# Patterns in Permutations and Involutions
## A Structural and Enumerative Approach

by

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To Carol and Fred Gropper, my grandparents

# Acknowledgements

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# Abstract

This dissertation presents a multifaceted look into the structural decomposition of permutation classes. The theory of permutation patterns is a rich and varied field, and is a prime example of how an accessible and intuitive definition leads to increasingly deep and significant line of research. The use of geometric structural reasoning, coupled with analytic and probabilistic techniques, provides a concrete framework from which to develop new enumerative techniques and forms the underlying foundation to this study.

This work is divided into five chapters. The first chapter introduces these techniques through working examples, both motivating the use of structural decomposition and showcasing the utility of their combination with analytic and probabilistic methods. The remaining chapters apply these concepts to separate aspects of permutation classes, deriving new enumerative, statistical, and structural results. These chapters are largely independent, but build from the same foundation to construct an overarching theme of building structure upon disorder.

The main results of this study are as follows. Chapter 2 investigates the average number of occurrences of patterns with permutation classes, and proves that the total number of 231-patterns is the same in the classes of 132- and 123-avoiding permutations. Chapter 3 applies structural decomposition to enumerate pattern avoiding involutions. Chapter 4 uses the theory of grid classes to develop an algorithm to enumerate the so-called polynomial permutation classes, and applies this to the biological problem of genetic evolutionary distance. Finally, we end in Chapter 5 with an exploration of pattern-packing, and determine the probability distribution for the number of distinct large patterns contained in a permutation.

# – Chapter 1 –

# Preliminaries

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Permutations are a fundamental mathematical concept used productively throughout the sciences to encode and understand disorder and rearrangement. The theory of permutation patterns captures this geometric notion of disorder, and has yielded a wide variety of productive and surprising research over the past several decades. This dissertation presents several interrelated projects within this interesting and rapidly developing field. Structural, analytic, and probabilistic combinatorics are central to this work, and combine to provide unique insight into pattern enumeration.

This dissertation is organized as follows: Chapter 1 provides an accessible introduction to the ideas and methods at play, followed by four illustrative examples which serve to motivate and introduce the material to come. The following four chapters represent self-contained projects utilizing these techniques. Each of these chapters is based partly on separate publications [26, 51, 52, 53], but together they speak to the utility of structural methods coupled with multivariate analysis. Recursive structural decomposition intersected with modern analytic and probabilistic techniques has proven exceptionally useful in investigating patterns within permutations, and each chapter focuses on a separate facet of this productive combination.

For an accessible introduction to the field of combinatorics, the reader is directed to Bóna [21]. Stanley [79, 80] provides a more advanced treatment to the subject as a whole, while Bóna [22] focuses on the combinatorics of permutations. Wilf [90] gives an excellent introduction to the theory of generating functions, while Petkovšek, Wilf, and Zeilberger [72] provide a survey of algorithmic methods. Finally, analytic methods in combinatorics are presented best by Flajolet and Sedgewick [43] and by Pemantle and Wilson [71], who

focus on single- and multi-variate methods, respectively.

# § 1.1   Permutation Classes

Permutations owe much of their rich structure to their variety of equivalent representations. In this section we establish some of the basic notation and definitions of permutations and permutation classes. Throughout this dissertation, let $\mathbb{N}$ denote the non-negative integers $\{0, 1, 2, 3, \dots\}$, $\mathbb{P}$ the positive integers $\{1, 2, 3, 4, \dots\}$, and, for a given integer $n \in \mathbb{P}$, let $[n]$ denote the integers $\{1, 2, \dots n\}$.

## Permutations and Patterns

Definition 1.1.1. For a given integer $n \in \mathbb{P}$, a permutation of length $n$ is a sequence $\pi = \pi_1 \pi_2 \dots \pi_n$ in which $\pi_i \in [n]$ and each integer of $[n]$ is used exactly once. There are $n!$ permutations of length $n$, the set of all of which is denoted $\mathfrak{S}_n$.

For example, the six permutations of length three are as follows:

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Permutations can be represented in many different ways, each leading to different generalizations. The above definition is known as the one-line representation in the literature, and this approach leads naturally to the theory of permutation patterns. We start by presenting formal definitions of patterns before providing a geometric motivation.

Definition 1.1.2. For a positive integer $n$ any two sequences of distinct numbers $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$ and $\beta = \beta_1 \beta_2 \dots \beta_n$, we say that $\alpha$ and $\beta$ are order isomorphic (denoted $\alpha \sim \beta$) if

$$\alpha_i < \alpha_j \quad \text{if and only if} \quad \beta_i < \beta_j.$$

For example, the sequences $\alpha = 9\,2\,4$ is order isomorphic to $\beta = 5\,1\,3$, because their entries share the same relative order: the first is the biggest, the second is smallest, and the third lies in between.
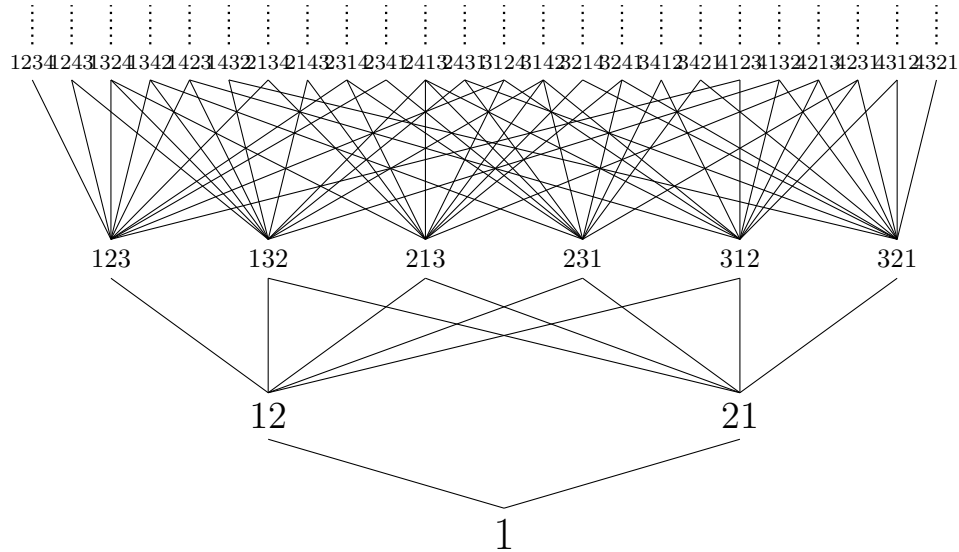
Figure 1.1.1: The first four levels of the permutation pattern poset. Two permutations are connected by a line if one is contained in the other as a pattern.

It follows that each sequence $\alpha$ of $n$ distinct numbers is order isomorphic to a unique permutation of length $n$, called the standardization of $\alpha$, and denoted $\text{st}(\alpha)$. For a given sequence $\alpha$, the standardization can be constructed by relabelling the smallest entry of of $\alpha$ by 1, the second smallest by 2, and so on (i.e., $\text{st}(9\ 2\ 4) = 3\ 1\ 2$). We can now present the formal definition of permutation patterns.

Definition 1.1.3. Let $n, k \in \mathbb{P}$ with $k \leq n$, and let $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$ and $\sigma = \sigma_1 \sigma_2 \ldots \sigma_k \in \mathfrak{S}_k$. Say that $\sigma$ is contained as a pattern in $\pi$ (denoted $\sigma \prec \pi$) if there is some subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that

$$\pi_{i_1} \pi_{i_2} \ldots \pi_{i_k} \sim \sigma_1 \sigma_2 \ldots \sigma_k.$$

Note that pattern containment is reflexive ($\pi \prec \pi$ for all permutations $\pi$), transitive ($\rho \prec \sigma$, $\sigma \prec \pi$ implies $\rho \prec \pi$), and anti-symmetric ($\sigma \prec \pi$ and $\pi \prec \sigma$ implies $\sigma = \pi$). These three properties mean that the set of all permutations, equipped with this ordering, forms a partially ordered set (a poset) known as the pattern poset.

The first four levels of this poset are shown in Figure 1.1.1. Note that the number of lines going up from each permutation depends only on the length of

the permutation, while the number going down varies. This will be a topic of study in Chapter 5, where we will establish the probability distribution for the number of large patterns contained within randomly selected permutations.

If a permutation $\pi$ does not contain a pattern $\sigma$, we say that $\pi$ avoids $\sigma$. The set of all permutations which avoid a fixed pattern $\sigma$ is denoted $\mathrm{Av}(\sigma)$. Transitivity of pattern containment implies that if $\pi \in \mathrm{Av}(\sigma)$ and $\rho \prec \pi$, then $\rho \in \mathrm{Av}(\sigma)$. This relationship motivates our next definition.

Definition 1.1.4. Let $\mathcal{P}$ be a poset. A subset $S \subseteq \mathcal{P}$ is called a downset if it is closed downwards. That is, if $x \in S$ and $y \prec x$, then $y \in S$. A downset of the permutation pattern poset is called a permutation class. For a permutation class $\mathcal{C}$, denote by $\mathcal{C}_n$ the set of permutations of length $n$ in $\mathcal{C}$.

The set of all patterns which avoid some specified set of patterns are known as the avoidance classes, and were first introduced by Knuth [61] in the context of stack sorting. The investigation of these and other classes has sparked a wide range of research over the past several decades, with a focus on enumeration. In particular, the question of 'which pattern is easiest to avoid?' has been a major open question for many years, and a variety of techniques have been developed to provide partial answers. The Marcus-Tardos Theorem [66] (which stood open as the Stanley-Wilf Conjecture for two decades) motivates much of this work.

Definition 1.1.5. Let $\mathcal{C}$ be a permutation class. The (upper) growth rate of $\mathcal{C}$ is defined as the limit
$$\limsup_{n \to \infty} \sqrt[n]{|\mathcal{C}_n|}.$$

Theorem 1.1.6 (Marcus, Tardos [66]). Every proper permutation class has a finite growth rate.

Wilf-Equivalence

Though Theorem 1.1.6 says that all proper permutation classes have a finite growth rate, finding and classifying these growth rates is difficult. Of particular interest is identifying those patterns which have the same enumeration, i.e., $\beta, \tau$ such that $\mathrm{Av}_n(\beta) = \mathrm{Av}_n(\tau)$ for all $n$. Such a pair $\beta, \tau$ are called Wilf-equivalent, and the set of all Wilf-equivalent permutations form a Wilf class. Though showing Wilf-equivalence can be hard in general, many equivalences arise from eight trivial symmetries.

Definition 1.1.7. Let $\pi = \pi_1 \pi_2 \ldots \pi_n$ a permutation. The reverse, the complement, and the inverse of $\pi$ (denoted $\pi^r$, $\pi^c$, and $\pi^{-1}$, respectively) are defined as follows:

$$
\begin{aligned}
(\pi^r)_i &= \pi_{n-i+1}, \\
(\pi^c)_i &= n - \pi_i + 1, \text{ and} \\
\left(\pi^{-1}\right)_{\pi_i} &= i.
\end{aligned}
$$

Each of these operations map the set of permutations to itself, and each preserves pattern containment. That is, if $\sigma \prec \pi$, then $\sigma^i \prec \pi^i$, for each $i \in \{r, c, -1\}$. It follows than that the class of permutations avoiding a pattern are in bijection with the class avoiding any symmetry of this pattern. These three symmetries thus generate an automorphism group of the pattern poset, which is isomorphic to the dihedral group of order eight. Of these three, only the inversion map has any fixed points; a permutation which is its own inverse is called an involution. It follows from Smith [77] that this is the complete set of automorphisms which respect pattern containment. Note that further order-respecting isomorphisms between classes are explored in Albert, Atkinson and Claesson [4]. Note further that Wilf-classes need not contain bases of the same size: Burstein and Pantone [30] recently showed the Wilf-equivalence of $\mathrm{Av}(1324, 3416725)$ and $\mathrm{Av}(2143, 3142, 246135)$.

For permutations of length three, 123 and 321 are complements (and reverses) of each other, and thus the classes $\mathrm{Av}(123)$ and $\mathrm{Av}(321)$ have the same enumeration (i.e., $|\mathrm{Av}_n(123)| = |\mathrm{Av}_n(321)|$ for all $n \in \mathbb{N}$). The permutation 132 can be reversed to obtain 231 or complemented to obtain 312, and 312 can be complemented to obtain 213. Therefore the permutations $\{132, 213, 231, 312\}$ are Wilf-equivalent, and so there are at most two Wilf classes for length 3 permutations.

MacMahon, in 1915/16 [65] enumerated the 123-avoiding permutations while Knuth, in 1968 [61], enumerated the 231-avoiding permutations, leading to the first non-trivial Wilf equivalence. A bijection between $123-$ and 132-avoiding permutations was presented by Simion and Schmidt [76] in 1985.

Theorem 1.1.8 (MacMahon, Knuth [61, 65]). The number of permutations of length $n$ avoiding 123 is equal to the number avoiding 231.

We explore this result further in Sections 1.3.1 and 1.3.2, and rederive this result using geometric constructions. Note that two Wilf-equivalent classes can have sharply contrasting structure, as we will soon see is the case for $\mathrm{Av}(123)$ and $\mathrm{Av}(132)$. Theorem 1.1.8 shows that there is only one Wilf class

Table 1.1.1: Enumerations of the three Wilf classes for patterns of length four.

| | | | | $\mid \mathrm{Av}_n(\beta)\mid$ | | | | |
|---|---|---|---|---|---|---|---|---|
| $n =$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $\beta = 1342$ | 1 | 2 | 6 | 23 | 103 | 512 | 2740 | 15485 |
| $\beta = 1234$ | 1 | 2 | 6 | 23 | 103 | 513 | 2761 | 15767 |
| $\beta = 1324$ | 1 | 2 | 6 | 23 | 103 | 513 | 2762 | 15793 |

for length three patterns, which gives false hope for longer patterns. As we see here, the situation becomes much more complicated as patterns get longer.

Of the twenty-four patterns of length four, the trivial symmetries show that there are at most eight Wilf classes. Non-trivial theorems from Babson and West [14] and West [89] (and generalized in Backelin, West, and Xin [15]) reduce this number to four, and a result of Stankova [78] shows that two of these remaining classes are Wilf-equivalent. This leaves the patterns of length four partitioned into three Wilf classes. That these three classes do in fact have different enumerations can be seen in the data presented in Table 1.1.1.

Note that the monotone pattern is neither the easiest nor hardest to avoid, as one might expect. These three cases speak to the complexity involved in enumerating permutation classes. The class Av(1342) was first counted by Bóna [18], and was found to have an algebraic generating function and an exponential growth rate of 8. The class Av(1234) was enumerated by Gessel [46] and Regev [74], who provided an exact formula the number of permutations of a given length in the class and showed that the exponential growth rate is 9, but showed that the generating function is D-finite but nonalgebraic. Finally, the class Av(1324) has not been enumerated and the growth rate is unknown, except that it is between 9.42 (Albert et. al. [8]) and 13.93 (Bóna [23]).

The permutation 1324 is a layered permutation, meaning it can be written as a sequence of decreasing runs, the entries of which are each larger than the previous layer. Layered permutations were conjectured by Arratia [12] to be the easiest to avoid, i.e., their avoidance classes have the fastest growth. This conjecture led to interest in these patterns [19, 34, 40], but was recently overturned by Fox [44], who showed that the situation is much more complex than small examples suggest. In Chapter 3 we consider the problem of finding growth rates of pattern avoiding involutions, and determine the growth rates of two such sets avoiding patterns of length four.
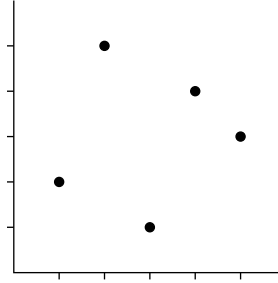
Figure 1.1.2: The plot of the permutation $\pi = 2\ 5\ 1\ 4\ 3$.

## Geometric Motivation

The investigation and classification of Wilf classes is a deep and complex research program. The primary focus of this dissertation, however, is on the geometric structure of permutation classes, and the use of this structure to understand and explore pattern containment. The concepts presented above can all be reconsidered in a geometric context which allows for a more intuitive description of permutations and their patterns and symmetries. This geometric approach helps to illuminate new directions of research, is central to this work.

Definition 1.1.9. The plot of the permutation $\pi$ of length $n$ is the set of points $(i, \pi_i) \in \mathbb{R}^2$ for each $i \in [n]$.

The plot of a permutation is shown in Figure 1.1.2. Say that a set of $n$ points in $\mathbb{R}^2$ is generic if no two points lie on the same horizontal or vertical line. Say that two generic sets $P$ and $T$ are order isomorphic (written $P \sim T$) if the axes can be stretched or shrunk in some way to transform one into the other.

It follows that every generic point set is order isomorphic to a unique permutation plot, and that order isomorphism is an equivalence relation. The set of all $n$-element generic point sets, modulo this relation, is therefore in bijection with the set of all permutations of length $n$. This correspondence allows us to identify a permutation with its plot, and provides an alternate geometric definition of permutation patterns, illustrated in Figure 1.1.3.

Definition 1.1.10. Let $n, k \in \mathbb{P}$ with $k \leq n$, and let $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$. Let $P, T$ be the points in the plots of $\pi$ and $\sigma$, respectively. Say that $\sigma \prec \pi$ if there is some subset $R \subseteq S$ for which $R \sim T$.
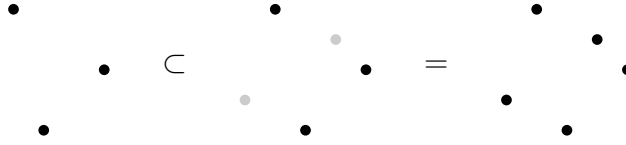
Figure 1.1.3: The permutation $\sigma = 312$ is contained in the permutation $\pi = 25143$.

Many operations on permutations are easier to understand through these geometric plots. For example, the plot of a permutation can be reflected and rotated to produce new permutations. Letting $\pi = \pi_1 \pi_2 \ldots \pi_n$ be a permutation, the reverse of $\pi$ is obtained by reflecting the dots across a vertical line, the complement by reflecting across a horizontal line, and the inverse is obtained by reflecting across the line $y = x$. That these operations generate a group of automorphisms isomorphic to the dihedral group of order eight is clear when viewing permutations as plots within a square. It is equally clear from this viewpoint that these operations respect pattern containment

We can also define operations which act on pairs of permutations, combining two or more permutations into a single new one, and these operations can also be described entirely at the geometric level. Two such examples are the direct sum and skew sum of permutations.

Definition 1.1.11. Let $n, k \in \mathbb{P}$, and let $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_k$. The direct sum of $\pi$ and $\sigma$, written $\pi \oplus \sigma$, is the permutation defined by

$$(\pi \oplus \sigma)_i = \begin{cases} \pi_i & \text{if } i \leq n \\ \sigma_{i-n} + n & \text{if } i > n. \end{cases}$$

The skew sum, written $\pi \ominus \sigma$ is defined similarly:

$$(\pi \oplus \sigma)_i = \begin{cases} \pi_i + k & \text{if } i \leq n \\ \sigma_{i-n} & \text{if } i > n. \end{cases}$$

A sum-indecomposable (resp. skew-indecomposable) permutation is one which cannot be written as a direct (resp. skew) sum,

Geometrically, $\pi \oplus \sigma$ is the permutation whose plot is represented by placing the plot of $\pi$ below and to the left of the plot of $\sigma$, while $\pi \ominus \sigma$ places the plot of $\pi$ above and to the left of $\sigma$, as shown in Figure 1.1.4.

These definitions will prove essential when describing permutation classes. In his thesis [82], Waton describes and explores classes defined entirely by points
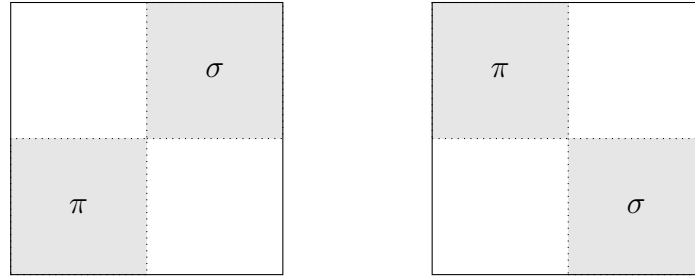
Figure 1.1.4: The plots of $\pi \oplus \sigma$ and $\pi \ominus \sigma$, respectively.

plotted on specified geometric shapes. We focus here, however, on more general classes.

Direct sums and skew sums are simple examples of the so called inflation operation. A non-geometric definition of inflation is technical and unillustrative, but is natural when viewed as an operation of permutation plots. Before defining inflations, we need another definition which will itself prove useful.

Definition 1.1.12. Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$. An interval of $\pi$ is a contiguous sequence of entries $\pi_i \pi_{i+1} \ldots \pi_{i+k}$ whose values form a contiguous sequence of integers.

For example, in the permutation $\pi = 2743516$, the third, fourth, and fifth entries (435) form an interval. Every permutation has an interval of size $n$ (the entire permutation) and intervals of size one (each entry). Permutations which have only these trivial intervals are especially significant.

Definition 1.1.13. An permutation $\pi \in \mathfrak{S}_n$ whose only intervals have size 1 and $n$ is called simple.

Simple intervals are useful for describing permutation classes, as we will see. Monotone intervals will be investigated further in Chapters 4 and 5, and simplicity will be a major topic of Chapter 3. We can now define inflations, which will used throughout this dissertation.

Definition 1.1.14. Let $\pi \in S_n$, and let $\alpha_1, \alpha_2 \ldots \alpha_n$ be permutations of any length. The inflation of $\pi$ by the permutations $\alpha_i$ is defined as the permutation obtained by replacing the $i$th entry of $\pi$ with an interval which is order isomorphic to the permutation $\alpha_i$. This inflation is denoted

$$\pi[\alpha_1, \alpha_2, \ldots \alpha_n].$$
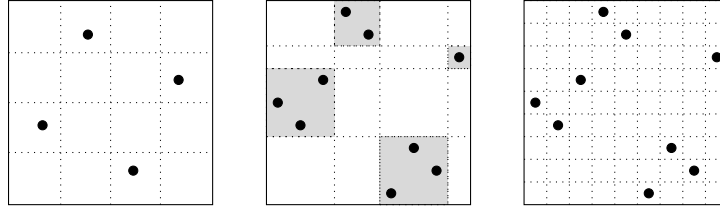
Figure 1.1.5: The simple permutation 2413 and its inflation $2413[213, 21, 132, 1] = 546\ 98\ 132\ 7$.

For example, for any two permutations $\pi$ and $\sigma$, $\pi \oplus \sigma = 12[\pi, \sigma]$ and $\pi \ominus \sigma = 21[\pi, \sigma]$. A more complicated example is shown in Figure 1.1.5. While simple permutations and inflations are useful for working with and describing permutations, their true utility is illustrated in the following theorem, which has generalizations to a wider range of combinatorial objects [67].

Theorem 1.1.15 (Substitution Decomposition [28]). Every permutation $\pi$ can be written as the inflation of a unique simple permutation. Further, if $\pi = \sigma[\alpha_1, \ldots \alpha_m]$, where each $\alpha_i$ is a permutation of length $\geq 1$ and $m \geq 4$, then the permutations $\alpha_i$ are uniquely determined as well.

## § 1.2   Dyck Paths and the Catalan Numbers

Before exploring two examples of permutation classes, we take a brief detour and investigate another set of combinatorial objects known as Dyck paths. These paths will be used throughout this dissertation, and provide a convenient and flexible means of encoding recursive and structural information.

These paths are enumerated by the so-called Catalan numbers, a ubiquitous and useful sequence of integers. Stanley [79] has famously collected a series a sixty-six examples of combinatorial objects, each enumerated by these numbers. Their pervasiveness is due in part to their multiple recursive descriptions.

### Paths on the Integer Lattice

At its most formal, a Dyck path of semilength $n$ is a sequence $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_{2n}$ of vectors $\vec{v}_i \in \{\langle 1, 1 \rangle, \langle 1, -1 \rangle\}$, satisfying $\sum_{n=1}^{2n} \vec{v}_i = \langle 2n, 0 \rangle$ and, for all integers $k \in [2n]$ and $\langle x, y \rangle = \sum_{n=1}^{k} \vec{v}_i$, we have that $y \geq 0$.
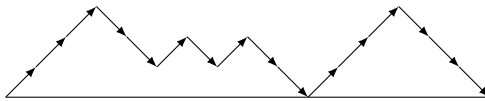
Figure 1.2.1: A Dyck path of semilength 8.

As usual, a more intuitive definition will be useful. Suppose that, starting from the point $(0,0) \in \mathbb{R}^2$, we want to travel to the point $(2n, 0)$. Suppose further that are only allowed to walk diagonally northeast (from a point $(x, y)$ to $(x+1, y+1)$) or southeast (from a point $(x, y)$ to $(x+1, y-1)$). Call a northeast step an upstep and a southeast step a downstep. The total number of walks from $(0,0)$ to $(2n, 0)$ is then $\binom{2n}{n}$, since the number of up steps must equal the number of down steps, and so we need only specify which of the $2n$ steps are up. Dyck paths can now be defined as follows.

Definition 1.2.1. A Dyck path of semilength $n$ (or of length $2n$) is path $p = s_1 s_2 \ldots s_{2n}$ from $(0,0)$ to $(2n, 0)$ using the steps $u = \langle 1, 1 \rangle$ and $d = \langle 1, -1 \rangle$ which never passes below the line $y = 0$.

These paths can be represented as a string of symbols from the alphabet $\{u, d\}$, representing upsteps and downsteps, respectively. The path $p = uuuddududduuddd$ is shown in Figure 1.2.1.

### Enumerating Dyck Paths

Dyck paths are a fundamental combinatorial object, and their properties have been studied extensively [31, 37, 38]. Their well understood structure makes them (and their generalizations) a useful intermediate object for building bijections between other objects [17, 35]. To illustrate their recursive structure, we derive their enumeration here.

In order to count Dyck paths, we first need to consider their structure, and how they can be broken down into smaller pieces. We focus on two separate decompositions, which lead to two different recursive descriptions, each of which leads to the Catalan numbers.

