THE SEQUENCE OF FRACTIONAL PARTS OF ROOTS

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ABSTRACT. We study the function $M_\theta(n) = \lfloor 1/\{\theta^{1/n}\}\rfloor$, where θ is a positive real number, $\lfloor\cdot\rfloor$ and $\{\cdot\}$ are the floor and fractional part functions, respectively. Nathanson proved, among other properties of M_θ, that if $\log\theta$ is rational, then for all but finitely many positive integers n, $M_\theta(n) = \lfloor n/\log\theta - 1/2\rfloor$. We extend this by showing that, without condition on θ, all but a zero-density set of integers n satisfy $M_\theta(n) = \lfloor n/\log\theta - 1/2\rfloor$. Using a metric result of Schmidt, we show that almost all θ have asymptotically $(\log\theta\log x)/12$ exceptional $n \le x$. Using continued fractions, we produce uncountably many θ that have only finitely many exceptional n, and also give uncountably many explicit θ that have infinitely many exceptional n.

1. Introduction

The author finds the identity (valid for any nonzero integer n)

$$\left\lfloor \frac{1}{e^{\sqrt{2}/n}-1}\right\rfloor = \left\lfloor \frac{n}{\sqrt{2}} - \frac{1}{2}\right\rfloor \tag{1}$$

breathtaking. Even more perplexing is that the similar expression (see [7])

$$\left\lfloor \frac{1}{2^{1/n}-1}\right\rfloor = \left\lfloor \frac{n}{\log(2)} - \frac{1}{2}\right\rfloor \tag{2}$$

holds for integers $1 < n < 777\,451\,915\,729\,368$, but fails at both of the given endpoints.

This identity and near-identity arise in our study of the sequence of fractional parts of roots, following Nathanson [5]. The distribution of $(\{\theta^n\})_{n\ge1}$, where $\theta > 1$, has been the object of much study [1] but remains enigmatic except for a few peculiar θ. The sequence $(\{\theta^{1/n}\})_{n\ge1}$ has been thought too simple to warrant study: trivially, for $\theta > 1$ one has $\theta^{1/n} > 1$ and $\theta^{1/n} \to 1$, and so $\{\theta^{1/n}\} \to 0$. Nevertheless, Nathanson found interesting phenomena

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in the regularity with which this convergence takes place. He introduced and derived the basic properties of

$$M_\theta(n) := \left\lfloor \frac{1}{\{\theta^{1/n}\}} \right\rfloor,$$

and identified symmetries that allow one to assume without loss of generality that $\theta > 1$ and that the integer n is positive. Surprisingly, he proved that for any real $\theta > 1$ and integer $n > \log_2 \theta$, either $M_\theta(n) = \lfloor n/\log\theta - 1/2 \rfloor$ or $M_\theta(n) = \lfloor n/\log\theta + 1/2 \rfloor$; moreover, if $\log\theta$ is rational, then $M_\theta(n) = \lfloor n/\log\theta - 1/2 \rfloor$ for all sufficiently large n.

We will show that the set

$$\left\{ n \in \mathbb{N} : M_\theta(n) \neq \left\lfloor \frac{n}{\log\theta} - \frac{1}{2} \right\rfloor \right\} \tag{3}$$

has density 0 for all $\theta > 1$, and for almost all $\theta > 1$ has counting function asymptotic to $\frac{\log\theta}{12}\log n$. For $\theta < e^6 \approx 400$, we give criteria for (3) to be finite or infinite in terms of the continued fraction expansions of $1/\log\theta$ and $2/\log\theta$. As a consequence, we are able to give explicit θ for which (3) is empty and is infinite. As mentioned above, Nathanson proved that for $\theta = e^{p/q}$, (3) is finite; we give another proof of this below that gives an explicit bound on the size in terms of p and q.

In the final sections of this article, we discuss the two displayed equations at the beginning of this introduction, and update Nathanson's list of open problems for $M_\theta(n)$.

2. Conventions, Results, Strategy

The set of positive integers is denoted $\mathbb{N}$. Throughout, we assume that $\theta > 1$ and that n is a positive integer. If $n > \log_2\theta$, then $1 < \theta^{1/n} < 2$, and so $\{\theta^{1/n}\} = \theta^{1/n} - 1$. Set

$$M'_\theta(n) := \left\lfloor \frac{1}{\theta^{1/n} - 1} \right\rfloor,$$

so that $M_\theta(n) = M'_\theta(n)$ if $n > \log_2\theta$. Although we don't use it here, this sort of expression arises [3, 6] in the following way. $M'_\theta(n)$ is the largest integer N such that $\theta N^n \leq (N+1)^n$; and $M'_\theta(n)$ is the largest integer N such that $(1 + \frac{1}{N})^n \geq \theta$. We call the elements of

$$\mathcal{A}_\theta := \left\{ n \in \mathbb{N} : M'_\theta(n) \neq \left\lfloor \frac{n}{\log\theta} - \frac{1}{2} \right\rfloor \right\}$$

the *atypical* numbers, terminology which we will justify later. Nathanson proved the following result, albeit in different notation.

