned around.

We will prove by contradiction that J_1 is non empty. Assume $J_1 = \emptyset$. Then the graph induced by K^D on the set $[n] \setminus I_1$ is a directed graph, where all vertices have at least one incoming edge (vertices in W have at least one incoming edge in this induced graph because we have assumed J_1 empty and vertices in $V \setminus I_1$ have an incoming edge because they do not belong to I_1). This graph should contain a directed cycle and we reach a contradiction.

Thus J_1 is non-empty and, by construction, $J_1 \times (V \setminus I_1)$ is included in $E(K^D)$.

Consider now the subset I_2 of $V \setminus I_1$ of elements x such that

$$\{(x,y), y \in (W \setminus J_1)\} \cap D = \emptyset.$$

The same proof as above (considering the graph induced on $[n] \setminus (I_1 \cup J_1)$) shows that, if $I_1 \subsetneq V$, then I_2 is non-empty. By construction, $I_2 \times (W \setminus J_1)$ is included in $E(K^D)$.

We keep going like this, defining, for each $m \geq 1$,

$$I_m = \left\{ x \in V_{m-1} \text{ s.t. } \{(x,y), y \in W_{m-1} \cap D\} = \emptyset \right\};$$
$$J_m = \left\{ y \in W_{m-1} \text{ s.t. } \{(x,y), x \in V_m\} \subseteq D \right\},$$

where we set $V_m = V \setminus (I_1 \cup \cdots \cup I_m)$ and $W_m = W \setminus (J_1 \cup \cdots \cup J_m)$. We stop the construction when $I_1 \sqcup \cdots \sqcup I_r = V$, which automatically implies $J_1 \sqcup \cdots \sqcup J_r = W$. Then the argument above shows that all sets I_m and J_m, except possibly J_r, are non-empty (which explains that the construction above always ends) and, by construction, if $1 \leq m \leq r$,

$$I_m \times W_{m-1} \subseteq E(K^D);$$
$$J_m \times V_m \subseteq E(K^D).$$

In other terms, the edge set of K^D contains the one of $H_{(\mathbf{I},\mathbf{J})}$. But for all (v,w) in $V \times W$, either (v,w) or (w,v) is an edge of $H_{(\mathbf{I},\mathbf{J})}$, so that K^D cannot have more edges. Thus $K^D = H_{(\mathbf{I},\mathbf{J})}$, as wanted.

The fact that each set composition with $\bigsqcup_{1 \leq k \leq r} I_k = V$ and $\bigsqcup_{1 \leq k \leq r} J_k = W$ corresponds to exactly one set D is trivial: just take D as the set of edges which are oriented from W to V in the graph $H_{(\mathbf{I},\mathbf{J})}$. □

5.4. **Another $\mathbb{Z}$-basis of $\mathbf{WQSym}$.** The purpose of this section is to prove that the word quasi-symmetric functions $\Gamma^{\text{nc}}(B_{(\mathbf{I},\mathbf{J})})$ form a $\mathbb{Z}$-basis of $\mathbf{WQSym}$, when $(\mathbf{I},\mathbf{J})$ runs over all set compositions.

As in Section 4.2, we use an intermediate family. If $(\mathbf{I},\mathbf{J})$ is a set composition of $[n]$, define

$$\mathbf{N}_{(\mathbf{I},\mathbf{J})} = \sum a_{k_1}\cdots a_{k_n}, \tag{9}$$

where the sum runs over lists $(k_1,\dots,k_n)$ that satisfy:

- if x is in I_m and y in J_m for some index $m \le r$, then $k_x \le k_y$;
- if x is in J_m and y in I_{m+1} for some index $m \le r-1$, then $k_x < k_y$.

For example, continuing Example 5.3, one has:

$$\mathbf{N}_{(\mathbf{I}_{\text{ex}},\mathbf{J}_{\text{ex}})} = \sum_{\substack{k_1,\dots,k_7 \\ \max(k_2,k_6)\le k_4<k_5\le\min(k_1,k_7) \\ \max(k_1,k_7)<k_3}} a_{k_1}\cdots a_{k_7}.$$

In general, denote $m(x)$ the index m such that x lies in I_m or J_m. Then the inequalities above on the indices k_x automatically imply that $k_x < k_y$ whenever $m(x) < m(y)$.

Remark 5.5. The family $(\mathbf{N}_{(\mathbf{I},\mathbf{J})})$ has been recently considered by the author and coauthors in [2] (our family $\mathbf{N}_{(\mathbf{I},\mathbf{J})}$ corresponds to $\mathbf{F}(\mathbf{P_K})$ with the notations of [2]). The commutative projection of $\mathbf{N}_{(\mathbf{I},\mathbf{J})}$ had appeared before: indeed, it coincides with a $\mathbb{Z}$-basis of $\mathbf{WQSym}$ introduced by K. Luoto in [13] (denoted N in Luoto's paper).