First, let $p = s_1 s_2 \ldots s_{2n}$ be a Dyck path, and let $s_i$ be the first step which brings it back to the line $y = 0$. Such a step must exist, since $s_{2n}$ always ends at this line. It follows then that $i$ is even, $s_1 = u$, $s_i = d$, and $s_{i+1} s_{i+2} \ldots s_{2n}$ is a Dyck path of length $2n-i$. Further, since $s_i$ is the first time the path touches the line $y = 0$, each of the steps $s_2, s_3, \ldots s_{i-1}$ have a height greater than or

equal to 1, which implies that $s_2 s_3 \ldots s_{i-1}$ is a Dyck path. This implies that for every Dyck path $p$, there exist two smaller Dyck paths $p_1, p_2$ such that

$$p = u p_1 d\, p_2.$$

It follows that if $\mathcal{P}$ is the language of Dyck paths (i.e., the set of all strings of the letters $u, d$ which represent valid Dyck paths), then $\mathcal{P} = u\mathcal{P}d\mathcal{P} + \epsilon$, where the $\epsilon$ represents the empty path. This leads immediately to a generating function relation: if we let $c_n$ be the number of Dyck paths of semilength $n$ and $C(z) = \sum_{n \geq 0} c_n z^n$, then this relation leads to the equation

$$C(z) = zC(z)^2 + 1. \tag{1.1}$$

Before investigating further, we present an alternate decomposition. Let $p = s_1 s_2 \ldots s_{2n}$ be a Dyck path, and let $i_1, i_2, \ldots i_k$ be all of the indices with the property that the step $s_i$ ends on the line $y = 0$. It follows then that each subword $s_{i_j+1} s_{i_j+2} \ldots s_{i_{j+1}-1}$ stays above the line $y = 1$, and is therefore itself a Dyck path. Therefore, for all Dyck paths $p$, there exist some integer $k$ and Dyck paths $p_1, p_2, \ldots p_k$ such that

$$p = u p_1 d\ u p_2 d\ \ldots u p_k d.$$

This gives an alternate relation for the generating function $C(z)$ enumerating Dyck paths:

$$C(z) = 1 + zC(z) + z^2 C(z)^2 + z^3 C(z)^3 + \cdots = \frac{1}{1 - zC(z)}. \tag{1.2}$$

The equivalence of equations 1.2 and 1.1 is immediately obvious — one can be rearranged into the other. It follows then that these two seemingly different recursions are in fact equivalent, and so any object exhibiting either of these recursive descriptions are counted by the same numbers. With Dyck paths, both recurrences are clear; with other objects, however, they are less transparent. Dyck paths are useful in part because of the simplicity of their decompositions, and Catalan numbers are ubiquitous because they capture so many of these recursions.

## The Catalan Numbers

The generating function presented above (equation 1.1) can be solved using the quadratic formula, yielding the following (note that the quadratic formula

actually yields two solutions, but we discard the one which does not have a series expansion with positive integer coefficients)

$$C(z) = \sum_{n \geq 0} c_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}. \tag{1.3}$$

The first few coefficients in the expansion of $C(z)$ are $1, 1, 2, 5, 14, 42, 132, \ldots$, and are sequence A000108 in the OEIS [84]. The generating function recurrence $C(z) = zC(z)^2 + 1$ translates to $c_0 = 1$ and $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$, and this uniquely defines this sequence. The binomial theorem can be used to obtain an exact formula for $c_n$ from equation 1.3 above:

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}. \tag{1.4}$$

We note that the generating function presented above (equation 1.3) has a singularity at $z = 1/4$. It follows that, when expanded as a power series about $z = 0$, $C(z)$ has a radius of convergence of $1/4$. The exponential growth rate of a sequence is equal to the reciprocal of the radius of convergence, which implies that $\lim_{n \to \infty} \sqrt[n]{c_n} = 4$. While Stirling's approximation for the factorials gives a simpler means of calculating this growth rate (and allow for the derivation of the subexponential growth rate), analytic techniques, summarized in the textbook of Flajolet and Sedgewick [43], provide a wide framework for deriving these exponential growth rates.

## § 1.3   Four Case Studies

The advantage to this geometric focus is best illustrated through examples. In this section we present four case studies, each of which corresponds roughly to the subject of a later chapter. Together these provide motivation and a gentle introduction to the methods used throughout this dissertation.

We begin by deriving the enumeration of the classes of 132- and 123-avoiding permutations. Though they share the same enumeration, these two classes present starkly different decompositions. We then combine these ideas and explore the class of 123- and 231-avoiding permutations, motivating the investigation of polynomial permutation classes. Finally, we examine an example

of the use of probabilistic techniques and structural decomposition in finding statistical information about classes.

## Permutations Avoiding 132

We start with the enumeration of the class Av(132). The study of simples within a permutation class has been a deep and productive line of research in recent years [2, 27, 28]. Further, this investigation has seen numerous applications in the enumeration of classes [5, 30, 70]. While the vast majority of this machinery is not needed for the class Av(132), but in the interest of exposition we hit a small nail with a large hammer. The enumeration of a class using its simples is the core idea of Chapter 3, where we apply it to sets of pattern-avoiding involutions.

A plot of a permutation within Av(132) has strict restrictions: every element to the left of the highest point must be higher than every element to the right, since otherwise we would have a 132 pattern with the highest element playing the role of the 3. This highest element then divides the plot into two sides. It follows that every entry after the peak forms an interval, which implies that the only simples in Av(132) are $\{1, 12, 21\}$.

By describing the simple permutations in the class, we can often obtain a full enumeration. The class Av(132) is uncomplicated enough to be described entirely using direct and skew sums, but it falls into a larger set of classes, those which have only finitely many simple permutations. Such permutation classes posess a number of useful properties, including the following theorem, due to Albert and Atkinson.

Theorem 1.3.1 (Albert, Atkinson [2]). If a class contains only finitely many simple permutations, then its enumeration is given by an algebraic generating function.

In addition to theoretical results, the investigation of simple permutations and decomposition has led to practical enumeration techniques. Once the simples of a class have been obtained, one needs only determine the manner in which each simple can be inflated in order to fully describe the class. While much of this work has focused on enumerating classes, it can also be used to obtain statistical information about the class. Section 1.3.4 gives an introductory example to this technique, while Chapter 2 explores the concept further.

Returning now to the class $\mathrm{Av}(132)$, note that arbitrary inflations of the simple permutations $\{1, 12, 21\}$ do not lead to 132-avoiding permutations. Letting $\pi \in \mathrm{Av}(132)$, recall that every entry after the maximal entry must have a smaller value than every entry before. The substitution decomposition (Theorem 1.1.15) implies that each permutation can be defined as an inflation of precisely one of these: the simple permutation 1 can only be inflated to the length 1 permutation, inflations of 12 are the sum-decomposable elements, and the skew-decomposable elements are the inflations of 21.

For an inflation of 12, the 2 can only be inflated by an increasing run of entries, or else would contain a 21 pattern, creating a 132 occurrence with any entry of the inflation of the 1, which can be inflated by any 132-permutation. Recall that the substitution decomposition does not guarantee uniqueness when inflating the simple permutations 12 and 21, so we have to be careful. To ensure uniqueness, only allow the 2 of 12 to be inflated by a single element (if there is an increasing run, take it to be part of the 1).

Finally, when inflating 21, the 1 can be inflated by any 132-permutation, while the 2 can be inflated by any 132-avoiding permutation which ends in its last element, which can be represented as the direct sum of a 132-avoiding permutation (or the empty permutation) with the permutation 1. We express this as follows, letting $\mathcal{C}$ denote $\mathrm{Av}(132)$ and $\epsilon$ denote the empty permutation:

$$\mathcal{C} = 1[1] \bigcup 12[\mathcal{C}, 1] \bigcup 21\big[(\mathcal{C} \cup \epsilon) \oplus 1, \mathcal{C}\big].$$

Letting $f$ denote the generating function $\sum_{n \geq 0} |\mathrm{Av}_n(132)| z^n$, this leads to the following expression

$$f = z + fz + z(f+1)f.$$

Solving for $f$ using the quadratic formula gives that $f$ is the generating function for the Catalan numbers with the constant term subtracted off. This gives an exact formula for the enumeration of $\mathrm{Av}(132)$, as originally derived by Knuth [61].

Theorem 1.3.2. The number of permutations of length $n$ avoiding 132 is the $n$th Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$.

Note that this result can be obtained using more elementary methods. It follows that a permutation is 132-avoiding if and only if it can be written
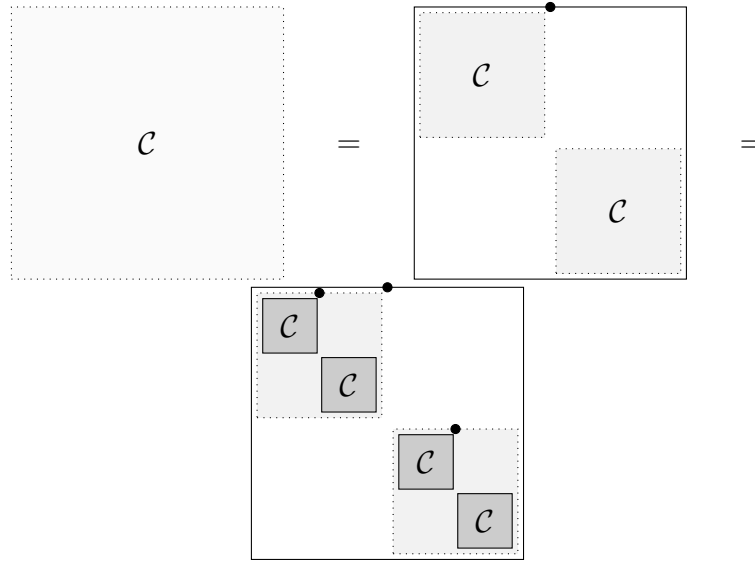
Figure 1.3.1: A geometric description of the class $\mathcal{C} = \mathrm{Av}\,132$.

as $(\pi \oplus 1) \ominus \sigma$, where $\pi$ and $\sigma$ are 132-avoiding permutations (or empty). Applying this characterization iteratively provides a recursive description of the 132-avoiding permutations, shown in Figure 1.3.1, and in fact characterizes this class.

This recursive decomposition can be used to generate a recursively defined bijection $\phi : \mathrm{Av}_n(132) \to \mathcal{D}_n$ from permutations in $\mathrm{Av}_n(132)$ to Dyck paths of semilength $n$, thus reproving Theorem 1.3.2 once again. Let $\pi \in \mathrm{Av}_n(132)$, and $\pi = (\pi_1 \oplus 1) \ominus \pi_2$ be the decomposition defined above. Then define

$$\phi(\pi) = u \; \phi(\pi_1) \; d \; \phi(\pi_2).$$

This recursive definition was originally presented by Knuth [61]. For example,

$$\begin{aligned}
\phi(74352681) &= u\phi(743526)d\phi(1) \\
&= u(ud\phi(43526))dud \\
&= uud(u\phi(4352)d)dud \\
&= uudu(u\phi(43)d)\phi(2)dud \\
&= uuduu(ud\phi(3))d(ud)dud \\
&= uuduuud(ud)duddud \\
&= uuduuududdudddud \in \mathcal{D}_n.
\end{aligned}$$
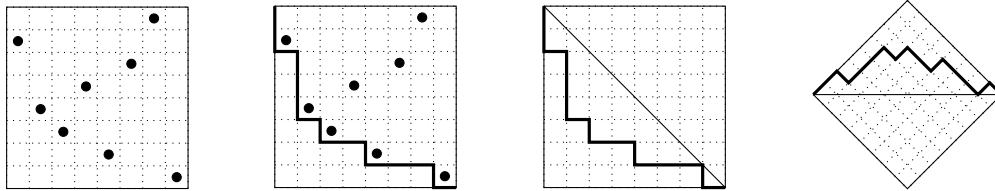
Figure 1.3.2: The construction of the Dyck path $\varphi(74352681)$.

There is an alternate, non-recursive bijection $\varphi$, first presented in an alternate, non-geometric form by Krattenthaler [62], whose equivalence to the above definition follows from the work of Claesson and Kitaev [35]. Let $\pi \in \mathrm{Av}_n(132)$, and define $\varphi(\pi)$ as follows. First, plot $\pi$ and define a lattice path from $(1, n)$ to $(n, 1)$ using the steps $\{\langle 0, -1 \rangle, \langle 1, 0 \rangle\}$. Take this to be the unique path using these steps which maximizes the area underneath the path, while remaining below and to the left of each entry of the plotted permutation. Finally, translate this to a Dyck path by mapping each $\langle 0, -1 \rangle$ to be an up step, and each $\langle 1, 0 \rangle$ to be a down step. See Figure 1.3.2 for an example.

## Permutations Avoiding 123

Despite having the same enumeration, the class $\mathrm{Av}(123)$ presents a stark contrast to the class $\mathrm{Av}(132)$. First, there are infinitely many simple permutations in the class, which prevents us from using many of the tools from the previous example. Enumerating and describing these simples is the central idea of Chapter 3. We first present a bijective enumeration of the class, before analyzing the structure.

As a further example highlighting the benefit of the geometric viewpoint note that, remarkably, the bijection $\varphi$ described in Figure 1.3.2 leads to a bijection $\varphi' : \mathrm{Av}_n(123) \to \mathcal{D}_n$, using exactly the same description. See Figure 1.3.3 for an example. Note that $\varphi^{-1} \circ \varphi'$ is a bijection from $\mathrm{Av}_n(123)$ to $\mathrm{Av}_n(132)$, which is equivalent to the one presented by Simion and Schmidt [76], and shows that the locations of left-to-right minima has the same distribution in both classes.

A modification of this bijection is central to Chapter 2, and will be used to count pattern occurrences within the class. Dyck paths can be used to encode structural information about the permutations they represent, and can be easily enumerated.

To see that $\mathrm{Av}(123)$ contains infinitely many simple permutations, we define the decreasing oscillations, a family of simples which are contained within the
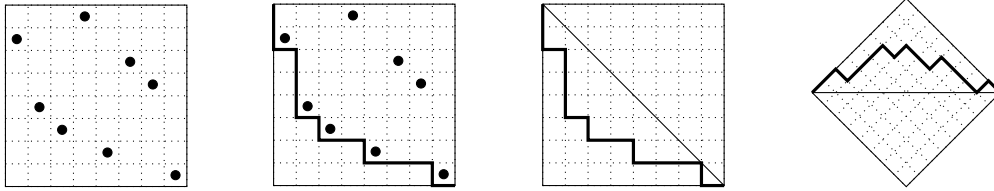
Figure 1.3.3: The construction of the Dyck path $\varphi'(74382651)$.
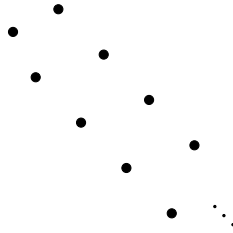


Figure 1.3.4: The decreasing oscillations.
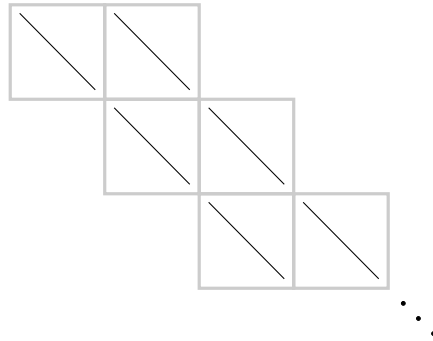


Figure 1.3.5: The class Av(123) is precisely those permutations which can be plotted on descending lines of the diagram.

class. Figure 1.3.4 gives a graphical description of these permutations. Though the simples are not as easily described as in our previous example, Av(123) exhibits a different kind of geometric structure which will be equally useful. Since a 123-avoiding permutation does not contain any three increasing entries, it follows that it can be written as the union of two decreasing sequences of entries. It follows further that we can partition the plot of such a permutation into an alternating sequence of monotone decreasing runs. We formalize this in Chapter 3, but for now present an diagram of the so-called staircase decomposition [3, 26] in Figure 1.3.5.

This decomposition will be used to enumerate and describe the simple permu-

tations within the class, which will then be used to enumerate pattern avoiding involutions in Chapter 3. We present one final method of enumerating the class Av(123), by inflating the (infinitely many) simples. In Chapter 3 we use the staircase decomposition to enumerate the simples of the class, and find that their generating function (equation 3.1) is given by

$$f = \sum_{\substack{\sigma \in \mathrm{Av}(123) \\ \sigma \text{ simple}}} z^{|\sigma|} = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}.$$

Each entry of a simple permutation in the class can be inflated only by decreasing runs, whose generating functions are given by $\frac{z}{1-z}$. It follows then that, since each $z$ in the above generating function represents an entry of a simple permutation, replacing $z$ by $\frac{z}{1-z}$, we obtain the generating function for all permutations of the class. Indeed, after simplifying, we find that this composition gives the generating function for the Catalan numbers, with the constant term (representing the empty permutation) removed:

$$f\left(\frac{z}{1-z}\right) = \frac{1 - 2z - \sqrt{1 - 4z}}{2z}.$$

## Permutations Avoiding 123 and 231

Our next example enumerates the class Av(123, 231) of permutations which avoid both 123 and 231, using a structural description of the class. This example motivates the exploration of the polynomial classes (the classes whose enumeration is given by a polynomial). This will be investigated more fully in Chapter 4, where an algorithm will be presented which, given a structural description, enumerates the class.

Since we have already shown that the only simples in the class Av(231) are $\{1, 12, 21\}$ (because it is a symmetry of Av(132)), the fact that Av(123, 231) $\subset$ Av(231) implies that these are the same simples in Av(123, 231). The added restriction of avoiding 123 changes the way these simples can be inflated. Both entries of 12 can only be inflated by decreasing runs, to avoid constructing an occurrence of 123. Finally, the first entry of a 21 can be inflated only by a decreasing run (to avoid 231), while the second can be inflated by any element from the class.

After accounting for uniqueness, it follows that every permutation in the class can be obtained by inflating the permutation 312 with (possibly empty) de-
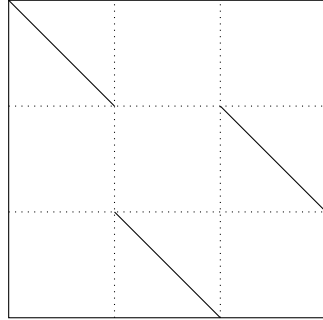
Figure 1.3.6: The class $\mathrm{Av}(123, 231)$ is precisely those permutations which can be plotted on descending lines of the diagram.

scending permutations. Therefore, this class is precisely those permutations which can be drawn on the diagram shown in Figure 1.3.6

This is a simple example of a grid class [68], a useful concept which has produced many new enumerations in recent years. It is known [6, 55], and is presented formally in Theorem 4.1.5, that a permutation class is enumerated by a polynomial if and only if it is a union or intersection of classes which can be represented with such a diagram, with only one nonempty cell per row and column.

Returning to Figure 1.3.6, it is trivial to enumerate those permutations which have at least one element in each block: the generating function for a single block is $z/(1 - z)$, and so the generating function for those with no empty blocks are $z^3/(1 - z)^3$. If the first block is empty, then we have the generating function $z^2/(1 - z)^2$. If either the second or third block is empty, the entire permutation is a single decreasing run, with generating function $z/(1 - z)$. Therefore, the generating function for the entire class is simply the sum of these three:

$$\sum_{n \geq 1} |\mathrm{Av}_n(123, 231)| z^n = \frac{z^3}{(1 - z)^3} + \frac{z^2}{(1 - z)^2} + \frac{z}{1 - z} = \frac{z^3 - z^2 + z}{(1 - z)^3}. \quad (1.5)$$

Equation 1.5 expanded using the binomial theorem to produce an exact equation for the number of permutations of each length in the class.

$$|\mathrm{Av}_n(123, 231)| = \frac{n^2 - n + 2}{2} = \binom{n}{2} + 1.$$

More complicated decompositions lead to a number of technical obstacles, but this same general idea can be used to calculate the polynomials enumerating all such classes. This will be presented in Chapter 4, and an implementation of the algorithm is available online [54].

## Ascents in 132-Avoiding Permutations

We end this chapter with an illustrative example which utilizes a class's structural decomposition to investigate the distribution of a permutation statistic. This example, while relatively simple, serves to showcase the techniques which will be used throughout the following chapters, and is particularly pertinent to Chapter 2.

A permutation statistic is any function $\chi : \mathfrak{S}_n \to \mathbb{R}$. In practice, we often consider statistics that map from permutations to non-negative integers which capture some structural trait of the permutation. Examples include the location of the largest element, number of cycles, value of the first entry, and number of inversions. In this section we consider the number of ascents of a permutation. An ascent of a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ is an index $i$ such that $\pi_i < \pi_{i+1}$, and the number of ascents in a permutation $\pi$ is denoted $\mathrm{asc}(\pi)$.

For a given permutation $\pi$ of length $n$, it follows that $\mathrm{asc}(\pi) \in \{0, 1, \ldots n-1\}$. If $i$ is an ascent of $\pi$ then $i$ is a descent of $\pi^c$, and so the number of permutations of lengh $n$ with $k$ ascents is equal to the number of such permutations with $k$ descents (or $n - k - 1$ ascents). This implies in particular that the average number of ascents in a randomly selected permutation from $\mathfrak{S}_n$ is $(n - 1)/2$. When we restrict to a proper permutation class, however, the distribution can be more difficult to compute.

For a finite set $S$ of permutations and a statistic $f$, the generating polynomial for $f$ on $S$ in indeterminate $u$ is

$$\sum_{\pi \in S} u^{f(\pi)}.$$

For example, if $\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}$ then the generating polynomial for the number of ascents is $u^2 + 4u + 1$, since there is one permutation with two ascents, four with one ascent, and one permutation with no ascents. There is one crucial observation: if we take the derivative (with respect to $u$) of the generating polynomial and set $u = 1$ we obtain a weighted sum which

evaluates to the expected value, or average, of the statistic on $S$. Further, by differentiating twice before setting $u = 1$, and then dividing by two, we obtain the first factorial moment of the statistic, which can be used to compute the variance. This process can be iterated to calculate higher moments of the distribution.

Extending to permutation classes, let $|\pi|$ denote the length of a permutation $\pi$ and define the generating function for a statistic $f$ across a class $\mathcal{C}$ as

$$\sum_{\pi \in \mathcal{C}} z^{|\pi|} u^{f(\pi)}.$$

The coefficient of $z^n$ in this bivariate generating function is precisely the generating polynomial for the statistic $f$ on the set $\mathcal{C}_n$, and so it follows that by differentiating with respect to $u$ and plugging in $u = 1$, we can obtain generating functions whose coefficients represent the moments of the distribution on $\mathcal{C}_n$. Asymptotic analysis can then be used to compute the limiting distribution as $n$ approaches infinity.

Throughout this section, let $a_{n,k}$ be the number of 132-avoiding permutations of length $n$ which contain exactly $k$ ascents, and let

$$f(z, u) = \sum_{\pi \in \mathrm{Av}(132)} z^{|\pi|} u^{\mathrm{asc}(\pi)} = \sum_{n \geq 0} \sum_{k \geq 0} a_{n,k} u^k z^n.$$

Our goal is to derive a closed expression for $f$, and use this to analyze the distribution of descents across $\mathrm{Av}(132)$. Consider the recursive description of the class, shown in Figure 1.3.1, and let $\pi = (\rho \oplus 1) \ominus \sigma$ be a 132-avoiding permutation. It follows that the number of ascents of $\pi$ is equal to the sum of ascents in $\rho$ and $\sigma$, plus one if $\rho$ is nonempty (otherwise the permutation starts with its biggest entry). This relationship leads to the following functional equation.

$$f = zf + uz(f - 1)f + 1.$$

The first term on the right hand side is the case where $\rho$ is empty, the second is when $\rho$ is non-empty, and the constant term accounts for the empty permutation. We can solve for $f$ above to find the following:

$$\begin{aligned} f(z, u) &= \frac{1 + (u - 1)z - \sqrt{(u^2 - 2u + 1)z^2 - 2(u + 1)z + 1}}{2uz} \\ &= 1 + z + (u + 1)z^2 + (u^2 + 3u + 1)z^3 + (u^3 + 6u^2 + 6u + 1)z^4 + \dots. \end{aligned}$$

Note that substituting $u = 1$ gives the generating function for the Catalan numbers, as expected. The coefficient of $z^3$ is $(u^2 + 3u + 1)$, as there is one 132-avoiding permutation with two ascents (123), three with one ascent (213, 231, 312), and one with no ascents (321). Finally, we can obtain the total number $a_n$ of ascents in all 132-avoiding permutations of length $n$ by differentiating with respect to $n$ and setting $u = 1$:

$$\sum_{n \geq 0} a_n z^n = \partial_u f(z, u)\big|_{u=1}$$
$$= \frac{1 - 3z - (z - 1)\sqrt{1 - 4z}}{1 + z\sqrt{1 - 4z}}$$
$$= \sum_{n \geq 0} \binom{2n - 1}{n - 2} z^n$$
$$= z^2 + 5z^3 + 21z^4 + 84z^5 + 330z^6 + 1287z^6 \dots.$$

It follows then that the average number of ascents in a randomly selected 132-avoiding permutation is given by this total divided by the total number of such permutations, the Catalan numbers. Therefore the average is given by

$$\binom{2n - 1}{n - 2} \frac{n + 1}{\binom{2n}{n}} = \frac{n - 1}{2}.$$

Note that this expectation is identical to the average number of ascents in a random permutation chosen from the set $\mathfrak{S}_n$, and so it follows that the property 'avoids 132' is independent from the random variable asc. This can also proven bijectively, by constructing a map from $\mathrm{Av}_n(132)$ to itself which maps ascents to descents (by mapping the permutations to unlabelled binary trees, and then reflecting the tree), but the above approach can be extended and generalized to other statistics and classes, as we will soon see.

In Chapter 2 we explore how pattern-avoidance changes the distribution of other statistics. These same techniques will be revisited in Chapter 5 and used to compute the distribution of intervals of size two, which relates to the number of distinct patterns within a permutation.

# – Chapter 2 –

# Pattern Expectation

In the set of all permutations of length $n$, all patterns of a fixed length occur the same number of times. However, if we restrict to smaller classes of permutation, the situation quickly becomes more interesting. The investigation of pattern occurrences within permutations is a recent and productive research topic. This chapter explores this new area, and uses it to develop connections between permutation classes.

In particular, we examine the classes of 123- and 132-avoiding permutations, and show that the number of 231 patterns is identical in each. This identity extends an earlier result of Miklós Bóna [24], and its derivation sheds further light on the distribution of pattern occurrences within permutation classes. Further, this chapter brings to light new equivalences between these classes, building on those presented by Elizalde [41], and forming a foundation for further study [29, 57, 75]. This chapter is based partly on [52].