Theorem 1. *If $n > \log_2\theta$, then either*

$$M_\theta(n) = \left\lfloor \frac{n}{\log\theta} - \frac{1}{2} \right\rfloor \qquad \text{and} \qquad n \notin \mathcal{A}_\theta$$

or

$$M_\theta(n) = \left\lfloor \frac{n}{\log\theta} + \frac{1}{2} \right\rfloor \qquad \text{and} \qquad n \in \mathcal{A}_\theta.$$

This shows that understanding $M_\theta(n)$ for nonsmall n is equivalent to understanding $\mathcal{A}_\theta$. Our results are presented as properties of $\mathcal{A}_\theta$. We state the theorems here, although we define some of the terminology, such as that relating to density and continued fractions, in the proofs.

Theorem 2 (Nathanson). *If $\log\theta = p/q > 1$ is rational, then*

$$\mathcal{A}_\theta \subseteq [1, \frac{p^2}{6q}).$$

Theorem 3. *For all $\theta > 1$, $\mathcal{A}_\theta$ has density 0.*

Theorem 4. *For almost all $\theta > 1$,*

$$\left|\mathcal{A}_\theta \cap [1,n]\right| \sim \frac{\log\theta}{12}\log n.$$

Theorem 5. *Let a_i be positive integers with $a_{2k} = 1$ for $k \geq 0$. Set ℓ to be the irrational number with simple continued fraction $[a_0; a_1, a_2, \dots]$, and set $\theta = e^{2/\ell}$. Then $\mathcal{A}_\theta = \emptyset$. In particular, if $c \in \mathbb{N}$ and $\theta = e^{-c+\sqrt{c(c+4)}}$, then $\mathcal{A}_\theta$ is empty.*

Theorem 6. *Let a_i be positive integers with $a_0 = 0$, $a_1 = 2$, $a_{2k} = 4$ for all $k \geq 1$. Set ℓ to be the irrational with simple continued fraction $[a_0; a_1, a_2, \dots]$, and set $\theta = e^{2/\ell}$. Then $\mathcal{A}$ is infinite. In particular, if $c \in \mathbb{N}$ and $\theta = e^{4-c+\sqrt{c(c+1)}}$, then $\mathcal{A}_\theta$ is infinite.*

The last two theorems give explicit uncountable families of θ with $\mathcal{A}_\theta$ empty and infinite, and also draw attention to even more explicit countable subfamilies. The simplest examples are that $\mathcal{A}_{e^{\sqrt{5}-1}}$ is empty and $\mathcal{A}_{e^{2\sqrt{5}}}$ is infinite. The actual results proved are inequalities on the partial quotients of the continued fraction, and the specific a_i given in these theorems are not the only a_i that satisfy the inequalities. Our countable families consist entirely of transcendental numbers; we do not know if there is an algebraic θ with $\mathcal{A}_\theta = \emptyset$, nor if there is an algebraic θ with $\mathcal{A}_\theta$ infinite.

We now outline our approach. We first obtain an asymptotic expansion

$$\frac{1}{\theta^{1/n}-1} = \frac{n}{\log\theta} - \frac{1}{2} + f\left(\frac{\log\theta}{n}\right)$$

for a very small positive function f. The floor of the left hand side is $M'_\theta(n)$, and the floor of the right hand side is $\lfloor n/\log\theta - 1/2\rfloor$ unless $n/\log\theta - 1/2$ is within $f(\log\theta/n)$ of an integer. We are thus led to a nonhomogeneous diophantine approximation problem that we can partially handle with continued fractions. In particular, we will need to know the simple continued fractions of both $1/\log\theta$ and $2/\log\theta$. By defining θ through the continued fraction of $2/\log\theta$ we are able to set, or at least control, the size of $\mathcal{A}_\theta$.

3. Bernoulli numbers and nth roots

We use the generating function for the sequence $(B_k)_{k=0}^{\infty}$ of Bernoulli numbers to obtain an asymptotic expansion of $M'_\theta(n)$. For $|t| < 2\pi$,

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \sum_{k=3}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k}. \tag{4}$$

For $t > 0$, we define the function

$$f(t) = \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}. \tag{5}$$

Lemma 7. *For $t > 0$, the function $f(t)$ is strictly increasing,* $\lim_{t\to 0^+} f(t) = 0$, *and* $\lim_{t\to\infty} f(t) = 1/2$. *If $0 < t < 1$, then*

$$\frac{t}{12} - \frac{t^3}{720} < f(t) < \frac{t}{12}. \tag{6}$$

Proof. The function $f(t)$ is strictly increasing (because $f'(t) > 0$), $\lim_{t\to 0^+} f(t) = 0$ (apply l'Hôpital's rule twice), and $\lim_{t\to\infty} f(t) = 1/2$.

For $0 < t < 2\pi$, we have the power series

$$f(t) = \frac{1}{12}t - \frac{1}{720}t^3 + \sum_{k=3}^{\infty} \frac{B_{2k}}{(2k)!} t^{2k-1}. \tag{7}$$

The Bernoulli numbers satisfy the classical identity ([2], [8, formula (9.1)])

$$\frac{B_{2k}}{(2k)!} = \frac{2(-1)^{k-1}}{(2\pi)^{2k}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}}.$$

It follows that for $0 < t < 1$ the sequence

$$\left(\frac{|B_{2k}|}{(2k)!} t^{2k-1} \right)_{k=1}^{\infty}$$

is strictly decreasing and tends to 0, hence (7) is an alternating series and (6) follows. This completes the proof. □

Lemma 8. *Either*

$$M'_\theta(n) = \left\lfloor \frac{n}{\log\theta} - \frac{1}{2} \right\rfloor \quad \text{or } M'_\theta(n) = \left\lfloor \frac{n}{\log\theta} + \frac{1}{2} \right\rfloor,$$

and $M'_\theta(n) = \lfloor n/\log\theta + 1/2 \rfloor$ *if and only if*

$$\frac{1}{2} - f\left(\frac{\log\theta}{n}\right) \le \left\{ \frac{n}{\log\theta} \right\} < \frac{1}{2}. \tag{8}$$

Note that Theorem 1 is a direct consequence of Lemma 8.