Proposition 5.6. *The family $(\mathbf{N}_{(\mathbf{I},\mathbf{J})})$, where $(\mathbf{I},\mathbf{J})$ runs over all set compositions, is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$.*

Proof. See [2, Proposition 5.4]. □

Remark 5.7. A surprising fact in this proof is that we have not been able to find some other $\mathbb{Z}$-basis of $\mathbf{WQSym}$ with a unitriangular change-of-basis matrix. The proof uses an evaluation on a virtual alphabet which turns $(\mathbf{N}_{(\mathbf{I},\mathbf{J})})$ into a two-alphabet version, whose linear independence is easy to observe.

Such a trick is not needed in the commutative setting – see [13, proof of Theorem 3.4]. Finding a more elementary proof in the noncommutative setting would certainly be interesting.

A nice feature of this basis is that, for any bipartite graph B, the associated word quasi-symmetric function $\Gamma^{\text{nc}}(B)$ can be written as a multiplicity-free sum of $\mathbf{N}$ function. A weaker version of the following proposition was announced in [2] (see Proposition 5.5 there).

Proposition 5.8. *Let B be a bipartite graph with vertex set $[n]$ and edge set E_B and consider* the *bipartition $[n] = V \sqcup W$ of its vertex set so that $E_B \subseteq V \times W$ and W contains no isolated vertex. Then*

$$\Gamma^{nc}(B) = \sum \mathbf{N}_{(\mathbf{I},\mathbf{J})},$$

where the sum runs over set compositions $(\mathbf{I},\mathbf{J})$ such that:

- $\bigsqcup_{1\le k\le r} I_k = V$ *and* $\bigsqcup_{1\le k\le r} J_k = W$;
- $(x,y)\in E_B \implies m(x)\le m(y)$.

Proof. We denote $\overline{E_B}$ the set of *non-edges* of B, that is $(V \times W) \setminus E_B$. Consider a B non-decreasing function $f : [n] \to \mathbb{N}$. For each non-edge $(x,y) \in \overline{E_B}$, one has either $f(x) \leq f(y)$ or $f(y) < f(x)$. This trivial remark allows us to decompose

$$\{f : [n] \to \mathbb{N},\ f \text{ is } B \text{ non-decreasing}\} = \bigsqcup_{D \subseteq \overline{E_B}} \mathscr{F}_D,$$

where $\mathscr{F}_D$ is the set of B non-decreasing functions that satisfy:

- $f(y) < f(x)$ for each (x,y) in D;
- $f(x) \leq f(y)$ for each (x,y) in $\overline{E_B} \setminus D$.

This decomposition yields the formula

$$\Gamma^{\text{nc}}(B) = \sum_{D \subseteq \overline{E_B}} \mathbf{N}_D, \tag{10}$$

where $\mathbf{N}_D = \sum_{f \in \mathscr{F}_D} a_{f(1)} \cdots a_{f(n)}$. We will prove that, for each set D, the word quasi-symmetric function $\mathbf{N}_D$ is either 0 or equal to one of the basis element $\mathbf{N}_{(\mathbf{I},\mathbf{J})}$.

Fix a subset D of $\overline{E_B}$. Note that $\overline{E_B}$, and thus D, seen as a subset of $V \times W$, satisfies the hypothesis of Lemma 5.4 as we assumed that W contains no isolated vertex. Applying Lemma 5.4, we are left with two cases.

- Either the graph K^D contains a directed cycle

  $$(x_1, y_1, x_2, y_2, \cdots, x_k, y_k),$$

  where x_ℓ, respectively y_ℓ, lies in V, respectively W (for $1 \leq \ell \leq k$). Then any function f in $\mathscr{F}_D$ satisfies

  $$f(x_1) \leq f(y_1) < f(x_2) \leq \cdots < f(x_k) \leq f(y_k) < f(x_1),$$

  which is clearly impossible. Thus $\mathscr{F}_D$ is empty and $\mathbf{N}_D = 0$.
- Or the graph K^D is identical to some $H_{(\mathbf{I},\mathbf{J})}$ for some set composition $(\mathbf{I},\mathbf{J})$. In this case, functions f in $\mathscr{F}_D$ fulfills by definition

$$\begin{cases} f(x) \leq f(y) & \text{if } (x,y) \in (V \times W) \setminus D \text{, that is if } x \in I_m \text{ and } y \in J_{m'} \text{ with } m \leq m'; \\ f(y) < f(x) & \text{if } (x,y) \in D \text{, that is if } y \in J_m \text{ and } y \in I_{m'} \text{ with } m < m'; \end{cases}$$

  These functions correspond to the lists $(k_1, \cdots, k_n)$ in the summation index in the definition of $\mathbf{N}_{(\mathbf{I},\mathbf{J})}$ in Eq. (9). Therefore $\mathbf{N}_D = \mathbf{N}_{(\mathbf{I},\mathbf{J})}$.

