# MRA-W ${ }^{\mathbf{a}^{\mathbf{v}}}{ }^{\mathbf{e}} \mathbf{e}_{\text {t subspace architecture for logic, probability, and }}$ symbolic sequence processing 

Daniel J. Greenhoe


#### Abstract

The linear subspaces of a multiresolution analysis (MRA) and the linear subspaces of the wavelet analysis induced by the MRA, together with the set inclusion relation $\subseteq$, form a very special lattice of subspaces which herein is called a primorial lattice. This paper introduces an operator $\mathbf{R}$ that extracts a set of $2^{N-1}$ element Boolean lattices from a $2^{N}$ element Boolean lattice. Used recursively, a sequence of Boolean lattices with decreasing order is generated-a structure that is similar to an MRA. A second operator, which is a special case of a "difference operator", is introduced that operates on consecutive Boolean lattices $L_{2}^{n}$ and $L_{2}^{n-1}$ to produce a sequence of orthocomplemented lattices. These two sequences, together with the subset ordering relation $\subseteq$, form a primorial lattice $\mathbb{P}$. A logic or probability constructed on a Boolean lattice $L_{2}^{N}$ likewise induces a primorial lattice $\mathbb{P}$. Such a logic or probability can then be rendered at $N$ different "resolutions" by selecting any one of the $N$ Boolean lattices in $\mathbb{P}$ and at $N$ different "frequencies" by selecting any of the $N$ different orthocomplemented lattices in $\mathbb{P}$. Furthermore, $\mathbb{P}$ can be used for symbolic sequence analysis by projecting sequences of symbols onto the sublattices in $\mathbb{P}$ using one of three lattice projectors introduced. $\mathbb{P}$ can be used for symbolic sequence processing by judicious rejection and selection of projected sequences. Examples of symbolic sequences include sequences of logic values, sequences of probabilistic events, and genomic sequences (as used in "genomic signal processing").


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## 1 Background：lattices

## 1．1 Order

## 1．1．1 Order relations

Definition $1.1^{1}$ Let $X$ be a set．Let $2^{X X}$ be the set of all relations on $X$ ．A relation $\leq$ is an order relation in $2^{X X}$ if
$\begin{array}{llll}\text { 1．} x \leq x & \forall x \in X & \text {（reflexive）} & \text { and } \\ \text { 2．} x \leq y \text { and } y \leq z \Longrightarrow x \leq z & \forall x, y, z \in X & \text {（transitive）} & \text { and } \\ \text { 3．} x \leq y \text { and } y \leq x \Longrightarrow x=y & \forall x, y \in X & \text {（anti－symmetric）}\end{array}$
An ordered set is the pair $(X, \leq)$ ．The set $X$ is called the base set of $(X, \leq)$ ．If $x \leq y$ or $y \leq x$ ，then elements $x$ and $y$ are said to be comparable，denoted $x \sim y$ ．Otherwise they are incomparable，denoted $x \| y$ ．The relation $\lesseqgtr$ is the relation $\leq \backslash=$（＂less than but not equal to＂），where $\backslash$ is the set difference operator，and $=$ is the equality relation．

Definition 1．2 ${ }^{2}$ Let（ $X, \leq$ ）be an ordered set（Definition 1.1 page 3）．Let $2^{X X}$ be the set of all relations on $X$ ．The relations $\geq,<,>\in 2^{X X}$ are defined as follows：

[^0]```
\(x \geq y \stackrel{\text { def }}{\Longleftrightarrow} y \leq x \quad \forall x, y \in X\)
\(x \leq y \quad \stackrel{\text { def }}{\Longleftrightarrow} x \leq y\) and \(x \neq y \quad \forall x, y \in X\)
\(x \geq y \quad \stackrel{\text { def }}{\Longleftrightarrow} \quad x \geq y\) and \(x \neq y \quad \forall x, y \in X\)
```

The relation $\geq$ is called the dual of $\leq$ ．

## Example 1.3

| order relation |  | dual order relation |  |
| :--- | :--- | :--- | :--- |
| $\leq \quad$（integer less than or equal to） | $\geq$ | （integer greater than or equal to） |  |
| $\subseteq$ | （subset） | （super set） |  |
| $\mid$ | （divides） |  | （divided by） |
| $\Longrightarrow$ | （implies） | （implied by） |  |

Definition $1.4{ }^{3}$ A relation $\leq$ is a linear order relation on $X$ if
1．$\leq$ is an order relation（Definition 1.1 page 3）and
2．$x \leq y$ or $y \leq x \quad \forall x, y \in X \quad$（comparable）．
A linearly ordered set is the pair（ $X, \leq$ ）．
A linearly ordered set is also called a totally ordered set，a fully ordered set，and a chain．

## 1．1．2 Representation

Definition $1.5{ }^{4} y$ covers $x$ in the ordered set $(X, \leq)$ if
1．$x \leq y$
2．$(x \leq z \leq y)$$\quad(z=x$ or $z=y) \quad \begin{aligned} & (y \text { is greater than } x) \\ & \text {（there is no element between } x \text { and } y) .\end{aligned}$ and
The case in which $y$ covers $x$ is denoted $x<y$ ．

An ordered set can be represented in any of three ways：
．Hasse diagram
（Definition 1.6 page 4）
解 a set of ordered pairs of order relations
（Definition 1.1 page 3）
2 a set of ordered pairs of cover relations
（Definition 1.5 page 4）

Definition 1．6 Let $(X, \leq)$ be an ordered pair．A diagram is a Hasse diagram of $(X, \leq)$ if it satisfies the following criteria：

6．Each element in $X$ is represented by a dot or small circle．
\＆For each $x, y \in X$ ，if $x<y$ ，then $y$ appears at a higher position than $x$ and a line connects $x$ and $y$ ．

[^1]Example 1.7 Here are three ways of representing the ordered set $\left(2^{\{x, y\}}, \subseteq\right)$;
(1) Hasse diagrams: If two elements are comparable, then the lesser of the two is drawn lower on the page than the other with a line connecting them.

(2) Sets of ordered pairs specifying order relations (Definition 1.1 page 3):

$$
\subseteq=\left\{\begin{array}{ccc}
(\varnothing, \varnothing), & (\{x\},\{x\}), & (\{y\},\{y\}), \\
(\varnothing,\{x\}), & (\varnothing,\{y\}), & (\varnothing,\{x, y\}), \\
(\{x\},\{x, y\}),(\{x, y\}), \\
(\{y\},\{x, y\})
\end{array}\right\}
$$

(3) Sets of ordered pairs specifying covering relations:

$$
<=\{(\varnothing,\{x\}), \quad(\varnothing,\{y\}), \quad(\{x\},\{x, y\}),(\{y\},\{x, y\})\}
$$

### 1.1.3 Decomposition

Definition 1.8 ${ }^{5}$ The tupple $(Y,())$ is a subposet of the ordered set $(X, \leq)$ if
$\begin{array}{llll}\text { 1. } & Y \subseteq X & (Y \text { is a subset of } X) & \text { and } \\ \text { 2. } & \subseteq & =\left(\left\langle\cap Y^{2}\right)\right. & (Q \text { is the relation } \leq \text { restricted to } Y \times Y)\end{array}$
2. $\otimes=\left(\leq \cap Y^{2}\right) \quad(\otimes$ is the relation $\leq$ restricted to $Y \times Y)$

## Example 1.9

Subposets of

include





Example 1.10 Let

$$
\begin{aligned}
&(X, \leq) \triangleq(\{0, a, b, c, p, 1\},\{(0,0),(a, a),(b, b),(c, c),(p, p),(1,1), \\
&(0, a),(0, b),(0, c),(0, p),(0,1), \\
&(a, b),(a, c),(a, 1),(p, 1), \\
&(Y, @) \triangleq(\{0, a, c, p, 1\},(b, 1),(c, 1),(p, 1)\}) \\
&\{(0,0),(a, a),(c, c),(p, p),(1,1), \\
&(0, a),(0, c),(0, p),(0,1), \\
&(a, c),(a, 1),(p, 1),(c, 1),(p, 1)\}) .
\end{aligned}
$$

Then $(Y, \otimes)$ is a subposet of $(X, \leq)$ because $Y \subseteq X$ and $\Theta=\left(\leq \cap Y^{2}\right)$.

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A chain is an ordered set in which every pair of elements is comparable（Definition 1.4 page 4）． An antichain is just the opposite－it is an ordered set in which no pair of elements is com－ parable（next definition）．

Definition $1.11{ }^{6}$ The subposet $(A, \otimes)$ in the ordered set $(X, \leq)$ is an antichain if all elements in $A$ are incomparable（Definition 1.1 page 3），such that

$$
x \| y \quad \forall x, y \in A
$$



Definition $1.12{ }^{7}$ The length $\ell(\boldsymbol{L})$ of a chain（Definition 1.4 page 4）$L$ with $N$ elements is $N-1$ ． The length of an ordered set（Definition 1.1 page 3）is the length of the longest chain in the ordered set．The width of an ordered set is the number of elements in the largest antichain in the ordered set．

Theorem 1.13 （Dilworth＇s theorem）${ }^{8}$ Let $(X, \leq)$ be an ordered set．
$\left\{\begin{array}{l}\text { width } N \text { of }(X, \leq) \\ \text { is } \operatorname{\text {FINITE}}\end{array}\right\} \Longrightarrow\left\{\begin{array}{ll}1 . & \text { there exists a partition of }(X, \leq) \text { into } N \text { chains and } \\ 2 . & \text { there does not exist any partition } \\ \text { of }(X, \leq) \text { into less than } N \text { chains }\end{array}\right\}$
Definition 1．14 ${ }^{9}$ Let $X$ and $Y$ be disjoint sets．Let $\boldsymbol{P} \triangleq(X, \Theta)$ and $\boldsymbol{Q} \triangleq(Y, \lessdot)$ be ordered sets on $X$ and $Y$ ．The direct sum of $P$ and $Q$ is defined as

$$
\boldsymbol{P}+\boldsymbol{Q} \triangleq(X \cup Y, \leq)
$$

where $x \leq y$ if

$$
\begin{array}{ll}
\text { 1. } x, y \in X & \text { and } \\
\text { 2. } x, y \in Y & \text { and } \\
x \lessdot y & \text { or } \\
\end{array}
$$

The direct sum operation is also called the disjoint union．The notation $n \boldsymbol{P}$ is defined as $n \boldsymbol{P} \triangleq \underbrace{\boldsymbol{P}+\boldsymbol{P}+\cdots+\boldsymbol{P}}_{n-1 \text {＂}+ \text {＂operations }}$.

Definition 1．15 ${ }^{10}$ Let $X$ and $Y$ be disjoint sets．Let $\boldsymbol{P} \triangleq(X, \Theta)$ and $\boldsymbol{Q} \triangleq(Y, \lessdot)$ be ordered sets on $X$ and $Y$ ．The direct product of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as
$\boldsymbol{P} \times \boldsymbol{Q} \triangleq(X \times Y, \leq)$
where $\left(x_{1}, y_{1}\right) \leq\left(x_{2}, y_{2}\right)$ if $x_{1} \otimes x_{2}$ and $y_{1} \otimes y_{2}$ ．