Proof. By the definition of f, we have

$$\begin{aligned} \frac{1}{\theta^{1/n} - 1} &= \frac{n}{\log\theta} - \frac{1}{2} + f\left(\frac{\log\theta}{n}\right) \\ &= \left\lfloor \frac{n}{\log\theta} - \frac{1}{2} \right\rfloor + \left\{ \frac{n}{\log\theta} - \frac{1}{2} \right\} + f\left(\frac{\log\theta}{n}\right) \end{aligned}$$

and so

$$M'_\theta(n) = \left\lfloor \frac{n}{\log\theta} - \frac{1}{2} \right\rfloor + \left\lfloor \left\{ \frac{n}{\log\theta} - \frac{1}{2} \right\} + f\left(\frac{\log\theta}{n}\right) \right\rfloor.$$

As fractional parts are between 0 and 1 while f is between 0 and $1/2$, we can now see that either $M'_\theta(n) = \lfloor n/\log\theta - 1/2 \rfloor$ or $M'_\theta(n) = \lfloor n/\log\theta - 1/2 \rfloor + 1$, which is the first claim of this lemma. Moreover, $M'_\theta(n) = \lfloor n/\log\theta + 1/2 \rfloor$ if and only if

$$1 \le \left\{ \frac{n}{\log\theta} - \frac{1}{2} \right\} + f\left(\frac{\log\theta}{n}\right) < 2.$$

By Lemma 7, for $t > 0$ we have $0 < f(t) < 1/2$, and if $1/2 < \{t - 1/2\}$ then $\{t - 1/2\} = \{t\} + 1/2$, and so

$$\frac{1}{2} < 1 - f\left(\frac{\log\theta}{n}\right) \le \left\{ \frac{n}{\log\theta} - \frac{1}{2} \right\} = \left\{ \frac{n}{\log\theta} \right\} + \frac{1}{2} < 1.$$

This implies (8).

Conversely, inequality (8) implies that

$$1 \le \left\{ \frac{n}{\log\theta} \right\} + \frac{1}{2} + f\left(\frac{\log\theta}{n}\right) = \left\{ \frac{n}{\log\theta} - \frac{1}{2} \right\} + f\left(\frac{\log\theta}{n}\right) < \frac{3}{2} < 2.$$

This completes the proof. □

Lemma 9. *If $0 \le a < b \le 1$, then*

$$\left\{ t \in \mathbb{R} : \frac{a}{2} \le \{t\} < \frac{b}{2} \right\} = \{t \in \mathbb{R} : a \le \{2t\} < b\} \setminus \left\{ t \in \mathbb{R} : \frac{a+1}{2} \le \{t\} < \frac{b+1}{2} \right\}.$$

Proof. If $\{t\} < \frac{1}{2}$, then $\{2t\} = 2\{t\}$ and so $a \le \{2t\} < b$ if and only if $a/2 \le \{t\} < b/2$. If $\{t\} \ge \frac{1}{2}$, then $\{2t\} = 2\{t\} - 1$ and so $a \le \{2t\} < b$ if and only if $(a+1)/2 \le \{t\} < (b+1)/2$. Thus,

$$\begin{aligned} \{t \in \mathbb{R} :& a \le \{2t\} < b\} \\ &= \left\{ t \in \mathbb{R} : \{t\} < \frac{1}{2} \text{ and } \frac{a}{2} \le \{t\} < \frac{b}{2} \right\} \\ &\qquad \cup \left\{ t \in \mathbb{R} : \{t\} \ge \frac{1}{2} \text{ and } \frac{a+1}{2} \le \{t\} < \frac{b+1}{2} \right\} \\ &= \left\{ t \in \mathbb{R} : \frac{a}{2} \le \{t\} < \frac{b}{2} \right\} \cup \left\{ t \in \mathbb{R} : \frac{a+1}{2} \le \{t\} < \frac{b+1}{2} \right\}. \end{aligned}$$

The Lemma follows from the observation that the two sets on the right side of this equation are disjoint. □

Combining Lemmas 8 and 9 proves the following result.

Lemma 10. *We have $n \in \mathcal{A}_\theta$ if and only if both*

$$\left\{ \frac{2n}{\log\theta} \right\} \ge 1 - 2f\left(\frac{\log\theta}{n}\right)$$

and

$$\left\{\frac{n}{\log\theta}\right\} < 1 - f\left(\frac{\log\theta}{n}\right).$$

4. Proofs of Theorems 2, 3, and 4

Nathanson proved that $\mathcal{A}_{e^{p/q}}$ is finite; while his proof is different from the one here, it could also be pushed to give this bound.