It remains to prove that each set composition $(\mathbf{I},\mathbf{J})$ with the conditions given in the Proposition appears exactly once. This is a consequence of the second part of Lemma 5.4: there is a one-to-one correspondence between subset $D \subseteq V \times W$ such that K^D is acyclic and set compositions $(\mathbf{I},\mathbf{J})$ with $\bigsqcup_{1 \leq k \leq r} I_k = V$ and $\bigsqcup_{1 \leq k \leq r} J_k = W$. In this correspondence, the fact that $D \subseteq \overline{E_B}$ translates as

$$(x,y) \in E_B \implies m(x) \leq m(y),$$

which concludes the proof of the proposition. □

Example 5.9. Consider the graph $B = B_{(\mathbf{I}_{\text{ex}},\mathbf{J}_{\text{ex}})}$ from Example 5.3. In this case $\overline{E_B} = \{(5,4),\ (3,4),\ (3,1),\ (3,7)\}$. It has 16 subsets D. Among these 16 sets D, exactly 3 of them lead to a graph K^D with a directed cycle: the one where D contains $(5,4)$ but not $(3,4)$ and either $(3,1)$ or $(3,7)$ or both. The other 13 sets D yield each a basis element $\mathbf{N}_{(\mathbf{I},\mathbf{J})}$ in the expansion of $\Gamma^{\text{nc}}(B_{(\mathbf{I}_{\text{ex}},\mathbf{J}_{\text{ex}})})$, which is:

$$\begin{aligned}\Gamma^{\text{nc}}(B_{(\mathbf{I}_{\text{ex}},\mathbf{J}_{\text{ex}})}) = \mathbf{N}_{(26|5|3,4|17|)} + \mathbf{N}_{(26|5|3,4|1|7)} + \mathbf{N}_{(26|5|3,4|7|1)} + \mathbf{N}_{(26|35,4|17)} \\ + \mathbf{N}_{(236|5,4|17)} + \mathbf{N}_{(256|3,147|)} + \mathbf{N}_{(256|3,14|7)} + \mathbf{N}_{(256|3,17|4)} \\ + \mathbf{N}_{(256|3,47|1)} + \mathbf{N}_{(256|3,1|47)} + \mathbf{N}_{(256|3,4|17)} + \mathbf{N}_{(256|3,7|14)} + \mathbf{N}_{(2356,147)}\end{aligned}$$

One can check that these 13 set compositions are exactly the ones that fulfill the condition from Proposition 5.8.

Corollary 5.10. *The family* $(\Gamma^{nc}(B_{(\mathbf{I},\mathbf{J})}))$*, when* $(\mathbf{I},\mathbf{J})$ *runs over all set compositions, is a* $\mathbb{Z}$*-basis of* $\mathbf{WQSym}$*.*

Proof. We endow set compositions $(\mathbf{I},\mathbf{J})$ with the lexicographic containment order on $(I_1, \overline{J_1}, I_2, \overline{J_2}, \dots)$ ($\overline{J_m}$ denotes here the complement of J_m in W) that is

$$(\mathbf{I},\mathbf{J}) \preceq (\mathbf{I}',\mathbf{J}') \text{ if and only if } \begin{cases} \quad I_1 \subsetneq I'_1 \\ \text{or } (I_1 = I'_1 \text{ and } J_1 \supsetneq J'_1) \\ \text{or } (I_1 = I'_1 \text{ and } J_1 = J'_1 \text{ and } I_2 \subsetneq I'_2) \\ \text{or } \dots \end{cases}$$

We use in this proof the following notations: for an element $x \in V$, we denote $m(x)$ (respectively $m'(x)$) the index m (resp m') such that $x \in I_m$ (respectively $x \in I'_{m'}$). The same notation will be used for $y \in W$, except that $\mathbf{I}$ and $\mathbf{I}'$ should be replaced by $\mathbf{J}$ and $\mathbf{J}'$. Besides, as in the proof of Lemma 5.4, we denote

$$\begin{aligned}V_m &= V \setminus (I_1 \cup \dots \cup I_m);\\ W_m &= W \setminus (J_1 \cup \dots \cup J_m).\end{aligned}$$

Analogous notations will be used for $\mathbf{I}'$ and $\mathbf{J}'$. We will prove that if $\mathbf{N}_{(\mathbf{I}',\mathbf{J}')}$ appears in the expansion (10) of $\Gamma^{\text{nc}}(B_{(\mathbf{I},\mathbf{J})})$, then $(\mathbf{I},\mathbf{J}) \preceq (\mathbf{I}',\mathbf{J}')$.