## § 2.1 Pattern Occurrences

Our primary concern in this chapter (and much of Chapter 5) will be the number of occurrences of a pattern within a permutation. The number of occurrences is the number of copies of the pattern we can find within a permutations; formally, we define this as follows:

Definition 2.1.1. Let $\sigma = \sigma_1\sigma_2\ldots\sigma_k$ be a pattern of length $k$, and $\pi = \pi_1\pi_2\ldots\pi_n$ a permutation of length $n$. An occurrence of the pattern $\sigma$ in

$\pi$ is a subsequence $i_1 < i_2 < \cdots < i_k$ such that

$$\pi_{i_1}\pi_{i_2}\ldots\pi_{i_k} \sim \sigma_1\sigma_2\ldots\sigma_k.$$

The number of occurrences of $\sigma$ in $\pi$, denoted by $\nu_\sigma(\pi)$, is the number of such subsequences.

For example, the permutation $\pi = 462513$ contains 2 occurrences of the pattern 213, since the first, third, and fourth, as well as the third, fifth, and sixth, entries of $p$ form 213 patterns. Thus, $\nu_{213}(462513) = 2$.

Clearly, for permutations $\pi$ of length $n$ and $\sigma$ of length $k$, we have that $\nu_q(\pi)$ is bounded below by 0 and above by $\binom{n}{k}$. This minimum value is realized by taking $\pi$ to be any $\sigma$-avoiding permutation, and the maximum is attained, for example, when both $\pi$ and $\sigma$ are ascending permutations. Our primary concern will be the average number of occurrences of a pattern over a set of permutations. In the interest of brevity, we will abuse the above notation to apply to sets:

Definition 2.1.2. For a given pattern $\sigma$ and a set $S$ of permutations, let $\nu_q(S)$ denote the total number of occurrences of $\sigma$ within the set $S$. That is,

$$\nu_\sigma(S) = \sum_{\pi \in S} \nu_\sigma(\pi).$$

For example, letting $S = \{2341, 4321, 1234\}$, we have that

$$\nu_{123}(S) = 1 + 0 + 4 = 5.$$

## Pattern Expectation

Counting the total number of occurrences of a pattern within a set of permutations has an alternate, probabilistic interpretation. The expectation of a pattern within a set is defined to be the average number of occurrences of the pattern within a randomly selected element from the set. Clearly, we have that the expectation of a pattern $\sigma$ in a set $S$ is equal to $\nu_\sigma(S)/|S|$.

This probabilistic interpretation motivates many questions, several of which have yielded interesting and surprising answers. We start with an illustrative example, whose derivation showcases some of the ideas which will be useful later. In particular, linearity of expectation will prove useful.

Proposition 2.1.3. Let $\sigma$ be any pattern of length $k$, and let $n \geq k$. Then

$$\nu_\sigma(\mathfrak{S}_n) = \frac{n!}{k!}\binom{n}{k}.$$

Proof. We show that the expectation of the pattern $\sigma$ is equal to $\binom{n}{k}/k!$, which will imply the desired result. Let $\pi$ be a (uniformly) randomly selected permutation in $\mathfrak{S}_n$, and let $X$ be the random variable denoting the number of occurrences of $\sigma$ within $\pi$.

There are $\binom{n}{k}$ sets of positions of $\pi$ in which a $\sigma$ pattern could possibly occur. For each set $P$, let

$$X_P = \begin{cases} 1 & \text{the entries of } P \text{ form a } \sigma \text{ pattern} \\ 0 & \text{otherwise} \end{cases}.$$

It now follows that $X = \sum_P X_P$, and so by linearity of expectation, we have that

$$\mathbb{E}[X] = \sum_P \mathbb{E}[X_P].$$

Finally, for any specified set of indices, all patterns are equally likely. Therefore, $\mathbb{E}[X_P] = 1/k!$. Combining, we see that

$$\mathbb{E}[X] = \sum_P \frac{1}{k!} = \binom{n}{k}\frac{1}{k!}.$$

Therefore, we have that

$$\nu_\sigma(\mathfrak{S}_n) = \frac{|\mathfrak{S}_n|}{k!}\binom{n}{k} = \frac{n!}{k!}\binom{n}{k}.$$

$\square$

Fact 2.1.3 shows that the total number of pattern occurrences within the set of all permutations depends only on the length of the pattern specified. This contrasts sharply with the fact that the numbers of permutations which avoid a given pattern varies widely based on the choice of pattern. This discrepancy can be explained in part by the fact that certain patterns are better able to overlap with themselves, so that a smaller number of permutations contains a higher concentration of pattern occurrences.

Table 2.1.1: Total number of pattern occurrences for length 3 patterns in 123-
and 132-avoiding permutations.

| $\mathrm{Av}_n(123)$ | | | | | | |
|---|---|---|---|---|---|---|
| length | $\nu_{123}$ | $\nu_{132}$ | $\nu_{213}$ | $\nu_{231}$ | $\nu_{312}$ | $\nu_{321}$ |
| 3 | 0 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0 | 9 | 9 | 11 | 11 | 16 |
| 5 | 0 | 57 | 57 | 81 | 81 | 144 |
| 6 | 0 | 312 | 312 | 500 | 500 | 1016 |
| 7 | 0 | 1578 | 1578 | 2794 | 2794 | 6271 |

| $\mathrm{Av}_n(132)$ | | | | | | |
|---|---|---|---|---|---|---|
| length | $\nu_{123}$ | $\nu_{132}$ | $\nu_{213}$ | $\nu_{231}$ | $\nu_{312}$ | $\nu_{321}$ |
| 3 | 1 | 0 | 1 | 1 | 1 | 1 |
| 4 | 10 | 0 | 11 | 11 | 11 | 13 |
| 5 | 68 | 0 | 81 | 81 | 81 | 109 |
| 6 | 392 | 0 | 500 | 500 | 500 | 748 |
| 7 | 2063 | 0 | 2794 | 2794 | 2794 | 4570 |

The problem of pattern packing will be discussed in more detail in Chapter 5.
In this chapter we examine the pattern expectation of of small patterns within
avoidance classes. In particular we seek insight to the following question, first
posed by Joshua Cooper: "How does the absence of one pattern affect the
expectation of another?"

## Background and Data

The total number of length 3 patterns in the sets $\mathrm{Av}_n(123)$ and $\mathrm{Av}_n(132)$ are
shown below, for $1 \leq n \leq 7$.

Since both 123 and 132 are involutions, inversion maps each set to itself, and
maps patterns to their inverse. This implies the identity $\nu_{231} = \nu_{312}$ in both
sets of permutations. Miklós Bóna [20, 24] investigated the set $\mathrm{Av}_n(132)$ and
enumerated the total occurrences of each length 3 pattern. In particular, he
established the identity $\nu_{213}(\mathrm{Av}_n(132)) = \nu_{231}(\mathrm{Av}_n(132))$.

This implies that the statistics $\nu_{213}$ and $\nu_{231}$ have the same expectations over
the set of 132-avoiding permutations of length $n$. This identity is surprising

in part because these two statistics have different distributions over this set, but share the same average value.

The main motivation for Section 2.2 is establishing the identity

$$\nu_{231}(\mathrm{Av}(132)) = \nu_{231}(\mathrm{Av}(123)).$$

This identity extends Bóna's result, and presents another example of two permutation statistics with different distributions having the same mean.

## § 2.2   123-avoiding Permutations

In this section, we derive exact and asymptotic values for $\nu_\sigma(\mathrm{Av}_n(123))$ for $|\sigma| \leq 3$ and $n \geq 0$. In addition, we show that for $k \geq 1$, the pattern $k \ (k - 1) \ (k - 2) \ \ldots 2 \ 1$ has a higher expectation than any other pattern of length $k$ for large enough permutations. Finally, applying recent results of Miklós Bóna, we show that the total number of 231 patterns is identical within the sets of 132-avoiding and 123-avoiding permutations of length $n$.

Throughout this sections, let $n$ be some fixed positive integer. For simplicity of notation, we use $\nu_\sigma$ to denote $\nu_\sigma(\mathrm{Av}_n(123))$.

### Class Structure

The class of 123-avoiding permutations has a rigid structure, which we will use to investigate pattern occurrences. Recall (Section 1.2.3) that $|\mathrm{Av}_n(123)| = c_n$, where $c_n$ is the $n$th Catalan number. . For a permutation $\pi = \pi_1\pi_2\ldots\pi_n$, we say that the entry $\pi_i$ is a left-to-right minimum (ltr-min) if it is smaller than all of the elements to its left, and a right-to-left maximum (rtl-max) if it is larger than all of the elements to its right.

In a 123-avoiding permutation $\pi$, every element is either a ltr-min or a rtl-max (or possibly both), since otherwise it would have a bigger element to its right and a smaller element to its left, which would form a 123 pattern. By definition, the sets of ltr-min and of rtl-max are both decreasing when read from left to right. Therefore, every 123-avoiding permutation is the union of two decreasing sequences of entries.

Breaking down permutations into these two decreasing sequences will prove useful in the following sections. However, the possibility of an element being both a ltr-min and a rtl-max poses problems. Further restricting our permutations will alleviate this issue.

Definition 2.2.1. A permutation $\pi = \pi_1 \pi_2 \ldots p_n$ is skew-decomposable if there exist permutations $\sigma$ and $\varphi$ for which $\pi = \sigma \ominus \varphi$. Otherwise, we say that $\pi$ is skew-indecomposable. Denote the set of indecomposable 123-avoiding permutation by $\mathrm{Av}_n^*(123)$. Sum (in)decomposability is defined similarly.

In this chapter we consider only skew-(in)decomposability, and so we drop the word 'skew' for the simplicity of notation.

Note that if any element of $\pi$ is both a ltr-min and a rtl-max, then $\pi$ is decomposable. It follows then that every indecomposable 123-avoiding permutation can be uniquely decomposed into its left-to-right minima and its left-to-right maxima. Further, it follows that every 123-avoiding permutation can be written as a skew sum of indecomposable 123-avoiding permutations. We use this fact to enumerate these permutations.

Proposition 2.2.2. The number of indecomposable 123-avoiding permutations is $c_{n-1}$, the $(n-1)$st Catalan number .

Proof. Let
$$C^*(x) = \sum_{n \geq 0} |\mathrm{Av}_n^*(123)| x^n.$$

We know that $|\mathrm{Av}_n(123)| = c_n$, and so

$$\sum_{n \geq 0} |\mathrm{Av}_n(123)| x^n = \frac{1 - \sqrt{1 - 4x}}{2x} = C(x).$$

Since every permutation $\pi \in \mathrm{Av}_n(123)$ can be written as $sg_1 \ominus \sigma_2 \ldots \sigma_k$ for some $\sigma_1, \sigma_2, \ldots \sigma_k \in \mathrm{Av}_n^*(123)$ and some $k \geq 1$, it follows that

$$C(x) = 1 + C^*(x) + (C^*)(x))^2 + (C^*(x))^3 + \cdots = \frac{1}{1 - C^*(x)}.$$

Rearranging this equation leads to

$$C^*(x) = \frac{C(x) - 1}{C(x)} = xC(x).$$

The second equality follows from the identity $C(x) = xC(x)^2 + 1$.

Therefore,

$$\sum_{n \geq 0} |\operatorname{Av}_n^*(123)| x^n = C^*(x) = xC(x) = \sum_{n \geq 1} c_{n-1} x^n.$$

$\square$

## Patterns of Length 2

To start, we compute the values $\nu_{12}$ and $\nu_{21}$. Since every pair of entries must form either a 12 or a 21 pattern, the sum $\nu_{12} + \nu_{21}$ is equal to the total number of pairs of entries amongst the set of all 123-avoiding permutations. Therefore, we have

$$\nu_{12} + \nu_{21} = \binom{n}{2}.$$

An inversion of a permutation is an occurrence of the pattern 21. Inversions are a well-known and well-studied permutation statistic, and the total number of inversions amongst the set $\operatorname{Av}_n(321)$ is known.

Theorem 2.2.3 (Cheng, Eu, Fu [32]). The total number of inversions in the set $\operatorname{Av}_n(321)$ is given by

$$\nu_{21}(\operatorname{Av}_n(321)) = 4^{n-1} - \binom{2n-1}{n}.$$

The generating function for this sequence is as follows:

$$\sum_{n \geq 0} \nu_{21}(\operatorname{Av}_n(321)) x^n = \frac{x^2 C(x)^2}{1 - 4x}.$$

By reversing permutations, we see that $\nu_{21}(\operatorname{Av}_n(321)) = \nu_{12}(\operatorname{Av}_n(123))$. This allows us to establish exact answers for the number of occurrences of length 2 patterns within $\operatorname{Av}_n(123)$.

Proposition 2.2.4. The total number of 12 patterns in $\operatorname{Av}_n(123)$ is given by

$$\nu_{12} = 4^{n-1} - \binom{2n-1}{n}.$$

Further, since $\nu_{21}(\operatorname{Av}_n(321)) = 4^{n-1} - \binom{2n-1}{n}$, it follows that

$$\nu_{21} = \binom{n}{2} c_n - 4^{n-1} + \binom{2n-1}{n}.$$

## Patterns of Length 3

Deriving the number of occurrences for length three patterns is considerably
more involved, but utilizes some of the same ideas. In this section we find
both the asymptotic and exact values for the total occurrences of $\nu_\sigma$ for each
$\sigma \in \mathfrak{S}_3$. The key idea will be derive the total number of occurrences of a single
pattern, and then use the class structures to develop the other values. Let

$$a_n = \nu_{213}, \quad b_n = \nu_{231}, \quad \nu_{321}.$$

We start by finding the generating function for the numbers $\nu_{213}(\mathrm{Av}_n^*(123))$.
While this may seem arbitrary, this will in fact lead to generating functions
for all other patterns. Let $\pi$ be a permutation in $\mathrm{Av}_n^*(123)$. Recall that each
entry in $\pi$ is either a ltr-min or a rtl-max, and no entry is both. An occurrence
of 213 within $\pi$ must consist of two left-to-right minima followed by a right-to-
left maximum. By counting the number of entries to the left and below each
rtl-max we can exactly determine the number of 213 patterns within $\pi$.

Lemma 2.2.5. The generating function $A^*(x)$ for the number of 213 patterns
in $\mathrm{Av}_n^*(123)$ is given by

$$A^*(x) = \sum_{n \geq 0} \nu_{213}(\mathrm{Av}_n^*(123)) = \frac{x^3 C(x)}{(1 - 4x)^{3/2}} = \frac{x^2}{2(1 - 4x)^{3/2}} - \frac{x^2}{2(1 - 4x)}.$$

Proof. The proof consists of three parts: First, we examine the structure of
permutations in $\mathrm{Av}_n^*(123)$, and find a simple way of counting the number of
213 patterns. Second, we build a bijection onto Dyck paths which maps 213
patterns to a path statistic. Finally, we find the weighted sum of all Dyck
paths with respect to this statistic.

The idea of the proof is as follows: We build a bijection from the set of
permutations $\mathrm{Av}_n^*(123)$ to the set $\mathcal{D}$ of elevated Dyck paths of semilength $n$,
find a statistic on these paths which corresponds to 213 patterns, and then
find the weighted sum of all Dyck paths with respect to this statistic.

Let $\pi$ be a permutation in $\mathrm{Av}_n^*(123)$, and consider the plot of $\pi$. Note that, by
the indecomposability of $\pi$, there is no entry which is simultaneously a ltr-min
and a rtl-max. Construct a Dyck path $\phi(\pi)$ of semilength $n-1$ as follows. First,
build a path from $(1, n)$ to $(n, 1)$ using the steps $\{\langle 1, 0 \rangle, \langle 0, 1 \rangle\}$. Let this path
be the be the unique path which minimizes the area underneath itself while
lying above all of the entries of $\pi$. This path, a variation of the construction
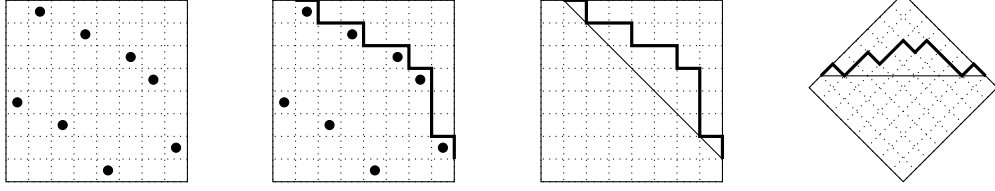
Figure 2.2.1: The construction of the Dyck path $\phi(48371652) = uduuduududddud$.

presented in Section 1.3.2, is then uniquely defined by the locations of the right-to-left maxima, which in turn uniquely define the permutation. Finally, rotate each $\langle 1, 0 \rangle$ step to be an up step, and each $\langle 0, -1 \rangle$ to become a downstep in the path $\phi(\pi)$. See Figure 2.2.1 for an example construction.

This path is a slight modification of the path given by Krattenthaler's bijection [62], taking advantage of the indecomposability of the permutation to yield a more geometric description. This geometric interpretation of the bijection gives some additional insight into the number of 213 patterns.

Note that each rtl-max in $\pi$ produces a peak in $P$. If $\pi_i$ is a rtl-max, let the span of $\pi_i$ (Sp $\pi_i$) denote the number of entries to the left and below this entry. It follows then that $\pi_i$ corresponds to a peak of height Sp $\pi_i$ above the $x$-axis in $P$. An occurrence of 213 must have a rtl-max as its 3 entry, and it follows then that the 21 entries must lie in the span of this entry. We therefore see that every rtl-max is involved in $\binom{\text{Sp} \pi_i}{2}$ occurrences of 213, since we need only choose any two elements in its span to act as the 21. Therefore, if we let $h_{n,k}$ denote the total number of peaks of height $k$ in all Dyck paths of semilength $n$, we have that

$$\nu_{213}(\text{Av}_n^*(123)) = \sum_{k=1}^{n-1} \binom{k}{2} h_{n-1,k}.$$

Finally, we can compute $H(x, u) = \sum_{n,k \geq 0} h_{n,k} x^n u^k$ as follows. First, note that since each Dyck path begins with an upstep it has a unique first point at which the path returns to the $x$-axis, so we can decompose each path $P$ of length $n$ into the concatenation of two shorter paths $Q$ and $R$. This gives that $P = uQdR$, where $u$ denotes an upstep and $d$ a downstep, and each peak of height $k - 1$ in $Q$ and height $k$ in $R$ leads to a peak of height $k$ in $P$. With this in mind, we have the following generating function relation:

$$H(x, u) = ux(H(x, u) + 1)C(x) + xH(x, u)C(x).$$

Here the first term counts the peaks from the $uQd$ part, including the case when $Q$ is empty. The second term counts the contribution from the $R$ part. Rearranging leads to

$$H(x, u) = \frac{uxC(x)}{1 - uxC(x) - xC(x)}.$$

Now, to count 213 patterns, we need to count each peak with weight $\binom{k}{2}$. By taking derivatives twice with respect to $u$, setting $u = 1$, dividing by two and scaling by $x$, we find that

$$\sum_{n,k\geq 0} \binom{k}{2} h_{n-1,k} x^n = x \frac{\partial_u^2 H(x, u)|_{u=1}}{2} = \frac{x^3 C(x)}{(1 - 4x)^{3/2}}$$

$$= x^3 + 7x^4 + 38x^5 + 187x^6 + 874x^7 + \ldots .$$

The sequence $0, 0, 1, 7, 38, 187 \ldots$ is sequence A000531 in the OEIS [84]. Finally, the correspondence between peaks and 213 patterns completes the proof. $\qquad\square$

Now, it is relatively simple to move from the set of indecomposable 123-avoiding permutations to the larger set of all 123-avoiding permutations.

**Theorem 2.2.6.** Let $a_n$ be the number of 213 patterns in $\mathrm{Av}_n\, 123$. Then

$$\sum_{n\geq 0} a_n x^n = \frac{x^3 C(x)^3}{(1 - 4x)^{3/2}} = \frac{x - 1}{2(1 - 4x)} - \frac{3x - 1}{2(1 - 4x)^{3/2}}.$$

**Proof.** Let $A(x)$ be the generating function for the numbers $a_n$, and let $A^*(x)$ denote the generating function for the number of 213 patterns in indecomposable 123-avoiding permutations.

Now, any permutation $\pi$ in $\mathrm{Av}(123)$ can be written uniquely as a skew sum of a nonempty indecomposable 123-avoiding permutation $\sigma$ and another, possibly empty, 123-avoiding permutation $\varphi$. Now, it is clear that any 213 pattern in $\pi$ must be contained entirely in either $\sigma$ or $\varphi$. This leads to the following relation:

$$A(x) = A^*(x)C(x) + xC(x)A(x).$$

Solving for $A$ gives

$$A(x) = \frac{A^*(x)C(x)}{1 - xC(x)} = C^2(x)A^*(x).$$

Lemma 2.2.5 now implies

$$A(x) = \frac{x^3 C(x)^3}{(1 - 4x)^{3/2}}.$$

$\square$

From here, we obtain the generating functions of the other patterns simply by relating their enumerations with the one already obtained. The following two observations provide linear relations between these numbers. The first follows from the simple fact that any three entries must form some 3-pattern.

Lemma 2.2.7. On the set $\mathrm{Av}_n(123)$, we have that

$$\nu_{132} + \nu_{213} + \nu_{231} + \nu_{312} + \nu_{321} = c_n \binom{n}{3}.$$

Proof. Both sides count the total number of 3-patterns within the class $\mathrm{Av}_n(123)$. The right-hand-side is the total number of ways of choosing three indices in any 123-avoiding permutation. Each of these choices is an occurrence of a 3-patterns other than 123, which is counted by the left-hand-side. $\square$

The next lemma provides a relationship between the numbers $\nu_{132}, \nu_{213}, \nu_{231}$, and $\nu_{312}$ by counting the total number of 3-patterns which contain a non-inversion (an occurrence of 12).

Lemma 2.2.8. The following equality holds on the set $\mathrm{Av}_n(123)$:

$$2\nu_{132} + 2\nu_{213} + \nu_{231} + \nu_{312} = (n - 2)\nu_{12}.$$

Proof. Rewrite this equation as

$$(n - 2)\nu_{12} - (\nu_{132} + \nu_{213}) = \nu_{132} + \nu_{213} + \nu_{231} + \nu_{312}.$$

Both sides count the total number of length 3 patterns which contain at least one non-inversion. Indeed, the right-hand-side counts all 3-patterns except for 321. The left-hand-side builds such a pattern by first choosing a 12 pattern, and then adding another entry to create a 3-pattern. However, this overcounts the patterns 132 and 213, since each of these contains two 12-patterns, so we subtract these off to correct the equality. $\square$

The generating functions for the numbers $c_n\binom{n}{3}$ and $(n-2)\nu_{12}$ can be determined from the generating functions we already have. These equations can be obtained using techniques explained in Section 1.3.4.

Lemma 2.2.9. Letting $J(x) = \sum_{n\geq 0}\nu_{12}(\mathrm{Av}_n(123))x^n$, the following identities hold:

$$\sum_{n\geq 0} c_n\binom{n}{3} = \frac{x^3\frac{\mathrm{d}^3}{\mathrm{d}x^3}(C(x))}{6}$$

$$\sum_{n\geq 0}(n-2)\nu_{12}(\mathrm{Av}_n(123)) = x^3\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{J(x)}{x^2}\right).$$

Lemmas 2.2.8 and 2.2.7, coupled with Lemma 2.2.9, establish a system of linear equations with three unknowns, $\nu_{213}, \nu_{231}$, and $\nu_{321}$. Any new linear relation or solution to one of these would solve the system, giving generating functions and exact formulas for the number of all length 3 patterns within $\mathrm{Av}_n(123)$.

The calculation of the $\nu_{213}$ provides that missing piece, but we note that there are many other identities which, once these lemmas are established, are equivalent to Theorem 2.2.6. We collect some of these in Corollary 2.2.13. A direct proof of any of them could help to simplify the arguments presented here while retaining all of the same results, and provide further insight into the connections between $\mathrm{Av}(123)$ and $\mathrm{Av}(132)$. While each of these seem tractable to bijective methods, they have resisted many attempts at a direct proof and we include them here partly out of spite. First, we present the generating functions for the occurrences of 231 and 321, which follow by routine (but technical) computation.

Theorem 2.2.10. The number of 231 (or 312) occurrences is given by

$$\sum_{n\geq 0}\nu_{231}(\mathrm{Av}_n(123))z^n = \frac{3z-1}{(1-4z)^2} - \frac{4z^2-5z+1}{(1-4z)^{5/2}}.$$

Corollary 2.2.11. The total number of 231 occurrences in $\mathrm{Av}_n(123)$ is equal to the number in $\mathrm{Av}_n(132)$.

Theorem 2.2.12. The number of 321 occurrences is given by

$$\sum_{n\geq 0}\nu_{321}(\mathrm{Av}_n(123))z^n = \frac{8z^3-20z^2+8z-1}{(1-4z)^2} - \frac{36z^3-34z^2+10z-1}{(1-4z)^{5/2}}.$$

Corollary 2.2.13. The following identities hold

$$\nu_{21}(\mathrm{Av}_n(123)) = 2\nu_{213}(\mathrm{Av}_n^*(123))$$

$$\nu_{213}(\mathrm{Av}_n(123)) + \nu_{231}(\mathrm{Av}_n(123)) = \nu_{231}(\mathrm{Av}_{n-1}^*(123))$$

$$C(z)\left(\sum_{n\geq 0}\nu_{213}(\mathrm{Av}_n(123))z^n\right) = zC'(z)\left(\sum_{n\geq 0}\nu_{12}(\mathrm{Av}_n(123))z^n\right)$$

$$\sum_{n\geq 0}\nu_{213}(\mathrm{Av}_n^*(132)z^n) = \sum_{n\geq 0}\left(\nu_{132}(\mathrm{Av}_n^*(123)) + \nu_{231}(\mathrm{Av}_n^*(123))\right)z^n.$$

Now we can do some analysis of the main sequences. Using some standard generating function analysis [43], we find that the asymptotic growth of the number of length 3 patterns are as follows:

$$\nu_{213}(\mathrm{Av}_n(123)) \sim \sqrt{\frac{n}{\pi}}4^{n-1}$$

$$\nu_{231}(\mathrm{Av}_n(123)) \sim \frac{n}{2}4^{n-1}$$

$$\nu_{321}(\mathrm{Av}_n(123)) \sim \frac{2}{3}\sqrt{\frac{n^3}{\pi}}4^{n-1}.$$

We see that the three sequences each differ by a factor of approximately $\sqrt{n}$. Surprisingly, this is the same factor that the sequences $\nu_{123}, \nu_{231}, \nu_{321}$ differ by in the class $\mathrm{Av}(132)$, as seen in [24].

Each of these generating functions are simple enough that exact formulas can be obtained with relatively little hassle. One could argue that the asymptotic values are more interesting and provide more insight than the complicated formulas, but we present them here for completeness.