Proof of Theorem 2. Assume that $n \geq p^2/(6q)$. Then by Lemma 7

$$f\left(\frac{\log\theta}{n}\right) < \frac{\log\theta}{12n} \leq \frac{1}{2p},$$

and so there is no rational strictly between $1/2 - f(\log\theta/n)$ and $1/2$ with denominator p. But clearly $\{n/\log\theta\}$ has denominator p, and so Lemma 8 tells us that $n \notin \mathcal{A}_\theta$; that is, $\mathcal{A}_\theta \subseteq [1, p^2/(6q))$. □

The set $\mathcal{X}$ of positive integers has *density* ϵ if

$$\lim_{N\to\infty} \frac{\operatorname{card}(\{x \in \mathcal{X} : x \leq N\})}{N}$$

exists and is equal to ϵ. We use the following results concerning density. If a set $\mathcal{X}$ is a subset of a set with density ϵ for every small $\epsilon > 0$, then $\mathcal{X}$ has density 0. Let $0 \leq a < b < 1$ and let α be any irrational; the set $\{n \in \mathbb{N} : a \leq \{n\alpha\} < b\}$ has density $b - a$.

Proof of Theorem 3. If $\log\theta$ is rational, then by Theorem 2, we know that $\mathcal{A}_\theta$ is finite. As finite sets have density 0, this case is handled.

Now suppose that $\log\theta$ is irrational. Take small $\epsilon > 0$, and take $n_0 = \log\theta/(12\epsilon)$ so that for all $n > n_0$ we have $f(\log\theta/n) < \frac{\log\theta}{12n} < \epsilon$. Lemma 8 now implies that

$$\mathcal{A}_\theta \subseteq \left\{n \in \mathbb{N} : \frac{1}{2} - \epsilon \leq \left\{\frac{n}{\log\theta}\right\} < \frac{1}{2}\right\}.$$

Since $\log\theta$ is irrational, $\{n/\log\theta\}$ is uniformly distributed, and so $\mathcal{A}_\theta$ is contained in a set with density ϵ. As ϵ was arbitrary, it follows that $\mathcal{A}_\theta$ has density 0. □

The main tool for Theorem 4 is a result of Schmidt [10, Theorem 1], which we state a special case of below. Let $[\![P]\!]$ be 1 if P is true, and 0 if P is false, and let $|I|$ be the length of the interval I.

Theorem (Schmidt)**.** *Let $I_1 \supseteq I_2 \supseteq \dots$ be a nested sequence of subintervals of $[0,1)$, $\epsilon > 0$, then for almost all α*

$$\sum_{k=1}^{n} [\![\{k\alpha\} \in I_k]\!] = \sum_{k=1}^{n} |I_k| + O\left((\sum_{k=1}^{n} |I_k|)^{1/2+\epsilon}\right).$$

Proof of Theorem 4. We apply Schmidt's theorem with α replaced by $1/\log\theta$, take $x > 1$ and intervals

$$I_k = \left[\frac{1}{2} - f\left(\frac{\log x}{k}\right), \frac{1}{2}\right),$$

which are properly nested since $f(t)$ is increasing for $t > 0$. Since $f(t) = t/12 + O(t^3)$ (as $t \to 0$), as $n \to \infty$ we have

$$\sum_{k=1}^{n} |I_k| = \sum_{k=1}^{n} f\left(\frac{\log x}{k}\right) = \sum_{k=1}^{n} \left(\frac{\log x}{12k} + O(1/k^3)\right) = \frac{\log x}{12} \log n + O(1).$$

By Lemma 8, for $\theta < x$,

$$\begin{aligned} \mathcal{A}_\theta &= \left\{k : \left\{\frac{k}{\log\theta}\right\} \in [\frac{1}{2} - f\left(\frac{\log\theta}{k}\right), \frac{1}{2})\right\} \\ &\subseteq \left\{k : \left\{\frac{k}{\log\theta}\right\} \in [\frac{1}{2} - f\left(\frac{\log x}{k}\right), \frac{1}{2})\right\} \\ &= \left\{k : \left\{\frac{k}{\log\theta}\right\} \in I_k\right\}, \end{aligned}$$

and so

$$|A_\theta \cap [1,n]| \le \sum_{k=1}^{n} \left[\!\left[\left\{\frac{k}{\log\theta}\right\} \in I_k \right]\!\right].$$

Similarly, for $\theta > x$

$$|A_\theta \cap [1,n]| \ge \sum_{k=1}^{n} \left[\!\left[\left\{\frac{k}{\log\theta}\right\} \in I_k \right]\!\right].$$

Set

$$g(\theta) := \limsup_{n\to\infty} \frac{\big|A_\theta \cap [1,n]\big|}{\log n},$$

which must be Lebesgue measurable since its definition makes no appeal to the axiom of choice. One may verify the measurability of g more directly by observing that, for fixed n, the preimages of $\theta \mapsto A_\theta \cap [1,n]$ are unions of half-open intervals, and so each $\theta \mapsto \frac{|A_\theta \cap [1,n]|}{\log n}$ is a simple measurable function, and so $g(\theta)$ is the lim sup of a sequence of simple measurable functions, and so is itself measurable.