Assume that $I_m = I'_m$ and $J_m = J'_m$ for all m smaller than an integer $m_0 \geq 1$. We shall prove that $I_{m_0} \subseteq I'_{m_0}$. Assume $I_{m_0} \neq \emptyset$.

- Either J'_{m_0} is empty, which forces $I'_{m_0} = V'_{m_0-1}$ (in particular, $(\mathbf{I}',\mathbf{J}')$ has the semi-length m_0). But as $I_m = I'_m$ for $m < m_0$, we have $V_{m_0-1} = V'_{m_0-1}$, so $I_{m_0} \subseteq I'_{m_0}$.
- Or J'_{m_0} contains an element y_0. As $J_m = J'_m$ for $m < m_0$, one has $W_{m_0-1} = W'_{m_0-1}$. Therefore y_0 belongs to W_{m_0-1} and for any $x \in I_{m_0}$ the pair (x, y_0) is an edge of $B_{(\mathbf{I},\mathbf{J})}$, thus, from Proposition 5.8, one has $m'(x) \leq m'(y_0) = m_0$. But elements x in I_{m_0} cannot belong to any of the $I'_m = I_m$ with $m < m_0$, therefore we have $x \in I'_{m_0}$. We have proved that $I_{m_0} \subseteq I'_{m_0}$, which is what we wanted.

Fix a positive integer m_0 as before and assume that $I_m = I'_m$ and $J_m = J'_m$ for $m < m_0$ and $I_{m_0} = I'_{m_0}$. We shall prove that $J_{m_0} \supseteq J'_{m_0}$. Again, we consider two cases.

- Either I_{m_0+1} is not defined (because $(\mathbf{I}, \mathbf{J})$ has semi-length m_0), which means that $J_{m_0} = W_{m_0-1}$. But, the hypothesis $J_m = J'_m$ for $m < m_0$ implies $W_{m_0-1} = W'_{m_0-1}$. Moreover, by definition, $J'_{m_0} \subseteq W'_{m_0-1}$ so that $J_{m_0} \supseteq J'_{m_0}$.
- Or I_{m_0+1} contains an element x_0. For each y in W_{m_0}, the pair (x_0, y) is an edge of $B_{(\mathbf{I},\mathbf{J})}$ and thus, from Proposition 5.8, one has $m'(x_0) \leq m'(y)$. But $m'(x_0) = m_0+1$. This implies $m'(y) \geq m_0+1$, that is $y \in W'_{m_0}$. We have proved that $W_{m_0} \subseteq W'_{m_0}$, which, together with $W_{m_0-1} = W'_{m_0-1}$, implies that $J_{m_0} \supseteq J'_{m_0}$, as wanted.

Finally, we have proved that, if $\mathbf{N}_{(\mathbf{I}',\mathbf{J}')}$ appears in the expansion (10) of the function $\Gamma^{\text{nc}}(B_{(\mathbf{I},\mathbf{J})})$, then $(\mathbf{I}, \mathbf{J}) \preceq (\mathbf{I}', \mathbf{J}')$. Note that, again from Proposition 5.8, the basis element $\mathbf{N}_{(\mathbf{I},\mathbf{J})}$ appears in this expansion with coefficient 1. In other terms the matrix of the family $(\Gamma^{\text{nc}}(B_{(\mathbf{I},\mathbf{J})}))$ in the $\mathbb{Z}$-basis $\mathbf{N}_{(\mathbf{I},\mathbf{J})}$ is unitriangular with respect to the order $\preceq$, which proves that $(\Gamma^{\text{nc}}(B_{(\mathbf{I},\mathbf{J})}))$ is also a $\mathbb{Z}$-basis of $\mathbf{WQSym}$. □

5.5. **A generating family of the quotient.** We will now show that $(B_{(\mathbf{I},\mathbf{J})})$, where $(\mathbf{I}, \mathbf{J})$ runs over all set compositions, is a generating family in the quotient $\mathscr{G}_b/\mathscr{C}_b$. As explained in Section 5.6, together with the results of Section 5.4 and Remark 3.4, this implies that the morphism $\Gamma^{\text{nc}}_b : \mathscr{G}_b/\mathscr{C}_b \to \mathbf{WQSym}$ is an isomorphism.

As in the non-restricted setting, the result follows from a combinatorial lemma (which is surprisingly simpler than in the non-restricted setting).

Lemma 5.11. *Let B be a bipartite graph on vertex set $[n]$. Then*

- *either $B = B_{(\mathbf{I},\mathbf{J})}$ for some set composition $(\mathbf{I}, \mathbf{J})$;*
- *or B can be written as linear combination of graphs with the same vertex set and more edges in $\mathscr{G}_b/\mathscr{C}_b$.*

Proof. Let $[n] = V \sqcup W$ the bipartition of the vertices of B. For $v \in V$, we denote $\mathscr{N}(v)$ the subset of W of vertices linked to v.