```
* [72], page 2
7\otimes [72], page 2, }\otimes\mathrm{ [18], page 5
8 塐 [47], page 161, 自 [48], 圊 [56], page 4
9}| [155], page 100
10}\otimes\mathrm{ [155], pages 100-101, }\otimes\mathrm{ [154], page 43
```

The direct product operation is also called the cartesian product. The order relation $\leq$ is called a coordinate wise order relation. The notation $\boldsymbol{P}^{n}$ is defined as

$$
\boldsymbol{P}^{n} \triangleq \underbrace{\boldsymbol{P} \times \boldsymbol{P} \times \cdots \times \boldsymbol{P}}_{n-1 \text { " } \times \text { " operations }} .
$$

Definition 1.16 ${ }^{11}$ Let $X$ and $Y$ be disjoint sets. Let $\boldsymbol{P} \triangleq(X, \Theta)$ and $Q \triangleq(Y, \lessdot)$ be ordered sets on $X$ and $Y$. The ordinal sum of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as
$\boldsymbol{P} \oplus \boldsymbol{Q} \triangleq(X \cup Y, \leq)$
where $x \leq y$ if

1. $x, y \in X$ and $x \otimes y$ or
2. $x, y \in Y$ and $x \lessdot y$ or
3. $x \in X \quad$ and $y \in Y$.

Definition 1.17 ${ }^{12}$ Let $X$ and $Y$ be disjoint sets. Let $\boldsymbol{P} \triangleq(X, \Theta)$ and $\boldsymbol{Q} \triangleq(Y, \lessdot)$ be ordered sets on $X$ and $Y$. The ordinal product of $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as
$\boldsymbol{P} \otimes \boldsymbol{Q} \triangleq(X \times Y, \leq)$
where $\left(x_{1}, y_{1}\right) \leq\left(x_{2}, y_{2}\right)$ if $\left\{\begin{array}{llll}1 . & x_{1} \neq x_{2} & \text { and } x_{1} \odot x_{2} & \text { or } \\ 2 . & x_{1}=x_{2} & \text { and } y_{1} \lessdot y_{2}\end{array}\right\}$
The order relation $\leq$ is called a lexicographical order relation, dictionary order relation, or alphabetic order relation.

Definition $1.18{ }^{13}$ Let $\boldsymbol{P} \triangleq(X, \leq)$ be an ordered set. Let $\geq$ be the dual order relation of $\leq$. The dual of $\boldsymbol{P}$ is defined as $\boldsymbol{P}^{*} \triangleq(X, \geq)$

Definition 1.19 ${ }^{14}$ Let $X$ and $Y$ be disjoint sets. Let $\boldsymbol{P} \triangleq(X, \Theta)$ and $\boldsymbol{Q} \triangleq(Y, \lessdot)$ be ordered sets on $X$ and $Y . \quad Q^{P} \triangleq\left(\left\{f \in Y^{X} \mid f\right.\right.$ is order preserving $\left.\}, \leq\right)$
where $\mathrm{f} \leq \mathrm{g}$ if $\mathrm{f}(x) \leq \mathrm{g}(x) \quad \forall x \in X$. The order relation $\leq$ is called a pointwise order relation.

Theorem 1.20 (cardinal arithmetic) ${ }^{15}$ Let $\boldsymbol{P} \triangleq(X, \leq)$ be an ordered set.

|  | $\boldsymbol{P}+\boldsymbol{Q}=$ | $Q+P$ | (commutative) |
| :---: | :---: | :---: | :---: |
| 2. | $P \times Q$ | $Q \times P$ | (commutative) |
| 3. | $(P+Q)+R=$ | $P+(Q+R)$ | (associative) |
| 4. | $(P \times Q) \times R=$ | $\boldsymbol{P} \times(\boldsymbol{Q} \times \mathbf{R})$ | (associative) |
|  | $P \times(Q+R)=$ | $(P \times Q)+(P \times R)$ | (distributive) |
| 6. | $R^{P+Q}$ | $R^{P} \times R^{Q}$ |  |
| 7. | $\left(P^{Q}\right)^{R}$ | $P^{Q \times R}$ |  |
| ${ }^{11} \otimes$ [155], page 100 |  |  |  |
| $12 \rightarrow$ [155], page 101, $\triangle$ [154], page 44, $\triangle$ [81], page 58, $\triangle$ [82], page 5413 [155], page 101 |  |  |  |
| $14 \otimes$ [155], page 101 |  |  |  |
| 15 [155], page 102 |  |  |  |

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Figure 1: Operations on ordered sets (Example 1.23 page 8)
Definition 1.21 The ordered set $L_{1}$ is defined as $(\{x\}, \leq)$, for some value $x$.
It is illustrated by the Hasse diagram to the right.
Definition 1.22 The ordered set $L_{2}$ is defined as $L_{2} \triangleq L_{1}^{2}$.
It is illustrated by the Hasse diagram to the right.

0

0

### 1.1.4 Decomposition examples

Example 1.23 Figure 1 (page 8) illustrates the four ordered set operations,$+ \times, \oplus$, and $\otimes$.
Example 1.24 ${ }^{16} \mathrm{The}$ ordered set $n L_{1}$ is the anti-chain with $n$ elements. The ordered set $4 L_{1}$ is illustrated to the right.
○ ○ ○ ○

Examples of the Boolean lattices (Definition 1.69 page 18) $\boldsymbol{L}_{2}^{1}, \boldsymbol{L}_{2}^{2}, \boldsymbol{L}_{2}^{3}, \boldsymbol{L}_{2}^{4}$ and $\boldsymbol{L}_{2}^{5}$ are illustrated in Example 1.74 (page 21).

[^3]

Figure 2: Lattice of width 4 and examples of minimal order partitions of chains (see Example 1.26 page 9 )

Example 1.26 ${ }^{17}$ The longest antichain (Definition 1.11 page 6) in the lattice illustrated in Figure 2 (page 9) has 4 elements giving this ordered set a width (Definition 1.12 page 6) of 4 . The longest chain also has 4 elements, giving the ordered set a length (Definition 1.12 page 6) of 3 . By Dilworth's theorem (Theorem 1.13 page 6), the smallest partition consists of four chains (Definition 1.4 page 4). Examples of such minimal order partitions those listed in Figure 2.

Definition 1.27 Let ( $X, \leq$ ) be an ordered set and $2^{X}$ the power set of $X$. For any set $A \in 2^{X}, c$ is an upper bound of $A$ in $(X, \leq)$ if

1. $x \leq c \quad \forall x \in A$.

An element $b$ is the least upper bound, or lub, of $A$ in $(X, \leq)$ if
2. $b$ and $c$ are upper bounds of $A \quad \Longrightarrow \quad b \leq c$.

The least upper bound of the set $A$ is denoted $\bigvee A$. It is also called the supremum of $A$, which is denoted sup $A$. The join $x \vee y$ of $x$ and $y$ is defined as $x \vee y \triangleq \bigvee\{x, y\}$.

Definition 1.28 Let $(X, \leq)$ be an ordered set and $2^{X}$ the power set of $X$. For any set $A \in 2^{X}, p$ is a lower bound of $A$ in $(X, \leq)$ if

1. $p \leq x \quad \forall x \in A$.

An element $a$ is the greatest lower bound, or GLB, of $A$ in $(X, \leq)$ if
2. $a$ and $p$ are lower bounds of $A \quad p \leq a$.

The greatest lower bound of the set $A$ is denoted $\bigwedge A$. It is also called the infimum of $A$, which is denoted $\inf A$. The meet $x \wedge y$ of $x$ and $y$ is defined as $x \wedge y \triangleq \bigwedge\{x, y\}$.

Proposition 1.29 $\operatorname{Let}(X, \vee, \wedge$; $\leq$ ) be an ORDERED SET (Definition 1.1 page 3).

$$
x \leq y \Longleftrightarrow\left\{\begin{array}{ll}
1 . & x \wedge y=x \quad \text { and } \\
2 . & x \vee y=y
\end{array}\right\} \quad \forall x, y \in X
$$

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Proposition 1.30 Let $2^{X}$ be the Power Set of a set $X$ ．
$A \subseteq B \Longrightarrow\left\{\begin{array}{l}1 . \bigvee A \leq \bigvee B \text { and } \\ 2 . \bigwedge A \leq \bigwedge B\end{array}\right\} \quad \forall A, B \in 2^{X}$

## 1．2 Lattices

## 1．2．1 Definition

The structure available in an ordered set（Definition 1.1 page 3）tends to be insufficient to en－ sure＂well－behaved＂mathematical systems．This situation is greatly remedied if every pair of elements in the ordered set has both a least upper bound and a greatest lower bound （Definition 1.28 page 9）in the set；in this case，that ordered set is a lattice（next definition）．Gian－ Carlo Rota（1932－1999）has illustrated the advantage of lattices over simple ordered sets by pointing out that the ordered set of partitions of an integer＂is fraught with pathologi－ cal properties＂，while the lattice of partitions of a set＂remains to this day rich in pleasant surprises＂．${ }^{18}$

Definition 1．31 ${ }^{19}$ An algebraic structure $L \triangleq(X, \vee, \wedge ; \leq)$ is a lattice if
1．（ $X, \leq$ ）is an ordered set $((X, \leq)$ is a partially or totally ordered set）and
2．$\exists x \vee y \in X \quad \forall x, y \in X \quad$（every pair of elements in $X$ has a least upper bound in $X$ ）and
3．$\exists x \wedge y \in X \quad \forall x, y \in X \quad$（every pair of elements in $X$ has a greatest lower bound in $X$ ）．
The algebraic structure $L^{*} \triangleq(X, \otimes, \otimes ; \geq)$ is the dual lattice of $L$ ，where $\otimes$ and $\otimes$ are determined by $\geq$ ．The lattice $L$ is linear if（ $X, \leq$ ）is a chain（Definition 1.4 page 4）．

Theorem $1.32{ }^{20}(X, \vee, \wedge ; \leq)$ is a LATTICE $\Leftrightarrow$

$$
\left\{\begin{array}{rl|l|lll}
x \vee x & =x & x \wedge x & =x & & \forall x \in X \\
x \wedge y & =y \wedge x & \text { (IDEMPOTENT) } & \text { and } \\
x \vee y & =y \vee x & \forall x, y \in X & & \text { (COMMUTATIVE) } & \text { and } \\
(x \vee y) \vee z & =x \vee(y \vee z) & (x \wedge y) \wedge z & =x \wedge(y \wedge z) & \forall x, y, z \in X & \\
\text { (ASSOCIATIVE) } & \text { and } \\
x \vee(x \wedge y) & =x
\end{array}\right.
$$

Lemma 1．33 ${ }^{21}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be Lattice（Definition 1.31 page 10）．
$x \leq y \quad \Longleftrightarrow \quad x=x \wedge y \quad \forall x, y \in L$
（2）Proof：

[^5]（1）Proof for $\Longrightarrow$ case：by left hypothesis and definition of $\wedge$（Definition 1.28 page 9）．
（2）Proof for $\Longleftarrow$ case：by right hypothesis and definition of $\wedge$（Definition 1.28 page 9 ）．
$\Leftrightarrow$

Proposition 1.34 （Monotony laws）${ }^{22} \operatorname{Let}(X, \vee, \wedge ; \leq)$ be a lattice．

$$
\left\{\begin{array}{l}
a \leq b \text { and } \\
x \leq y
\end{array}\right\} \quad \Longrightarrow \quad\left\{\begin{array}{l}
a \wedge x \leq b \wedge y \text { and } \\
a \vee x \leq b \vee y
\end{array}\right\}
$$

Theorem 1．35（Minimax inequality）${ }^{23} \operatorname{Let}(X, \vee, \wedge ; \leq)$ be a lattice．

maxmini：largest of the smallest minimax：smallest of the largest

Special cases of the minimax inequality include three distributive inequalities（next the－ orem）．If for some lattice any one of these inequalities is an equality，then all three are equalities（Theorem 1.54 page 15）；and in this case，the lattice is a called a distributive lattice（Defi－ nition 1.53 page 15）．

Theorem 1.36 （distributive inequalities）${ }^{24}(X, \vee, \wedge ; \leq)$ is a lattice $\Longrightarrow$
$\left\{\begin{array}{rlrll}x \wedge(y \vee z) & \geq(x \wedge y) \vee(x \wedge z) & \forall x, y, z \in X & \text {（JOIN SUPER－DISTRIBUTIVE）and } \\ x \vee(y \wedge z) & \leq(x \vee y) \wedge(x \vee z) & \forall x, y, z \in X & \text {（MEET SUB－DISTRIBUTIVE）and } \\ (x \wedge y) \vee(x \wedge z) \vee(y \wedge z) & \leq(x \vee y) \wedge(x \vee z) \wedge(y \vee z) & \forall x, y, z \in X & & \text {（MEDIAN INEQUALITY）．}\end{array}\right.$
Besides the distributive property，another consequence of the minimax inequality is the modularity inequality（next theorem）．A lattice in which this inequality becomes equality is said to be modular（Definition 1.47 page 14）．

Theorem 1．37（Modular inequality）${ }^{25} \operatorname{Let}(X, \vee, \wedge ; \leq)$ be a LATTICE（Definition 1.31 page 10）． $x \leq y \quad \Longrightarrow \quad x \vee(y \wedge z) \leq y \wedge(x \vee z)$

Theorem 1.32 （page 10）gives 4 necessary and sufficient pairs of properties for a structure $(X, \vee, \wedge ; \leq)$ to be a lattice．However，these 4 pairs are actually overly sufficient（they are not independent），as demonstrated next．

[^6]Theorem $1.38{ }^{26}$