Schmidt's theorem implies: for all $x > 1$, almost all $\theta < x$ satisfy

$$g(\theta) \le \frac{\log x}{12},$$

and almost all $\theta > x$ satisfy

$$g(\theta) \ge \frac{\log x}{12}.$$

Now consider the integral

$$\int_1^x \left(g(\theta) - \frac{\log\theta}{12}\right) d\theta. \tag{9}$$

Let $1 = x_0 < x_1 < \cdots < x_N = x$ be evenly spaced from 1 to x. We have

$$\int_1^x \left(g(\theta) - \frac{\log\theta}{12} \right) d\theta = \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \left(g(\theta) - \frac{\log\theta}{12} \right) d\theta$$
$$\leq \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \left(\frac{\log x_{i+1}}{12} - \frac{\log\theta}{12} \right) d\theta,$$

which goes to 0 as $N \to \infty$ since $\log\theta/12$ is Riemann integrable over $[1, x]$. Similarly, we find that (9) is at least 0, whence $g(\theta) = \log\theta/12$ for almost all θ less than x, and x is arbitrary. □

We note that LeVeque [4] constructed α with

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} [\![\{k\alpha\} < 1/k]\!] = \infty,$$

showing that the "almost all" in Schmidt's theorem cannot be improved to "all". While LeVeque's construction, using continued fractions, does not immediately carry over to intervals that do not contain 0, we believe that the same phenomenon affects us. That is, we believe that for any function $g(n) \to 0$, there is a θ so that $\left|A_\theta \cap [1, n]\right| > n \cdot g(n)$ for infinitely many n.

5. Continued fractions and the proofs of Theorems 5 and 6.

The continued fraction algorithm produces a positive integer from a real number $\alpha > 1$ by taking the integer part of the reciprocal of the fractional part of α. This is exactly how the function $M_\theta(n)$ operates on the nth root of a real number $\theta > 1$, so it is, perhaps, not surprising that there is a relationship between continued fractions and the fractional parts of roots.

We shall consider infinite continued fractions of the form $[a_0; a_1, a_2, \ldots]$ with partial quotients $a_0 \in \mathbb{Z}$ and $a_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Then $\alpha = [a_0; a_1, a_2, \ldots]$ is a real irrational number whose *kth convergent* is the rational number

$$\frac{A_k}{B_k} = [a_0; a_1, a_2, \ldots, a_k]$$

where A_k and B_k are relatively prime positive integers. Also, set

$$\lambda_k := [0; a_{k-1}, a_{k-2}, \ldots, a_1] + [a_k; a_{k+1}, a_{k+2}, \ldots].$$

We follow the notation of Rockett and Szüsz [9], and use some results that are found there but not in the other standard references. The sequence of denominators, sometimes called *continuants*, satisfies $B_k \geq F_{k+1}$, the $(k+1)$th Fibonacci number. Further,

$$\frac{A_{2k-2}}{B_{2k-2}} < \frac{A_{2k}}{B_{2k}} < \alpha < \frac{A_{2k+1}}{B_{2k+1}} < \frac{A_{2k-1}}{B_{2k-1}} \tag{10}$$

and

$$\alpha - \frac{A_k}{B_k} = \frac{(-1)^k}{B_k^2 \lambda_{k+1}}. \tag{11}$$

This is often used in conjuction with the trivial bounds

$$a_{k+1} < \lambda_{k+1} < a_{k+1} + 2.$$

If m and n are positive integers and

$$\left|\alpha - \frac{m}{n}\right| \le \frac{1}{2n^2}, \tag{12}$$

then [9, Theorem II.5.1] there are integers $k \ge 0, c \ge 1$ such that $m = cA_k$ and $n = cB_k$ and $\lambda_{k+1} > 2c^2$.

Lemma 11. *Let $1 < \theta < e^3$ with $\log\theta$ irrational, and a_k, B_k, λ_k be associated to the continued fraction of $2/\log\theta$. For each $n \in \mathcal{A}_\theta$, there exist positive integers c, k such that $n = cB_{2k-1}$ and $\lambda_{2k} > \frac{6c^2}{\log\theta}$.*

Proof. Let $n \in \mathcal{A}_\theta$, i.e., $M'_\theta(n) = \lfloor n/\log\theta + 1/2\rfloor$. By Theorem 10

$$1 - 2f\left(\frac{\log\theta}{n}\right) \le \left\{\frac{2n}{\log\theta}\right\} < 1.$$

Let $m = 1 + \lfloor 2n/\log\theta\rfloor$. Applying the upper bound in Lemma 7 with $t = \log\theta/n$, we obtain

$$0 < 1 - \left\{\frac{2n}{\log\theta}\right\} = m - \frac{2n}{\log\theta} \le 2f\left(\frac{\log\theta}{n}\right) < \frac{\log\theta}{6n} < \frac{1}{2n}$$

and so

$$0 < \frac{m}{n} - \frac{2}{\log\theta} < \frac{1}{2n^2}.$$

Properties (10) and (12) of continued fractions imply that m/n is an odd convergent to $2/\log\theta$. Thus, there exist positive integers k and c with $\lambda_{2k} > 2c^2$ such that $m = cA_{2k-1}$ and $n = cB_{2k-1}$. It follows from property (11) that

$$\frac{1}{B_{2k-1}^2\lambda_{2k}} = \frac{A_{2k-1}}{B_{2k-1}} - \frac{2}{\log\theta} < \frac{\log\theta}{6c^2 B_{2k-1}^2}$$

and so $\lambda_{2k} > 6c^2/\log\theta$, which makes the earlier restriction $\lambda_{2k} > 2c^2$ redundant. This completes the proof. □