First case: let us suppose that for all v and v' in V, we have either $\mathscr{N}(v) \subseteq \mathscr{N}(v')$ or $\mathscr{N}(v') \subseteq \mathscr{N}(v)$. Then one can label the vertices in V by $\{v_1, \dots, v_s\}$ such that

$$\mathscr{N}(v_1) \supseteq \mathscr{N}(v_2) \cdots \supseteq \mathscr{N}(v_s).$$

We group together vertices v_i which have the same neighbourhood $\mathscr{N}(v_i)$. This gives a set composition $(I_1, \dots, I_r)$ of V such that

$$\mathscr{N}(I_1) \supsetneq \mathscr{N}(I_2) \cdots \supsetneq \mathscr{N}(I_r),$$

where $\mathscr{N}(I_m)$ denotes the common value of $\mathscr{N}(v)$ for $v \in I_m$. Then we define $J_k = \mathscr{N}(I_k) \setminus \mathscr{N}(I_{k+1})$ for $k < r$ (these sets are nonempty by definition) and

$J_r = \mathscr{N}(I_r)$ so that, for all $m \leq r$,

$$\mathscr{N}(I_m) = \bigsqcup_{k \geq m} J_k.$$

This equation precisely says that B is the graph $B_{(\mathbf{I},\mathbf{J})}$.

We consider now the second case: there exist v, v' in V and w, w' in W such that (v, w) and (v', w') belong to edge-set E_B but neither (v, w') nor (v', w). Let B_0 be the graph obtained from B by adding edges from v to w' and from v' to w (note that it is still bipartite as a directed graph, and hence is acyclic). The undirected version of this graph contains a cycle $C : v \to w' \to v' \to w \to v$, whose corresponding set C^+ is $\{(v, w'),\ (v', w)\}$ (with the notations of Section 3.1). Then $B = B_0 \setminus C^+$ is the smallest graph appearing in $\mathrm{CIE}_{B_0,C}$ and thus, in the quotient, $\mathscr{G}_b/\mathscr{C}_B$, the graph B can be written as a linear combination of bigger graphs (*i.e.* with the same set of vertices and more edges). □

Let B be a bipartite directed graph with vertex set $[n]$. Iterating Lemma 5.11, one can write B as an integral linear combination of $B_{(\mathbf{I},\mathbf{J})}$ in the quotient space $\mathscr{G}_b/\mathscr{C}_b$. So $(B_{(\mathbf{I},\mathbf{J})})$, where $(\mathbf{I}, \mathbf{J})$ runs over all set compositions is a generating family for $\mathscr{G}_b/\mathscr{C}_b$.

5.6. **Third main result.** We are now ready to prove the following statement.

Theorem 3. *The space $\mathscr{C}_b$, spanned by cyclic-inclusion elements, is the kernel of the surjective morphism Γ_b^{nc} from $\mathscr{G}_b$ to $\mathbf{WQSym}$.*

Proof. The proof is completely similar to that of Theorem 1.

Denote $\mathscr{K}_b$ the kernel of Γ_b^{nc}. By Proposition 3.2, it contains $\mathscr{C}_b$. On the one hand (Section 5.5), we know that $\mathscr{G}_b/\mathscr{C}_b$ is spanned by the family $(B_{(\mathbf{I},\mathbf{J})})$. On the other hand (Corollary 5.10), the family $\Gamma^{\mathrm{nc}}\left(B_{(\mathbf{I},\mathbf{J})}\right)$ is a basis of $\mathbf{WQSym}$, which implies in particular that the $(B_{(\mathbf{I},\mathbf{J})})$ are linearly independent in $\mathscr{G}_b/\mathscr{K}_b$ and hence in $\mathscr{G}_b/\mathscr{C}_b$.

Therefore $(B_{(\mathbf{I},\mathbf{J})})$ is a basis of $\mathscr{G}_b/\mathscr{C}_b$ and Γ_b^{nc} is an isomorphism from $\mathscr{G}_b/\mathscr{C}_b$ to $\mathbf{WQSym}$ (it sends a basis on a basis), which concludes the proof. □

5.7. **Unlabeled commutative framework and fourth main result.** We will use the following obvious notations for the commutative bipartite framework: $\overline{\mathscr{G}_b}$ is the subspace of $\overline{\mathscr{G}}$ spanned by unlabeled bipartite graph and Γ_b is the restriction of Γ to $\overline{\mathscr{G}_b}$.

Moreover, we denote $\overline{\mathscr{C}_b}$ the space spanned by $\mathrm{CIE}_{\overline{G},\overline{C}}$, where $\overline{G}$ runs over unlabeled bipartite directed graphs and $\overline{C}$ over cycles in the undirected version of $\overline{G}$. Equivalently, $\overline{\mathscr{C}_b}$ is the image of $\mathscr{C}_b$ by φ_u.