### 1.2.2 Bounded lattices

Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice. By the definition of a lattice (Definition 1.31 page 10), the upper bound $(x \vee y)$ and lower bound $(x \wedge y)$ of any two elements in $X$ is also in $X$. But what about the upper and lower bounds of the entire set $X(\bigvee X$ and $\bigwedge X)$ (Definition 1.27 page 9 , Definition 1.28 page 9)? If both of these are in $X$, then the lattice $L$ is said to be bounded (next definition). All finite lattices are bounded (next proposition). However, not all lattices are bounded-for example, the lattice $(\mathbb{Z}, \leq)$ (the lattice of integers with the standard integer ordering relation) is unbounded.

Definition 1.39 Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice. Let $\bigvee X$ be the least upper bound of $(X, \leq)$ and let $\wedge X$ be the greatest lower bound of ( $X, \leq$ ).
$L$ is upper bounded if $(\bigvee X) \in X$.
$L$ is lower bounded if $(\bigwedge X) \in X$.
$L$ is bounded if $L$ is both upper and lower bounded.
A bounded lattice is optionally denoted $(X, \vee, \wedge, 0,1 ; \leq)$, where $0 \triangleq \wedge X$ and $1 \triangleq \bigvee X$.
Proposition 1.40 Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice. $\{L$ is FINITE $\} \quad \Longrightarrow \quad\{L$ is bOUNDED $\}$

Proposition 1.41 ${ }^{27}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice with $\bigvee X \triangleq 1$ and $\wedge X \triangleq 0$.

$$
\left.\{\boldsymbol{L} \text { is BOUNDED }\} \quad \Longrightarrow \quad \begin{array}{llllll}
x \vee 1 & = & 1 & \forall x \in X & \text { (upper bounded) } & \text { and } \\
x \wedge 0 & = & 0 & \forall x \in X & \text { (lower bounded) } & \text { and } \\
x \vee 0 & = & x & \forall x \in X & \text { (join-identity) } & \text { and } \\
x \wedge 1 & = & x & \forall x \in X & \text { (meet-identity) } &
\end{array}\right\}
$$

Definition $1.42{ }^{28}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition 1.39 page 12). The height $h(x)$ of a point $x \in L$ is the least upper bound of the lengths (Definition 1.12 page 6) of all the chains that have 0 and in which $x$ is the least upper bound. The height $h(L)$ of the lattice $L$ is defined as

$$
h(L) \triangleq h(1) .
$$

[^7]


Figure 3：Selected atomic，anti－atomic，and neither atomic nor anti－atomic lattices（see Exam－ ple 1.45 page 13）

Example 1．43 The height of the lattice illustrated in Figure 2 （page 9）is 3 because

$$
\begin{aligned}
\mathrm{h}(L) & \triangleq \mathrm{h}(1) \\
& \triangleq \bigvee\{\ell(\boldsymbol{C}) \mid \boldsymbol{C} \text { is a chain in } L \text { containing both } 0 \text { and } 1\} \\
& =\bigvee\{\ell(\{0, a, p, 1\}, \leq), \ell(\{0, b, p, 1\}, \leq), \ell(\{0, c, p, 1\}, \leq), \ell(\{0, c, q, 1\}, \leq), \\
& \quad \ell(\{0, c, r, 1\}, \leq),\} \\
& =\bigvee\{4-1,4-1,4-1,4-1,4-1\} \\
& =\bigvee\{3,3,3,3,3\} \\
& =3
\end{aligned}
$$

## 1．2．3 Atomic lattices

Definition $1.44{ }^{29}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition 1.39 page 12）． $x$ is an atom of $L$ if $x$ covers（Definition 1.5 page 4） 0 ． $x$ is an anti－atom of $L$ if $x$ is covered by 1 ．
$L$ is atomic if every $x \in X \backslash 0$ can be represented as joins of atoms of $L$ ．
$L$ is anti－atomic if every $x \in X \backslash 1$ can be represented as meets of anti－atoms of $L$ ．

Example 1．45 Figure 3（page 13）illustrates some examples of lattices that are atomic，anti－ atomic，both，and neither．

[^8]
## 1．2．4 Modular Lattices

Definition $1.46{ }^{30}$ Let $\left(X, \vee, \wedge\right.$ ；$\leq$ ）be a lattice．Let $2^{X X}$ be the set of all relations in $X^{2}$ ． The modularity relation $\Perp \in 2^{X X}$ and the dual modularity relation $@^{*} \in 2^{X X}$ are defined as

$$
\begin{array}{lll}
x \bowtie y & \stackrel{\text { def }}{\Longleftrightarrow}\left\{(x, y) \in X^{2} \mid a \leq y \Rightarrow y \wedge(x \vee a)=(y \wedge x) \vee a \quad \forall a \in X\right\} \\
x 囚^{*} y & \stackrel{\operatorname{def}}{\Longleftrightarrow}\left\{(x, y) \in X^{2} \mid a \geq y \Rightarrow y \vee(x \wedge a)=(y \vee x) \wedge a\right. & \forall a \in X\}
\end{array}
$$

A pair $(x, y) \in @$ is alternatively denoted as $(x, y) \Perp$ ，and is called a modular pair．A pair $(x, y) \in @^{*}$ is alternatively denoted as $(x, y) @^{*}$ ，and is called a dual modular pair．A pair $(x, y)$ that is not a modular pair $((x, y) \notin \mathbb{M})$ is denoted $x \boxtimes y$ ．A pair $(x, y)$ that is not a dual modular pair is denoted $x \otimes^{*} y$ ．

Modular lattices are a generalization of distributive lattices（Definition 1.53 page 15）in that all distributive lattices are modular，but not all modular lattices are distributive（Example 1.61 page 16，Example 1.62 page 17）．

Definition $1.47{ }^{31}$ A lattice $(X, \vee, \wedge ; \leq)$ is modular if $\quad x \bowtie y \quad \forall x, y \in X$ ．
Theorem 1．48 ${ }^{32}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice．

$$
\begin{array}{rlcccc}
L \text { is MODULAR } & \Longleftrightarrow & \{x \leq y \Longrightarrow & x \vee(z \wedge y)=(x \vee z) \wedge y\} & \forall x, y, z \in X \\
& \Longleftrightarrow & x \vee[(x \vee y) \wedge z]=(x \vee y) \wedge(x \vee z) & \forall x, y, z \in X \\
& \Longleftrightarrow & x \wedge[(x \wedge y) \vee z]=(x \wedge y) \vee(x \wedge z) & \forall x, y, z \in X
\end{array}
$$

Definition 1.49 （N5 lattice／pentagon）${ }^{33}$ The N5 lattice is the or－ dered set（ $\{0, a, b, p, 1\}, \leq$ ）with cover relation

$$
<=\{(0, a),(a, b),(b, 1),(p, 1),(0, p)\} .
$$

The N5 lattice is also called the pentagon．The N5 lattice is illustrated by the Hasse diagram to the right．


Theorem 1．50 ${ }^{34}$ Let L be a LATTICE（Definition 1.31 page 10）．
$L$ is MODULAR（Definition 1.47 page 14）$\quad \Longleftrightarrow \quad L$ does Not contain the N5 LATTICE（Definition 1.49 page 14）．

Theorem 1．51 ${ }^{35}$ Let $\boldsymbol{A} \triangleq(X, \vee, \wedge ; \leq)$ be an algebraic structure．

$$
\left\{\begin{array}{l}
(x \wedge y) \vee(x \wedge z)=[(z \wedge x) \vee y] \wedge x \quad \forall x, y, z \in X \quad \text { and } \\
{[x \vee(y \vee z)] \wedge z=z}
\end{array}\right\} \Longleftrightarrow\left\{x, y, z \in X \quad\left\{\begin{array}{l}
\text { A is a } \\
\text { modular lattice }
\end{array}\right\}\right.
$$

[^9]Examples of modular lattices are provided in Example 1.61 （page 16）and Example 1.62 （page 17）．

## 1．2．5 Distributive Lattices

Definition $1.52{ }^{36}$ Let $(X, \vee, \wedge ; \leq)$ be a lattice（Definition 1.31 page 10）．Let $2^{X X X}$ be the set of all relations in $X^{3}$ ．The distributivity relation ©（D $\in 2^{X X X}$ and the dual distributivity rela－ tion $\mathbb{D O}^{*} \in 3^{X X X}$ are defined as
（D）$\triangleq\left\{(x, y, z) \in X^{3} \mid x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z)\right\} \quad$（each $(x, y, z)$ is disjunctive distributive）and
（D）$\triangleq\left\{(x, y, z) \in X^{3} \mid x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)\right\} \quad$（each $(x, y, z)$ is conjunctive distributive）．
A triple $(x, y, z) \in ®(®)$ is alternatively denoted as $(x, y, z)(\mathbb{D}$ ，and is a distributive triple．A triple $(x, y, z) \in \mathbb{D}^{*}$ is alternatively denoted as $(x, y, z) \mathbb{D}^{*}$ ，and is a dual distributive triple．

Definition 1．53 ${ }^{37}$ A lattice $(X, \vee, \wedge ; \leq)$ is distributive if $(x, y, z) \in$（D）$\forall x, y, z \in X$

Not all lattices are distributive．But if a lattice $L$ does happen to be distributive（Definition 1.53 page 15）－that is all triples in $L$ satisfy the distributive property（Definition 1.53 page 15）－then all triples in $L$ also satisfy the dual distributive property，as well as another property called the median property．The converses also hold（next theorem）．

Theorem 1．54 ${ }^{38}$ Let $L \triangleq(X, \vee, \wedge$ ；$\leq$ ）be $a$ LATTICE（Definition 1.31 page 10）．
$L$ is DISTRIbUTIVE（Definition 1.53 page 15）

$$
\left.\left.\begin{array}{lll}
\Longleftrightarrow & x \wedge(y \vee z)=(x \wedge y) \vee(x \wedge z) & \forall x, y, z \in X
\end{array}\right) \quad \begin{array}{ll}
\Longleftrightarrow & \text { (DISJUNCTIVE DISTRIBUTIVE) } \\
\Longleftrightarrow & x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)
\end{array}\right) \text { (CONJUNCTIVE DISTRIBUTIVE) }
$$

Definition 1.55 （M3 lattice／diamond）${ }^{39}$ The M3 lattice is the ordered set $(\{0, p, q, r, 1\}, \leq)$ with covering relation

$$
<=\{(p, 1),(q, 1),(r, 1),(0, p),(0, q),(0, r)\} .
$$

The M3 lattice is also called the diamond，and is illustrated by the Hasse diagram to the right．


[^10]Lemma $1.56{ }^{40}$

$$
\left\{\begin{array}{l}
\text { Lis an } \\
\text { M3 lattice }
\end{array}\right\} \quad \Longrightarrow \quad\left\{\begin{array}{lll}
1 . & \text { L is } \text { NOT distributive } & \text { (Definition } 1.53 \text { page } 15 \text { ) } \\
\text { 2. } & \text { and } \\
\text { IS modular } & \text { (Definition 1.47 page 14) }
\end{array}\right\}
$$

Theorem 1.57 （Birkhoff distributivity criterion）${ }^{41}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be $a$ Lattice． $L$ is Distributive $\Longleftrightarrow\left\{\begin{array}{l}\text { L does not contain N5 as a sublattice } \\ \text { L does not contain M3 as a sublattice }\end{array}\right.$

Distributive lattices are a special case of modular lattices．That is，all distributive lattices are modular，but not all modular lattices are distributive（next theorem）．An example is the M3 lattice－it is modular，but yet it is not distributive．

Theorem 1．58 ${ }^{42} \operatorname{Let}(X, \vee, \wedge ; \leq)$ be a lattice．

$$
\{(X, \vee, \wedge ; \leq) \text { is DISTRIBUTIVE }\} \quad \Longrightarrow \quad\{(X, \vee, \wedge ; \leq) \text { is MODULAR }\}
$$

Theorem 1．59 ${ }^{43}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be $a$ LATTICE（Definition 1.31 page 10）．

$$
\left\{\begin{array}{lll}
\text { 1. } & L \text { is } \operatorname{\text {DISTRIBUTIVE}} & \text { and } \\
\text { 2. } & x \vee a=x \vee b & \text { and } \\
\text { 3. } & x \wedge a=x \wedge b &
\end{array}\right\} \quad \Longrightarrow \quad\{a=b\} \quad \forall x, a, b \in X
$$

Proposition 1．60 ${ }^{44}$ Let $X_{n}$ be a finite set with order $n=\left|X_{n}\right|$ ．Let $l_{n}$ be the number of unlabeled lattices on $X_{n}, m_{n}$ the number of unlabeled modular lattices on $X_{n}$ ，and $d_{n}$ the number of unlabeled distributive lattices on $X_{n}$ ．

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $l_{n}$ | 1 | 1 | 1 | 1 | 2 | 5 | 15 | 53 | 222 | 1078 | 5994 | 37622 | 262776 | 2018305 | 16873364 |
| $m_{n}$ | 1 | 1 | 1 | 1 | 2 | 4 | 8 | 16 | 34 | 72 | 157 | 343 | 766 | 1718 | 3899 |
| $d_{n}$ | 1 | 1 | 1 | 1 | 2 | 3 | 5 | 8 | 15 | 26 | 47 | 82 | 151 | 269 | 494 |

Example 1．61 ${ }^{45}$ There are a total of 5 unlabeled lattices on a five element set．Of these， 3 are distributive（Proposition 1.60 page 16，and thus also modular），one is modular but non－