Proof of Theorem 5. Let $a_0 \ge 1$ and $a_{2k} \le 3a_0 - 2$ for $k \ge 1$, $\ell = [a_0; a_1, \ldots]$, $\theta = e^{2/\ell}$. Then $0 < \log\theta < 2/a_0 \le 2$, and so θ satisfies the hypotheses of Lemma 11. Consequently, for each $n \in \mathcal{A}_\theta$, there are positive integers c, k such that $n = cB_{2k-1}$ and $\lambda_{2k} > 6c^2/\log\theta$. But

$$\lambda_{2k} = [0; a_{2k-1}, a_{2k-2}, \ldots, a_1] + [a_{2k}; a_{2k+1}, \ldots] < a_{2k} + 2 \le 3a_0$$

while

$$\frac{6c^2}{\log\theta} = 3c^2\ell \ge 3a_0.$$

Therefore, there are no n in $\mathcal{A}_\theta$.

Set $a_{2k} = 1$ for $k \ge 0$, and let the a_{2k-1} be arbitrary positive integers, to see the first family stated in Theorem 5. Set $a_{2k+1} = c$, an arbitrary positive integer, for $k \ge 0$ to get

$$\ell = [1; c, 1, c, 1, \ldots] = \frac{c + \sqrt{c(c+4)}}{2c}$$

and

$$\theta = e^{-c+\sqrt{c(c+4)}}.$$

□

Set

$$\mathcal{Q}_\theta \coloneqq \left\{ cB_{2i-1} : i \text{ and } c \text{ positive integers, } 2c^2 < \lambda_{2i} \right\}$$

where B_i, λ_i correspond to the continued fraction of $1/\log\theta$ (not of $2/\log\theta$). By properties (11) and (12) of continued fractions,

$$\mathcal{Q}_\theta = \left\{ n : n \geq 1, \text{ there exists an integer } m \text{ with } 0 < m - \frac{n}{\log\theta} < \frac{1}{2n} \right\}.$$

In particular, $\mathcal{Q}_\theta$ is a set of good denominators for approximating $1/\log\theta$.

Our next lemma identifies continuants of $2/\log\theta$ that are either also good denominators for $1/\log\theta$ or are exceptional. When we apply the lemma in the proof of Theorem 6, we will have additional constraints that prevent the continuants from also being good denominators for $1/\log\theta$, and thereby force them to be exceptional.

Lemma 12. *Let $1 < \theta < e^6$ with $\log\theta$ irrational, and a_k, B_k, λ_k be associated to the continued fraction of $2/\log\theta$. For $0 < \delta < \log\theta$, choose $k_0 = k_0(\delta) \geq 3$ such that*

$$B_{2k-1}^2 > \frac{(\log\theta)^3}{60\delta} \tag{13}$$

for all $k \geq k_0$. If $k \geq k_0$ and

$$\lambda_{2k} \geq \frac{6}{\log\theta - \delta} \tag{14}$$

then

$$B_{2k-1} \in \mathcal{Q}_\theta \cup \mathcal{A}_\theta.$$

Proof. As $k \geq 3$, we have $B_{2k-1} \geq B_5 \geq 8$ and $0 < \log\theta / B_{2k-1} \leq 6/8 < 1$. Continued fraction inequalities (10) and (11) give

$$0 < \frac{A_{2k-1}}{B_{2k-1}} - \frac{2}{\log\theta} = \frac{1}{\lambda_{2k} B_{2k-1}^2},$$

whence, with $\lambda_{2k} > a_{2k} \geq 1$,

$$0 < A_{2k-1} - \frac{2B_{2k-1}}{\log\theta} = \frac{1}{\lambda_{2k} B_{2k-1}} < \frac{1}{8}.$$

It follows that $2B_{2k-1}/\log\theta$ is slightly less than an integer, and therefore

$$\left\{ \frac{2B_{2k-1}}{\log\theta} \right\} = 1 - \frac{1}{\lambda_{2k} B_{2k-1}}. \tag{15}$$

Assuming that λ_{2k} and B_{2k-1} satisfy the inequalities in (13) and (14), we have

$$\begin{aligned}\frac{1}{\lambda_{2k}B_{2k-1}} &\le \frac{1}{B_{2k-1}}\left(\frac{\log\theta-\delta}{6}\right)\\ &= \frac{\log\theta/B_{2k-1}}{6} - \left(\frac{60\delta B_{2k-1}^2}{(\log\theta)^3}\right)\frac{(\log\theta/B_{2k-1})^3}{360}\\ &< \frac{\log\theta/B_{2k-1}}{6} - \frac{(\log\theta/B_{2k-1})^3}{360}\\ &= 2\left(\frac{\log\theta/B_{2k-1}}{12} - \frac{(\log\theta/B_{2k-1})^3}{720}\right)\\ &< 2f\left(\frac{\log\theta}{B_{2k-1}}\right),\end{aligned}$$

where the last inequality uses the lower bound in Lemma 7 with $t = \log\theta/B_{2k-1}$. Combining this with (15) gives

$$\left\{\frac{2B_{2k-1}}{\log\theta}\right\} > 1 - 2f\left(\frac{\log\theta}{B_{2k-1}}\right).$$