Corollary 2.2.14. Let $a_n = \nu_{132}(\mathrm{Av}_n(123))$, $b_n = \nu_{213}(\mathrm{Av}_n(123))$, and $d_n = \nu_{321}(\mathrm{Av}_n(123))$. Then we have that

$$a_n = \frac{n+2}{4}\binom{2n}{n} - 3\cdot 2^{2n-3}$$

$$b_n = (2n-1)\binom{2n-3}{n-2} - (2n+1)\binom{2n-1}{n-1} + (n+4)\cdot 2^{2n-3}$$

$$d_n = \frac{1}{6}\binom{2n+5}{n+1}\binom{n+4}{2} - \frac{5}{3}\binom{2n+3}{n}\binom{n+3}{2} + \frac{17}{3}\binom{2n+1}{n-1}\binom{n+2}{2}$$
$$- 6\binom{2n-1}{n-2}\binom{n+1}{2} - (n+1)\cdot 4^{n-1}.$$

## Larger Patterns

Some of these same techniques are applicable to larger patterns. For example, we can easily modify Lemmas 2.2.8 and 2.2.7 to apply to patterns of all sizes. This leads to increasingly complicated expressions, but this simple idea can be used to prove the following proposition.

Proposition 2.2.15. Let $k \in \mathbb{Z}^+$, and $\sigma$ be any permutation in $\mathfrak{S}_k$ other than the decreasing permutation. Then for $n$ large enough, we have that

$$\nu_{k\ldots 321}(\mathrm{Av}_n(123)) > \nu_\sigma(\mathrm{Av}_n(123)).$$

Proof. Let $\mathcal{D}$ be the set of permutation in $\mathfrak{S}_k$ which are not the decreasing permutation. As in Lemma 2.2.8, we can express the number $\binom{n-2}{k-2}\nu_{12}(\mathrm{Av}_n(123))$ as a positive linear combination of all of $\nu_\sigma(\mathrm{Av}_n(123))$ where $\sigma \in \mathcal{D}$. As in Lemma 2.2.7, we can express $\binom{n}{k}c_n$ as the sum of all $\nu_\rho(\mathrm{Av}_n(123))$ where $\rho \in \mathfrak{S}_n$. It follows that there is a positive integer $m$ and positive integers $e_i$ such that

$$\binom{n}{k}c_n - m\binom{n-2}{k-2}\nu_{12}(\mathrm{Av}_n(123)) = \nu_{k\ldots 321} - \sum_{\sigma \in \mathcal{D}} e_i\nu_\sigma(\mathrm{Av}_n(123)).$$

Asymptotic analysis shows that the left hand side is eventually positive, and so the first term on the right side eventually outgrows the second term, which completes the proof. $\square$

# – Chapter 3 –

# Pattern Avoiding Involutions

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In this chapter, we investigate sets of pattern-avoiding involutions . While the enumeration of pattern-avoiding permutations has become a major topic of research in recent years, involutions have been largely overlooked. In particular, we focus on finding the Stanley-Wilf limit for sets of involutions which avoid patterns of length four.

Pattern-avoiding involutions were first considered by Simion and Schmidt [76], who enumerated the involutions avoiding any length three pattern. As in the case for permutations, the situation quickly becomes more complicated for longer patterns. We begin this chapter by examining the simple 123 involutions, which will be our primary tool. This chapter is based in part on [26].

## § 3.1  Definitions and Context

Definition 3.1.1. For a given permutation $\beta$, let $\mathrm{Av}^I(\beta)$ denote the set of $\beta$-avoiding involutions, the set of involutions (permutations which are their own inverse) which do not contain $\beta$. Let $\mathrm{Av}_n^I(\beta)$ be the set of permutations of length $n$ within this set.

Note that $\mathrm{Av}^I(\beta)$ is not necessarily a class, as the set of all involutions is not closed under the pattern ordering. However we can apply many of the same ideas in order to enumerate these sets. Clearly, $\mathrm{Av}^I(\beta) \subseteq \mathrm{Av}(\beta)$, and so the Marcus-Tardos theorem states that each set has a finite upper growth rate.

Note that due to the symmetry of inversion ($\sigma \prec \pi$ if and only if $\sigma^{-1} \prec \pi^{-1}$), these classes are not principally based in the classical sense. Indeed, $\mathrm{Av}_n^I(\beta) = \mathrm{Av}_n^I(\beta, \beta^{-1})$ for any permutation $\beta$. For simplicity of notation, and to parallel the work done in permutations, we write such a set with only a single basis element.

## Previous Results

Two patterns $\beta, \tau$ are involution Wilf-equivalent if $|\mathrm{Av}_n^I(\beta)| = |\mathrm{Av}_n^I(\tau)|$. Simion and Schmidt completed the classification of the involution Wilf-equivalence classes of patterns of length three in their 1985 paper [76] by showing that, for all patterns $\beta \in \{123, 132, 213, 321\}$ and $\sigma \in \{231, 312\}$,

$$|\mathrm{Av}_n^I(\beta)| = \binom{n}{\lfloor n/2 \rfloor} \quad \text{and} \quad |\mathrm{Av}_n^I(\sigma)| = 2^{n-1}.$$

Extending the work of Guibert, Pergola, and Pinzani [49], Jaggard [56] classified the eight involution Wilf-equivalence classes for length four patterns. Of these classes, only two have been successfully enumerated: Gessel [46] counted the set $\mathrm{Av}_n^I(1234)$, while Brignall, Huczynska, and Vatter [28] provided the enumeration for $\mathrm{Av}_n^I(2413)$. In this chapter we enumerate two of these unknown sets ($\mathrm{Av}_n^I(1342)$ and $\mathrm{Av}_n^I(2341)$), and provide bounds for a third ($\mathrm{Av}_n^I(1324)$).

Jaggard [56] computed the values $|\mathrm{Av}_n^I(\beta)|$ for each $\beta$ of length four, up to $n = 11$. This data (Table 3.1.1) suggests an ordering on the eight classes, which we will show is misleading. For example, it seems clear from his data that there are more involutions avoiding 2341 than avoiding 1234. However, there are exponentially more 1234 avoiding involutions, as we will soon show.

## Simple Involutions

Our primary tool will be the substitution decomposition. Inflations and involutions are linked by the following theorem, which provides a recipe for constructing new involutions from simples.

Theorem 3.1.2 (Brignall, Huczynska, Vatter [28]). Let $\sigma \neq 21$ be a simple permutation of length $m$, and $\alpha_1, \alpha_2, \ldots \alpha_m$. Then $\pi = \sigma[\alpha_1, \alpha_2, \ldots \alpha_m]$ is an involution if and only if $\sigma$ is an involution and $\alpha_i = \alpha_{\sigma_i}^{-1}$. Further, the skew

Table 3.1.1: The enumerations of involutions avoiding a pattern $\beta$ of length 4 for $n = 5, \ldots, 11$, as presented by Jaggard [56] (ordered by the last row).

|                        | 1324 | 1234 | 4231 | 2431 | 1342 | 2341 | 3421 | 2413 |
|------------------------|------|------|------|------|------|------|------|------|
| $|\text{Av}_5^I(\beta)|$  | 21   | 21   | 21   | 24   | 24   | 25   | 25   | 24   |
| $|\text{Av}_6^I(\beta)|$  | 51   | 51   | 51   | 62   | 62   | 66   | 66   | 64   |
| $|\text{Av}_7^I(\beta)|$  | 126  | 127  | 128  | 154  | 156  | 170  | 173  | 166  |
| $|\text{Av}_8^I(\beta)|$  | 321  | 323  | 327  | 396  | 406  | 441  | 460  | 456  |
| $|\text{Av}_9^I(\beta)|$  | 820  | 835  | 858  | 992  | 1040 | 1124 | 1218 | 1234 |
| $|\text{Av}_{10}^I(\beta)|$ | 2160 | 2188 | 2272 | 2536 | 2714 | 2870 | 3240 | 3454 |
| $|\text{Av}_{11}^I(\beta)|$ | 5654 | 5798 | 6146 | 6376 | 7012 | 7273 | 8602 | 9600 |

decomposable involutions are either of the form $21[\alpha_1, \alpha_2]$ with $\alpha_1 = \alpha_2^{-1}$ or $321[\alpha_1, \alpha_2, \alpha_3]$ with $\alpha_1 = \alpha_3^{-1}$ and $\alpha_2 = \alpha_2^{-1}$.

Describing classes as restricted inflations of their simple permutations is a new and useful method for enumerating classes of permutations [5], and we adapt this method to pattern-avoiding involutions. As we will show, the simple 2341-avoiding and 1342-avoiding involutions are (almost) the same as the simple 123-avoiding involutions. The enumerations of these sets can then be obtained by appropriately inflating these 123-avoiding involutions.

## § 3.2   Simple 123-Avoiding Permutations

We step back from involutions briefly, and investigate the simple 123-avoiding permutations. This investigation, while interesting on its own, provides a gentle introduction to the generating function techniques of Section 3.3. In particular, we mirror the techniques used by Albert and Vatter [10] to construct and analyze a generating function for the 123-avoiding permutations.
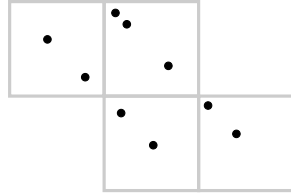
Figure 3.2.1: The staircase decomposition for the permutation 759381642.

## The Staircase Decomposition

In Section 1.3.2 we investigated the geometric structure of the class Av 123, and showed that it contains infinitely many simple permutations. While this class is not a grid class [6], it can be defined using similar language. The staircase decomposition of Av 123 allows one to utilize many of the specialized techniques which are typically only applicable to grid classes, and is central to our study.

Every permutation $\pi \in$ Av 123 can be written as a union of two increasing sequences of entries (the left-to-right minima and the right-to-left maxima). The plot of such a permutation can be fit into a descending staircase of blocks, the contents of which are monotone decreasing. See Figure 3.2.1. In general, such a decomposition is not unique, but for simple 123-avoiding permutations we can define a unique gridding as follows: let the first cell contain the longest decreasing prefix of the permutation, each eastward cell contain all entries whose value is greater than the smallest in the previous cell, and each southward cell contain all entries to the left of the rightmost entry of the previous cell.

This staircase decomposition was first introduced in [3] in the study of subclasses of Av 321. As 123 is the complement of 321, our decomposition is a mirror image of theirs. Note that this decomposition separates the left-to-right minima and right-to-left maxima. We will use this fact later to build a bivariate generating function that keeps track of these entries separately.

## Iterative Process

Let $f = \sum_{\pi \in \mathrm{Av}_n 123} x^n$. We follow the exposition presented by Albert and Vatter in [10] by first giving an almost correct derivation, then fixing two small errors to obtain the correct result.
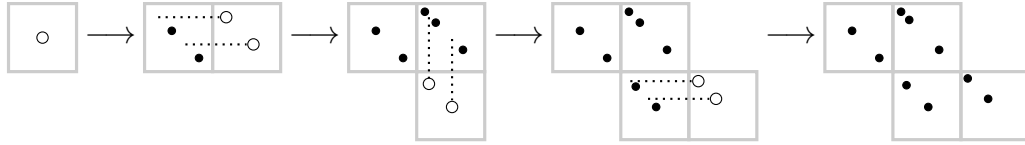
Figure 3.2.2: The evolution of the permutation 759381642 by our recurrence.

We can build a simple 123-avoiding permutations iteratively using the staircase decomposition by filling one cell at the time. We must, however, be careful to ensure simplicity at each step along the way. To this end, we fill up an infinite staircase with filled dots and hollow dots; a filled dot represents an entry of the permutation, while a hollow dot represents a region which must be filled by at least one entry in order to maintain simplicity. Filled dots can be filled with a monotone run of entries, but each pair must be split by a hollow dot in the next cell. Such a diagram with no hollow dots represents a simple 123-avoiding permutations, while a diagram with hollow dots is still a work in progress. Since there are only two cells 'active' at a time (the current one, and the next one), we can represent this process as an iterative system, and our goal is then to find a fixed point of the iteration.

We build $f$ one cell at a time. At step one, we have a single hollow dot in the first cell. At step two, we can fill this hollow dot with a descending run of filled dots, but each pair of these necessitates a hollow dot in the next cell to split them. During step three, each hollow dot in cell two can be filled with a descending run, but again we must place hollow dots in cell three to maintain simplicity. See Figure 3.2.2 for an example of this development.

Let $f_i$ be the generating function at stage $i$ of this evolution, with the exponent of $x$ indicating the number of filled dots and the exponent of $y$ indicating hollow dots (so $f_1 = y$). A hollow dot can be filled with a run of filled dots, each pair of which requires a hollow dot, and we have the option of placing a new hollow dot above the run. It follows then that in each step, each occurrence of $u$ will be replaced by

$$x(1+y) + x^2(y+y^2) + x^3(y^2+y^3) + \cdots = \frac{x(1+y)}{1-xy}.$$

Thus, we have

$$f_1(x,y) = y, \quad f_2 = f_1\left(x, \frac{x(1+y)}{1-xy}\right) = \frac{x(1+y)}{1-xy}, \quad f_{i+1} = f_i\left(x, \frac{x(1+y)}{1-xy}\right)\ldots.$$

Since we are interested in permutations with arbitrarily many staircase cells, we want to find the limit $f = \lim_{n \to \infty} f_i$. It follows then that $f$ is a fixed point of the iteration $x \to x; y \to \frac{x(y+1)}{1-xy}$. Since $f(x, y) = f\left(x, \frac{x(y+1)}{1-xy}\right)$, can solve for $x$ to find

$$y = \frac{x(y+1)}{1-xy} \implies y = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}.$$

Thus we have

$$f = f_1\left(x, \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}\right)$$
$$= \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x}$$
$$= x + x^2 + 2x^3 + 4x^4 + 9x^5 + 21x^6 + \ldots.$$

These coefficients are the Motzkin numbers, a well-studied and understood sequence (sequence A001006 in the OEIS [84]), but are unfortunately not the number of simple 123-avoiding permutations. This is due to the aforementioned errors, which we will now correct.

## Correcting the Errors

Our iteration was correct, but there are some slight discrepancies arising in the first two steps of the iteration which must be accounted for. In the second step, the 'optional' hollow dot above the topmost element is actually required, else the permutation will start with its largest entry (and therefore not be simple). Furthermore, when this required dot is inflated in the third step, the optional dot is in fact forbidden, else we will violate the greediness of the gridding. See Figure 3.2.3 for an illustration.

Fortunately, however, these issues only affect the first three iterations: afterwards, the iteration works as initially described. We can therefore compensate by simply computing the first three by hand, and then plug in the value of $y$ which leads to the fixed point, as found above. As above, we have $f_1 = y$. Since the next optional point is required and will be treated differently in the next step, we mark it with a $t$ to differentiate it from the standard hollow dots. Thus

$$f_2(x, y, t) = \frac{xt}{1 - xy}.$$
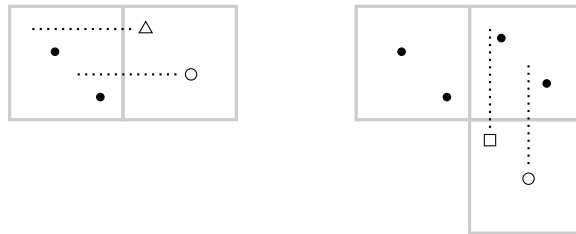
Figure 3.2.3: The hollow triangle represents the location of the hollow dot which is required, and the hollow square represents the location of the hollow dot which is forbidden.

To compute $f_3$, we perform the standard iteration on the variable $y$, and change the variable $t$ into a generating function representing runs of filled dots with no option to place one above. This leads to

$$f_3 = f_2 \left( x, \frac{x(y+1)}{1 - xy}, \frac{x}{1 - xy} \right).$$

At this point the standard iteration, taken to infinity, produces the correct generating function, which can be used to enumerate the class Av(123), as shown in Section 1.3.2.

$$
\begin{aligned}
f(z) = f_3 & \left( x, \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x} \right) \\
& = \frac{2x^2}{1 + x^2 + (1 + x)\sqrt{1 - 2x - 3x^2}} \\
& = x^2 + 2x^4 + 2x^5 + 7x^6 + 14x^7 + 37x^8 + \dots .
\end{aligned}
\tag{3.1}
$$

The coefficients are sequence A187306 in the OEIS [84].

## § 3.3   Simple 123-Avoiding Involutions

We return now to the problem of enumerating the simple 123-avoiding involutions. Though this is more difficult, the iterative development of the generating

function for the simple 123-avoiding permutations presented above forms the basis for our study. As we will eventually be inflating these involutions to enumerate the avoiding sets, we want to keep track of left-to-right minima (ltrmin), right-to-left maxima (rtlmax), and fixed points (fp) separately. Our goal here will be to find the generating functions $s^{(i)}(u, v)$, defined below

$$s^{(i)}(u, v) := \sum_{\substack{\text{simple } \sigma \in \text{Av}_n^I \, 123 \\ \text{with } \text{fp}(\sigma)=i}} u^{\text{ltrmin}(\sigma)} v^{\text{rtlmax}(\sigma)}.$$

## Extending the Iteration

We proceed defining an iterative process similar to the development presented in Section 3.2. This iterative process can be extended in a variety of ways, as we will soon see. Note, for example, that we could have used a two-part recurrences to keep track of the top cells and bottom cells separately; it follows then that this process can be used to enumerate the left-to-right minima separately from the right-to-left maxima with a more technical (but no more conceptually difficult) computation. The following sections will rely on some tedious and technical calculations, but the core ideas are relatively easy to express.

Geometrically, an involution is a permutation whose plot is symmetric about the line $y = x$ through the plane. As such, we can build a simple 123-avoiding involution using the staircase decomposition starting from the center, and building out in both directions. Figure 3.3.1 shows the two possible cases. When there is a single fixed point, the case is uniquely determined by considering whether the fixed point is a rtl-max or ltr-min.

As in Section 3.2, we start with a single hollow dot in the center cell, and proceed outwards in both directions simultaneously while mainaining symmetry. However, the number of fixed points determines how we proceed from here. In the interest of clarity, we develop the single fixed point case in detail, and give a sketch of the details of the other cases.

## Single Fixed Point

We first develop the generating function $\widehat{s}^{(1)}(z) = s^{(1)}(z, z)$, which only keeps track of the number of such permutations of each length and ignores the ltr-min and rtl-max, and then indicate how to obtain the more general $\widehat{s}^{(1)}(u, v)$.
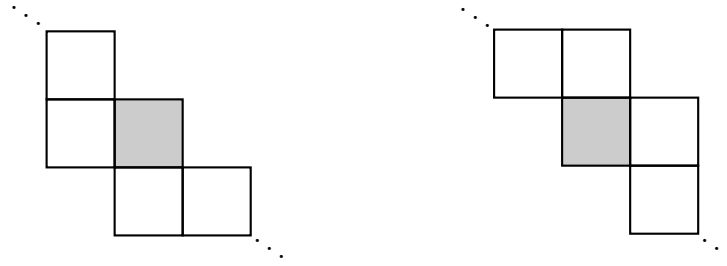
Figure 3.3.1: The diagrams on which we can draw simple permutations $\sigma \in$ $\text{Av}^I(123)$ that contain a single fixed point. The starting point of the iteration is the shaded cell.
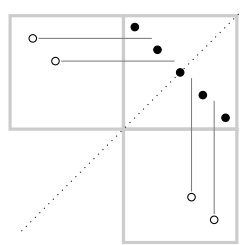


Figure 3.3.2: An example of a bad placement of splitting entries that leads to a skew decomposable permutation.

The set of all simple (123)-avoiding involutions with exactly one fixed point can be partitioned based on whether the fixed point is a ltr-min or a rtl-max. These two sets are in bijection with each other, as mapping a permutation to its reverse complement maps one set to the other. Therefore it suffices to enumerate those in which the fixed point is a rtl-max, and then simply double the result to obtain the full generating function (or in the case of $s^{(1)}$, add the result to itself with the rtl-max and ltr-min switched).

Assume that the fixed point is a rtl-max. The first hollow dot must then be inflated by an odd number of filled dots (with the fixed point at the center). The hollow dots here behave a bit differently than in the previous section: each pair of filled dots can be split either below or to the left, or both. Of these possible splittings, one of them (see Figure 3.3.2 yields a skew decomposable permutation, violating the simplicity condition. We can account for this with a calculation which takes the symmetry into account.

Suppose that the initial cell (which contains the fixed point) contains a total of $2k+1$ entries. It follows that $k$ of these entries lie below and to the right of

the fixed point. Because $\sigma$ is simple, each of the $2k$ adjacent pairs of entries in this cell must be separated by entries in the cell below, by entries in the cell to the left, or by both types of entries. Each adjacent pair lying above and to the left of the fixed point has a corresponding adjacent pair (its image under inversion) which lies below and to the right of the fixed point; if we split the former to the left, then the inverse-image of the separating entry splits the latter below, and vice versa.

We can split each adjacent pair with as few as $k$ entries in the cell below the fixed point, and this can be done in $2^k$ ways by picking which of each two corresponding pairs of entries to split below. Similarly, the number of ways to have $k+i$ separating entries in the cell below is given by $2^{k-i}\binom{k}{i}$, since we can first pick which of the $i$ corresponding pairs of gaps between entries are split both to the left and below, then we choose which of each of the remaining $k-i$ corresponding pairs are split below or to the left.

As in the derivation in Section 3.2.3, there are a few slight difficulties we must take into account, but again they only arise in the first three steps of the iteration. We therefore construct these three steps by hand, before letting the iteration go to infinity.

Every choice of separating entries leads to a simple permutation except one: if we split all of the pairs of entries to the right of the fixed point by entries below the initial cell and split no other pairs, then the resulting permutation will be skew decomposable, as shown in Figure 3.3.2. We compensate for these "bad cases" by subtracting the term $x/(1-x^2y)$.

It follows that

$$\widehat{s}_2^{(1)}(x,y,z) = \frac{2z}{y}\left(\sum_{k=0}^{\infty}\left(x^{2k+1}\sum_{i=0}^{k}2^{k-i}\binom{k}{i}y^{k+i}\right) - \frac{x}{1-x^2y}\right)$$
$$= \frac{2x^3z(1+y)}{(1-x^2y)(1-2x^2y-x^2y^2)}.$$

The 2 in $\widehat{s}_2^{(1)}$ accounts for both cases, where the fixed point is a rtl-max and a ltr-min, while the $z/y$ factor counts the topmost hollow dot in the cell below the fixed point by $z$ instead of $y$, as it will require special care. By our definition of greediness, this topmost hollow dot, shown as a hollow square in Figure 3.3.3, is not allowed to produce an hollow dot above it in the next cell. Therefore, when substituting for $z$ to obtain $\widehat{s}_3^{(1)}$, we substitute $x^2/(1-x^2y)$ instead of $x^2(1+y)/(1-x^2y)$. As such, we obtain

$$s_1^{(1)} \qquad\qquad s_2^{(1)} \qquad\qquad s_3^{(1)}$$
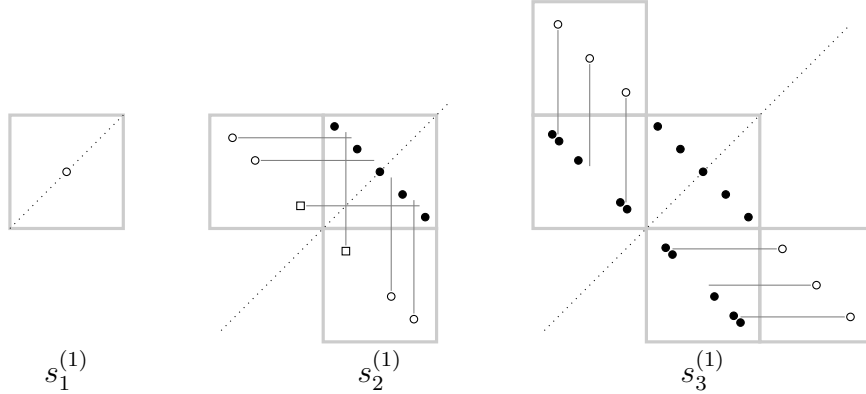
Figure 3.3.3: Three stages of the recurrence, in the case when the single fixed point is a right-to-left maximum.

$$\widehat{s}_3^{(1)}(x, y) = s_2\left(x, \frac{x^2(1+y)}{1 - x^2y}, \frac{x^2}{1 - x^2y}\right).$$

After this point, the same iteration leads from $\widehat{s}_i^{(1)}$ for $\widehat{s}_{i+1}^{(1)}$ for all $i \geq 3$. Since the filled dots above the center cell are completely determined by those below, we need only consider the expansion of hollow dots in the bottom cell. Their expansion is exactly as in Section 3.2, except that each expansion of a hollow dot adds dots in both the bottommost cell and the topmost. Letting $i \geq 3$, this leads to the relation

$$\widehat{s}_{i+1}^{(1)}(x, y) = \widehat{s}_i^{(1)}\left(x, \frac{x^2(y+1)}{1 - x^2y}\right). \tag{3.2}$$

To find the limit of this iteration, it suffices to find at fixed point, and plug it in for $y$ in the expression $\widehat{s}_3^{(1)}(x, y)$. This leads to

$$\widehat{s}^{(1)}(x) = \widehat{s}_3^{(1)}\left(x, \frac{1 - x^2 - \sqrt{1 - 2x^2 - 3x^4}}{2x^2}\right)$$

$$= \frac{2x^5\left(1 + x^2 + \sqrt{1 - 2x^2 - 3x^4}\right)}{(1 + x^2)^2\left(1 - 3x^2 + (1 - 2x^2)\sqrt{1 - 2x^2 - 3x^4}\right)} \tag{3.3}$$

$$= 2x^5 + 2x^7 + 10x^9 + 22x^{11} + 68x^{13} + 184x^{15} + 530x^{17} + \ldots$$

Note that an involution with only a single fixed point is necessarily of odd length, and so the power series in equation 3.3 contains no terms with even powers.

Rather than repeat this full derivation to find $s^{(1)}(u,v)$, we simply indicate the changes to make to the above calculation. Recall that $u$ (resp. $v$) represents a filled dot which is a ltr-min (reps. rtl-max), and introduce new variables $y_u$ and $y_v$ which represent hollow dots which are ltr-min and rtl-max, respectively. We can assume that the fixed point is a rtl-max, because then we can just add this generating function to itself with the $u$ and $v$ swapped to obtain the full generating function $s^{(1)}(u,v)$.