Theorem 4. *The ideal $\overline{\mathscr{C}_b}$, spanned by inclusion-exclusion elements, is the kernel of the surjective morphism Γ_b from $\overline{\mathscr{G}_b}$ to $QSym$.*

Proof. The proof is identical to that of Theorem 2, using Theorem 3 instead of Theorem 1. □

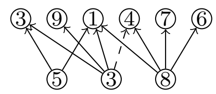


FIGURE 5. Example of non-expander (without the dashed edge) and expander (with the dashed edge) graphs.

6. APPLICATION OF THE MAIN RESULT TO KEROV CHARACTER POLYNOMIALS

In this section, we present our application of Theorem Theorem 4 to the theory of Kerov character polynomials. We do not obtain new results, but are able to significantly simplify some existing proofs.

6.1. **A family of invariant functionals.** We start by defining combinatorially a family of linear functions $I_\nu : \mathscr{G}_b \to \mathbb{C}$ indexed by integer partitions[10], whose kernels contain inclusion-exclusion elements.

Definition 6.1.

- A *decorated bipartite graph* is a pair (B, h) where B is a graph with vertex set bipartition $V \sqcup W$ and h a function $V \to \{1, 2, \cdots\}$ such that
$$\sum_{v \in V} h(v) = |W|.$$
- A connected decorated bipartite graph is said to be *expander* if, for any non-empty proper subset U of V (that is $U \neq \emptyset, V$),
$$|\mathscr{N}(U)| > \sum_{u \in U} h(u),$$
where $\mathscr{N}(U)$ is the neighbourhood of U, *i.e.* the set of vertices of W having at least one neighbour in U.
- A decorated bipartite graph is said to be *expander* if all its connected components are (in particular, if $V \sqcup W$ is the vertex set of a connected component, then $\sum_{v \in V} h(v) = |W|$).
- The *type* of a decorated bipartite graph is the integer partition obtained by sorting the multiset $h(V)$ in non-increasing order.

Example 6.2. Consider the bipartite graph B of Fig. 5 (without the dashed edge) and let h be given by $h(5) = 1$, $h(3) = 2$ and $h(8) = 3$. Then (B, h) is a decorated bipartite graph of type $(3, 2, 1)$. It is *not* expander as the neighbourhood of $\{3, 5\}$ has size 3 while $h(3) + h(5) = 3$ (notice the strong inequality in the definition of expander). If we add the dashed edge, we get an expander graph.

Remark 6.3. There are many variants of the definition of *expander graphs* in the literature. The one given here is a generalization of having *left-vertex expansion ratio* at least h (for a given integer h), see [16, Definition 12.7]. Expander graphs have found a lot of applications in analysis of communication networks, in the

[10] As usual, an *integer partition* is a non-increasing list of positive integers.

theory of error correcting codes and in the theory of pseudorandomness: we refer to [16] for a survey article. However, the way they appear here seems very different to what is usually done in the literature.

Expander graphs are known to encode some kind of strong connectivity of the graphs. In particular, trees (here, a tree is connected graph whose undirected version does not contain cycles) are not expanders (except for trivial cases), which is stated in the following lemma.

Lemma 6.4. *Let B be tree with vertex set bipartition $V \sqcup W$ and $h : V \to \{1, 2, \cdots\}$. Then (B, h) is expander if and only if every connected component of B contains exactly one vertex in V and h associates to each vertex in V its number of neighbours.*

Proof. It is enough to prove that (B, h) can not be expander unless B has one vertex of V per connected component or, equivalently, unless all vertices in W have degree 1. The remaining part of the lemma then follows easily.

Let us do a proof by contradiction and assume there is a vertex w of W of degree at least 2. Without loss of generality, we may assume that B is connected. As B is a tree, if we remove w, the graph obtained from B has several connected components: denote $V_1, \dots, V_r$ the intersections of V with these connected components ($r \geq 2$).

The union of the neighbourhoods $\mathscr{N}(V_1), \dots, \mathscr{N}(V_r)$ is clearly W, while two sets in this list have only w in common, so that

$$\sum_{i=1}^{r} |\mathscr{N}(V_i)| = |W| + (r-1).$$

But, by hypothesis,

$$\sum_{i=1}^{r} \left(\sum_{v \in V_i} h(v) \right) = \sum_{v \in V} h(v) = |W|$$

which is incompatible with the strict inequalities ($V_1, \dots, V_r$ are non-empty by definition and proper subsets of V because $r \geq 2$):

$$\text{for every } i \text{ in } \{1, \cdots, r\},\ |\mathscr{N}(V_i)| > \sum_{v \in V_i} h(v). \qquad \square$$

We can now define the functions I_ν.