If, further, $\{B_{2k-1}/\log\theta\} < 1 - f(\log\theta/B_{2k-1})$, then by Lemma 10, we have $B_{2k-1} \in \mathcal{A}_\theta$. We therefore assume that

$$\left\{\frac{B_{2k-1}}{\log\theta}\right\} \ge 1 - f\left(\frac{\log\theta}{B_{2k-1}}\right) \tag{16}$$

and need to show that $B_{2k-1} \in \mathcal{Q}_\theta$. Define b_i through

$$\frac{1}{\log\theta} = [b_0; b_1, b_2, \ldots],$$

and denote the convergents of $1/\log\theta$ by R_i/S_i, and set

$$\tau_k = [0; b_{k-1}, \ldots, b_1] + [b_k; b_{k+1}, b_{k+2}, \ldots].$$

We need to prove that $B_{2k-1} = cS_{2i-1}$ for some $c, i \in \mathbb{N}$ and $2c^2 < \tau_{2i}$.

Inequality (16) implies that

$$0 < 1 - \left\{\frac{B_{2k-1}}{\log\theta}\right\} \le f\left(\frac{\log\theta}{B_{2k-1}}\right) < \frac{\log\theta}{12B_{2k-1}}.$$

Let $r = 1 + \lfloor B_{2k-1}/\log\theta\rfloor$. Then

$$0 < r - \frac{B_{2k-1}}{\log\theta} < \frac{\log\theta}{12B_{2k-1}} < \frac{1}{2B_{2k-1}}$$

because $\log\theta < 6$, and so

$$0 < \frac{r}{B_{2k-1}} - \frac{1}{\log\theta} < \frac{1}{2B_{2k-1}^2}.$$

This implies that r/B_{2k-1} is an oddth convergent to $1/\log\theta$, i.e., there are positive integers c, i with $B_{2k-1} = cS_{2i-1}$ and $2c^2 < \tau_{2i}$. This completes the proof of this lemma. □

Proof of Theorem 6. Set $a_0 = 0, a_1 = 2$, $a_{2k} = 4$ for all $k \geq 1$, and let the a_{2k+1} be arbitrary positive integers, giving us uncountably many options; the choices $a_{2k+1} = c$ lead to $\theta = e^{4-c+\sqrt{c(c+1)}}$. Define θ through

$$\frac{2}{\log\theta} = [a_0; a_1, a_2, \dots]$$

which is clearly irrational, and let B_k be its continuants. Now,

$$[0;2,4] < \frac{2}{\log\theta} = [a_0; a_1, a_2, \dots] = [0; 2, 4, a_3, 4, a_5, \dots] < [0;2]$$

and so $e^4 < \theta < e^{9/2} < e^6$. We take $\delta = 2$, and since

$$\frac{(\log\theta)^3}{60\delta} < 1$$

we may take $k_0 = 3$. As

$$\lambda_{2k} > a_{2k} = 4 > \frac{6}{4-2} > \frac{6}{\log\theta - 2},$$

Lemma 12 tells us that B_{2k-1} (for $k \geq 3$) is in $\mathcal{Q}_\theta \cup \mathcal{A}_\theta$. We will show that B_{2k-1} is not in $\mathcal{Q}_\theta$, and this will prove that $\mathcal{A}_\theta$ is infinite.

Let S_k denote the kth convergent to $1/\log\theta$. Since a_{2k} is always even, we have

$$\frac{1}{\log\theta} = \frac{1}{2}\frac{2}{\log\theta} = \frac{1}{2}\cdot[a_0; a_1, a_2, a_3, \dots] = [\frac{a_0}{2}; 2a_1, \frac{a_2}{2}, 2a_3, \dots].$$

That is, the simple continued fraction of $1/\log\theta = [b_0; b_1, \dots]$ where $b_0 = 0$, $b_1 = 4$, $b_{2k} = 2$ and $b_{2k+1} = 2a_{2k+1}$ for $k \geq 1$. We have $S_0 = B_0 = 1$, $S_1 = 2B_1 = 4$, $S_2 = B_2 = 9$, and the recursion relations for $k \geq 2$

$$\begin{aligned} B_{2k} &= 4B_{k-1} + B_{k-2} \\ B_{2k-1} &= a_{2k-1}B_{2k-2} + B_{2k-3} \\ S_{2k} &= 2S_{2k-1} + S_{2k-2} \\ S_{2k-1} &= 2a_{2k-1}S_{2k-2} + S_{2k-3}. \end{aligned}$$

These imply that $B_{2k} = S_{2k}$ and $2B_{2k+1} = S_{2k+1}$ for all $k \geq 0$.