A hollow dot in a lower cell, represented by $y_u$, then leads to filled dots in the lower cell (represented by $u$) and hollow dots in an upper cell (represented by $y_v$s). A similar description of hollow dots in an upper cell leads to the iterations

$$
\begin{aligned}
y_u &\mapsto \frac{u^2(1+y_v)}{1-u^2 y_v} \\
y_v &\mapsto \frac{v^2(1+y_u)}{1-v^2 y_u}.
\end{aligned}
\tag{3.4}
$$

To find the fixed point of this iteration, we can compute two iterations and solve. That is, solve for $y_v$ in the expression

$$
\begin{aligned}
y_v &= \frac{v^2(1+y_u)}{1-v^2 y_u} \\
&= \frac{v^2\left(1+\frac{u^2(1+y_v)}{1-u^2 y_v}\right)}{1-v^2\frac{u^2(1+y_v)}{1-u^2 y_v}} \\
&= \frac{v^2(1+u^2}{1-u^2 v^2 - u^2 y_v - u^2 v^2 y_v}.
\end{aligned}
$$

Solving this system yields the fixed point of the iteration:

$$
y_v = \frac{1-u^2 v^2 - \sqrt{1-6u^2 v^2 - 4u^2 v^4 - 4u^2 v^2 - 3u^4 v^4}}{2u^2(1+v^2)}.
\tag{3.5}
$$

Mirroring the construction of $\widehat{s}^{(1)}$, we can derive $s_3^{(1)}(u,v,y_u,y_v)$ by hand using these extra variables. Note that there will be no $y_u$ terms in this expression, because at the third stage the only hollow dots will be in cells corresponding to left-to-right minima. The limit of the iteration is then given by plugging in the fixed point to this expression. This gives the generating function for the

case when the fixed point is a rtl-max, but by swapping occurrences of $u$ and $v$ and then adding it back to itself, we obtain the full generating function $s^{(1)}$.

$$s^{(1)}(u,v) = \frac{u^2 v^3 (1+u^2)(1+2v^2+u^2v^2+r)}{(1+v^2)(1-6u^2v^2-4u^2v^4-4u^4v^2-3u^4v^4+(1-3u^2v^2-2u^4v^2)r)}$$
$$\text{where} \quad r := \sqrt{1-6u^2v^2-4u^2v^4-4u^4v^2-3u^4v^4}.$$

$$(3.6)$$

## Zero and Two Fixed Points

We now turn to the remaining two cases, in which the involution has no fixed points or two fixed points. The derivation is largely the same as the single fixed point case, so we simply sketch the changes that must be made. Each of these has their own idiosyncrasies, but they can be dealt with easily.

First, consider the case of involutions with no fixed points. Such a permutation cannot be uniquely gridded, because the diagonal line on which the fixed points would lie can be taken to pass through either a lower or upper central cell. It follows, however, that every involution with no fixed points can be decomposed in both ways, and so it suffices to assume that the diagonal line passes through an upper cell, and take this to be our initial cell.

Since there is no fixed point, this initial cell must have an even number of elements. We build the first three iterations by hand, in the same manner as the one fixed point case, before substituting the fixed point of the iteration. A similar bad case (Figure 3.3.2) must be accounted for, and the same restriction applies to the topmost hollow dot of the second cell, as shown in Figure 3.3.3.

The generating function $\widehat{s}^{(0)}$ enumerating the class according to length, and the corresponding bivariate generating function $s^{(0)}$ enumerating the ltr-min and ltr-max entries are given below.

$$\widehat{s}^{(0)}(x) = \frac{2x^6(1+x^2-\sqrt{1-2x^2-3x^4})}{2-2x^2-10x^4-6x^6+(2-6x^4-4x^6)\sqrt{1-2x^2-3x^4}} \qquad (3.7)$$
$$= x^8 + 2x^{10} + 8x^{12} + 22x^{14} + 68x^{16} + 198x^{18} + 586x^{20} + \cdots.$$

$$s_1^{(1)} \qquad\qquad\qquad s_2^{(1)} \qquad\qquad\qquad s_3^{(1)}$$
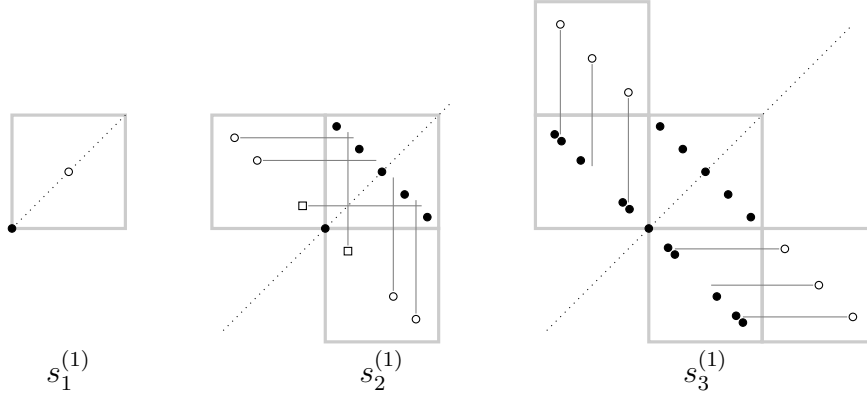
Figure 3.3.4: The decomposition of an involution with two fixed points.

$$s^{(0)}(u,v) = \frac{2u^2v^4(1+u^2)(1+2u^2+u^2v^2-r)}{(1-u^2v^2+r)(1-6u^2v^2-4u^2v^4-4u^4v^2-3u^4v^4+(1+2v^2+u^2v^2)r)}$$
$$\text{where} \quad r := \sqrt{1-6u^2v^2-4u^2v^4-4u^4v^2-3u^4v^4}.$$

$$(3.8)$$

Finally, we consider the case of involutions with two fixed points. As with the case of no fixed points, such a permutation can be drawn on either of the two diagrams shown in Figure 3.2.1. To ensure uniqueness, break our own rules slightly to say that the topmost fixed point is the center of the initial cell, while the bottom fixed point lies on the southwest corner of this cell. See Figure 3.3.4 for an example, and note that in this case, the hollow square is allowed to produce a hollow dot above itself in the next cell, as this no longer violates the greediness of the decomposition (because of the lower fixed point).

Note also that the 'bad case' (Figure 3.3.2) is no longer a bad case, as the lower fixed point maintains simplicity. Also, we are now allowed to add a hollow dot in the second cell immediately to the right of the lower fixed point, as long as we insert a hollow dot above this entry in the third cell. Taking these factors into consideration, we have the following generating functions for $\widehat{s}^{(0)}$ and $s^{(0)}$.

$$\widehat{s}^{(0)}(x) = \frac{x^4(2+5x^2+3x^4-(2+x^2)\sqrt{1-2x^2-3x^4})}{1-x^2-5x^4-3x^6+(1+2x^2+x^4)\sqrt{1-2x^2-3x^4}} \qquad (3.9)$$
$$= 3x^6 + 4x^8 + 15x^{10} + 36x^{12} + 105x^{14} + 288x^{16} + 819x^{18} + \cdots.$$

$$s^{(0)}(u,v) = \frac{uv^3\left(2 + 7u^2 + 4u^2v^2 + 4u^4 + 3u^4v^2 - (2+u^2)r\right)}{1 - 6u^2v^2 - 4u^2v^4 - 4u^4v^2 - 3u^4v^4 + (1 + 2v^2 + u^2v^2)r} \quad (3.10)$$

$$\text{where} \quad r := \sqrt{1 - 6u^2v^2 - 4u^2v^4 - 4u^4v^2 - 3u^4v^4}.$$

We can now combine the generating functions $s^{(0)}, s^{(1)}, s^{(2)}$ to obtain a generating function for all simple 123-avoiding permutations, enumerated by number of left-to-right minima and right-to-left maxima. However, it will be convenient to keep these separate, because in the next section we will explore inflations of these permutations, and oftentimes fixed points have different inflation rules from other entries.

## § 3.4  Enumerating Pattern Avoiding Involutions

We are now in position to enumerate the sets $\mathrm{Av}^I(1342)$ and $\mathrm{Av}^I(2341)$. Our tool for both of these is to first show that the simples in each set (almost) coincides with the simples within $\mathrm{Av}^I(123)$. This allows us to describe each of these sets by inflations of these simples, and so we need only determine what inflations are allowed to enumerate the sets.

### Involutions Avoiding 1342

Clearly, every involution avoiding 1342 must also avoid $1342^{-1} = 1423$. We first show that the set of simples in this set are precisely the 123-avoiding simple involutions. This will be easy once we establish suitable notation.

Definition 3.4.1. Given a permutation class $\mathcal{C}$, define its substitution closure $\langle \mathcal{C} \rangle$ to be the largest class with the same simple permutations as $\mathcal{C}$.

By definition, since $\mathrm{Av}(123) \subseteq \mathrm{Av}(1342, 1423)$, we have that the 123-avoiding simples are contained in $\mathrm{Av}(1342, 1423)$. Atkinson, Ruškuc, and Smith [13] investigated substitution closures, and found that

$$\langle \mathrm{Av}(123) \rangle = \mathrm{Av}(24153, 25314, 31524, 41352, 246135, 415263).$$

Each of these basis elements contains either 1342 or 1423, and so we have the following relation and its consequences.

$$\mathrm{Av}(1342, 1423) \subseteq \langle \mathrm{Av}(123) \rangle.$$

Proposition 3.4.2. The simple permutations within $\mathrm{Av}(1342, 1423)$ are precisely the same as the simple permutations within $\mathrm{Av}(123)$.

Corollary 3.4.3. The simple involutions within $\mathrm{Av}(1342, 1423)$ are precisely the same as the simple involutions within $\mathrm{Av}(123)$.

To enumerate the set we now need only describe the allowable inflations which maintain pattern avoidance and involutionicity. We divide the simples into three classes: first we have the inflations of 1, which themselves must be simple. Then come the inflations of 12 and 21, the sum- and skew-decomposable permutations, respectively. Finally we consider inflations of simples of length greater than three.

We begin by by defining $f$ to be the generating function for the class $\mathrm{Av}(1342, 1423)$ and $f_\oplus$ (resp., $f_\ominus$) the generating function for the sum (resp., skew) decomposable permutations of this class. We then define $g$ to be the generating function for the set $\mathrm{Av}^I(1342)$ and $g_\oplus$ (resp., $g_\ominus$) the generating function for the sum (resp., skew) decomposable 1342-avoiding involutions.

First we describe the sum decomposable permutations $\pi = \alpha_1 \oplus \alpha_2$ counted by $g_\oplus$. By Proposition 1.1.15, we can assure uniqueness of decomposition by requiring that $\alpha_1$ is sum indecomposable. To produce an involution, $\alpha_1$ and $\alpha_2$ must be involutions as well. In order for $\pi$ to avoid the patterns 1342 and 1423, it is required that $\alpha_1$ avoids these patterns, and that $\alpha_2$ avoids the patterns 231 and $312 = 231^{-1}$.

In fact, the class $\mathrm{Av}(231, 312)$, known as the class of layered permutations, consists entirely of involutions because a permutation lies in $\mathrm{Av}(231, 312)$ if and only if it can be expressed as a sum of some number of decreasing permutations. The layered permutations of length $n$ are in bijection with compositions of $n$, and hence there are $2^{n-1}$ permutations of length $n$ in $\mathrm{Av}(231, 312)$. Therefore, $g_\oplus$ satisfies the equation

$$g_\oplus = (g - g_\oplus)\left(\frac{x}{1 - 2x}\right).$$

From this expression it follows that

$$g_\oplus = \frac{gx}{1-x}. \tag{3.11}$$

Next we must briefly consider the permutation class $\mathrm{Av}(1342, 1423)$. Kremer [63, 64] showed that this class is counted by the large Schröder numbers, sequence A006318 in the OEIS [84], and has generating function

$$f(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2}.$$

Since this permutation class is skew closed (because both 1342 and 1423 are skew indecomposable), it follows by Proposition 1.1.15 that, since $f_\ominus = (f - f_\ominus)f$ and $f_\ominus = \frac{f^2}{1+f}$,

$$f - f_\ominus = \frac{f}{1+f} = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4}.$$

This is the generating function for the small Schröder numbers, sequence A001003 in the OEIS [84].

Returning our attention to $\mathrm{Av}^I(1342)$, which is also skew closed, we note that skew indecomposable permutations in this set are of the form $\alpha_1 \ominus \alpha_2 \ominus \alpha_1^{-1}$ where $\alpha_1$ is a skew decomposable member of $\mathrm{Av}(1342, 1423)$ and $\alpha_2$ is an arbitrary (and possibly empty) member of $\mathrm{Av}^I(1342)$. Therefore we see that

$$g_\ominus = \left(f(x^2) - f_\ominus(x^2)\right)(1 + g). \tag{3.12}$$

Lastly, we must enumerate 1342-avoiding involutions which are inflations of simple permutations of length at least four. Any such simple permutation must have at least two right-to-left maxima and by simplicity every right-to-left maximum must have some entry both below it and to the left. Hence to avoid creating a copy of 1342 or 1423, we may only inflate right-to-left maxima by decreasing intervals. An entry which is a left-to-right minimum can be inflated by any permutation in the class $\mathrm{Av}(1342, 1432)$. However, to ensure that the inflated permutation is an involution, we must inflate each fixed point by an involution. Additionally, if we inflate the entry with value $\sigma(i)$ by the permutation $\alpha$, we must make sure to inflate the entry with value $i$ by $\alpha^{-1}$.

Consider $s^{(0)}(u, v)$, which is the generating function for simple involutions of length at least four which avoid 123 and have zero fixed points. To inflate

each right-to-left maximum by a decreasing permutation in a way that yields an involution, we substitute

$$v^2 = \frac{x^2}{1 - x^2}.$$

This follows because if $\sigma(i)$ is a right-to-left maximum of the simple 123-avoiding involution $\sigma$ then the entry with value $i$ will also be a right-to-left maximum, and we must substitute a permutation and its inverse into this pair of entries of $\sigma$. Because the class $\mathrm{Av}(1342, 1423)$ is counted by the large Schröder numbers, the inflations of the simple involutions of length at least four with zero fixed points are counted by

$$s^{(0)}(u, v)\big|_{u^2 = f(x^2),\ v^2 = x^2/(1-x^2)}. \tag{3.13}$$

Recall that $s^{(1)}(u, v)$ counts only those simple involutions whose single fixed point is a right-to-left maximum. Since this fixed point must be inflated by a decreasing permutation, we count inflations of such permutations by

$$\left(\frac{s^{(1)}(u, v)}{v}\bigg|_{u^2 = f(x^2),\ v^2 = x^2/(1-x^2)}\right) \cdot \frac{x}{1 - x}. \tag{3.14}$$

To count those simple involutions whose single fixed point is a left-to-right minimum, we need only swap $u$ and $v$. Thus, inflations of these are counted by the generating function

$$\left(\frac{s^{(1)}(v, u)}{u}\bigg|_{u^2 = f(x^2),\ v^2 = x^2/(1-x^2)}\right) \cdot g. \tag{3.15}$$

Finally, we must account for inflations of those simple involutions which contain exactly two fixed points, one of which is a right-to-left maximum while the other is a left-to-right minimum. These permutations are counted by

$$\left(\frac{s^{(2)}(u, v)}{uv}\bigg|_{u^2 = f(x^2),\ v^2 = x^2/(1-x^2)}\right) \cdot \frac{gx}{1 - x}. \tag{3.16}$$

By summing the contributions of (3.11)–(3.16) and accounting for the single permutation of length 1, one finds that

$$g(x) = \frac{x\left(1 - 2x + x^2 + \sqrt{1 - 6x^2 + x^4}\right)}{2\left(1 - 3x + x^2\right)}.$$

It can then be computed that the growth rate of involutions avoiding 1342 is 1 plus the golden ratio,

$$1 + \frac{1 + \sqrt{5}}{2} \approx 2.62.$$

### Involutions Avoiding 2341

We turn our attention now to enumerating the 2341-avoiding involutions. Note that each involution avoiding 2341 must also avoid $2341^{-1} = 4123$. We begin by examining the simple involutions which avoid these patterns. Note that, in this case, the simple permutations of the class $\mathrm{Av}(2341, 4123)$ are not the same as the simples of $\mathrm{Av}(123)$. When we restrict to involutions, however, we find that the simples of $\mathrm{Av}^I(2341)$ are almost the same as the simples of $\mathrm{Av}^I(123)$.

Theorem 3.4.4. The simple 2341-avoiding involutions are precisely the union of set of 123-avoiding simple involutions along with the permutation 5274163.

We delay the technical proof of this theorem to the end of this section.

Now that we know the simples, we need only determine the ways in which they can be inflated. As in the previous section, we enumerate the 2341-avoiding involutions by separately enumerating the sum decomposable permutations, the skew decomposable permutations, and the inflations of simple permutations of length at least four. Again we define $g$ to be the generating function for the set $\mathrm{Av}^I(2341)$ and $g_\oplus$ (resp., $g_\ominus$) the generating function for the sum (resp., skew) decomposable 2341-avoiding involutions.

In this case we see that $\mathrm{Av}^I(2341)$ is sum closed, so we have

$$g_\oplus = (g - g_\oplus)g.$$

This then leads that

$$g_\oplus = \frac{g^2}{1 + g}. \tag{3.17}$$

By Proposition 3.1.2, the skew decomposable permutations must have the form $321[\alpha_1, \alpha_2, \alpha_1^{-1}]$, where $\alpha_1$ is skew indecomposable and $\alpha_2$ is a (possibly empty) involution. Furthermore, to avoid the occurrence of a 2341 or a 4123 pattern, we must also have that $\alpha_1, \alpha_2 \in \mathrm{Av}(123)$.

Recall that the 123-avoiding permutations are enumerated by the Catalan numbers, which have generating function

$$c(x) = \frac{1 - 2x - \sqrt{1 - 4x}}{2x}.$$

Since the class $\mathrm{Av}(123)$ is skew closed, when we denote the generating function for the skew decomposable 123-avoiding permutation, it follows (as in the previous section) that

$$c - c_\ominus = \frac{c}{1 + c} = x(c + 1).$$

As mentioned in the Section 3.1, Simion and Schmidt [76] proved that

$$|\mathrm{Av}_n^I(123)| = \binom{n}{\lfloor n/2 \rfloor}.$$

These terms are known as the central binomial coefficients, sequence A001405 in the OEIS [84]. These permutations thus have the generating function

$$\frac{1 - 4x^2 - \sqrt{1 - 4x^2}}{4x^2 - 2x}.$$

Therefore, the generating function which counts our choices for the pair $(\alpha_1, \alpha_1^{-1})$ is $x^2(c(x^2) + 1)$, and the generating function for all skew decomposable 2341-avoiding involutions is

$$g_\ominus = \left(x^2 \left(c(x^2) + 1\right)\right) \cdot \left(\frac{1 - 4x^2 - \sqrt{1 - 4x^2}}{4x^2 - 2x} + 1\right) \qquad (3.18)$$

Next, we consider inflations of the simple permutations in $\mathrm{Av}^I(123)$. In both cases, every entry of such a simple permutation can only be inflated by a decreasing permutation, as any inflation by a permutation with an increase would create a copy of 2341 or 4123. Thus inflations of the simple permutations counted by $s^{(0)}$ contribute

$$s^{(0)}(u, v)\big|_{u^2=v^2=x^2/(1-x^2)}. \tag{3.19}$$

Inflations of the simple permutations counted by $s^{(1)}$ contribute

$$2\left(\frac{s^{(1)}(u, v)}{v}\bigg|_{u^2=v^2=x^2/(1-x^2)}\right) \cdot \frac{x}{1-x}. \tag{3.20}$$

Next, inflations of simple permutations counted by $s^{(2)}$ contribute

$$\left(\frac{s^{(2)}(u, v)}{uv}\bigg|_{u^2=v^2=x^2/(1-x^2)}\right) \cdot \left(\frac{x}{1-x}\right)^2. \tag{3.21}$$

Lastly, we consider inflations of 5274163. Because this permutation has three fixed points, the 2341-avoiding involutions formed by inflations of 5274163 are counted by

$$\left(\frac{x^2}{1-x^2}\right)^2 \left(\frac{x}{1-x}\right)^3. \tag{3.22}$$

By combining the contributions (3.17)–(3.22) and accounting for the single permutation of length 1, it can be computed that $g$ has minimal polynomial Therefore, $b$ satisfies the functional equation

$$b = x + \frac{b^2}{1+b} + x^2(c(x^2) + 1)\left(\frac{1 + x + xc(x^2)}{\sqrt{1-4x^2}}\right) + I(x).$$

From this it follows that $b$ has minimal polynomial shown below.

$$t^2g^2 + (48x^{16} - 158x^{15} + 101x^{14} + 334x^{13} - 627x^{12} + 60x^{11} + 801x^{10} - 684x^9 - 231x^8$$
$$+624x^7 - 221x^6 - 162x^5 + 151x^4 - 24x^3 - 17x^2 + 8x - 1)tg$$
$$+(18x^{15} - 51x^{14} + 16x^{13} + 125x^{12} - 169x^{11} - 48x^{10} + 256x^9 - 130x^8 - 131x^7$$
$$+159x^6 - 11x^5 - 60x^4 + 28x^3 + 3x^2 - 5x + 1)tx$$

In the expression above, $t$ is defined as

$$t = 32x^{16} - 120x^{15} + 113x^{14} + 206x^{13} - 540x^{12} + 223x^{11} + 561x^{10} - 725x^9$$
$$+ 26x^8 + 514x^7 - 326x^6 - 55x^5 + 141x^4 - 50x^3 - 4x^2 + 6x - 1.$$

Note that though this minimal polynomial looks complicated, it is in fact quadratic in $g$, so it is not difficult to solve it explicitly. While the explicit solution is even more complicated than the minimal polynomial, this makes it relatively easy to compute the minimal polynomial for the growth rate of $\text{Av}^I(2341, 4123)$, which is

$$x^{16} - 6x^{15} + 4x^{14} + 50x^{13} - 141x^{12} + 55x^{11} + 326x^{10} - 514x^9 - 26x^8 + 725x^7$$
$$-561x^6 - 223x^5 + 540x^4 - 206x^3 - 113x^2 + 120x - 32.$$

The growth rate itself is approximately 2.54.

We now return to the proof of Theorem 3.4.4. The proof is rather technical, and relies on listing and eliminating a variety of cases. This was greatly assisted by Albert's PermLab [1] software.

Proof of Theorem 3.4.4. The proof of this theorem consists of the investigation of many cases relating to the placement of the fixed points in a 2341-avoiding simple involution. Recall that such an involution must also avoid $2341^{-1} = 4123$. To better understand these permutations, we utilize permutation diagrams, depicted in Figures 3.4.1, 3.4.2, and 3.4.3. Each of these diagrams consists of the plot of a permutation, together with a coloring of the cells. A cell is white if we are allowed to insert an entry without creating an occurrence of 2341 or 4123, and dark gray otherwise. A cell is light gray if we explicitly forbid any entries through the course of our arguments. The rectangular hull of a set $S$ of entries is defined to be the smallest axis-parallel rectangle which contains all points of $S$. Finally, the inverse image of a point $(x, y)$ is the point $(y, x)$, equivalent to the image of the point when reflecting across the line $y = x$. These tools will be useful in describing and understanding the various cases of this proof.

Let $\sigma$ be a 2341-avoiding simple involution, and claim that either $\sigma$ avoids 123 or $\sigma = 5274163$. Suppose that $\sigma$ contains at least one 123 pattern. Of all of the possible occurrences of 123, we focus on a single occurrence of this pattern, the one in which the 3 is the topmost possible entry, the 1 is the bottommost for the chosen 3, and the 2 is the rightmost for the chosen 1 and 3. It follows then that $\sigma$ can be drawn on the diagram shown in Figure 3.4.1a. Note that, despite the apparent symmetry, these three entries are not necessarily fixed points, because each white cell could be inflated by different numbers of entries.
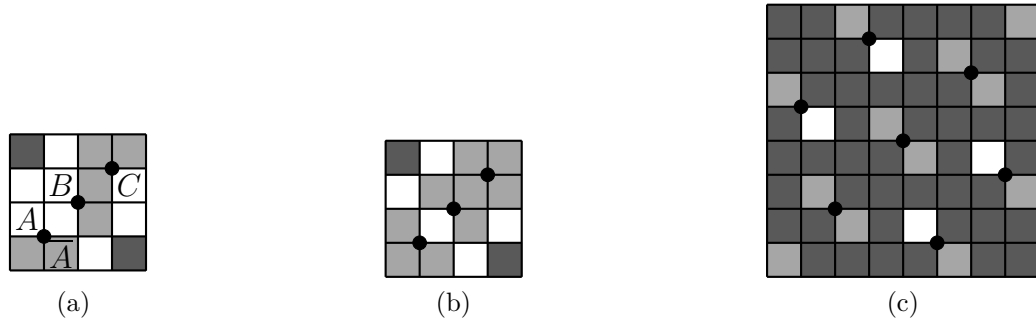
Figure 3.4.1: Permutation diagrams referenced in the proof of Theorem 3.4.4.

Thus, we must consider separate cases in which some combination of these entries lie on the diagonal.

Case 1: For our first case, assume that each of these entries are in fact fixed points. Then, since $\sigma$ is an involution, the cells labelled $A, B, C$ must all be empty, since otherwise the plot would not be symmetric about the line passing through the diagonal. It follows then that $\sigma$ can be plotted on the diagram shown in Figure 3.4.1b. We now claim that $\sigma = 5274163$.

By simplicity, the rectangular hull of the leftmost two entries shown in Figure 3.4.1b must be split by an entry either in the white cell above it or in the white cell to its right. Since $\sigma$ is an involution, it follows then that there are in fact splitting entries in both of these cells. Assume that the splitting entry in the cell above is the topmost possible entry and the one to the right is the rightmost possible. A similar argument applied to the rectangular hull of the rightmost two entries produces a permutation diagram depicted in Figure 3.4.1c.

We now claim that we can go no further. There are only four remaining white cells in Figure 3.4.1c, and no two of these cells shares a row or column. It follows then that by placing entries in any of these cells, we would be creating intervals which cannot be split by any other entry, thus violating simplicity. It follows then that the only simple 2341-avoiding involution which contains an occurrence of 123 in which each entry is a fixed point is the permutation 5274163, as desired.

Case 2: Now suppose that the rightmost entry of our specified 123 occurrence is not a fixed point. It therefore must lie either above or below the reflection line, i.e., it must be either above and to the left or below and to the right of its inverse image. Suppose first that it is below this line of reflection, and so
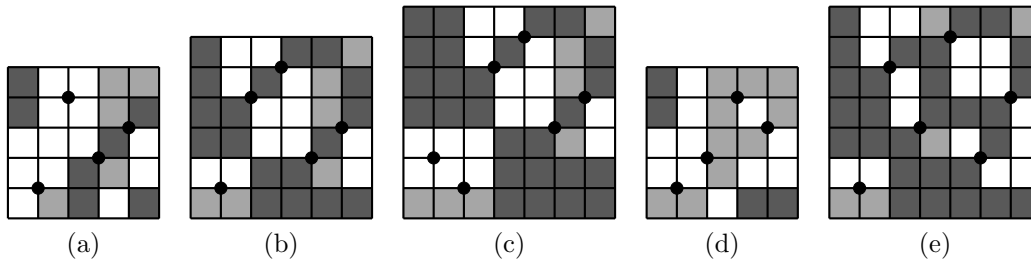
Figure 3.4.2: Permutation diagrams referenced in the proof of Theorem 3.4.4.

its inverse image must lie above and to the left. There is only one candidate cell, the result is shown in Figure 3.4.2a.