Definition 6.5. Let ν be an integer partition and B a bipartite graphs with c connected components. Then $(-1)^c I_\nu(B)$ is, by definition, the number of functions $h : V \to \{1, 2, \cdots\}$ such that (B, h) is an expander decorated bipartite graph of type ν.

The function I_ν is then extended by linearity to the bipartite graph algebra $\mathscr{G}_b$.

Proposition 6.6. *For any bipartite graph B and cycle C of B, one has:*

$$I_\nu(\mathrm{CIE}_{B,C}) = 0.$$

Proof. See [6, Lemma 8.3]. □

Remark 6.7. While all elements in the statement of Proposition 6.6 are combinatorial, the proof given in [6] involves computations of Euler characteristic. An *elementary* proof would certainly be interesting.

6.2. **Background on Kerov character polynomials.** We only present here what is strictly necessary to explain our application of Theorem 4. As this is not central in the paper, we assume some familiarity of the reader with representation theory of symmetric groups. Details and motivations can be found in [6] and references therein.

Let μ be fixed integer partition. Consider the function

$$\mathrm{Ch}_\mu(\lambda) = \begin{cases} |\lambda|(|\lambda|-1)\cdots(|\lambda|-|\mu|+1)\frac{\chi^\lambda_{\mu\,1^{|\lambda|-|\mu|}}}{\dim(\lambda)} & \text{if } |\lambda| \geq |\mu|; \\ 0 & \text{if } |\lambda| < |\mu|. \end{cases}$$

Here λ is a Young diagram, $\dim(\lambda)$ the dimension of the associated irreducible representation of the symmetric group and $\chi^\lambda_{\mu\,1^{|\lambda|-|\mu|}}$ the associated character evaluated on a permutation of cycle-type $\mu \cup (1^{|\lambda|-|\mu|})$.

Consider a diagram given by its modified multirectangular coordinates $(p_1, \cdots, p_m)$ and $(q_1, \cdots, q_m)$, that is

$$\lambda(\mathbf{p}, \mathbf{q}) := \underbrace{\sum_{i\geq 1} q_i, \dots, \sum_{i\geq 1} q_i}_{p_1 \text{ times}}, \underbrace{\sum_{i\geq 2} q_i, \dots, \sum_{i\geq 2} q_i}_{p_2 \text{ times}}, \dots$$

It has been shown (see *e.g.* [8, Theorem 1.5.1]) that

$$\mathrm{Ch}_\mu(\lambda(\mathbf{p}, \mathbf{q})) = \sum_{\substack{\sigma,\tau \in S_k \\ \sigma\,\tau = \pi}} (-1)^{\kappa(\tau)+r} \Delta(B(\sigma,\tau))(\mathbf{p}, \mathbf{q}), \tag{11}$$

where:

- k and r are respectively the size and the length of μ and S_k the symmetric group of size k;
- π is a fixed (arbitrary) permutation of cycle-type μ;
- $\kappa(\tau)$ is the number of cycles of τ;
- $B(\sigma,\tau)$ is a bipartite graph associated to the pair of permutations σ and τ (its precise definition is not important here);
- $\Delta(B)$ is a two-alphabet version of $\Gamma(B)$, namely:

$$\Delta(B)(\mathbf{p}, \mathbf{q}) = \sum_{\substack{f: V \sqcup W \to \mathbb{N} \\ f \ B \text{ non-decreasing}}} \left(\prod_{v \in V} p_{f(v)} \cdot \prod_{w \in W} q_{f(w)} \right),$$

  where $V \sqcup W$ is the proper bipartition of vertices of B without isolated vertices in W.

Another family of functions of interest is the family of *free cumulants*, which can be defined as follows:

$$R_{k+1}(\lambda(\mathbf{p},\mathbf{q})) = \sum_{\substack{\sigma,\tau\in S_k\\ \sigma\,\tau=(1\ 2\ \dots\ k)\\ \kappa(\sigma)+\kappa(\tau)=k+1}} (-1)^{\kappa(\tau)+1}\Delta(B(\sigma,\tau))(\mathbf{p},\mathbf{q}). \tag{12}$$

The restriction $\kappa(\sigma)+\kappa(\tau)=k+1$ imposed in the summation index is in fact equivalent (under the assumption $\sigma\,\tau=(1\ 2\ \dots\ k)$) to the fact that $B(\sigma,\tau)$ has no cycles.