If $B_{2k-1} \in \mathcal{Q}_\theta$, then there are positive integers c, i with $B_{2k-1} = cS_{2i-1}$ and $\tau_{2i} > 2c^2$, where

$$\tau_i := [0, b_{i-1}, \dots, b_1] + [b_i; b_{i+1}, b_{i+2}, \dots].$$

Clearly, $\tau_{2i} < b_{2i} + 2 = 4$, so that necessarily $c = 1$. If $k \geq 3$ and $B_{2k-1} = S_{2i-1}$ for some $i \geq 1$, then $B_{2k-1} = S_{2i-1} = 2B_{2i-1}$ and so $i < k$. But then

$$2B_{2i-1} = B_{2k-1} > B_{2k-2} + B_{2k-3} = (4B_{2k-3} + B_{2k-4}) + B_{2k-3} > 5B_{2k-3} \geq 5B_{2i-1}$$

which is absurd. Therefore, there are no such c, i, and therefore $B_{2k-1} \notin \mathcal{Q}_\theta$. □

6. The identities stated in the first paragraph, $\theta = 2$ and $\theta = e^{\sqrt{2}}$

Set $\theta = e^{\sqrt{2}}$. Because

$$\frac{2}{\log\theta} = \sqrt{2} = [1; 2, 2, 2, \ldots]$$

and for all $k \geq 1$

$$\lambda_{2k} < 4 < 3\sqrt{2} = \frac{6}{\log\theta},$$

Lemma 11 tells us that $\mathcal{A}_\theta$ is empty. By definition, $M'_\theta(n) = \lfloor n/\log\theta - 1/2\rfloor$ for all $n \in \mathbb{N}$, and $M_\theta(n) = \lfloor n/\log\theta - 1/2\rfloor$ for all $n \geq 3 > \log_2\theta = \sqrt{2}/\log 2$.

In the first sentence of this paper, we claimed that $M'_\theta(n) = \lfloor n/\log\theta - 1/2\rfloor$ for all nonzero n, which we deduce now from the positive n case. Assume $n > 0$. Since $\lfloor -x\rfloor = -\lfloor x\rfloor - 1$ for positive nonintegers x,

$$\begin{aligned}
M'_\theta(-n) &= \left\lfloor \frac{1}{\theta^{1/-n} - 1}\right\rfloor \\
&= \left\lfloor \frac{-\theta^{1/n}}{\theta^{1/n} - 1}\right\rfloor \\
&= \left\lfloor \frac{-\theta^{1/n}}{\theta^{1/n} - 1} + \frac{\theta^{1/n} - 1}{\theta^{1/n} - 1} - 1\right\rfloor \\
&= \left\lfloor -\frac{1}{\theta^{1/n} - 1}\right\rfloor - 1 \\
&= -\left\lfloor \frac{1}{\theta^{1/n} - 1}\right\rfloor - 2 \\
&= -M'_\theta(n) - 2,
\end{aligned}$$

making of use of the Gelfand-Schneider Theorem to be certain that

$$\frac{1}{\theta^{1/n} - 1} = \frac{1}{e^{\sqrt{2}/n} - 1}$$

is not an integer. Continuing,

$$\begin{aligned}
M'_\theta(-n) = -M'_\theta(n) - 2 &= -\left\lfloor \frac{n}{\log\theta} - \frac{1}{2}\right\rfloor - 2 \\
&= \left\lfloor -\left(\frac{n}{\log\theta} - \frac{1}{2}\right)\right\rfloor - 1 \\
&= -\left\lfloor \frac{-n}{\log\theta} - \frac{1}{2}\right\rfloor,
\end{aligned}$$

where we have used the value and irrationality of $\log\theta = \sqrt{2}$ to guarantee that $n/\log\theta - 1/2$ is positive and not an integer. This establishes (1) for negative n.

Now, set $\theta = 2$, and take n so that $n \in \mathcal{A}_\theta$. As $\log 2$ is irrational and $\frac{2}{\log 2} < 3$, we can apply Theorem 11 to deduce that there are positive integers c, k such that $n = cB_{2k-1}$ and $\lambda_{2k} > 6c^2/\log 2$ (where B_i, λ_i correspond to the continued fraction of $2/\log 2$). It is not difficult to compute $\lambda_2, \lambda_4, \dots, \lambda_{34}$ and find that only λ_2 is greater than $6/\log 2$. Therefore, our only candidate for $\mathcal{A}_2$ less than $B_{35} = 777\,451\,915\,729\,368$ is $B_1 = 1$ (we have to consider the multiples cB_1 with $8.73 > \lambda_2 > 6c^2/\log 2 > 8.65c^2$, that is, $c = 1$). Direct calculation shows that in fact B_1 and B_{35} are both in $\mathcal{A}_\theta$. This completes our justification of the claims made in our opening paragraph. This is essentially the same as the computation of sequence A129935 in the OEIS [7].

7. More Problems

Nathanson [5, Section 5] gives a list of problems concerning $M_\theta(n)$. Several of these problems are solved (explicitly or implicitly) in the current work, but those concerning small n or letting θ vary are not addressed here. To his list, we add the following problems:

(1) Is $\mathcal{A}_{e^e}$ infinite?
(2) Are there θ, τ with both $\mathcal{A}_\theta$ and $\mathcal{A}_\tau$ infinite, but the symmetric difference $\mathcal{A}_\theta \triangle \mathcal{A}_\tau$ finite?
(3) For every θ_0, are there uncountably many $\theta > \theta_0$ with $\mathcal{A}_\theta$ finite?
(4) What is the Hausdorff dimension of $\{\theta > 1 : \mathcal{A}_\theta \text{ is finite}\}$?
(5) Is there any algebraic θ for which $\mathcal{A}_\theta$ can be proved finite? Infinite?

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