Note that, in a general involution, if two entries from an increase (resp., a decrease) then their inverse image also forms an increase (resp., a decrease). It follows then that the third entry from the left shown in Figure 3.4.2a (the 2 of the original 123 pattern) cannot lie above or on the reflection line, and so must lie below. Therefore its inverse image lies above. There is only one appropriate white cell in which this entry can lie, as shown in Figure 3.4.2b. If the leftmost entry in this figure were a fixed point, then the permutation would begin with its smallest entry, violating simplicity. This entry therefore lies below the reflection line, and has an inverse above and to its left. This leads to Figure 3.4.2c, but we see that the this leads to a non simple, and in fact sum decomposable, permutation, because the bottom-leftmost three by three rectangular hull cannot be split by any other entries. This case therefore leads to a contradiction, and can be eliminated.

Suppose now that instead of lying below the reflection line, the 3 of our 123 pattern shown in Figure 3.4.1a lies above, and so its inverse image is in a cell below and to the right, of which there are two. If the inverse image, however, is in the lower of these two then by an argument analogous to the paragraph above we reach a violation of simplicity. Therefore the inverse image of the rightmost entry must lie in the cell directly below and to its right. The fact that $\sigma$ is an involution allows us to forbid the placing of entries into cells where the inverse image would create a forbidden pattern, leading to Figure 3.4.2d. Now the rectangular hull of the rightmost two entries must be split to preserve simplicity, and in fact must be split below and to the left to preserve involutionicity, leading to Figure 3.4.2e. Our situation is now analagous to that shown in Figure 3.4.2c, in that any placement of entries will lead to a sum decomposable (and hence non simple) permutation. Therefore
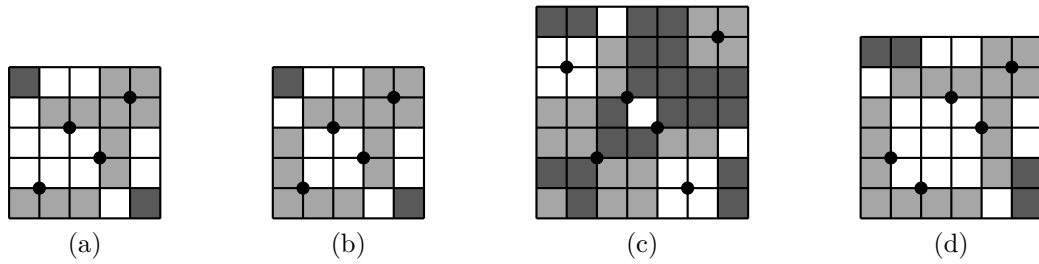
Figure 3.4.3: Permutation diagrams referenced in the proof of Theorem 3.4.4.

this case (in which the last entry of the original 123 is not a fixed point), can be discarded.

Case 3: Finally, we consider the case where the 3 of the 123 is a fixed point, but some other entry is not. Suppose first that the middle entry of Figure 3.4.1a lies above the reflection line. We are then forced into a situation identical to that shown in Figure 3.4.2e except rotated by 180 degrees, leading to a contradiction. Assuming that the middle entry is above the reflection line leads, and recalling that the inverse image of two increasing points are themselves increasing, leads to Figure 3.4.3a.

First assume that the leftmost entry shown in Figure 3.4.3a is a fixed point, leading to Figure 3.4.3b. Simplicity then requires that there be an entry in the bottommost white cell whose inverse image is in the leftmost white cell, yielding Figure 3.4.3c. The center of this diagram, however, contains an interval which cannot be split, contradicting our assumption that the leftmost entry of Figure 3.4.3a is a fixed point. Letting this entry lie above the reflection line leads to a contradiction analogous to Figure 3.4.2c, and so let this entry lie below the line, with its inverse image above and to the left. Inspecting the various cases shows that this inverse image must lie in the cell immediately above and to the left, producing Figure 3.4.3d. The rectangular hull of the leftmost two entries can be split in two ways, but one of them leads to a sum decomposable permutation and the other leads to a non involution.

Our final remaining case is when the middle entry of Figure 3.4.1a is a fixed point. Using similar methods to those presented above, we find that the that the leftmost entry must also be a fixed point. However, this case has already been investigated.

It therefore follows that there is precisely one simple 2341-avoiding involution. Since every 123-avoiding simple involution must also avoid 2341, it follows

that the set of all simple 2341-avoiding involutions is equal to the set of simple 123-avoiding involutions together with the permutation 5274163.

$\square$

# – Chapter 4 –

# Polynomial Classes and Genomics

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This chapter examines the so called polynomial classes, those permutation classes whose enumeration is given by a polynomial for large enough sizes. Much research in the area of permutation classes focuses on characterizing exponential growth rates, with a particular focus on the principally based classes. Considerably less attention has been paid to the small permutation classes [85, 86] of which the polynomial classes, having subexponential growth, are an example.

These classes have recently found biological applications to the field of genomics. Evolution and mutation of organisms can be modelled as a rearrangement of a sequence of genes, and permutations have recently been applied to model these rearrangements [42]. The physical mechanics of genome rearrangement have led to a variety of operations on permutations, and the theory of geometric grid classes [6] provides a geometric foundation from which to study these various operations. The polynomial classes are a subset of these grid classes, and arise when modelling the evolutionary distance.

Polynomial classes can characterized in a number of ways, but determining the actual polynomial which enumerates such a class can be computationally difficult. While there are several established methods for enumerating permutation classes, many of these are inefficient and none take advantage of the inherent structure in these classes. In this chapter, we introduce an algorithm which quickly and efficiently enumerates a polynomial class from a structural description of the class. This allows for an extension of existing genomic data, as well as a framework for further investigation. This chapter is based in part on [53], and the algorithm, implemented in Python, is freely available online [54].

## § 4.1   Class Structure

Definition 4.1.1. A permutation class $\mathcal{C}$ is a polynomial class if and only if the function $p(n) = |\mathcal{C}_n|$ is given by a polynomial for large enough $n$.

It is not obvious that this definition gives way to a strict geometric description, as we shall soon see. Geometric grid classes provides a range of tools for analyzing the geometric properties of permutation class structure, and has produced new enumerative techniques for classes. To describe polynomial classes, however, we don't need the full machinery of geometric grid classes; these classes can be defined entirely using inflations (Definition 1.1.14).

Note first that the polynomial classes fall under the purview of several established approaches, which could theoretically be used to enumerate the classes [2,6,9,28,87]. However, each of these approaches has its own drawbacks, and none provides an enumeration directly from a structural description of the class. Further, the work presented here illuminates some of the preliminary obstacles preventing a similar algorithmic approach to geometric grid classes.

### Peg Permutations

Polynomial classes can be viewed by considering a set of restricted inflations of a finite set of permutations. In order to properly analyze these inflations, we introduce an additional structure on permutations which will be used to specify which inflations are allowed.

Definition 4.1.2. A peg permutation $\tilde{\rho}$ is a permutation $\rho = \rho_1 \rho_2 \ldots \rho_n$ in which each entry is decorated with either a $+$, $-$, or $\bullet$. The length of a peg permutation $\tilde{\rho}$ is just the length of the underlying permutation $\rho$.

For example, $\tilde{\rho} = 3^+ 1^- 2^\bullet 4^+$ is a peg permutation of length 4, and there are $3^n n!$ peg permutations of length $n$. We denote peg permutations with a tilde, while the underlying permutation (with decoration removed) is written without.

We allow peg permutations to be inflated with monotone intervals. The entries marked with a $+$ (resp. $-$) can be inflated with ascending (resp. decreasing) runs. Entries marked with a $\bullet$ can be inflated with a single entry. Note that we go against tradition and allow empty inflations. It follows then that
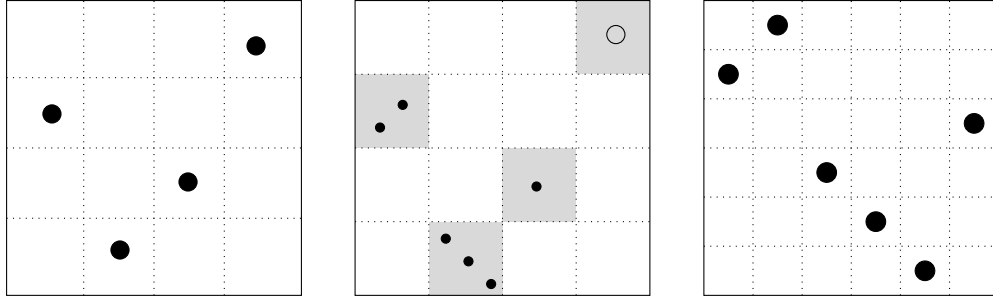
Figure 4.1.1: The peg permutation $\tilde{\rho} = 3^+1^-2^\bullet4^+$ inflated by the vector $\vec{i} = (2, 3, 1, 0)$ is the permutation 563214.

such an inflation can be described simply as a peg permutation together with a sequence of integers which represent the number of elements by which to inflate each entry. We formalize this below.

Definition 4.1.3. Let $\tilde{\rho} = \tilde{\rho}_1\tilde{\rho}_2\ldots\tilde{\rho}_n$ be a peg permutation of length $n$, and $\vec{i} = (i_1, i_2, \ldots i_n)$. Then let $\tilde{\rho}(I)$ be the permutation obtained by inflating entry $\tilde{\rho}_k$ by an interval of size $i_k$ according to the decoration of $\tilde{\rho}_k$: an ascending run if the decoration is a $+$, a descending run if it is a $-$, and a single entry if a $\bullet$. If $\tilde{\rho}_k$ has a dot, then $i_k$ must be 0 or 1, otherwise $i_k \in \mathbb{N}$.

Recall, for example, the class $\mathrm{Av}(123, 231)$ examined in Section 1.3.3. The decomposition of this class was shown in Figure 1.3.6, and can be described as inflations of the peg permutation $3^+1^+2^+$.

Like many definitions in this dissertation, this one is best illustrated with a graphic example. Figure 4.1.1 shows a peg permutation being inflated and then standardized into a permutation. The following definition and theorem provide our desired characterization of polynomial classes.

Definition 4.1.4. For a peg permutation $\tilde{\rho}$, denote by $\mathcal{I}(\tilde{\rho})$ the set of all valid inflations of $\tilde{\rho}$. Similarly, for a set $\tilde{S}$ of peg permutations, let

$$\mathcal{I}(\tilde{S}) = \cup_{\tilde{\rho} \in \tilde{S}} \mathcal{I}(\tilde{\rho}).$$

It follows that for a permutation $\pi \in \mathcal{I}(\tilde{\rho})$, there exists some partition $P$ of the entries of $\pi$ into monotone intervals which are compatible with $\tilde{\rho}$. This partition is referred to as a $\tilde{\rho}$-partition of $\pi$.

It can be easily shown that, for a peg permutation $\tilde{\rho}$ of length $n$, if $\vec{v} = (v_1, v_2 \ldots v_n) \in \mathbb{N}^n$ and $\vec{w} = (w_1, w_2, \ldots w_n) \in \mathbb{N}^n$ are two vectors such that
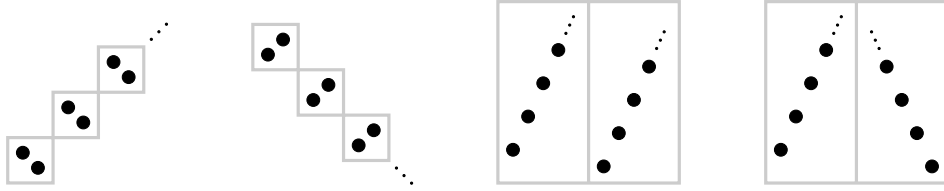
Figure 4.1.2: If a class contains arbitrarily long patterns of any of these forms, it is not a polynomial class.

$v_i \leq w_i$ for all $i \in [n]$, then $\tilde{\rho}(\vec{v}) \prec \tilde{\rho}(\vec{w})$ as permutations. Also, note that $\mathcal{I}(\tilde{\rho})$ forms a permutation class, and in fact, as we shall soon see, a polynomial class.

Theorem 4.1.5 ( [6,55]). For a permutation class $\mathcal{C}$, the following are equivalent.

1) $\mathcal{C}$ is a polynomial class,

2) $\mathcal{C}_n < f_n$ for some $n$, where $f_n$ is the $n$th Fibonacci number,

3) $\mathcal{C}$ does not contain arbitrarily long patterns of the forms shown in Figure 4.1.2,

4) $\mathcal{C} = \mathcal{I}(\tilde{S})$ for some set $\tilde{S}$ of peg permutations.

## Peg Patterns

Analogous to the permutation pattern ordering, we can define an ordering on peg permutations. Essentially, we say that a peg permutation is contained in another if it can be obtained by deleting entries and changing signs to dots.

Definition 4.1.6. Let $\tilde{\rho} = \tilde{\rho}_1 \tilde{\rho}_2 \ldots \tilde{\rho}_n$ and $\tilde{\tau} = \tilde{\tau}_1 \tilde{\tau}_2 \ldots \tilde{\tau}_k$ be peg permutations. Say that $\tilde{\tau}$ is contained within $\tilde{\rho}$ if there is a subsequence $\tilde{\rho}_{i_1} \tilde{\rho}_{i_2} \ldots \tilde{\rho}_{i_k}$, whose entries lie in the same relative order as those of $\tilde{\tau}$ and whose decorations are compatible, meaning that $\tilde{\rho}_{i_j}$ either have the same decoration or $\tilde{\tau}_j$ is decorated with a dot.

It follows from the definitions that if $\tilde{\tau} \prec \tilde{\rho}$, then $\mathcal{I}(\tilde{\tau}) \subset \mathcal{I}(\tilde{\rho})$. However, the converse is not true. For example, letting $\tilde{\tau} = 1^\bullet 2^\bullet$ and $\tilde{\rho} = 1^+$, we see that $\mathcal{I}(\tilde{\tau}) = \{1, 12\} \subset \mathcal{I}(\tilde{\rho})$, but $\tilde{\tau} \not\prec \tilde{\rho}$. The core idea of the algorithm is the partition all permutations of the class according to peg permutation, and then enumerate these by enumerating integer vectors.

Definition 4.1.7. For a peg permutation $\tilde{\rho}$ and a permutation $\pi$, say that $\pi$ fills $\tilde{\rho}$ if $\pi = \tilde{\rho}(\vec{v})$ such that $\vec{v}_i = 1$ whenever $\tilde{\rho}_i$ is decorated with a dot, and $\vec{v}_i \geq 2$ otherwise. Every peg permutation $\tilde{\rho}$ has a unique minimal filling permutation, denoted $m_{\tilde{\rho}}$.

## Integer Vectors

Peg permutations provide a way of translating between integer vectors and permutations. The underlying idea of the algorithm is to formalize this correspondence in a way which preserves the ordering, converting permutation posets into posets of integer vectors. We will now establish some machinery for working with and enumerating integer vector posets.

Downsets in the integer vector poset are easier to work with than permutation classes in part because of Higman's Theorem [50], which implies that every downset has a finite basis. The union and intersection of these downsets is easy to compute as well.

Definition 4.1.8. For two vectors $\vec{v}, \vec{w} \in \mathbb{N}^n$, say that $\vec{v} \prec \vec{w}$ if $v_i \leq w_i$ for each $i \in [n]$. For a downset $\mathcal{V} \subset \mathbb{N}^n$, denote by $\mathcal{B}_{\mathcal{V}}$ the set of minimal vectors in the complement of $\mathcal{V}$. It follows then that $\mathcal{V}$ can be described as precisely those vectors which avoid the vectors of $\mathcal{B}_{\mathcal{V}}$, that is,

$$\mathcal{V} := \{\vec{v} \in \mathbb{N}^n : \vec{b}_i \not\prec \vec{v}, \ \forall \ \vec{b}_i \in \mathcal{B}_{\mathcal{V}}\}.$$

For two vectors $\vec{v}, \vec{w} \in \mathbb{N}^n$, denote by $\vec{v} \vee \vec{w}$ the minimal vector for which $\vec{v} \prec \vec{v} \vee \vec{w}$ and $\vec{w} \prec \vec{v} \vee \vec{w}$. It follows that $(\vec{v} \vee \vec{w})_i = \max(\vec{v}_i, \vec{w}_i)$ for each $i \in [n]$.

Proposition 4.1.9. Let $\mathcal{V}, \mathcal{W}$ be downsets in $\mathbb{N}^n$ with corresponding downsets $\mathcal{B}_{\mathcal{V}}, \mathcal{B}_{\mathcal{W}}$. Letting $\mathcal{B}_{\mathcal{M}}$ be the minimal vectors of the set $\{\vec{v} \vee \vec{w} : \vec{v} \in \mathcal{V}, \vec{w} \in \mathcal{W}\}$, $\mathcal{B}_{\mathcal{U}}$ the minimal vectors of the union $\mathcal{B}_{\mathcal{V}} \cup \mathcal{B}_{\mathcal{W}}$, and $\mathcal{M}$ and $\mathcal{U}$ the downsets which avoid $\mathcal{B}_{\mathcal{M}}$ and $\mathcal{B}_{\mathcal{U}}$, respectively. We have that

$$\mathcal{V} \cap \mathcal{W} = \mathcal{U},$$

$$\mathcal{V} \cup \mathcal{W} = \mathcal{M}.$$

Proof. Clearly, any vector in $\mathcal{V} \cap \mathcal{W}$ must avoid all basis elements of both $\mathcal{B}_{\mathcal{V}}$ and $\mathcal{B}_{\mathcal{W}}$, and so the basis for $\mathcal{V} \cap \mathcal{W}$ is the set of minimal elements of the set

$\mathcal{B}_\mathcal{V} \cup \mathcal{B}_\mathcal{W}$. For unions, we proceed using De Morgan's laws:

$$\mathcal{V} \cup \mathcal{W} = \left( \bigcap_{\vec{v} \in \mathcal{B}_\mathcal{V}} \{\vec{v}\text{-avoiding vectors}\} \right) \bigcup \left( \bigcap_{\vec{w} \in \mathcal{B}_\mathcal{W}} \{\vec{w}\text{-avoiding vectors}\} \right)$$

$$= \bigcap_{\substack{\vec{v} \in \mathcal{B}_\mathcal{V} \\ \vec{w} \in \mathcal{B}_\mathcal{W}}} \left( \{\vec{v}\text{-avoiding vectors}\} \cup \{\vec{w}\text{-avoiding vectors}\} \right)$$

$$= \bigcap_{\substack{\vec{v} \in cBV \\ \vec{w} \in \mathcal{B}_\mathcal{W}}} \{\vec{v} \vee \vec{w}\text{-avoiding vectors}\}.$$

Therefore the basis for $\mathcal{V} \cup \mathcal{W}$ consists of the set $\mathcal{B}_\mathcal{M}$, completing the proof.   $\square$

Proposition 4.1.9 can also be used to enumerate downsets of integer vector classes, using inclusion exclusion. It will be useful to consider these downsets as collections of point-sets on an integer lattice, and to enumerate the classes based on the number of $n$-element sets they contain. We formalize this below.

We define the weight of a vector as the sum of its entries. A peg permutation, inflated by a vector of weight $k$, produces a permutation of length $k$. Counting integer vectors according to weight is relatively simple, and is equivalent to counting ordered compositions. Letting $a_{n,k}$ denote the number of $k$-weight vectors in $\mathbb{N}^n$, we have

$$\sum_{k \geq 0} a_{n,k} z^k = \frac{1}{(1-z)^n}.$$

Similarly, the generating function for the number of permutations which contain a given vector $\vec{v} \in \mathbb{N}^n$ is given by

$$\frac{z^{\text{wt}(\vec{v})}}{(1-z)^n}.$$

It follows from this and Proposition 4.1.9 that that downsets can be enumerated by adding and subtracting generating functions of this form. This leads to the following lemma.

Lemma 4.1.10. Let $\tilde{\rho}$ be a peg permutation, and let $s$ be the number of signs in the decoration of $\tilde{\rho}$, and $d$ the number of dots. Then the generating function for the filling permutations of $\tilde{\rho}$ is given by

$$\frac{z^{d+2s}}{(1-z)^s}.$$

Lemma 4.1.10 will ultimately be our enumeration scheme for these classes. The main barrier is partitioning the class into categories based on which peg permutation they fill. The bulk of the algorithm, described in the next section, will be performing this partitioning.

# § 4.2  The Algorithm

This section gives an overview of the enumeration algorithm, given a set of peg permutations as an input, and outputting a disjoint set of integer vector downsets, which can then be enumerated. The algorithm consists of three parts. First the set is completed, then compacted, and finally cleaned, at which point we have a set of peg permutations which partition the class. Letting $\tilde{S}$ be a set of peg permutations, we describe each part in detail below, with the goal of enumerating the class $\mathcal{I}(\tilde{S})$. A pseudocode overview of the algorithm is shown in Figure 4.2.1.

## Completing the Set

Say that a set $\tilde{S}$ is complete if every permutation $\pi \in \mathcal{I}(\tilde{S})$ fills at least one element $\tilde{\rho} \in tS$. For example, the set $\{2^+1^+\}$ is not complete, because $1\ 2\ 3 \in \mathcal{I}(2^+1^+)$ (since $1\ 2\ 3 = 2^+1^+(3,0)$), but doesn't fill $2^+1^+$. It follows from the definition of peg patterns, however, that every permutation in $\mathcal{I}(\tilde{S})$ must fill some pattern within an element in $\tilde{S}$.

Therefore, the downset of any peg pattern is a complete set. The first step of the algorithm completes the set $\tilde{S}$ by, for each $\tilde{\rho} \in \tilde{S}$, we add all patterns of $\tilde{\rho}$ into the set $\tilde{S}$. After this step, the set $\tilde{S}$ is complete.

## Compacting the Set

The next obstacle in the enumeration is ensuring that every permutation in the class fills a unique peg permutation in the set. Given a permutation, we can divide its entries up into monotone intervals in a number of ways. The following lemma will help to ensure uniqueness, and allow for enumeration.

Lemma 4.2.1. If two monotone intervals intersect, then their union and intersection are also monotone intervals.

Proof. Suppose we have two monotone intervals with a non-empty intersection. Without loss of generality, suppose that one of them is increasing, and so their intersection is either increasing or consists of a single element. Since each interval consists of contiguous entries, the second entry must also be increasing, and so the union of the two is an increasing interval. □

Lemma 4.2.1 implies that by greedily choosing the largest possible intervals, we can ensure that for each permutation $\pi$, there is a unique smallest peg permutation $\tilde{\rho}$ for which $\pi$ is in $\mathcal{I}(\tilde{\rho})$, but not in $\mathcal{I}(\tilde{\tau})$ for any $\tilde{\tau} \prec \tilde{\rho}$. However, not all peg permutations are able to fulfill this role.

Say that a peg permutation $\tilde{\rho}$ is compact if, for all $\tilde{\tau} \prec \tilde{\rho}$, we have that $\mathcal{I}(\tilde{\tau}) \neq \mathcal{I}(\tilde{\rho})$. For example, $2^{\bullet}1^{-}$ is not compact, since $\mathcal{I}(2^{\bullet}1^{-}) = \mathcal{I}(1^{-})$. The following lemma and proposition characterizes these peg permutations.

Proposition 4.2.2. For a peg permutation $\tilde{\rho}$, the following are equivalent:

1) $\tilde{\rho}$ is compact,

2) $\tilde{\rho}$ does not contain the patterns $1^{+}2^{+}, 1^{+}2^{\bullet}, 1^{\bullet}2^{+}$ or, symmetrically, $2^{-}1^{-}, 2^{-}1^{\bullet}$ or $2^{\bullet}1^{-}$,

3) every permutation $\pi$ which fills $\tilde{\rho}$ has a unique vector $\vec{v}$ for which $\tilde{\rho}(\vec{v}) = \pi$.

Proof. First we show that (1) and (2) are equivalent. Clearly (1) implies (2), so to show the reverse implication, let $\tilde{\rho}$ be a noncompact peg permutation. By definition, there exists some $\tilde{\tau} \prec \tilde{\rho}$ such that $\mathcal{I}(\tilde{\tau}) = \mathcal{I}(\tilde{\rho})$. Let $\pi$ be a permutation which fills $\tilde{\rho}$, with $P$ the $\tilde{\rho}$-partition and $P'$ the $\tilde{\tau}$ partition. Because $\tilde{\tau}$ is shorter than $\tilde{\rho}$, it follows that there must be some part of $P'$ which intersects two parts of $P'$. By Lemma 4.2.1 these two form a monotone interval, and so must be of one of the forms listed in (2).

Now, we show that (2) and (3) are equivalent. If a peg permutation $\tilde{\rho}$ contains one of the patterns specifies in (2), it is clear that a permutation can fill $\tilde{\rho}$ in at least two different ways, so (3) implies (2). Suppose that the permutation $\pi$ fills $\tilde{\rho}$ with two different $\tilde{\rho}$-partitions $P$ and $P'$. It follows then that a block of one partition must intersect two blocks of the other. However, this implies (Lemma 4.2.1) that intersection and unions are also monotone, and so must be of one of the forms given in (2). □

By simply removing each of the peg permutations which contain one of the intervals listed in Proposition 4.2.2, our set of peg permutations becomes a

set of compact peg permutations. Further, since our set is a full and complete downset, the definition of compact implies that the new set will still be complete.

## Cleaning the Set

The final step in the algorithm is bijecting our complete and compact set of peg permutations to a set of downsets of integer vectors. Our final obstacle in this bijection will be peg permutations which have intervals of dotted entries. For example, the peg permutation $1^\bullet 2^\bullet 3^\bullet 4^\bullet$ produces a class which is strictly contained in $1^+$, but there is no containment at the level of peg permutations. We remedy this by using forbidden vectors: the peg permutation $1^\bullet 2^\bullet 3^\bullet 4^\bullet$ is mapped to the inflations of $1^+$ which avoid the vector $\langle 5 \rangle$.

Definition 4.2.3. Say that a peg permutation $\tilde{\rho}$ is clean if $\mathcal{I}(\tilde{\rho}) \not\subset \mathcal{I}(\tilde{\tau})$ for any shorter permutation $\tilde{\tau}$.