```
    40 # [17], page 6, }\otimes\mathrm{ [23], page 11, }|\mathrm{ [105], page 157, <cf Salii1988 p. 37 >
    41 [23], page 12, }\otimes\mathrm{ [17], page 134, 圊 [19] | [78], <Theorem 6.2>
```



```
    43}| [113], pages 484-48
    44 ᄆ.. [2] 〈http://oeis.org/A006966\rangle, .ᅩ. [2] 〈http://oeis.org/A006982\rangle, 田 [2] 〈http://
```



```
    45 [54], pages 4-5, ® [78], <Example 6.2\rangle
```

distributive, and one is non-distributive (and non-modular).
distributive (and modular) $\quad$ modular non-distributive

Example 1.62 ${ }^{46}$ There are a total of 15 unlabeled lattices on a six element set. Of these, 5 are distributive (Proposition 1.60 page 16, and modular), 3 are modular but non-distributive, and 7 are non-distributive (and non-modular).



### 1.2.6 Complemented lattices

Definition 1.63 ${ }^{47}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition 1.39 page 12). An element $x^{\prime} \in X$ is a complement of an element $x$ in $L$ if

1. $x \wedge x^{\prime}=0$ (non-contradiction) and
2. $x \vee x^{\prime}=1$ (excluded middle).

An element $x^{\prime}$ in $L$ is the unique complement of $x$ in $L$ if $x^{\prime}$ is a complement of $x$ and $y^{\prime}$ is a complement of $x \Longrightarrow x^{\prime}=y^{\prime}$. L is complemented if every element in $X$ has a complement in $X . L$ is uniquely complemented if every element in $X$ has a unique complement in $X$. A complemented lattice that is not uniquely complemented is multiply complemented.

Example 1.64 Here are some examples:

```
46 [78], <Example 5.6 >
\(47 \ominus\) [157], page 9, \(\ominus\) [17], page 23
```



Example 1．65 Of the 53 unlabeled lattices on a 7 element set， 0 are uniquely comple－ mented， 17 are multiply complemented，and 36 are non－complemented．
Theorem 1.66 （next）is a landmark theorem in mathematics．
Theorem $1.66{ }^{48}$ For every lattice $\mathbf{L}$ ，there exists a lattice $\boldsymbol{U}$ such that
1．$L \subseteq \boldsymbol{U}(\mathbf{L}$ is a sublattice of $\boldsymbol{U})$ and
2． $\boldsymbol{U}$ is UNIQUELY COMPLEMENTED．
Corollary 1．67 ${ }^{49}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice．
$\left\{\begin{array}{lll}\text { 1．} & L \text { is DISTRIBUTIVE } & \text { and } \\ \text { 2．} & L \text { is COMPLEMENTED }\end{array}\right\} \quad \Longrightarrow \quad\{\boldsymbol{L}$ is UNIQUELY COMPLEMENTED $\}$
Theorem 1.68 （Huntington properties）${ }^{50}$ Let $L$ be a lattice．


## 1．2．7 Boolean lattices

Definition 1．69 ${ }^{51}$ A lattice（Definition 1.31 page 10）$L$ is Boolean if
1．$L$ is bounded（Definition 1.39 page 12）and
2．$L$ is distributive（Definition 1.53 page 15）and
3．$L$ is complemented（Definition 1.63 page 17）．

[^11]In this case，$L$ is a Boolean algebra or a Boolean lattice．
In this paper，a Boolean lattice with $2^{N}$ elements is sometimes denoted $L_{2}^{N}$ ．
The next theorem presents the classic properties of any Boolean algebra．The first 4 pairs of properties are true for any lattice（Theorem 1.32 page 10）．The bounded，distributive，and com－ plemented properties are true by definition of a Boolean lattice（Definition 1.69 page 18）．

Theorem 1.70 （classic 10 Boolean properties）${ }^{52}$ Let $\boldsymbol{A} \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be an alge－ braic structure．In the event that $\boldsymbol{A}$ is $a$ BOUNDED Lattice（Definition 1.39 page 12），let $x^{\prime}$ represent $a$ COMPLEMENT（Definition 1.63 page 17）of an element $x$ in $A$ ．
$A$ is a Boolean algebra $\Longleftrightarrow \quad \forall x, y, z \in X$

| $\begin{array}{ll} x \vee x & =x \\ x \vee y & =y \vee x \\ x \vee(y \vee z) & =(x \vee y) \vee z \\ x \vee(x \wedge y) & =x \\ x \vee 1 & =1 \\ x \vee 0 & =x \\ x \vee(y \wedge z) & =(x \vee y) \wedge(x \vee z) \\ x \vee x^{\prime} & =1 \\ (x \vee y)^{\prime} & =x^{\prime} \wedge y^{\prime} \end{array}$ | $\begin{array}{ll} x \wedge x & =x \\ x \wedge y & =y \wedge x \\ x \wedge(y \wedge z) & =(x \wedge y) \wedge z \\ x \wedge(x \vee y) & =x \\ x \wedge 0 & =0 \\ x \wedge 1 & =x \\ x \wedge(y \vee z) & =(x \wedge y) \vee(x \wedge z) \\ x \wedge x^{\prime} & =0 \\ (x \wedge y)^{\prime} & =x^{\prime} \vee y^{\prime} \\ =x & \end{array}$ | （IDEMPOTENT） and <br> （COMMUTATIVE） and <br> （ASSOCIATIVE） and <br> （ABSORPTIVE） and <br> （BOUNDED） and <br> （IDENTITY） and <br> （DISTRIBUTIVE） and <br> （COMPLEMENTED） and <br> （DE MORGAN） and <br> （INVOLUTORY）  |
| :---: | :---: | :---: |
| disjunctive properties | conjunctive properties | property name |

Proposition 1.71 （Huntington＇s fourth set）${ }^{53}$ Let $\boldsymbol{A} \triangleq(X, \vee, \wedge$ ；$\leq$ ）be an algebraic structure．A is a Boolean algebra

## 1．3 Orthocomplemented Lattices

Orthocomplemented lattices（Definition 1.72 page 20）are a kind of generalization of Boolean al－ gebras．The relationship between lattices of several types，including orthocomplemented and Boolean lattices，is stated in Theorem 1.86 （page 26）and illustrated in Figure 4 （page 20）．

[^12]

Figure 4: relationships between selected lattice types (see Theorem 1.86 page 26)

### 1.3.1 Definition

Definition $1.72{ }^{54}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition 1.39 page 12). An element $x^{\perp} \in X$ is an orthocomplement of an element $x \in X$ if

1. $x^{\perp \perp}=x$
$\forall x \in X$ (involutory) and
2. $x \wedge x^{\perp}=0 \quad \forall x \in X \quad$ (non-contradiction) and
3. $x \leq y \quad \Longrightarrow \quad y^{\perp} \leq x^{\perp} \quad \forall x, y \in X \quad$ (antitone).

The lattice $L$ is orthocomplemented ( $L$ is an orthocomplemented lattice) if every element $x$ in $X$ has an orthocomplement. The elements $\{x, y\}$ are orthocomplemented pairs in $L$ if $y=x^{\perp}$.

Definition $1.73{ }^{55}$
The $\mathbf{O}_{6}$ lattice is the ordered set $\left(\left\{0, p, q, p^{\perp}, q^{\perp}, 1\right\}, \leq\right)$ with cover relation $<=\left\{(0, p),(0, q),\left(p, q^{\perp}\right),\left(q, p^{\perp}\right),\left(p^{\perp}, 1\right),\left(q^{\perp}, 1\right)\right\}$.
The $O_{6}$ lattice is illustrated by the Hasse diagram to the right.


[^13][^14]Example 1．74 ${ }^{56}$ There are a total of 10 orthocomplemented lattices with 8 elements or less．These 10 ，along with 3 other orthocomplemented lattices with 10 elements，are illus－ trated next：
Lattices that are orthocomplemented but non－orthomodular and hence also not modular orthocomplemented and non－Boolean：


1．$O_{6}$ lattice

5.


2．$O_{8}$ lattice

6.

3.

7.

Lattices that are orthocomplemented and orthomodular but not modular orthocomplemented and hence also non－Boolean：


Lattices that are orthocomplemented，orthomodular，and modular orthocomplemented but non－Boolean：


10．$\quad M_{4}$ lattice


11．$M_{6}$ lattice

Lattices that are orthocomplemented，orthomodular，modular orthocomplemented and Boolean：
${ }^{56} \otimes$［12］，pages 33－42，圈［117］，page 250，$\theta$［98］，page 24，〈Figure 3．2 $\rangle, \nabla$［157］，page 12，白［88］， page 50


Example 1.75 The structure $\left(2^{\mathbb{R}^{N}},+, \cap, \varnothing, \boldsymbol{H} ; \subseteq\right)$ is an orthocomplemented lattice where
\& $\quad \mathbb{R}^{N}$ is an Euclidean space with dimension $N$
6. $2^{\mathbb{R}^{N}}$ is the set of all subspaces of $\mathbb{R}^{N}$
5. $\quad V+W$ is the Minkowski sum of subspaces $V$ and $W$
\& $\quad V \cap W$ is the intersection of subspaces $V$ and $W$.


Example 1.76 The structure $\left(2^{H}, \oplus, \cap, \varnothing, \boldsymbol{H} ; \subseteq\right)$ is an orthocomplemented lattice where $\boldsymbol{H}$ is a Hilbert space, $2^{\boldsymbol{H}}$ is the set of all closed subspaces of $\boldsymbol{H}, \boldsymbol{X}+\boldsymbol{Y}$ is the Minkowski sum of subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}, \boldsymbol{X} \oplus \boldsymbol{Y} \triangleq(\boldsymbol{X}+\boldsymbol{Y})^{-}$is the closure of $\boldsymbol{X}+\boldsymbol{Y}$, and $\boldsymbol{X} \cap \boldsymbol{Y}$ is the intersection of subspaces $X$ and $Y$.

### 1.3.2 Properties

Theorem 1.77 ${ }^{57}$ Let $x^{\perp}$ be the ORTHOCOMPLEMENT (Definition 1.72 page 20) of an element $x$ in a BOUNDED LATTICE $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$.
$57 \triangleq$ [12], pages 30-31, 嘖 [20], page 830, $\langle\mathrm{L} 74\rangle, \otimes$ [29], page 37, 〈3B.13. Theorem $\rangle$

QProof: Let $x^{\perp} \triangleq \neg x$, where $\neg$ is an ortho negation function (Definition 2.14 page 29). Then this theorem follows directly from Theorem 2.21 (page 30).

Corollary 1.78 Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be $a$ LATTICE (Definition 1.31 page 10).

$$
\left\{\begin{array}{l}
L \text { is orthocomplemented } \\
(\text { Definition } 1.72 \text { page 20) }
\end{array}\right\} \Longrightarrow\left\{\begin{array}{l}
L \text { is complemented } \\
\text { (Definition } 1.63 \text { page 17) }
\end{array}\right\}
$$

*Proof: This follows directly from the definition of orthocomplemented lattices (Definition 1.72 page 20) and complemented lattices (Definition 1.63 page 17).

Example 1.79


The $O_{6}$ lattice (Definition 1.73 page 20) illustrated to the left is both orthocomplemented (Definition 1.72 page 20) and multiply complemented (Definition 1.63 page 17). The lattice illustrated to the right is multiply complemented, but is non-orthocomplemented.

*Proof:
(1) Proof that $O_{6}$ lattice is multiply complemented: $b$ and $q$ are both complements of $p$.
(2) Proof that the right side lattice is multiply complemented: $a, p$, and $q$ are all complements of $r$.

### 1.3.3 Restrictions resulting in Boolean algebras

Proposition $1.80{ }^{58}$ Let $L=(X, \vee, \wedge, 0,1 ; \leq)$ be $a$ BOUNDED LATTICE (Definition 1.39 page 12).


$$
58 \boxtimes \text { [98], page } 22
$$

## *Proof:

$$
\begin{array}{rlr}
\left\{\begin{array}{ll}
L \text { is orthocomplemented } \text { and } \\
L \text { is distributive }
\end{array}\right\} & \Longrightarrow\left\{\begin{array}{ll}
L \text { is complemented } & \text { and } \\
L \text { is distributive }
\end{array}\right\} & \text { by Corollary } 1.78 \\
& \Longrightarrow\{L \text { is Boolean }\} & \text { by Definition } 1.69
\end{array}
$$

$$
\Leftrightarrow
$$

The center of an orthocomplemented lattice is defined later, but here is a characterization involving it now anyways.

Proposition 1.81 Let $L=(X, \vee, \wedge, 0,1 ; \leq)$ be a LATTICE (Definition 1.31 page 10). $\left\{\begin{array}{lll}\text { 1. } & L \text { is orthocomplemented } & \text { (Definition } 1.72 \text { page 20) } \\ \text { 2. } & \text { Every } x \in L \text { is in the center of } L & \text { (Definition 3.15 page 37) }\end{array}\right\} \Longleftrightarrow\left\{\begin{array}{l}L \text { is } \\ \text { Boolean }\end{array}\right\}$
*Proof:
(1) Proof that $(1,2) \Longrightarrow$ Boolean: $L$ is Boolean because it satisfies Huntington's Fourth Set (Proposition 1.71 page 19), as demonstrated by the following ...
(a) Proof that $x \vee x=x$ (idempotent): $L$ is a lattice (by definition of $L$ ), and all lattices are idempotent (Definition 1.31 page 10).
(b) Proof that $x \vee y=y \vee x$ (commutative): Lis a lattice (by definition of $L$ ), and all lattices are commutative (Definition 1.31 page 10).
(c) Proof that $(x \vee y) \vee z=x \vee(y \vee z)$ (associative): $L$ is a lattice (by definition of $L$ ), and all lattices are associative (Definition 1.31 page 10).
(d) Proof that $\left(x^{\perp} \vee y^{\perp}\right)^{\perp} \vee\left(x^{\perp} \vee y\right)^{\perp}=x$ (Huntington's axiom):

$$
\left(x^{\perp} \vee y^{\perp}\right)^{\perp} \vee\left(x^{\perp} \vee y\right)^{\perp}
$$

$$
=\left(x^{\perp} \perp \wedge y^{\perp} \perp\right) \vee\left(x^{\perp} \perp \wedge y^{\perp}\right) \quad \text { by de Morgan property (Theorem } 1.77 \text { page 22) }
$$

$$
=(x \wedge y) \vee\left(x \wedge y^{\perp}\right) \quad \text { by involution property (Definition } 1.72 \text { page 20) }
$$

$$
=x \quad \text { by def. of center (Definition } 3.15 \text { page 37) }
$$

(2) Proof that (1) $\Longleftarrow$ Boolean:
(a) Proof that $x \vee x^{\perp}=1$ : by definition of Boolean algebras (Definition 1.69 page 18).
(b) Proof that $x \wedge x^{\perp}=0$ : by definition of Boolean algebras (Definition 1.69 page 18).
(c) Proof that $x^{\perp \perp}=x$ : by involutory property of Boolean algebra (Theorem 1.70 page 19).
(d) Proof that $x \leq y \Longrightarrow y^{\perp} \leq x^{\perp}$ :

$$
\begin{array}{rlrl}
y^{\perp} \leq x^{\perp} & \Longleftrightarrow y^{\perp} & =y^{\perp} \wedge x^{\perp} & \\
\text { by Lemma } 1.33 \text { page } 10 \\
& \Longleftrightarrow y^{\perp \perp} & =\left(y^{\perp} \wedge x^{\perp}\right)^{\perp} & \\
& \Longleftrightarrow y^{\perp \perp} & & =y^{\perp \perp} \vee x^{\perp \perp} \\
& & \text { by de Morgan property (Theorem 1.70 page 19) } \\
& \Longleftrightarrow y & & =y \vee x
\end{array} \begin{array}{ll} 
& \text { by involutory property (Theorem 1.70 page 19) } \\
& \Longleftrightarrow y
\end{array}
$$

(3) Proof that (2) $\Longleftarrow$ Boolean: for all $x, y \in L$

$$
\begin{aligned}
(x \wedge y) \vee\left(x \wedge y^{\perp}\right) & =[(x \wedge y) \vee x] \wedge\left[(x \wedge y) \vee y^{\perp}\right] & & \text { by distributive property (Theorem } 1.70 \text { page 19) } \\
& =x \wedge\left[(x \wedge y) \vee y^{\perp}\right] & & \text { by absorptive property (Theorem 1.70 page 19) } \\
& =x \wedge\left[\left(x \vee y^{\perp}\right) \wedge\left(y \vee y^{\perp}\right)\right] & & \text { by distributive property (Theorem 1.70 page 19) } \\
& =x \wedge\left(x \vee y^{\perp}\right) \wedge 1 & & \text { by complement property (Theorem 1.70 page 19) } \\
& =x & & \text { by absorptive property (Theorem 1.70 page 19) } \\
& \Longrightarrow x @ y \forall x, y \in L & & \text { by Definition 3.9 page 36 } \\