In 2001, S. Kerov proved that, for each partition μ, there exists a polynomial K_μ, now called Kerov polynomial, such that, for every Young diagram λ, one has

$$\mathrm{Ch}_\mu(\lambda) = K_\mu(R_2(\lambda), R_3(\lambda), \dots, R_{|\mu|+1}(\lambda)). \tag{13}$$

He then conjectured – see [4] – that $K_{(k)}$ has non-negative coefficients for any positive integer k. This result was proved by the author in [8] and an explicit combinatorial interpretation of the coefficients was given in [6]. We explain in next Section how Theorem 4 and the invariants I_ν may be used to simplify the arguments in these papers.

6.3. **Application of our main result.** Similarly to Γ, the two-alphabet version Δ can be extended by linearity to the bipartite graph algebra $\mathscr{G}_b$. Consider elements G_{Ch_μ} and G_{R_k} in the graph algebra such that

$$\mathrm{Ch}_\mu(\lambda(\mathbf{p},\mathbf{q})) = \Delta(G_{\mathrm{Ch}_\mu})(\mathbf{p},\mathbf{q}), \quad R_k(\lambda(\mathbf{p},\mathbf{q})) = \Delta(G_{R_k})(\mathbf{p},\mathbf{q}),$$

that is

$$G_{\mathrm{Ch}_\mu} = \sum_{\substack{\sigma,\tau\in S_k\\ \sigma\,\tau=\pi}} (-1)^{\kappa(\tau)+r} B(\sigma,\tau);$$

$$G_{R_{k+1}} = \sum_{\substack{\sigma,\tau\in S_k\\ \sigma\,\tau=(1\ 2\ \dots\ k)\\ \kappa(\sigma)+\kappa(\tau)=k+1}} (-1)^{\kappa(\tau)+1} B(\sigma,\tau).$$

Then observe that the Eq. (13) for any Young diagram λ implies that

$$\mathrm{Ch}_\mu(\lambda(\mathbf{p},\mathbf{q})) = K_\mu(R_2(\lambda(\mathbf{p},\mathbf{q})), R_3(\lambda(\mathbf{p},\mathbf{q})), \dots, R_{|\mu|+1}(\lambda(\mathbf{p},\mathbf{q})))$$

as polynomials in infinitely many variables $p_1, q_1, p_2, q_2, \cdots$, so that

$$\Delta(G_{\mathrm{Ch}_\mu}) = K_\mu\big(\Delta(G_{R_2}), \dots, \Delta(G_{R_k})\big) = \Delta\big(K_\mu(G_{R_2}, \dots, G_{R_k})\big).$$

Recall indeed that the product in the graph algebra is given by disjoint union of graphs and that Δ is clearly an algebra morphism with respect to this product.

But sending $p_i, q_i \to x_i$ sends $\Delta(B)$ to $\Gamma(B)$, thus the difference

$$A := G_{\mathrm{Ch}_\mu} - K_\mu(G_{R_2}, \dots, G_{R_k})$$

lies in $\mathscr{K}(\Gamma)$. By Theorem 4, it lies in $\mathscr{C}_b$ and thus Proposition 6.6 implies that $I_\nu(A) = 0$ for any partition ν.

But, one can easily seen from Lemma 6.4 (recall that graphs appearing in G_{R_k} have no cycles) that

$$I_\nu(G_{R_{i_1}} \cdots G_{R_{i_\ell}}) = \begin{cases} (-1)^\ell & \text{if } \nu \text{ is obtained by antisorting } (i_1 - 1, \dots, i_\ell - 1) \text{ in decreasing order;} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $I_\nu(K_\mu(G_{R_2}, \dots, G_{R_k}))$ is, up to a sign, the coefficient of the monomial

$$\prod_{i=1}^{\ell(\nu)} R_{\nu_i+1}$$

in K_μ. From the relation $I_\nu(A) = 0$, we get that it is also equal to $I_\nu(G_{\mathrm{Ch}_\mu})$. This last quantity is a signed enumeration of expander graphs, so that we obtain a signed combinatorial interpretation for coefficients of Kerov polynomials.

This signed combinatorial interpretation is equivalent to [6, Theorem 1.6]. In the case $\mu = (k)$, the signs disappear and the non-negativity of the coefficients of $K_{(k)}$ follows.

6.4. **Comparison with the proofs given in** [6]**.** In [6], two proofs of the result above were given. The first one is quite different from the one sketched above. The second one also used cyclic inclusion-exclusion and Proposition 6.6, but a huge part of the proof was dedicated to proving the fact that the quantity A belongs to $\mathscr{C}_b$ – see [8, Sections 3 and 4]. With Theorem 4, it follows immediately from the fact that $\Delta(A) = 0$.

Besides, the proof that $A \in \mathscr{C}_b$ given in [8, Sections 3 and 4], uses the structure of the symmetric group, while the argument that we use here works if we replace Ch_μ by any function that has an expression similar to Eq. (11) – for instance the zonal characters studied in [9]. Note that the first proof of paper [6] also extends readily to zonal characters, so the result that we obtain that way is not new.

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