Proposition 4.2.4. The compact peg permutation $\tilde{\rho}$ is clean if and only if it does not contain an interval order isomorphic to $1^\bullet 2^\bullet$ or $2^\bullet 1^\bullet$.

Proof. If $\tilde{\rho}$ contains one of the specified intervals, then letting $\tilde{\tau}$ be the shorter peg permutation obtained by contracting these two entries into a single entry with the appropriate sign, we find that $\mathcal{I}(\tilde{\rho}) \subseteq \mathcal{I}(\tilde{\tau})$.

For the other direction, suppose that $\mathcal{I}(\tilde{\rho}) \subseteq \mathcal{I}(\tilde{\tau})$ for some shorter peg permutation $\tilde{\tau}$. Let $\pi$ be any permutation which fills $\tilde{\rho}$. In any $\tilde{\tau}$-partition of $\pi$ there must be a monotone interval formed from entries in different parts of any $\tilde{\rho}$ partition. Because $\tilde{\rho}$ is compact, it follows (from Proposition 4.2.2) that $\tilde{\rho}$ must contain either $1^\bullet 2^\bullet$ or $2^\bullet 1^\bullet$, completing the proof. □

Given a complete and compact set $\tilde{S}$ of peg permutations, it is not possible in general to find a clean set which inflates to the same class. To see this, let $\tilde{\rho} = 1^\bullet 2^\bullet 3^\bullet$. Then there is no clean set whose inflation is equal to $\mathcal{I}(\tilde{\rho})$. However, we can put the set $\tilde{S}$ in bijection with a clean set together with a set of allowable inflation vectors. We formalize this below.

Definition 4.2.5. For a peg permutation $\tilde{\rho}$ and a set $\mathcal{V}$ of vectors of the same length, let $\mathcal{I}(\tilde{\rho}; \mathcal{V})$ denote the inflations of $\tilde{\rho}$ using vectors from the set $\mathcal{V}$.

Lemma 4.2.6. For each peg permutation $\tilde{\rho}$, there exists a clean permutation $\tilde{\tau}$ and a vector set $\mathcal{V}$ such that the set of all inflations of $\tilde{\rho}$ is equal to $\mathcal{I}(\tilde{\tau}, \mathcal{V})$.

Proof. To construct $\tilde{\tau}$, simply contract all of the intervals of dotted entries in $\tilde{\rho}$ into signed entries. To construct $\mathcal{V}$, build a vector $\vec{v}$ such that, if the entry $\tilde{\tau}_i$ arose from a dotted interval of length $k$, let $\vec{v}_i = k+1$, and take $\mathcal{V}$ to be the set of vectors avoiding $\vec{v}$. This ensures that this entry will never be inflated by a run longer than the original sequence of dotted entries. $\qquad\square$

The final step of the algorithm can be described as follows. First, let $\mathfrak{V}$ be an empty set, which will be the output. For each peg permutation $\tilde{\rho} \in \tilde{S}$, compute the pair $(\tilde{\tau}, \vec{W})$ as described in Lemma 4.2.6, and let $\mathcal{V}$ be the vector downset with basis $\mathcal{B}_{\mathcal{V}} = \{\vec{v}\}$. If there is no pair $(\tilde{\tau}, \mathcal{W})$ in the set $\mathfrak{V}$, add $(\tilde{\tau}, \mathcal{V})$ to $\mathfrak{V}$. Otherwise, replace $(\tilde{\tau}, \mathcal{W})$ with $(\tilde{\tau}, \mathcal{W} \cup \mathcal{V})$.

Since every permutation in the class fills a unique clean and compact peg permutation, and since each permutation which fills a compact permutation has a unique partition, it follows that the polynomial class is in bijection with the set

$$\biguplus_{(\tilde{\rho}, \mathcal{V}) \in \mathfrak{V}} \mathcal{I}(\tilde{\rho}, \mathcal{V}).$$

Letting $\vec{m}_{\tilde{\rho}}$ be the vector defined by $(\vec{m}_{\tilde{\rho}})_i = 1$ if $\tilde{\rho}_i$ is decorated with a dot, and $(\vec{m}_{\tilde{\rho}})_i = 2$ otherwise, and let $s(\tilde{\rho})$ denote the number of signed (non-dotted) entries of $\tilde{\rho}$. The generating function for $\mathcal{I}(\tilde{\rho})$ is then given by inclusion exclusion in conjunction with Proposition 4.1.9, and allows us to efficiently enumerate these classes.

$$\sum_{B \subseteq \mathcal{B}_{\mathcal{V}}} (-1)^{|B|} \frac{z^{\text{wt}\left(\vec{m}_{\tilde{\rho}} \vee (\bigvee B)\right)}}{(1-z)^{s(\tilde{\rho})}}.$$

# § 4.3  Genomics

The field of computational biology is a new and rapidly developing field. The vast quantities of sequencing data produced by modern geneticists necessitate the use of complex mathematical techniques for analysis. A common problem, given two related genetic sequences, is to determine the most recent evolutionary ancestor. This is generally solved by determining the number of mutations required to rearrange one sequence into the other, allowing a researcher to determine the midpoint between the two. Determining this distance, however, is

Input:   Set $\tilde{S}$ of Peg Permutations
Output:   Integer vectors in bijection with the class
// Complete $\tilde{S}$
for $\tilde{\rho} \in \tilde{S}$ do
 | Add to $\tilde{S}$ all peg permutations which can be realized by deleting
 | entries of $\tilde{\rho}$, or changing a signs to dots
end
// Remove all non-compact elements from $\tilde{S}$
for $\tilde{\rho} \in \tilde{S}$ do
 | if $\tilde{\rho}$ contains any of the consecutive permutations $1^+2^+, 1^\bullet2^+, 1^+2^\bullet$
 | or their symmetries then
 |  | Remove $\tilde{\rho}$ from $\tilde{S}$
 | end
end
// Clean $\tilde{S}$ and construct a vector set
Initialize the set $\mathfrak{V}$, which will contain pairs $(\tilde{\rho}, \mathcal{V})$, where $\tilde{\rho}$ is a
peg permutation and $\mathcal{V}$ is a set of integer vectors of the same
length as $\tilde{\rho}$
for $\tilde{\rho} \in \tilde{S}$ do
 | if $\tilde{\rho}$ contains intervals of the form $1^\bullet2^\bullet$ or $2^\bullet1^\bullet$ then
 |  | Let $\tilde{\tau}$ denote the cleaned $\tilde{\rho}$, and $\mathcal{V}$ the set of integer vectors
 |  | for which $\{\tilde{\tau}[\vec{v}] : \vec{v} \in \mathcal{V}\} = \{\tilde{\rho}[\vec{v}] : \vec{v} \in \mathcal{F}_{\tilde{\rho}}\}$
 |  | Let $(\tilde{\rho}', \mathcal{V}') \leftarrow (\tilde{\tau}, \mathcal{V})$
 | else
 |  | Let $(\tilde{\rho}', \mathcal{V}') \leftarrow (\tilde{\rho}, \mathcal{F}_{\tilde{\rho}})$
 | end
 | if $(\tilde{\rho}', \mathcal{W}) \in \mathfrak{V}$ for some $\mathcal{W}$ then
 |  | Replace the element $(\tilde{\rho}', \mathcal{W})$ with $(\tilde{\rho}', \mathcal{W} \cup \mathcal{V}')$
 | else
 |  | Add $(\tilde{\rho}', \mathcal{V}')$ to $\mathfrak{V}$
 | end
end
The permutation class is now in bijection with the disjoint union
$\biguplus_{(\tilde{\rho}, \mathcal{V}) \in \mathfrak{V}} \{\tilde{\rho}[\vec{v}] : \vec{v} \in \mathcal{V}\}.$

Figure 4.2.1: A pseudocode overview of the algorithm.

computationally difficult, but the work presented in this chapter can be used to effectively and efficiently perform these and other computations.

This section applies the theory of polynomial classes to the problem of evolutionary distance. While the focus is on the combinatorial aspects of genome rearrangement, we begin with a rough overview of the biological mechanics. For a more complete introduction, see the surveys [73] or [42].

## Chromosomes and Mutation

Every living organism encodes its hereditary information in molecules called chromosomes, the set of which is known as the organism's genome. The information carried in the genome is passed down from organism to organism, and undergoes mutations which can cause both subtle and dramatic change between generations.

Each chromosome is composed of double strands of deoxyribonucleic acid (DNA), each strand of which is in turn composed of a sequence of nucleotides. Nucleotides come in four types (A, C, G, and T), and the two strands, arranged in a double helix, are complementary, i.e., a A's are always coupled with a T, and G's with C. It follows that DNA can be defined as a single sequence - a word on the alphabet {A,C,G,T}. A DNA sequence is some consecutive piece of this word, while genes are the smallest sequences which have some independent biological function.

The genome is made up of chromosomes, which are in turn made up of coiled DNA strands, which can be broken down into genes sequences, which themselves are simply sequences of nucleotides. This complexity leads to a variety of errors which can be introduced during replication, and these inaccuracies are the basis for genetic evolution. Many of these mutations can be viewed as rearranging sequences of genes, and can be effectively modelled using permutations.

The physical properties of chromosomes lead to a variety of rearrangement operations, but they share a common theme: some contiguous segment of the gene sequence is removed, reversed and/or relocated, then replaced back in the sequence. While there are other mutations possible at both the larger and smaller scales, these so called genome-rearrangements have received much attention in recent research and, most importantly, fall under the purview of polynomial classes.

## Block Transformations

Permutations are apt models for rearrangement, and can be used to study genetic mutations. Mutations happen in various ways, and a variety of permutation transformations have been studied. These operations are known collectively as block transformations, as each of them acts on contiguous subsequences of permutations, henceforth referred to as blocks. Each of these operations can be viewed as a set of allowable moves which transform one permutation into another.

Treating block transformations as mutations, the basic problem is as follows: given two permutations, what is the shortest sequence of moves which can transform one into the other? By relabelling the entries, we can assume, without any loss of generality, that the target permutation is the identity permutation. In this light, the question becomes a sorting sorting problem, and asks how quickly a sequence can be sorted. We present here some of the more commonly studied operations, but note that other varieties and models are biologically significant.

Definition 4.3.1. Let $\pi = \pi_1 \pi_2 \ldots \pi_n$ be a permutation written in one-line notation. A block of $\pi$ is some contiguous string of entries $\pi_i \pi_{i+1} \ldots \pi_{i+k}$. A prefix is a block which starts at $\pi_1$.

Blocks of permutations are models for gene sequences, and each of the block permutations below differ only in their treatment of blocks. We define each type of sorting by defining a single allowable operation.

Definition 4.3.2 (Block Reversal). A block reversal operation consists of reversing the entries of any block of the permutation. This operation was first studied by Watterson, Ewens, Hall, and Morgan [88] and further investigated by Alpar-Vajk [11].

Definition 4.3.3 (Block Transposition). A block transposition operation consists of moving one block from its current position to any other location in the permutation. This operation was first studied by Bafna and Pevzner [16].

Definition 4.3.4 (Block Interchange). A block interchange operation consists of selecting two non-intersecting blocks of the permutation and interchanging them. This operation was first studied by Christie [33], and further investigated by Bóna and Flynn [25].

Definition 4.3.5 (Prefix Transposition). A prefix transposition operation consists of moving a prefix of the permutation to any other location in the permutation. This operation was first studied by Dias and Meidanis [39].

Definition 4.3.6 (Prefix Reversal). A prefix reversal operation consists of reversing the entries of a prefix of the permutation. This is sometimes referred to as the 'pancake flipping operation', and was first studied by "Harry Dweighter" (actually, Jacob E. Goodman) as a Monthly problem [60] (and was also studied by Gates [45]).

Definition 4.3.7 (Cut-Paste Sorting). A cut-paste operation consists of moving a block of the permutation, with the option to reverse its entries. This operation was first studied by Cranston, Sudborough, and West [36].

For a given block transformation, we refer to the distance between two permutations $\pi$ and $\sigma$ as the minimum number of operations needed to transform one into the other. Finding the maximal distance between two permutations of a given length is equivalent to finding the maximal distance from the identity to any permutation. Further, since each of these operations is reversible — if $\pi$ can be transformed into $\sigma$, then $\sigma$ can be transformed into $\pi$ — this is equivalent to finding the distance from the identity to any permutation.

Biologically, two permutations with a small distance represent closely related organisms, as each transformation represents a mutation which can occur from one generation to the next. Understanding the sets of permutations at each fixed distance from the identity can help to understand how different genomes are related. For any $k \in \mathbb{N}$, the set of permutations which are at distance $\leq k$ from the identity forms a polynomial class, and thus can be enumerated by our algorithm.

Theorem 4.3.8. For each of the operations presented above and for a positive integer $k$, the set of permutations with distance at most $k$ from the identity forms a polynomial class.

Proof. The class of identity permutations is the inflations of the peg permutation $1^+$, which can be represented geometrically as a diagonal line parallel to $y = x$. Each block transformation can be viewed as taking some piece of this line and moving or reversing it. Such an array of lines can be translated back into a peg permutation, and it follows that the set of distance $\leq k$ permutation can be represented as the union of all peg permutations obtained in this way. See Figures 4.3.1 and 4.3.2 for graphical examples. $\qquad\square$
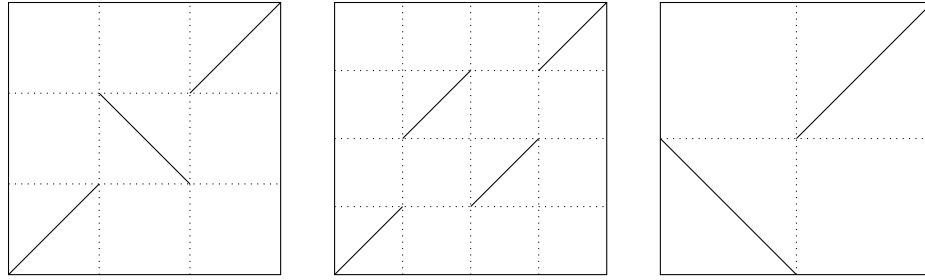
Figure 4.3.1: The classes of permutations which are at most one block reversal, block transposition, and prefix reversal away from the identity are given by $\mathcal{I}(1^+2^-3^+)$, $\mathcal{I}(1^+3^+2^+4^+)$, and $\mathcal{I}(1^-2^+)$, respectively.
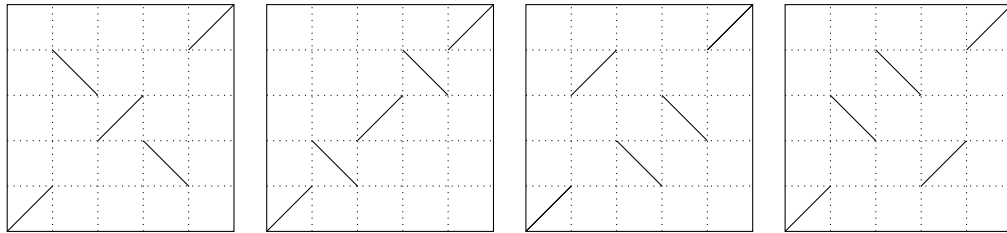


Figure 4.3.2: The class of permutations which are at most two block reversals from the identity is given by inflations of the four peg permutations $1^+4^-3^+2^-5^+$, $1^+2^-3^+4^-5^+$, $1^+4^+2^-3^-5^+$, and $1^+3^-4^-2^+5^+$.

## Data

Calculating the number of permutations of length $n$ which are at most $k$ operations away from the identity helps to understand how these block transformations differ, and how accurately they model biological mutation. The following tables show the numbers of these permutation in each radii from the identity, and build on the data presented in [42]. The polynomials (in the variable $n$) enumerating these classes have integer coefficients when presented with the basis $\{\binom{n}{k}\}_{k\geq0}$ (as implied by [58]). These enumerations are presented in the tables below.

Table 4.3.1: Number of permutations of length $n$ within $k$ block transpositions of the identity.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | OEIS [84] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 5 | 11 | 21 | 36 | 57 | 85 | 121 | 166 | A000292 |

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{3}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 6 | 23 | 89 | 295 | 827 | 2017 | 4405 | 8812 | A228392 |

$$\binom{n}{0} + \binom{n}{2} + 2\binom{n}{3} + 8\binom{n}{4} + 18\binom{n}{5} + 11\binom{n}{6}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2 | 6 | 24 | 120 | 675 | 3527 | 15484 | 56917 | 179719 | A228393 |

$$\binom{n}{0} + \binom{n}{2} + 2\binom{n}{3} + 9\binom{n}{4} + 44\binom{n}{5} + 220\binom{n}{6} + 656\binom{n}{7} + 841\binom{n}{8} + 369\binom{n}{9}$$

Table 4.3.2: Number of permutations of length $n$ within $k$ prefix transpositions of the identity.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | OEIS [84] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 7 | 11 | 16 | 22 | 29 | 37 | 46 | A000124 |

$$\binom{n}{0} + \binom{n}{2}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 6 | 21 | 61 | 146 | 302 | 561 | 961 | 1546 | A228394 |

$$\binom{n}{0} + \binom{n}{2} + 2\binom{n}{3} + 6\binom{n}{4}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2 | 6 | 24 | 116 | 521 | 1877 | 5531 | 13939 | 31156 | A228395 |

$$\binom{n}{0} + \binom{n}{2} + 2\binom{n}{3} + 9\binom{n}{4} + 40\binom{n}{5} + 90\binom{n}{6}$$

Table 4.3.3: Number of permutations of length $n$ within $k$ block reversals of the identity.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | OEIS [84] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 4 | 7 | 11 | 16 | 22 | 29 | 37 | 46 | A000124 |

$$\binom{n}{0} + \binom{n}{2}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 2 | 6 | 22 | 63 | 145 | 288 | 516 | 857 | 1343 | A228396 |

$$8\binom{n}{0} - 3\binom{n}{1} + \binom{n}{2} + 4\binom{n}{3}$$

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 2 | 6 | 24 | 118 | 534 | 1851 | 5158 | 12264 | 25943 | A228397 |

$$318\binom{n}{0} - 214\binom{n}{1} + 131\binom{n}{2} - 61\binom{n}{3} + 20\binom{n}{4} + 70\binom{n}{5} + 35\binom{n}{6}$$

Table 4.3.4: Number of permutations of length $n$ within $k$ prefix reversals of the identity.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | OEIS [84] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | A000027 |
| | | | | | $\binom{n}{1}$ | | | | | | |
| 2 | 1 | 2 | 5 | 10 | 17 | 26 | 37 | 50 | 65 | 82 | A002522 |
| | | | $2\binom{n}{0} - 1\binom{n}{1} + 2\binom{n}{2}$ | | | | | | | | |
| 3 | 1 | 2 | 6 | 21 | 52 | 105 | 186 | 301 | 456 | 657 | A228398 |
| | | | $-3\binom{n}{0} + 3\binom{n}{1} - 2\binom{n}{2} + 6\binom{n}{3}$ | | | | | | | | |

Table 4.3.5: Number of permutations of length $n$ within $k$ cut-paste moves of the identity.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | OEIS [84] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 6 | 16 | 35 | 66 | 112 | 176 | 261 | 370 | A060354 |
| | | | | | $\binom{n}{1} + 3\binom{n}{3}$ | | | | | | |
| 2 | 1 | 2 | 6 | 24 | 120 | 577 | 2208 | 6768 | 17469 | 39603 | A228399 |
| | | | $-18\binom{n}{0} + 45\binom{n}{1} - 61\binom{n}{2} + 70\binom{n}{3} - 53\binom{n}{4} + 88\binom{n}{5} + 107\binom{n}{6}$ | | | | | | | | |
| 3 | 1 | 2 | 6 | 24 | 120 | 720 | 5040 | 36757 | 223898 | 1055479 | A228400 |
| | | | $508264\binom{n}{0} - 280036\binom{n}{1} + 140012\binom{n}{2} - 57622\binom{n}{3} + 13839\binom{n}{4}$ | | | | | | | | |
| | | | $+4136\binom{n}{5} - 5368\binom{n}{6} + 531\binom{n}{7} + 21125\binom{n}{8} + 12615\binom{n}{9}$ | | | | | | | | |

Table 4.3.6: Number of permutations of length $n$ within $k$ block interchanges of the identity.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | OEIS [84] |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 6 | 16 | 36 | 71 | 127 | 211 | 331 | 496 | A145126 |
| | | | | | $\binom{n}{0} + \binom{n}{2} + 2\binom{n}{3} + \binom{n}{4}$ | | | | | | |
| 2 | 1 | 2 | 6 | 24 | 120 | 540 | 1996 | 6196 | 16732 | 40459 | A228401 |
| | | | $\binom{n}{0} + \binom{n}{2} + 2\binom{n}{3} + 9\binom{n}{4} + 44\binom{n}{5} + 85\binom{n}{6} + 70\binom{n}{7} + 21\binom{n}{8}$ | | | | | | | | |

# – Chapter 5 –

# Fixed-Length Patterns

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The set of all permutations, equipped with the pattern ordering, forms an infinite graded poset. While much research in this area (and within this dissertation) focuses on infinite downsets of this poset, this chapter focuses on finite subsets. In particular, we examine the downset induced by a single permutation, and investigate the number of distinct patterns.

In 2003, Herb Wilf raised the question of finding the maximum number of distinct patterns which could be contained within a single permutation of length $n$, and classifying those permutations which maximize this number. In [7], the authors showed that the maximum number of patterns for a length $n$ pattern is asymptotic to $2^n$, and provided a construction which achieves this number.

In this chapter we examine the number of distinct patterns of a specified length which can be contained within a permutation. In the language of posets, Wilf's question asks to find which permutations which maximize the size of their downset, while here we seek to maximize the width of the downset. This chapter can be divided into two parts: in the first, we examine the number of $(n-1)$-patterns contained in a random permutation of length $n$, and obtain the expectation and variance for this statistic by extending a 1945 result of Kaplansky and Wolfowitz [59,91]. In the second part, we examine the number of patterns of any fixed size within a permutation, and provide a construction which maximizes this number. This chapter is based partly on [51].
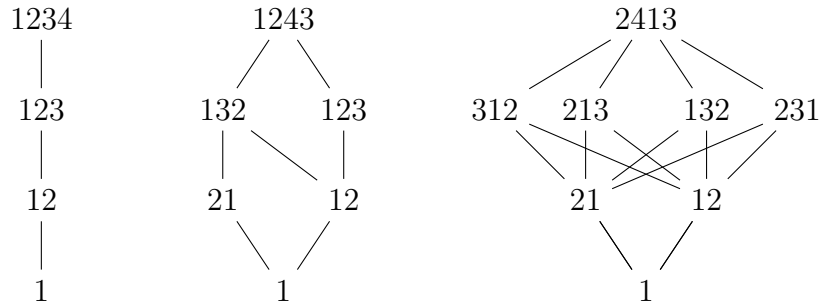
```
1234              1243                        2413
 |               /    \                    / |  |   \
123           132      123          312  213   132   231
 |             |  \    |             |   |  \ /  |    |
12            21      12             21  | / \ | 12
 |             \     /                \  |/   \| /
 1              \   /                  \ |    |/
                  1                      1
```

Figure 5.1.1: Downsets of 1234, 1243, and 2413

## § 5.1   Large Patterns

The set of all permutations, equipped with the pattern ordering, forms an infinite partially ordered set (see Figure 1.1.1). We focus here on the local properties of this poset, namely the number of patterns containing and contained in a given pattern. The more general topology of this poset was studied by McNamara and Steingrímsson [81].

The set of patterns contained within any fixed permutation forms a partially ordered set, in fact a finite downset of the full pattern poset. To examine these downsets, we use a top-down approach: deleting entries one at a time from the permutation to obtain the full set of patterns. Figure 5.1.1 shows several examples of these downsets.

Definitions and Notation

It will be convenient to establish some machinery for dealing with large patterns. Fix $n \geq 2$, let $\pi$ be a permutation of length $n$, and let $\sigma$ be an $(n-1)$-permutation. If $\sigma$ is contained as a pattern within $\pi$, then it follows that 1 can be obtained by deleting one entry from $\pi$, and relabelling with respect to order. Similarly, it follows that $\pi$ can be obtained by inserting an appropriate entry into $\sigma$. We formalize these ideas with the following pair of definitions.

Definition 5.1.1. For any permutation $\pi \in \mathfrak{S}_n$, define the function $\nabla_\pi : [n] \to \mathfrak{S}_{n-1}$, where $\nabla_\pi(i)$ is the permutation obtained by deleting the $i$th entry of $\pi$, and standardizing the remaining entries. Let $\nabla_\pi = \{\nabla_\pi(i) : i \in [n]\}$ denote the image of $\nabla_\pi$.

Definition 5.1.2. For any permutation $\sigma \in \mathfrak{S}_{n-1}$, define the function $\Phi_\sigma : [n] \times [n] \to \mathfrak{S}_n$, where $\Phi_\sigma(i,j)$ is the permutation obtained by inserting the entry $j - 1/2$ immediately to the left of the $i$th entry of $\sigma$, and then standardizing the entries. Let $\Phi_\sigma = \{\Phi_\sigma(i,j) : i, j \in [n]\}$ denote the image of $\Phi_\sigma$.

Letting $\pi$ and $\sigma$ be an $n$- and $(n-1)$-permutation, respectively, it follows from their definitions that these functions that $\nabla_\pi$ is the set of all $n - 1$-patterns contained in $\pi$, and $\Phi_\sigma$ is the set of permutations of length $n$ which contain $\sigma$. In addition, these functions satisfy the following inverse relationship:

$$\nabla_{\Phi_\sigma(i,j)}(i) = \sigma \text{ and } \Phi_{\nabla_\pi(i)}(i, \pi_i).$$

## § 5.2   Plentiful Permutations

Fix $n \in \mathbb{Z}_p$, and let $\pi$ be a permutation of length $n$. Since every pattern within $\pi$ can be obtained by deleting elements of $\pi$ one by one, the relationship between $\nabla_\pi$ and $\pi$ can be applied iteratively to understand the full downset of $\pi$. It follows directly from the definition that $|\nabla_\pi| \leq n$, and that $|\nabla_\pi| = n$ if and only if $\nabla_\pi$ is a one-to-one function, i.e., $\nabla_\pi(i) = \nabla_\pi(j)$ if and only if $i = j$. Before investigating further, we introduce another pair of definitions.

Definition 5.2.1. Let $\pi$ be a permutation of length $n$. Say that $\pi$ is plentiful if it contains $n$ distinct $(n - 1)$-patterns. Equivalently, $\pi$ is plentiful if and only if $\nabla_\pi$ is a one-to-one function.