& \Longrightarrow x \text { is in the center of } L & & \text { by Definition 3.15 page 37 }
\end{aligned}
$$

Example 1.82 The $O_{6}$ lattice (Definition 1.73 page 20) illustrated to the left is orthocomplemented (Definition 1.72 page 20) but non-join-distributive (Definition 1.53 page 15), and hence nonBoolean. The lattice illustrated to the right is orthocomplemented and distributive and hence also Boolean (Propo-
 sition 1.80 page 23 ).

### 1.3.4 Orthomodular lattices

Definition $1.83{ }^{59}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition 1.39 page 12). $L$ is orthomodular if

1. L is orthocomplemented and
2. $x \leq y \quad \Longrightarrow \quad x \vee\left(x^{\perp} \wedge y\right)=y \quad \forall x, y \in X \quad$ (orthomodular identity)

Theorem 1.84 ${ }^{60}$ Let $L=(X, \vee, \wedge, 0,1 ; \leq)$ be an algebraic structure.

Definition 1.85 Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition 1.39 page 12). $L$ is a modular orthocomplemeted lattice if

1. L is orthocomplemented (Definition 1.72 page 20) and
2. $L$ is modular (Definition 1.47 page 14)

59 [98], page 22, $\theta$ [110], page 90, 围 [91]
${ }^{60}$ 圈 [144], page 72

Theorem $1.86{ }^{61}$ Let $L$ be a lattice．
$\{L$ is BOOLEAN $\} \Longrightarrow$ \｛L is MODULAR ORTHOCOMPLEMENTED（Definition 1.85 page 25）\}
$\Longrightarrow$ \｛L is ORTHOMODULAR $\quad$（Definition 1.83 page 25）$\}$
$\Longrightarrow$ \｛L is ORTHOCOMPLEMENTED
（Definition 1.72 page 20）\}

## 2 Background：functions on lattices

## 2．1 Valuations

Definition 2．1 ${ }^{62}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be a lattice（Definition 1.31 page 10）． A function $\mathrm{v} \in \mathbb{R}^{\boldsymbol{X}}$ is a valuation on $L$ if

$$
\mathrm{v}(x \vee y)+\mathrm{v}(x \wedge y)=\mathrm{v}(x)+\mathrm{v}(y) \quad \forall x, y \in X
$$

Proposition 2.2 Let $\mathrm{v} \in \mathbb{R}^{\boldsymbol{X}}$ be $a$ FUNCTION on $a$ LATTICE $L \triangleq(X, \vee, \wedge ; \leq)$（Definition 1.31 page 10）．
$\{L$ is LINEAR（Definition 1.31 page 10）$\} \quad \Longrightarrow \quad\{\mathrm{v}$ is a VALUATION（Definition 2.1 page 26）$\}$
QProof：Let $x, y \in X$ such that $x \leq y$ or $y \leqq x$ ．

$$
\mathrm{v}(x \vee y)+\mathrm{v}(x \wedge y)=\mathrm{v}(x)+\mathrm{v}(y) \quad \text { because } L \text { is linear }
$$

Example 2．3 ${ }^{63}$ Consider the real valued lattice $L \triangleq(\mathbb{R}$ ，max， $\min ; \leq)$ ． The absolute value function $|\cdot|$ is a valuation on $L$ ．

QProof：$\quad L$ is linear（Definition 1.31 page 10），so $v$ is a valuation by Proposition 2.2 （page 26）．$\Leftrightarrow$
Definition 2．4 ${ }^{64}$ Let $X$ be a set and $\mathbb{R}^{\vdash}$ the set of non－negative real numbers． A function $\mathrm{d} \in \mathbb{R}^{\vdash X \times X}$ is a metric on $X$ if

1． $\mathrm{d}(x, y) \geq 0 \quad \forall x, y \in X \quad$（non－negative）and
2． $\mathrm{d}(x, y)=0 \Longleftrightarrow x=y \quad \forall x, y \in X \quad$（nondegenerate）$\quad$ and
3． $\mathrm{d}(x, y)=\mathrm{d}(y, x) \quad \forall x, y \in X \quad$（symmetric）and
4． $\mathrm{d}(x, y) \leq \mathrm{d}(x, z)+\mathrm{d}(z, y) \quad \forall x, y, z \in X \quad$（subadditive／triangle inequality）．${ }^{65}$
A metric space is the pair（ $X, \mathrm{~d}$ ）．A metric is also called a distance function．

[^15]Definition 2．5 ${ }^{66}$ Let（ $X$ ，d）be a metric space（Definition 2.4 page 26）．
An open ball centered at $x$ with radius $r$ is the set $\mathrm{B}(x, r) \triangleq\{y \in X \mid \mathrm{d}(x, y) \lesseqgtr r\}$ ．
A closed ball centered at $x$ with radius $r$ is the set $\overline{\mathrm{B}}(x, r) \triangleq\{y \in X \mid \mathrm{d}(x, y) \leq r\}$ ．
A unit ball centered at $x \quad$ is the set $\mathrm{B}(x, 1)$ ．
A closed unit ball centered at $x$
is the set $\overline{\mathrm{B}}(x, 1)$ ．

Theorem 2．6 ${ }^{67}$ Let $\mathrm{v} \in \mathbb{R}^{X}$ be a function on $a \operatorname{LATtICE} L \triangleq(X, \vee, \wedge ; \leq)$（Definition 1.31 page 10）．


Definition 2．7 ${ }^{68}$ Let v be a valuation（Definition 2.1 page 26）on a lattice $L \triangleq(X, \vee, \wedge ; \leq)$（Defini－ tion 1.31 page 10）．Let $\mathrm{d}(x, y)$ be the metric defined in Theorem 2.6 （page 27）．
The pair（ $L, \mathrm{~d}$ ）is called a metric lattice．

For finite modular lattices，the height function $h(x)$（Definition 1.42 page 12）can serve as the iso－ tone valuation that induces a metric（next proposition）．

Proposition $2.8{ }^{69}$ Let $\mathrm{h}(x)$ be the HEIGHT（Definition 1.42 page 12）of a point $x$ in a BOUNDED LATtice（Definition 1.39 page 12）$L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ．
\｛ 1．L is MODULAR and 2．Lis FINITE \}

$$
\Longrightarrow\left\{\begin{array}{llll}
\Longrightarrow & \mathrm{h}(x \vee y)+\mathrm{h}(x \wedge y)=\mathrm{h}(x)+\mathrm{h}(y) & \forall x, y \in X & \text { (valuation) and } \\
\text { 2. } & x \leq y \Longrightarrow \mathrm{~h}(x) \leq \mathrm{h}(y) & \forall x, y \in X & \text { (Positive) } \\
\text { 1. } & \mathrm{h}(x \vee y)+\mathrm{h}(x \wedge y)=\mathrm{h}(x)+\mathrm{h}(y) & \forall x, y \in X & \text { (valuation) and } \\
\text { 2. } & x \leq y \Longrightarrow \mathrm{~h}(x) \leq \mathrm{h}(y) & \forall x, y \in X & \text { (isotone) }
\end{array}\right\}
$$

Theorem $2.9{ }^{70}$ Let v be $a$ Valuation（Definition 2.1 page 26）on $a$ Lattice $L \triangleq(X, \vee, \wedge$ ；$\leq)$ （Definition 1.31 page 10）．Let $\mathrm{d}(x, y)$ be the METRIC defined in Theorem 2.6 （page 27）．

$$
\left\{\begin{array}{c}
(L, \mathrm{~d}) \text { is a METRIC LATTICE } \\
(\text { Definition } 2.7 \text { page 27) }
\end{array}\right\} \quad \Longrightarrow \quad\left\{\begin{array}{c}
L \text { is MODULAR } \\
\text { (Definition } 1.47 \text { page 14) }
\end{array}\right\}
$$

[^16]
## Example 2.10

The function h on the Boolean（and thus also modular）lattice $L_{2}^{3}$ illustrated to the right is a valuation（Definition 2.1 page 26）that is positive（and thus also isotone，Proposition 2.8 page 27）．Therefore

$$
\mathrm{d}(x, y) \triangleq \mathrm{h}(x \vee y)-\mathrm{h}(x \wedge y) \quad \forall x, y \in X
$$

is a metric（Definition 2.7 page 27）on $L_{2}^{3}$ ．For example，

$$
\mathrm{d}(b, q) \triangleq \mathrm{h}(b \vee q)-\mathrm{h}(b \wedge q)=\mathrm{h}(1)-\mathrm{h}(0)=3-0=3 .
$$

The closed unit ball centered at $b$（Definition 2.5 page 27）and illustrated with solid dots to the right is


$$
\mathrm{B}(b, 1) \triangleq\{x \in X \mid \mathrm{d}(b, x) \leq 1\}=\{b, p, r, 0\}
$$

## Example 2.11

The height function h （Definition 1.42 page 12）on the orthocomple－ mented but non－modular lattice $\mathrm{O}_{6}$ illustrated to the right is not a valuation because for example
$\mathrm{h}(a \vee c)+\mathrm{h}(a \wedge c)=\mathrm{h}(1)+\mathrm{h}(0)=3+0=3 \neq 2=1+1=\mathrm{h}(a)+\mathrm{h}(b)$. Moreover，we might expect the＂distance＂from $a$ to $c$ to be 2 ． However，if we attempt to use $h(x)$ to define a metric on $\mathrm{O}_{6}$ ，then we get


$$
\mathrm{d}(a, c) \triangleq \mathrm{h}(a \vee c)-\mathrm{h}(a \wedge c)=\mathrm{h}(1)-\mathrm{h}(0)=3-0=3 \neq 2 .
$$

## 2．2 Negation

## 2．2．1 Definitions

Definition 2．12 ${ }^{71}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition 1.39 page 12）． A function $\neg \in X^{X}$ is a subminimal negation on $L$ if ${ }^{72}$

$$
x \leq y \quad \Longrightarrow \quad \neg y \leq \neg x \quad \forall x, y \in X \quad \text { (antitone). }
$$

Definition 2．13 ${ }^{73}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice（Definition 1.39 page 12）．
${ }^{71} \boxminus$［51］，pages 4－6，$\bullet$［52］，pages 24－26，〈 2 The Kite of Negations 〉
${ }^{72}$ In the context of natural language，D．Devidi has argued that，subminimal negation（Definition 2.12 page 28）is＂difficult to take seriously as＂a negation．For further details see［40］，page 511，［39］， page 568，園［77］，$\langle § 2.1 .1\rangle$ ，［78］，$\langle § 11.1\rangle$
$73 \boxminus$［51］，pages 4－6，$\forall$［52］，pages 24－26，$\langle 2$ The Kite of Negations $\rangle, ~[161]$ ，page 4，〈1．6 Intuitionism．（b）$\rangle$ ，圈［162］，page 11，〈Definition 16〉，［70］，page 21，〈Definition 3．3〉，$\theta$ ［132］，page 50，〈DEFINItION 2．26〉，$\oslash$［131］，PAGES 98－99，〈5．4 NEGATIONS $\rangle$ ，圏［10］，PAGES 155－156， $\langle(\mathrm{N} 1) ~ \neg 0=1$ AND $\neg 1=0$ ，（N3）$\neg \neg x=x\rangle$
Monday $13^{\text {TH }}$ October， 2014 MRA－Wa $\mathrm{v}^{\mathrm{e}} \mathrm{e}_{\mathrm{e}}$ subspace architecture for logic，probability，and symbolic sequence processing


Figure 5: lattice of negations
A function $\neg \in X^{X}$ is a negation, or minimal negation, on $L$ if

1. $x \leq y \quad \Longrightarrow \quad \neg y \leq \neg x \quad \forall x, y \in X \quad$ (antitone) and
2. $x \leq \neg \neg x \quad \forall x \in X \quad$ (weak double negation).

A minimal negation $\neg$ is an intuitionistic negation on $L$ if
3. $x \wedge \neg x=0 \quad \forall x, y \in X \quad$ (non-contradiction).

A minimal negation $\neg$ is a fuzzy negation on $L$ if
4. $\neg 1=0 \quad$ (boundary condition).

Definition 2.14 ${ }^{74}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a bounded lattice (Definition 1.39 page 12).
A minimal negation $\neg$ is a de Morgan negation on $L$ if
5. $x \quad=\quad \neg \neg x \quad \forall x \in X \quad$ (involutory).

A de Morgan negation $\neg$ is a Kleene negation on $L$ if
6. $x \wedge \neg x \leq y \vee \neg y \quad \forall x, y \in X \quad$ (Kleene condition).

A de Morgan negation $\neg$ is an ortho negation on $L$ if
7. $x \wedge \neg x=0 \quad \forall x, y \in X \quad$ (non-contradiction). A de Morgan negation $\neg$ is an orthomodular negation on $L$ if
8. $x \wedge \neg x=0 \quad \forall x, y \in X \quad$ (non-contradiction) and
9. $x \leq y \quad \Longrightarrow \quad x \vee\left(x^{\perp} \wedge y\right)=y \quad \forall x, y \in X \quad$ (orthomodular).
${ }^{74} \boxminus$ [52], pages 24-26, 〈2 The Kite of Negations $\rangle, ~$ [96], page 283, $\otimes$ [98], page 22, $\Delta$ [110], PAGE 90, 圏 [91]

Remark 2．15 ${ }^{75}$ The Kleene condition is a weakened form of the non－contradiction and excluded middle properties in the sense

$$
\underbrace{x \wedge \neg x=0}_{\text {non-contradiction }} \leq \underbrace{1=y \vee \neg y}_{\text {excluded middle }} .
$$

Definition 2．16 Let $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be a bounded lattice（Definition 1.39 page 12）with a function $\neg \in X^{X}$ ．If $\neg$ is a negation（Definition 2.13 page 28），then $L$ is a lattice with negation．

## 2．2．2 Properties of negations

Theorem 2．17 ${ }^{76}$ Let $\neg \in X^{X}$ be a function on $a$ bOUNDED Lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ． $\left\{\begin{array}{l}\neg \text { is } a \\ \text { FUZZY NEGATION }\end{array}\right\} \Longrightarrow\{\neg 0=1 \quad$（Boundary condition）$\}$

Theorem 2．18 ${ }^{77}$ Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ． $\left\{\begin{array}{l}\neg \text { is an } \\ \text { INTUITIONISTIC NEGATION }\end{array}\right\} \Longrightarrow\left\{\begin{array}{lllll}\text {（a）} \neg 1=0 & \text {（BOUNDARY CONDITION）} & \text { and } \\ \text {（b）} \neg 0=1 & \text {（BOUNDARY Condition）} & \text { and } \\ \text {（c）} & \neg \text { is } a \text { FUZZY NEGATION }\end{array}\right\}$

Theorem 2．19 ${ }^{78}$ Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ． $\left\{\begin{array}{l}\neg \text { is a } \\ \text { minimal } \\ \text { negation }\end{array}\right\} \Longrightarrow\left\{\begin{array}{lllll}\neg x \vee \neg y \leq \neg(x \wedge y) & \forall x, y \in X & \text {（conjunctive de Morgan inequality）} & \text { and } \\ \neg(x \vee y) & \leq \neg x \wedge \neg y & \forall x, y \in X & \text {（disfunctive de Morgan inequaltit）}\end{array}\right\}$

Theorem 2．20 ${ }^{79}$ Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ． $\left.\begin{array}{l}\neg \text { is a } \\ \text { de Morgan negation }\end{array}\right\} \Longrightarrow\left\{\begin{array}{lllll}\neg(x \vee y) & =\neg x \wedge \neg y & \forall x, y \in X & \text {（disfunctive de Morgan）} & \text { and } \\ \neg(x \wedge y) & =\neg x \vee \neg y & \forall x, y \in X & \text {（conjunctive de Morgan）}\end{array}\right.$

Theorem $2.21{ }^{80}$ Let $\neg \in X^{X}$ be a function on $a$ bounded lattice $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ．

```
75 [26], page 78
76 塐 [77], \langle$2.1.2\rangle, - [78], <§11.2\rangle
77 圊 [77], \langle§2.1.2\rangle, & [78], \langle§11.2\rangle
78圊 [77], \langle§2.1.2\rangle, 囚 [78], \langle§11.2\rangle
79圊 [77], \langle$2.1.2\rangle, 日 [78], \langle§11.2\rangle
80圊 [77], \langle§2.1.2\rangle, [78], <§11.2\rangle
```


## 2．3 Projections

Definition 2．22 ${ }^{81}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be an orthocomplemented lattice（Definition 1.72 page 20）．A function $\phi_{x} \in X^{X}$ is a Sasaki projection on $x \in X$ if $\phi_{x}(y) \triangleq\left(y \vee x^{\perp}\right) \wedge x$ ． The Sasaki projections $\phi_{x}$ and $\phi_{y}$ are permutable if $\phi_{x} \circ \phi_{y}(u)=\phi_{y} \circ \phi_{x}(u) \quad \forall u \in X$ ．

Proposition 2．23 Let $\phi_{x}(y)$ be the SASAKI PROJECTION OF $y$ ONTO $x$（Definition 2.22 page 31）in an ORTHOCOMPLEMENTED LATTICE $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ ．

| （1）．$x \leq y$ | $\Longrightarrow$ |
| :--- | :--- |
| （2）．$y \leq x$ | $\left.\Longrightarrow y \leq \begin{array}{l}\phi_{x}(y)\end{array}\right)=x \quad \forall x, y \in X$ |
| $\phi_{x}(y) \leq x \quad \forall x, y \in X$ |  |
| （3）．$y \leq x$ and $L$ is BOOLEAN | $\Longrightarrow$ |

（Proof：

$$
\begin{array}{rlrl}
(1) \Longrightarrow \phi_{x}(y) & \triangleq\left(y \vee x^{\perp}\right) \wedge x & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& =1 \wedge x & & \text { by } x \leq y \text { hypothesis and Proposition } 3.1 \text { page } 34 \\
& =x & & \text { by property of bounded lattices (Proposition } 1.41 \text { page 12) } \\
(2) \Longrightarrow \boxed{y} & =y \wedge x & & \text { by } y \leq x \text { hypothesis } \\
& \leq\left(y \vee x^{\perp}\right) \wedge x & & \text { by definition of } \vee \text { (Definition } 1.27 \text { page } 9) \\
& =\phi_{x}(y) & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& \leq\left(y \vee x^{\perp}\right) \wedge x & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& \leq x & & \text { by definition of } \wedge \text { (Definition 1.28 page 9) } \\
(3) \Longrightarrow \phi_{x}(y) & =\left(y \vee x^{\perp}\right) \wedge x & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& =(y \wedge x) \vee\left(x^{\perp} \wedge x\right) \\
& =(y \wedge x) \vee 0 & & \text { by distributive property of Boolean lattices (Theorem 1.70 page 19) } \\
& =(y \wedge x) & & \text { by non-contradiction of Boolean lattices (Theorem 1.70 page 19) } \\
& =y & & \text { by boundary property of bounded lattices (Proposition 1.41 page 12) } \\
& & \text { by } y \leq x \text { hypothesis and definition of } \wedge \text { (Definition } 1.28 \text { page } 9)
\end{array}
$$

$\Leftrightarrow$
Proposition 2．24 Let $\phi_{x}(y)$ be the SASAKI PROJECTION OF $y$ ONTO $x$（Definition 2．22 page 31）in an orthocomplemented lattice（ $X, \vee, \wedge, 0,1 ; \leq$ ）．
（1）．$\quad \phi_{0}(y)=0 \quad \forall y \in X$
（2）．$\phi_{x}(0)=0 \quad \forall x \in X$
（3）．$\phi_{1}(y)=1 \quad \forall y \in X$
（4）．$\phi_{x}(1)=x \quad \forall x \in X$
（5）．$\phi_{x}\left(x^{\perp}\right)=0 \quad \forall x \in X$
${ }^{81} \otimes$［127］，pages 158－159，〈equation $\left.(\mathrm{S})\right\rangle, \nabla$［152］，page 300，〈Def．5．1，cf Foulis 1962〉，$\Delta$［98］， page 117

## *Proof:

$$
\begin{aligned}
\phi_{0}(y) & =0 & & \text { because } 0 \leq y \text { and by Proposition } 2.23 \text { page } 31 \\
\phi_{x}(0) & \triangleq\left(0 \vee x^{\perp}\right) \wedge x & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& =x^{\perp} \wedge x & & \text { by property of bounded lattices (Proposition } 1.41 \text { page 12) } \\
& =0 & & \text { by definition of orthocomplemented (Definition 1.72 page 20) } \\
\phi_{1}(y) & \triangleq\left(y \vee 1^{\perp}\right) \wedge 1 & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& =(y \vee 0) \wedge 1 & & \text { by boundary condition (Theorem 2.21 page 30) } \\
& =y \wedge 1 & & \text { by property of bounded lattices (Proposition } 1.41 \text { page 12) } \\
& =1 & & \text { by property of bounded lattices (Proposition } 1.41 \text { page 12) } \\
\phi_{x}(1) & =x & & \text { because } x \leq 1 \text { and by Proposition } 2.23 \text { page } 31 \\
\phi_{x}\left(x^{\perp}\right) & \triangleq\left(x^{\perp} \vee x^{\perp}\right) \wedge x & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& =x^{\perp} \wedge x & & \text { by idempotency of lattices (Theorem 1.32 page 10) } \\
& =0 & & \text { by non-contradiction prop. of } \text { orthocomplemented lattice (Definition } 1.72 \text { page 20) }
\end{aligned}
$$

Example 2.25 Here are some examples of projections in the $O_{6}$ lattice onto the element $x$ :

| $\phi_{p}(q)$ | $\triangleq \quad(q$ | $\left.\vee p^{\perp}\right)$ | $\wedge p$ | $=p$ | $p^{\perp}$ | $\wedge p$ | $=0$ | (because $p \perp q$ ) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\phi_{p}\left(p^{\perp}\right)$ | $\triangleq \quad\left(p^{\perp}\right.$ | $\left.\vee p^{\perp}\right)$ | $\wedge p$ | $=p$ | $p^{\perp}$ | $\wedge p$ | 0 | ${ }^{\perp}$ ) |
| $\phi_{p}\left(q^{\perp}\right)$ | $\triangleq \quad\left(q^{\perp}\right.$ | $\vee p^{\perp}$ ) | $\wedge p$ | $=1$ | 1 | $\wedge$ | $p$ | (because $p \leq q^{\perp}$ ) |
| $\phi_{q^{ \pm}}(p)$ | $\triangleq \quad(p$ | $\vee q)$ | $\wedge q^{\perp}$ | $=1$ | 1 | $\wedge q^{\perp}$ | $=q^{\perp}$ | (because |
| $\phi_{p}(1)$ | $\triangleq \quad(1$ | $\vee p^{\perp}$ ) | $\wedge p$ | $=1$ |  | $\wedge$ | $p$ | (because $p \leq 1$ ) |
| $\phi_{p}(0)$ | (0) | $\vee p^{\perp}$ ) | $\wedge p$ | $=p$ | $p^{\perp}$ | $\wedge p$ | $=0$ | (because $p \perp 0$ ) |



## Example 2.26

Let $\mathbb{R}^{3}$ be the 3-dimensional Euclidean space (Example 1.75 page 22) with subspaces $\boldsymbol{Z}$ and $\boldsymbol{V}$. Then the projection operator $P_{\boldsymbol{Z}^{\perp}}$ onto $\boldsymbol{Z}^{\perp}$ is a sasaki projection $\phi_{Z^{\perp}}$. In particular

$$
\begin{aligned}
P_{Z^{\perp}} V & \triangleq \phi_{Z^{\perp}}(V) \\
& \triangleq\left(V+Z^{\perp \perp}\right) \cap Z^{\perp} \\
& =(V+Z) \cap Z^{\perp}
\end{aligned}
$$

as illustrated to the right.



Figure 6：lattice of logics

## 2．4 Logics

Definition 2．27 ${ }^{82}$ Let $\rightarrow$ be an implication function defined on a lattice with negation $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$（Definition 2.16 page 30）．

| $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a logic | if $\neg$ is a minimal negation． |
| :--- | :--- |
| $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a fuzzy logic | if $\neg$ is a fuzzy negation． |
| $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is an intuitionalistic logic | if $\neg$ is an intuitionalistic negation． |
| $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a de Morgan logic | if $\neg$ is a de Morgan negation． |
| $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a Kleene logic | if $\neg$ is a Kleene negation． |
| $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is an ortho logic | if $\neg$ is an ortho negation． |
| $(X, \vee, \wedge, \neg, 0,1 ; \leq, \rightarrow)$ is a Boolean logic | if $\neg$ is an ortho negation and |
|  | $L$ is Boolean．. |

For examples and a definition of implication，see 圊［77］，$\langle \$ 3.1\rangle$ ．

## 3 Background：relations on lattices

The relations in this section are typically defined on an orthocomplemented lattice（Defini－ tion 1.72 page 20）．Here，some relations are generalized to a lattice with negation（Definition 2.16 page 30）．A lattice（Definition 1.31 page 10）with an ortho negation successfully defined on it is an orthocomplemented lattice（Definition 1.72 page 20 ）．In many cases，these relations only work

[^17]well on an orthocomplemented lattice, and thus many results are restricted to orthocomplemented lattices.

### 3.1 Orthogonality

Proposition 3.1 Let $(X, \vee, \wedge, 0,1 ; \leq)$ be an Orthocomplemented lattice (Definition 1.72 page 20).

$$
x \leq y \quad \Longrightarrow \quad\left\{\begin{array}{rlllll}
x^{\perp} & \vee & y & = & 1 \\
x & \wedge & y^{\perp} & = & 0
\end{array}\right\} \quad \forall x, y \in X
$$

*Proof:

$$
\begin{aligned}
x \leq y & \Longrightarrow x \vee x^{\perp} \leq y \vee x^{\perp} & & \text { by monotone property of lattices (Proposition } 1.34 \text { page 11) } \\
& \Longrightarrow 1 \leq y \vee x^{\perp} & & \text { by excluded middle property (Definition } 1.72 \text { page 20) } \\
& \Longrightarrow x^{\perp} \vee y=1 & & \text { by upper bounded property of bounded lattices (Definition 1.39 page 12) } \\
x \leq y & \Longrightarrow x \wedge y^{\perp} \leq y \wedge y^{\perp} & & \text { by monotone property of lattices (Proposition } 1.34 \text { page 11) } \\
& \Longrightarrow x \wedge y^{\perp} \leq 0 & & \text { by non-contradiction property (Definition } 1.72 \text { page 20) } \\
& \Longrightarrow x \wedge y^{\perp}=0 & & \text { by lower bounded property of bounded lattices (Definition } 1.39 \text { page 12) }
\end{aligned}
$$

Definition 3.2 ${ }^{83}$ Let ( $X, \vee, \wedge, \neg, 0,1 ; \leq$ ) be a lattice with negation (Definition 2.16 page 30). The orthogonality relation $\perp \in 2^{X X}$ is defined as

$$
x \perp y \quad \stackrel{\text { def }}{\Longleftrightarrow} \quad x \leq \neg y
$$

If $x \perp y$, we say that $x$ is orthogonal to $y$.
Lemma 3.3 Let $(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be $a$ LATtice with NEGATION (Definition 2.16 page 30). $\{x \perp y \quad$ (ORTHOGONAL Definition 3.2 page 34) $\} \quad \Longrightarrow \quad \begin{cases}y \perp x & \text { (SYMMETRIC) }\}\end{cases}$

PProof:

$$
\begin{aligned}
x \perp y & \Longrightarrow x \leq \neg y & & \text { by definition of } \perp \text { (Definition } 3.2 \text { page } 34 \text { ) } \\
& \Longrightarrow(\neg \neg y) \leq \neg x & & \text { by antitone property (Definition } 1.72 \text { page 20) } \\
& \Longrightarrow y \leq \neg x & & \text { by weak double negation property of negation (Definition } 2.13 \text { page 28) } \\
& \Longrightarrow y \perp x & & \text { by definition of } \perp \text { (Definition } 3.2 \text { page } 34)
\end{aligned}
$$

[^18]Monday $13^{\text {TH }}$ October, 2014 MRA-Wa $\mathrm{v}^{\mathrm{e}} \mathrm{e}_{\mathrm{e}}$ subspace architecture for logic, probability, and symbolic sequence processing

Lemma 3．4 ${ }^{84} \operatorname{Let}(X, \vee, \wedge, 0,1 ; \leq)$ be an ORTHOCOMPLEMENTED LATTICE（Definition 1.72 page 20）．

$$
\underbrace{x \perp y}_{\text {ORTHOGONAL (Definition 3.2 page 34) }} \quad \Longrightarrow \quad\left\{\begin{array}{llll}
1 . & x \wedge y & =0 & \text { and } \\
2 . & x^{\perp} \vee y^{\perp}=1
\end{array}\right\}
$$

Remark 3．5 In an orthocomplemented lattice $L$ ，the orthogonality relation $\perp$ is in general non－associative．That is，

$$
\left\{\begin{array}{llll}
x & \perp & y & \text { and } \\
y & \perp & z &
\end{array}\right\} \Rightarrow x \perp z
$$

＊Proof：Consider the $L_{2}^{4}$ Boolean lattice in Example 1.74 （page 21）．
与 $\quad a^{\perp} \perp p$ because $a^{\perp} \leq p^{\perp}$ ．
\＆$p \perp r$ because $p \leq r^{\perp}$ 。 日
＊But yet $a^{\perp}$ is not orthogonal to $r$ because $a^{\perp} \not r^{\perp}$ ．

Example 3．6 In the $O_{6}$ lattice（Definition 1.73 page 20），there are a total of $\binom{6}{2}=\frac{6!}{(6-2)!2!}=\frac{6 \times 5}{2}=15$ distinct unordered（the $\perp$ relation is symmetric by Lemma 3.3 page 34 so the order doesn＇t matter）pairs of elements．
Of these 15 pairs， 8 are orthogonal to each other， and 0 is orthogonal to itself，making a total of 9 or－ thogonal pairs：

| $x$ | $\perp$ | $y$ | $x$ | $\perp$ | 0 | $y^{\perp}$ | $\perp$ | 0 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $x$ | $\perp$ | $x^{\perp}$ | $y$ | $\perp$ | 0 | 1 | $\perp$ | 0 |
| $y$ | $\perp$ | $y^{\perp}$ | $x^{\perp}$ | $\perp$ | 0 | 0 | $\perp$ | 0 |

Example 3．7 In lattice 5 of Example 1.74 （page 21），there are a total of $\binom{10}{2}=\frac{10!}{(10-2)!2!}=$ $\frac{10 \times 9}{2}=45$ distinct unordered pairs of elements．
Of these 45 pairs， 18 are orthogonal to each other，and 0 is orthogonal to it－ self，making a total of 19 orthogonal pairs：