Definition 5.2.2. Let $\pi = \pi_1 p_2 \ldots \pi_n$ be a permutation, and let $i \in [n-1]$. Say that the pair $(\pi_i, \pi_{i+1})$ is a bond, of entries of $\pi$ if $\pi_i - \pi_{i+1} = \pm 1$. We say that the sequence $(\pi_i, \pi_{i+1}, \ldots \pi_{i+k-1})$ is a run of length $k$ if, for $1 \leq j \leq k - 2$, the pair $(\pi_{i+j}, \pi_{i+j+1})$ is a bond. Denote by $\beta(\pi)$ the number of bonds in $\pi$.

Note that runs are necessarily either increasing or decreasing, and that a run of length $k$ contains $k-1$ bonds. We can now establish a fundamental relationship between bonds and $(n - 1)$-patterns.

Lemma 5.2.3. Let $\pi = \pi_1 \pi_2 \ldots \pi_n$. For any $j, k \in [n]$ with $j \neq k$, $\nabla_\pi(i) = \nabla_\pi(j)$ if and only if $\pi_j$ and $\pi_k$ are part of the same run.

Proof. The forward direction is clear, since removing any element of a run simply results in a shorter run.

The reverse implication takes a bit more work. Suppose that there exist $j, k$ with $1 \le j < k \le n$ and $\nabla_\pi(j) = \nabla_\pi(k)$. We proceed by induction on $k - j$.

For the base case, suppose that $k = j + 1$. Assume first that $\pi_j < \pi_{j+1}$, and consider the $j$th entry of $\nabla_\pi(j) = \nabla_\pi(j + 1)$. By the definition of $\nabla$, the $j$th entry of $\nabla_\pi(j)$ is $\pi_{j+1} - 1$, and the same entry in $\nabla_\pi(j + 1)$ is $\pi_j$. Therefore, we see that $\pi_{j+1} - 1 = \pi_j$, which means that $(\pi_j, \pi_{j+1})$ is a bond. Again, the case where $\pi_{j+1} < \pi_j$ follows similarly.

Now assume by way of induction that the statement holds when $k = j + m - 1$, and suppose there exists $1 \le j < k \le n$ such that $k - j = m$ and $\nabla_\pi(j) = \nabla_\pi(k)$. Assume first that $\pi_j < \pi_k$. $\nabla_\pi(j) = \nabla_\pi(k)$ implies, in particular, that the $(k-1)$st entries on both sides of the equality are equal. By definition, the $k - 1$ entry of $\nabla_\pi(j)$ is $\pi_k - 1$, while the $k - 1$ entry of $\nabla_\pi(k)$ is either $\pi_{k-1}$ or $\pi_{k-1} - 1$. The latter case would imply that $\pi_{k-1} = \pi_k$, a contradiction, and so it follows that $\pi_{k-1} = \pi_k$.

By what has already been proved, $\nabla_\pi(k - 1) = \nabla_\pi(k)$ since these entries form a bond. But then $\nabla_\pi(j) = \nabla_\pi(k) = \nabla_\pi(k - 1)$, and so by the induction hypothesis the entries $(\pi_j \pi_{j+1} \ldots \pi_{k-1})$ form a run. Finally, $\pi_k - 1 = \pi_{k-1}$ implies that $(\pi_j \pi_{j+1} \ldots \pi_{k-1} \pi_k)$ is a length $m$ run. Once more, the case where $\pi_j > \pi_k$ follows similarly, and the lemma is proved. $\qquad\square$

The size of the set $\nabla_\pi$ then depends entirely on $\beta(\pi)$, since each bond decreases by one the number of distinct $(n - 1)$-patterns contained in $\pi$. This leads to the following theorem, and its immediate corollary.

**Theorem 5.2.4.** Let $\pi \in \mathfrak{S}_n$. Then $|\nabla_\pi| = n - \beta(\pi)$.

**Corollary 5.2.5.** A permutation is plentiful if and only if it contains no bonds.

Theorem 5.2.4 also provides a simple proof of the following local property of the permutation pattern poset .

**Corollary 5.2.6.** If $\sigma \in \mathfrak{S}_{n-1}$, then $|\Phi_\sigma| = n^2 - 2n + 2 = (n - 1)^2 + 1$. In other words, every permutation of length $n$ is contained in exactly $n^2 + 1$ $(n + 1)$-permutations.

Proof. By definition, the set $\Phi_\sigma = \{ins_\sigma(j, k) : 1 \le j, k \le n\}$, so we see that $|\Phi_\sigma| \le n^2$.

Now, a permutation $\pi \in \mathfrak{S}_n$ is contained in $\Phi_\sigma$ more than once exactly when $\sigma$ can be obtained in more than one way by deleting a entry of $\pi$. It follows that $\sigma$ is contained in a permutation $\pi \in \mathfrak{S}_n$ more than once exactly when $\Phi_\sigma(j, k) = \Phi_\sigma(j', k')$ where $(j, k) \neq (j', k')$. By the lemma, this happens exactly when the $j$th entry of $\Phi_\sigma(j, k)$ is a part of the same run as the $j'$ entry of $\Phi_\sigma(j', k')$. We can prevent this from occurring by never inserting an element just to the right and directly above or below an existing element of $\sigma$, as this ensures that any new bonds can be created in exactly one way.

This eliminates exactly $2(n-1)$ choices for inserting an entry into $\sigma$, and so therefore $|\Phi_\sigma| = n^2 - 2(n-1) = (n-1)^2 + 1$, and the proof is complete. $\square$

## § 5.3   Distribution of the Number of Patterns

We now consider let $\pi$ be a (uniformly) randomly chosen permutation of length $n$, and examine the distribution of the statistic $|\nabla_\pi|$. The correlation presented in Theorem 5.2.4 allows us to investigate this distribution by analyzing the distribution of bonds. This distribution has been examined previously in other contexts, most notably by Kaplansky and Wolfowitz [59, 91]. In this section we extend their asymptotic results by finding exact values for the expectation and variance of $\beta(\pi)$, and therefore of $|\nabla_\pi|$.

Throughout this section, fix $n$ and let $\delta_n$ and $\beta_n$ be random variables denoting the number of distinct $(n-1)$-patterns and the number of bonds in a random permutation of length $n$, respectively. Our primary tool in this investigation will be multivariate generating functions, but first we note that $\mathbb{E}[\delta]$ can be obtained directly using results from the previous section.

Proposition 5.3.1. The expectation of $\delta$ is equal to $n - \frac{2(n-1)}{n}$, which approaches $n - 2$ as $n$ increases.

Proof. By the definition of expectation, we have

$$\mathbb{E}[\delta] = \frac{\sum_{\pi \in \mathfrak{S}_n} |\nabla_\pi|}{n!}.$$

The proposition then follows immediately from Corollary 5.2.6 and the identity

$$(n^2 - 2n + 2)(n-1)! = \left(n - \frac{2(n-1)}{n}\right) n!.$$

$\square$

## Generating Functions

Generating functions allow us to go several steps further, and obtain higher moments for the distributions of these variables. It follows from Theorem 5.2.4 and the linearity of expectation that

$$\mathbb{E}\left[\delta\right] = n - \mathbb{E}\left[\beta\right].$$

Therefore we can translate the distribution of $\beta$ to that of $\delta$. We start by building a multivariate generating function which keeps track of the distribution of bonds throughout all permutations. We use a method similar to the cluster method of Goulden and Jackson [47,48], described by Noonan and Zeilberger [69]. Note that this generating function converges nowhere, but still yields useful algebraic information.

Theorem 5.3.2. Let $a_{n,k}$ be the number of permutations of length $n$ which contain exactly $k$ bonds, and let $a_{0,0} = 1$. Then the numbers $a_{n,k}$ have the following generating function

$$\sum_{n\geq 0}\sum_{k\geq 0} a_{n,k} z^n u^k = \sum_{m\geq 0} m! \left( z + \frac{2z^2(u-1)}{1-z(u-1)} \right)^m.$$

Denote this function by $f(z, u)$.

Proof. First we construct a related generating function, then translate it into ours using the technique of inclusion-exclusion. Say that a bond in a permutation can be arbitrarily marked, and then a marked permutation is one in which each bond is either marked or unmarked. Let $b_{n,k}$ be the number of permutations of length $n$ which contain exactly $k$ marked bonds. For example, $b_{n,0} = n!$, since every permutation can be written with no bonds marked, and no permutation is counted more than once. Similarly, $b_{n,n-1} = 1$, since the only marked permutation with $n-1$ marked bonds is the decreasing permutation with all of its bonds marked.

Let

$$g(z, u) := \sum_{n\geq 0}\sum_{k\geq 0} b_{n,k} z^n u^k.$$

This generating function is easier to construct, as we can build a permutation of length $n$ with $k$ marked bonds by first specifying our marked runs, then permuting these runs with the remaining entries. The benefit to this method is that we don't have to worry about bonds forms between these runs, as we

have already specified which ones are marked. A marked run of length $j$ can be either ascending or descending, and contains $j - 1$ bonds. It follows that

$$g(z, v) = \sum_{m \geq 0} m! \left( z + \frac{2z^2 v}{1 - zv} \right)^m.$$

Now, we can use this generating function to obtain $f(z, u)$. The variable $v$ keeps track of marked bonds, while $u$ keeps track of all bonds. Since every bond can either be marked or unmarked, it follows that by substituting $u$ for $v + 1$ we can translate $f(z, u)$ to $g(z, v)$. Therefore, we have the relation $f(z, v + 1) = g(z, v)$, from which we see that

$$f(z, u) = g(z, u - 1) = \sum_{m \geq 0} m! \left( z + \frac{2z^2(u - 1)}{1 - z(u - 1)} \right)^m.$$

$\square$

The following corollary is immediate, and follows from the relationship between $\delta$ and $\beta$.

Corollary 5.3.3. Let $d_{n,k}$ denote the number of permutations of length $n$ containing exactly $k$ distinct $(n - 1)$ patterns, and let $d_{0,0} = 1$. Then

$$h(z, u) := \sum_{n \geq 0} \sum_{k \geq 0} h_{n,k} z^n u^k = \sum_{m \geq 0} m! \left( zu + \frac{2zu^2(1/u - 1)}{1 - zu(1/u - 1)} \right)^m.$$

Proof. Since $\delta = n - \beta$, it follows that $h(z, u) = f(zu, 1/u)$. $\square$

The remainder of this section will consist of the analysis of the function $F(z, u)$, and the translation of this analysis into facts about permutations. First, we compute the number of permutations which have no bonds (and are therefore plentiful).

Proposition 5.3.4. Let $b_n$ be the number of permutations of length $n$ with no bonds. Then

$$\sum_{n \geq 0} b_n z^n = \sum_{m \geq 0} m! z^m \frac{(1 - z)^m}{(1 + z)^m}$$
$$= 1 + z + 2z^4 + 14z^5 + 90z^6 + 646z^7 + 5242z^8 + \ldots$$

Proof. This follows immediately by setting $u = 0$ in $f(z, u)$.  □

The numbers $b_n$ in Corollary 5.3.4 are sequence A002464 in the OEIS [84]. These numbers are also equal to the number of ways of placing $n$ non-attacking kings on an $n \times n$ chessboard with one king per each row and column, as can be seen by plotting the permutations. It was shown in [83] that this sequence is asymptotic to $n!/e^2$, and so Corollary 5.2.5 implies the following corollary.

Corollary 5.3.5. The probability that a randomly selected $n$ permutation is plentiful tends to $1/e^2$ as $n$ tends to infinity.

In addition to exact results, we can use the function $f(z, u)$ to determine the expected number of bonds within a randomly selected permutation of length $n$, Using techniques described in Chapter 1 and in [43].

Theorem 5.3.6. The expectation and variance of the random variable $\beta_n$ are as follows:

$$\mathbb{E}[\beta_n] = 2\frac{(n-1)}{n}$$

$$\mathbb{V}[\beta_n] = 4\frac{(n-2)^2}{n(n-1)} + 2\frac{n-1}{n} - 4\frac{(n-1)^2}{n^2}.$$

Proof. The expectation is obtained by taking the partial derivative with respect to $u$, then plugging in $u = 0$ as shown below.

$$\sum_{n \geq 0} \mathbb{E}[\beta_n] z^n = \frac{\partial_u f(z, u)\big|_{u=0}}{n!}$$

$$= \sum_{n \geq 0} 2(n-1)! \cdot (n-1)z^n.$$

The second factorial moment $\mathbb{E}[\beta_n(\beta_n - 1)]$ can be computed from the generating function as follows:

$$\sum_{n \geq 0} \mathbb{E}[\beta_n(\beta_n - 1)] z^n = \frac{\partial_u^2 f(z, u)\big|_{u=1}}{n!}.$$

The variance can then be computed using linearity of expectation:

$$\mathbb{V}[\beta_n^2] = \mathbb{E}[\beta_n^2] - \mathbb{E}[\beta_n]^2 = \mathbb{E}[\beta_n(\beta_n - 1)] + \mathbb{E}[\beta_n] - \mathbb{E}[\beta_n]^2.$$

From here, a tedious and technical computation finishes the proof.  □

Higher moments can be computed iteratively. The relationship between the variables $\beta_n$ and $\delta_n$ immediately provides the corresponding expectation and variance for $\delta_n$. Taking the limit as $n \to \infty$ gives asymptotic values for this distribution, which leads to the results found in [59, 91]. We summarize these ideas in the following corollaries.

Corollary 5.3.7. The expectation and variance for the variable $\delta_n$ are as follows:

$$\mathbb{E}\left[\beta_n\right] = n - \frac{2(n-1)}{n}$$
$$\mathbb{V}\left[\beta_n\right] = 4\frac{(n-2)^2}{n(n-1)} + 2\frac{n-1}{n} - 4\frac{(n-1)^2}{n^2}.$$

Corollary 5.3.8. For large $n$, we have that

$$\mathbb{E}\left[\beta_n\right] \sim n-2 \quad \text{and} \quad \mathbb{V}\left[\beta_n\right] \sim 2.$$

## § 5.4   Patterns of Other Sizes

In this section, we examine the number of distinct $(n-k)$-patterns contained in a permutation of length $n$. For a given permutation of length $n$ $\pi$, $\nabla_\pi$ denotes the image of the function $\nabla_\pi$, which is exactly the set of $(n-1)$-patterns contained in $\pi$. The following definitions generalize the Definitions 5.1.1 and 5.2.1.

Definition 5.4.1. Let $S = \{i_1, i_2, \ldots i_k\} \subseteq [n]$, with $i_1 < i_2 < \cdots < i_k$. We denote by $\nabla_\pi(S)$ the permutation obtained by deleting the entries in positions $i_1, \ldots i_k$, and standardizing the remaining entries. Denote by $\nabla_\pi^k$ the set of all permutations which can be obtained by deleting $k$ entries from $\pi$ and standardizing.

Definition 5.4.2. Say that a permutation of length $n$ $\pi$ is $k$-plentiful if it has the maximal number of distinct $(n-k)$-patterns, i.e., if

$$|\nabla_\pi^k| = \binom{n}{k}.$$

Characterizing k-plentiful Permutations

We seek to characterize those permutations which are $k$-plentiful, for an arbitrary $k \in [n]$. In Section 5.3 we found that a permutation is plentiful if and only if it contains no bonds. By generalizing our notion of bonds, we obtain an analogous result here.

Definition 5.4.3. Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$. For any two integers $i, j \in [n]$, define the distance $d_\pi(i, j)$ between $i$ and $j$ to be

$$d_\pi(i, j) = |i - j| + |\pi_i - \pi_j|.$$

The minimum gap of $\pi$, denoted by $\Gamma(\pi)$, is defined to be the minimum distance between any two entries. Formally:

$$\Gamma(\pi) = \min\{d_\pi(i, j) : 1 \leq i, j \leq n\}.$$

If we plot a permutation $\pi$, then the function $d_\pi$ is just the usual taxicab metric on $\{(i, \pi_i) : 1 \leq i \leq n\} \subset \mathbb{R}^2$. It is easy to see that $(\pi_i, \pi_j)$ is a bond if and only if $d_\pi(i, j) = 2$. It follows then that $\pi$ is plentiful if and only if $\Gamma(\pi) \geq 3$. This idea allows us to generalize Corollary 5.2.5. We start with one more definition, and a simple lemma which will prove useful.

Definition 5.4.4. Let $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$ and let $i, j \in [n]$ with $i < j$. The span of the indices $i$ and $j$, denoted $\sigma_\pi(i, j)$, is defined as the set of indices corresponding to entries which are between $(i, \pi_i)$ and $(j, \pi_j)$ either horizontally and vertically. Formally, when $\pi_i < \pi_j$ we have

$$\sigma_\pi(i, j) = \{k : i < k < j\} \cup \{k : \pi_i < \pi_k < \pi_j\}.$$

The case when $\pi_i > \pi_j$ is defined analogously.

Lemma 5.4.5. Let $\pi \in \mathfrak{S}_n$ be such that $\Gamma(\pi) = m$, and let $i, j$ be such that $d_\pi(i, j) = m$. Then $|\sigma_\pi(i, j)| = m - 2$. Further, deleting one entry can reduce the minimum gap by at most one, i.e., $\Gamma(\nabla_\pi(k)) \geq k - 1$ for all $k \in [n]$.

Proof. Clearly $|\sigma_\pi(i, j)| \leq m - 2$, since otherwise this would contradict $\Gamma(\pi) = m$. The only way in which $|\sigma_\pi(i, j)|$ could be less than $k - 2$ is if there exists an entry $\pi_k$ which lies between $(i, \pi_i)$ and $(j, \pi_j)$ both vertically and horizontally. However, this would imply that $d_\pi(i, m) < d_\pi(i, j) = m - 2$, which contradicts the minimality of $d_\pi(i, j)$. Therefore, $|\sigma_\pi(i, j)| = m - 2$.

For the second part, note that the only way that deleting a single entry could reduce the minimum gap by more than one is if that entry lies between two minimally separated entries. However, we have just seen that no such entry exists. $\qquad\square$

We are now able to give a partial characterization of the $k$-plentiful permutations in the following generalization of Corollary 5.2.5.

**Theorem 5.4.6.** A permutation $\pi$ is $k$-plentiful if and only if $\Gamma(\pi) \geq k + 2$.

*Proof.* First let $\pi = \pi_1 \pi_2 \ldots \pi_n$ be a $k$-plentiful permutation, and assume by way of contradiction that $\Gamma(\pi) = m < k + 2$. Let $i < j$ be such that $d_\pi(i, j) = m$. By Lemma 5.4.5, we have that $\sigma_\pi(i, j) = \{s_1, s_2, \ldots s_{m-2}\}$. Let $\sigma = \nabla_\pi(\sigma_\pi(i, j) \in \mathfrak{S}_{n-m+2}$, the permutation obtained by removing the entries with indices $s_i$ and standardizing the remaining entries. If follows then that $\Gamma(\sigma) = 2$ and so $\sigma$ has a bond $(\sigma_i, \sigma_j)$ and is therefore not plentiful. It follows then that $\nabla_\sigma(i) = \nabla_\sigma(j)$, and so there are two sets of indices $S$ and $S'$ for which $\nabla_\pi(S) = \nabla_\pi(S')$. Therefore $|\nabla_\pi^k| < \binom{n}{k}$, contradicting the plentifulness of $\pi$.

For the other direction, we proceed using induction. We have already shown that the theorem holds when $k = 1$ (Corollary 5.2.5), so let $k > 1$ and assume that the statement holds for all positive integers less than $k$. Let $\pi \in \mathfrak{S}_n$ be such that $\Gamma(\pi) \geq k + 2$. We know by induction that this permutation is $m$-plentiful for all $1 \leq m < k$.

Suppose by way of contradiction that $\sigma \in \mathfrak{S}_{n-k}$ can be obtained by deleting two different sets of entries from $\pi$. That is, suppose that there exist $A = \{a_1, a_2, \ldots a_k\} \neq B = \{b_1, b_2, \ldots b_k\}$, with $a_i < a_j$ and $b_i < b_j$ for $i < j$, such that $\nabla_\pi(A) = \nabla_\pi(B) = \sigma$. Claim that $A \cap B = \emptyset$. To see this, suppose that $a_i = b_j$, and note that since $A - \{a_i\} \neq B - \{b_j\}$, $\sigma$ is contained in $\nabla_\pi(a_i)$ in two different ways. However, by Lemma 5.4.5, $\Gamma(\nabla_\pi(a_i)) \geq k + 1$, and so by induction $\nabla_\pi(a_i)$ is $(k-1)$-plentiful, a contradiction. Therefore $A$ and $B$ must be disjoint.

Assume without loss of generality that $a_1 < b_1$. Let $j \in [n]$ be the smallest integer such that $j > a_1$ but $j \notin A$. Since $\nabla_\pi(A) = \nabla_\pi(B) = \sigma = \sigma_1 \sigma_2 \ldots \sigma_{n-k}$, it follows that the entries $p_{a_1}$ will move to fulfill the role of $\sigma_{a_1}$ once the $B$ entries are deleted. However, the entry $a_j$ will also move to fulfill this role once the $A$ entries are deleted. However, this implies that every entry in the
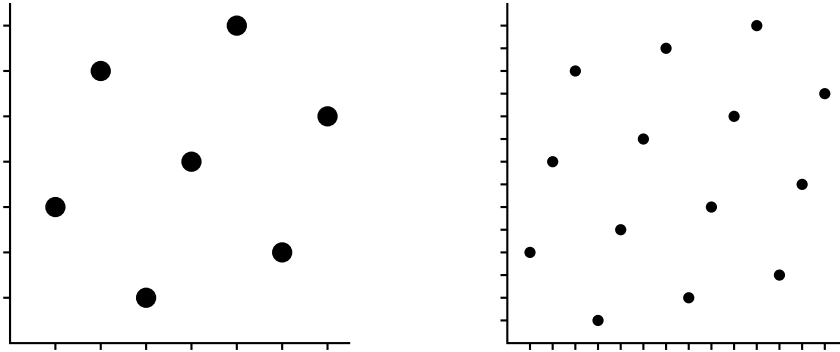
Figure 5.4.1: The plots of the permutations $\Theta^{(4)}$ and $\Theta^{(5)}$.

span of $\pi_{a_1}$ and $\pi_j$ must be deleted, but there must be at least $k$ such entries by Lemma 5.4.5. Therefore, $A$ must contain $a_1$ and $k$ additional entries, contradicting $|A| = k$ and proving the theorem.                    □

## Constructing k-plentiful Permutations

It is not immediately obvious that there exist permutations with arbitrarily large minimum gaps. In [7], the authors constructed a permutation of length $(k-1)^2$ which has a minimum gap equal to $k$. We conclude this section with a construction that gives a slightly smaller permutation which achieves the same gap size, and prove that this construction is the best possible.

Definition 5.4.7. Let $\pi \in \mathfrak{S}_{(k-1)^2}$ be defined by

$$\pi_{i(k-1)+j+1} = i + j(k-1) + 1, \quad 0 \leq i, j \leq k - 2.$$

Then let $\Theta^{(k)} \in \mathfrak{S}_{(k-1)^2-2}$ be defined by removing the first and last entries of $\pi$.

The permutations $\Theta^{(4)}$ and $\Theta^{(5)}$ are shown in Figure 5.4.1. It is clear from the figure, and can be shown from the definition (with some tedious but simple calculation) that $\Gamma(\Theta^{(k)}) = k$. It also follows that $\Theta^{(k)}$ is an involution, and its reverse is equal to its complement, so its orbit under the automorphism group of the pattern poset consists of only two elements.

By embedding a permutation $\pi$ into the plane, the function $d_\pi$ can be extended to the usual taxicab metric $d_1$ on $\mathbb{R}^2$. If $\pi$ has a minimum gap size of $k$, then $\pi$ defines a tiling of the plane with angled bricks of uniform size and centered on

the points of $\mathbb{Z}^2$. It is clear that a minimal such permutation will correspond to a maximal tiling of this form, with the property that no two centers lie on the same horizontal or vertical line. There are exactly two such tilings, corresponding to the permutation $\Theta^{(k)}$ and its reverse. We summarize this in the following theorem.

**Theorem 5.4.8.** The permutation $\Theta^{(k)}$ and its reverse are the shortest permutations with minimum gap size equal to $k$.

We end this chapter with one last theorem, generalizing Theorem 5.2.4.

**Theorem 5.4.9.** Let $\pi \in \mathfrak{S}_n$ have $\Gamma(\pi) = k + 1$, and let $p_k$ be the number of pairs $(i, j)$ such that $d_\pi(i, j) = k$. Then

$$|\nabla_\pi^k| = \binom{n}{k} - p_k.$$

*Proof.* Let $\pi \in \mathfrak{S}_n$ be such that $\Gamma(\pi) = k + 1$, and let $i, j \in [n]$ be such that $d_\pi(i, j) = k + 1$ (i.e., $|\sigma_\pi(i, j)| = k - 1$). If we let $S = \sigma_\pi \cup i$ and $S' = \sigma_\pi \cup j$, we see that $\nabla_\pi(S) = \nabla_\pi(S')$, and so

$$\nabla_\pi^k \le \binom{n}{k} - p_k.$$

To show equality, let $A = \{a_1, a_2, \ldots a_k\} \neq B = \{b_1, b_2, \ldots b_k\}$, with $a_i < a_j$ and $b_i < b_j$ when $i < j$, and suppose that $\nabla_\pi(A) = \nabla_\pi(B)$.

Claim that $|A \cap B| = k - 1$, i.e., that the two sets differ by exactly one element. . Suppose first that $a_1 \neq b_1$, and let $s$ be the smallest integer greater than $a_1$ such that $s \notin A$. Then, as in the proof of Theorem 5.4.6, we have $d_\pi(a_1, s) = k + 1$, and $A - a_1 = B - b_1 = \sigma_\pi(a_1, s)$. In the case where $a_1 = b_1$, let $\pi' = \nabla_\pi(a_1)$, $A' = A - \{a_1\}$, and $B' = B - \{b_1\}$. Since $\nabla_{\pi'}(A') = \nabla_{\pi'}(B')$, by Lemma 5.4.5 and Theorem 5.4.6 imply that that $\Gamma(\pi') = k$. We now find that either $a_2 = b_2$ or $A' - \{a_2\} = B' - \{b_2\}$. Iterating this argument shows that the two sets differ by at most one element.

Finally, let $i, j$ be such that $a_i \in A - B$ and $b_j \in B - A$. It follows then that $A - \{a_i\} = B - \{b_j\} - \{\sigma_\pi(i, j)\}$. But since their span has size $k - 1$, their distance must be equal to $k + 1$, an element in between them both horizontally and vertically would contradict the size of the minimum gap. Thus, each pair $i, j$ for which $d_\pi(i, j) = k + 1$ reduces the number of $(n - k)$-patterns by exactly one, which completes the proof. $\square$

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