| $p$ | $\perp$ | $p^{\perp}$ | $x$ | $\perp$ | $x^{\perp}$ | $y$ | $\perp$ | $z$ | $x^{\perp}$ | $\perp$ | 0 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $p$ | $\perp$ | $x^{\perp}$ | $x$ | $\perp$ | $y$ | $y$ | $\perp$ | 0 | $y^{\perp}$ | $\perp$ | 0 |
| $p$ | $\perp$ | $y$ | $x$ | $\perp$ | $z$ | $z$ | $\perp$ | $z^{\perp}$ | $z^{\perp}$ | $\perp$ | 0 |
| $p$ | $\perp$ | $z$ | $x$ | $\perp$ | 0 | $z$ | $\perp$ | 0 | 0 | $\perp$ | 0 |
| $p$ | $\perp$ | 0 | $y$ | $\perp$ | $y^{\perp}$ | $p^{\perp}$ | $\perp$ | 0 |  |  |  |

Example 3．8 In the $\mathbb{R}^{3}$ Euclidean space illustrated in Example 1.75 （page 22），
$X \subseteq Y^{\perp} \quad \Longrightarrow X \perp Y \quad Y \subseteq X^{\perp} \quad \Longrightarrow Y \perp X$ $X \subseteq Z^{\perp} \quad \Longrightarrow X \perp Z \quad Y \subseteq Z^{\perp} \quad \Longrightarrow \quad Y \perp Z$ $X \wedge Y=X \wedge Z=Y \wedge Z=0$
${ }^{84} \because$［87］，page 67，$\theta$［78］，〈Lemma 13．2〉

## 3．2 Commutativity

The commutes relation is defined next．Motivation for the name＂commutes＂is provided by Proposition 3.14 （page 36）which shows that if $x$ commutes with $y$ in a lattice $L$ ，then $x$ and $y$ commute in the Sasaki projection $\phi_{x}(y)$ on $L$ ．

Definition 3．9 ${ }^{85}$ Let $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be a lattice with negation（Definition 2.16 page 30 ）． The commutes relation © is defined as

$$
x @ y \quad \stackrel{\text { def }}{\Longleftrightarrow} \quad x=(x \wedge y) \vee(x \wedge \neg y) \quad \forall x, y \in X,
$$ in which case we say，＂$x$ commutes with $y$ in $L$＂．

That is，© is a relation in $2^{X X}$ such that

$$
\bigcirc \triangleq\left\{(x, y) \in X^{2} \mid x=(x \wedge y) \vee(x \wedge \neg y)\right\}
$$

Proposition 3．10 ${ }^{86}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be an ORTHOCOMPLEMENTED LATTICE．

| $x \bigcirc 0$ | and | $0 \mathrm{C} x$ | $\forall x \in X$ | $x \bigcirc y$ |  | $x \bigcirc y^{\perp}$ | $\forall x, y \in X$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x$ © 1 | and | 1－x | $\forall x \in X$ | $\leq y$ |  | $x ® y$ | $\forall x, y \in X$ |
| $x$ © $x$ |  |  | $\forall x \in X$ | $x \perp y$ |  | $x \bigcirc y$ | $\forall x, y \in X$ |

Definition 3．11 Let © be the commutes relation（Definition 3.9 page 36）on lattice with negation $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$（Definition 2.16 page 30）．$L$ is symmetric if $x \bigcirc y \quad y \bigcirc x \quad \forall x, y \in X$

In general，the commutes relation is not symmetric．But Proposition 3.12 （next）describes some conditions under which it is symmetric．
Proposition 3．12 ${ }^{87} \operatorname{Let}(X, \vee, \wedge, 0,1 ; \leq)$ be an ORTHOCOMPLEMENTED LATTICE（Definition 1.72 page 20）．

Theorem 3．13 ${ }^{88}$ Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be an ORTHOCOMPLEMENTED LATTICE（Defini－ tion 1.72 page 20）．
$\{x @ c \quad \forall x \in X\} \quad \Longleftrightarrow \quad\left\{L\right.$ is ISOMORPHIC to $\left.[0, c] \times\left[0, c^{\perp}\right]\right\}$
with isomorphism $\theta(x) \triangleq\left([0, c],\left[0, c^{\perp}\right]\right)$ ．
Proposition 3．14 ${ }^{89} \operatorname{Let}(X, \vee, \wedge, 0,1 ; \leq)$ be an orthomodular lattice．

[^19]

Figure 7: Lattices with centers marked with solid dots (see Example 3.17 page 37)

$$
x \bigcirc y \quad \Longleftrightarrow \quad \phi_{x}(y)=\phi_{y}(x)=x \wedge y \quad \forall x, y \in X
$$

### 3.3 Center

An element in an orthocomplemented lattice (Definition 1.72 page 20) is in the center of the lattice if that element commutes (Definition 3.9 page 36) with every other element in the lattice (next definition). All the elements of an orthocomplemented lattice are in the center if and only if that lattice is Boolean (Proposition 1.81 page 24).

Definition 3.15 ${ }^{90}$ Let © be the commutes relation (Definition 3.9 page 36) on a lattice with negation $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ (Definition 2.16 page 30). The center of $L$ is defined as
$\{x \in X \mid x \odot y \quad \forall y \in X\}$
Proposition 3.16 Let $L \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be an orthocomplemented lattice (Definition 1.72 page 20). The elements 0 and 1 are in the center of $L$.

QProof: This follows directly from Definition 3.9 (page 36) and Proposition 3.10 (page 36).
Example 3.17 The centers of the lattices in Figure 7 (page 37) are illustrated with solid dots. Note that in the case of the Boolean lattice in (D), every dot is in the center (Proposition 1.81 page 24).

### 3.4 D-Posets

Definition 3.18 ${ }^{91}$ Let 1 be the upper bound of an ordered set ( $X, \leq$ ).
An operation $\backslash$ is a difference on $(X, \leq)$ if
${ }^{90} \otimes$ [88], page 80
${ }^{91}$ 圊 [104], page 22,24, 〈DEFINITIONS 1,2$\rangle$

1．$x \leq y \quad \Longrightarrow y \backslash x \leq y \quad \forall x, y \in X \quad$ and
2．$x \leq y \quad \Longrightarrow \quad y \backslash(y \backslash x)=x \quad \forall x, y \in X \quad$ and
3．$x \leq y \leq z \quad \Longrightarrow \quad z \backslash y \leq z \backslash x \quad \forall x, y, z \in X \quad$ and
4．$x \leq y \leq z \quad \Longrightarrow \quad(z \backslash x) \backslash(z \backslash y)=y \backslash x \quad \forall x, y, z \in X$
The structure $(X, \leq, \backslash, 1)$ is called a D－poset．
Proposition 3．19 ${ }^{92}$ Let $X$ be a SET．

$$
\left\{\begin{array}{l}
(X, \leq, \backslash, 1) \text { is } a \\
\text { D-POSET } \\
\text { (Definition 3.18 page 37) }
\end{array}\right\} \Longrightarrow\left\{\begin{array}{llll}
1 . & x \leq y \leq z & \Longrightarrow y \backslash x \leq z \backslash x & \forall x, y, z \in X \\
2 . & \text { and } \\
2 . & x \leq y \leq z & \Longrightarrow x \leq z \backslash(y \backslash x) & \forall x, y, z \in X \\
\text { and } \\
3 . & x \leq y \leq z & \Longrightarrow & \Longrightarrow z \backslash x) \backslash(y \backslash x)=z \backslash y \\
4 . & \forall x, y, z \in X & \text { and } \\
4 . & x \leq y \leq z & \Longrightarrow[z \backslash(y \backslash x)] \backslash x=z \backslash y & \forall x, y, z \in X
\end{array}\right\}
$$

Example 3．20 ${ }^{93}$ The structure $\left(\mathbb{R}^{+},-, \leq\right)$is a $D$－poset where $\mathbb{R}^{+}$is the set of positive real numbers，－is the standard subtraction operation on $\mathbb{R}$ ，and $\leq$ is the standard ordering relation on $\mathbb{R}^{+}$．

Example 3．21 ${ }^{94}$ The structure $\left(2^{X}, \backslash \subseteq\right)$ is a D－poset where $2^{X}$ is the power set of a set $X$ ， $\backslash$ is the set difference operator，and $\subseteq$ is the set inclusion relation．

## 4 Background：MRA－wavelet analysis

## 4．1 Transversal Operators

## Definition $4.1{ }^{95}$

1． $\mathbf{T}$ is the translation operator on $\mathbb{C}^{\mathbb{C}}$ defined as

$$
\mathbf{T}_{\tau} \mathrm{f}(x) \triangleq \mathrm{f}(x-\tau) \quad \text { and } \quad \mathbf{T} \triangleq \mathbf{T}_{1} \quad \forall \mathrm{f} \in \mathbb{C}^{\mathbb{C}}
$$

2． $\mathbf{D}$ is the dilation operator on $\mathbb{C}^{\mathbb{C}}$ defined as


[^20]Proposition 4.2 ${ }^{96}$ Let $\mathbf{T}$ be the translation operator (Definition 4.1 page 38).

$$
\sum_{n \in \mathbb{Z}} \mathbf{T}^{n} \mathbf{f}(x)=\sum_{n \in \mathbb{Z}} \mathbf{T}^{n} \mathrm{f}(x+1) \quad \forall f \in \mathbb{R}^{\mathbb{R}} \quad\left(\sum_{n \in \mathbb{Z}} \mathbf{T}^{n f}(x) \text { is PERIoDic with period } 1\right)
$$

Proposition 4.3 ${ }^{97}$ Let $\mathbf{T}$ and $\mathbf{D}$ be as defined in Definition 4.1 page 38.
$\mathbf{T}$ has an inverse $\mathbf{T}^{-1}$ in $\mathbb{C}^{\mathbb{C}}$ expressed by the relation

$$
\mathbf{T}^{-1} \mathbf{f}(x)=\mathrm{f}(x+1) \quad \forall \mathrm{f} \in \mathbb{C}^{\mathbb{C}} \quad \text { (translation operator inverse). }
$$

$\mathbf{D}$ has an inverse $\mathbf{D}^{-1}$ in $\mathbb{C}^{\mathbb{C}}$ expressed by the relation

$$
\mathbf{D}^{-1} \mathbf{f}(x)=\frac{\sqrt{2}}{2} \mathrm{f}\left(\frac{1}{2} x\right) \quad \forall \mathrm{f} \in \mathbb{C}^{\mathbb{C}} \quad \text { (dilation operator inverse). }
$$

Proposition $4.4{ }^{98}$ Let $\mathbf{T}$ and $\mathbf{D}$ be as defined in Definition 4.1 page 38. Let $\mathbf{D}^{0}=\mathbf{T}^{0} \triangleq \mathbf{I}$ be the IDENTITY OPERATOR.

$$
\mathbf{D}^{j} \mathbf{T}^{n} \mathbf{f}(x)=2^{j / 2} \mathbf{f}\left(2^{j} x-n\right) \quad \forall j, n \in \mathbb{Z}, f \in \mathbb{C}^{\mathbb{C}}
$$

Example 4.5 (linear functions) ${ }^{99}$ Let $\mathbf{T}$ be the translation operator (Definition 4.1 page 38). Let $\mathcal{L}(\mathbb{C}, \mathbb{C})$ be the set of all linear functions in $L_{\mathbb{R}}^{2}$.

1. $\{x, \mathbf{T} x\}$ is a basis for $\mathcal{L}(\mathbb{C}, \mathbb{C})$ and
2. $\mathrm{f}(x)=\mathrm{f}(1) x-\mathrm{f}(0) \mathbf{T} x \quad \forall f \in \mathcal{L}(\mathbb{C}, \mathbb{C})$
*Proof: By left hypothesis, f is linear; so let $\mathrm{f}(x) \triangleq a x+b$

$$
\begin{aligned}
\mathrm{f}(1) x-\mathrm{f}(0) \mathbf{T} x & =\mathrm{f}(1) x-\mathrm{f}(0)(x-1) \\
& =\left.(a x+b)\right|_{x=1} x-\left.(a x+b)\right|_{x=0}(x-1) \\
& =(a+b) x-b(x-1) \\
& =a x+b x-b x+b \\
& =a x+b
\end{aligned}
$$

by Definition 4.1 page 38
by left hypothesis and definition of $f$

$$
=f(x) \quad \text { by left hypothesis and definition of } f
$$

$$
\Leftrightarrow
$$

Example 4.6 (Cardinal Series) Let $\mathbf{T}$ be the translation operator (Definition 4.1 page 38). The Paley-Wiener class of functions $P W_{\sigma}^{2}$ are those functions which are "bandlimited" with respect to their Fourier transform. The cardinal series forms an orthogonal basis for such a space. The Fourier coefficients for a projection of a function $f$ onto the Cardinal series basis elements is particularly simple-these coefficients are samples of $\mathrm{f}(x)$ taken at regular intervals. In fact, one could represent the coefficients using inner product notation with

[^21]the Dirac delta distribution $\delta$ as follows:
$\left\langle\mathrm{f}(x) \mid \mathbf{T}^{n} \delta(x)\right\rangle \triangleq \int_{\mathbb{R}} \mathrm{f}(x) \delta(x-n) \mathrm{d} x \triangleq \mathrm{f}(n)$

1. $\left\{\left.\mathbf{T}^{n} \frac{\sin (\pi x)}{\pi x} \right\rvert\, n \in \mathbb{N}\right\}$ is a basis for $\boldsymbol{P} W_{\sigma}^{2}$ and
2. $\mathrm{f}(x)=\underbrace{\sum_{n=1}^{\infty} \mathrm{f}(n) \mathrm{T}^{n} \frac{\sin (\pi x)}{\pi x}}_{\text {Cardinal series }}$

Example 4.7 (Fourier Series)

1. $\left\{\mathbf{D}_{n} e^{i x} \mid n \in \mathbb{Z}\right\}$ is a basis for $L(0,2 \pi)$ and
2. $\mathrm{f}(x)=\frac{1}{\sqrt{2 \pi}} \sum_{n \in \mathbb{Z}} \alpha_{n} \mathbf{D}_{n} e^{i x} \quad \forall x \in(0,2 \pi), \mathrm{f} \in L(0,2 \pi) \quad$ where
3. $\quad \alpha_{n} \triangleq \frac{1}{\sqrt{2 \pi}} \int_{0}^{2 \pi} \mathrm{f}(x) \mathbf{D}_{n} e^{-i x} \mathrm{~d} x \quad \forall f \in L(0,2 \pi)$

## Example 4.8 (Fourier Transform)

1. $\left\{\mathbf{D}_{\omega} e^{i x} \mid \omega \in \mathbb{R}\right\}$ is a basis for $\boldsymbol{L}_{\mathbb{R}}^{2} \quad$ and
2. $\mathrm{f}(x)=\frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} \tilde{\mathrm{f}}(\omega) \mathbf{D}_{x} e^{i \omega} \mathrm{~d} \omega \quad \forall f \in L_{\mathbb{R}}^{2} \quad$ where
3. $\tilde{f}(\omega) \triangleq \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} \mathrm{f}(x) \mathbf{D}_{\omega} e^{-i x} \mathrm{~d} x \quad \forall \mathrm{f} \in L_{\mathbb{R}}^{2}$

Example 4.9 (Gabor Transform) 100

1. $\left\{\left.\left(\mathbf{T}_{\tau} e^{-\pi x^{2}}\right)\left(\mathbf{D}_{\omega} e^{i x}\right)\right|_{\tau, \omega \in \mathbb{R}}\right\}$ is a basis for $\boldsymbol{L}_{\mathbb{R}}^{2} \quad$ and
2. $\mathrm{f}(x)=\int_{\mathbb{R}} \mathrm{G}(\tau, \omega) \mathbf{D}_{x} e^{i \omega} \mathrm{~d} \omega \quad \forall x \in \mathbb{R}, \mathrm{f} \in L_{\mathbb{R}}^{2} \quad$ where
3. $\mathrm{G}(\tau, \omega) \triangleq \int_{\mathbb{R}} \mathrm{f}(x)\left(\mathbf{T}_{\tau} e^{-\pi x^{2}}\right)\left(\mathbf{D}_{\omega} e^{-i x}\right) \mathrm{d} x \quad \forall x \in \mathbb{R}, \mathrm{f} \in L_{\mathbb{R}}^{2}$

Example 4.10 (wavelets) Let $\psi(x)$ be a mother wavelet.

1. $\left\{\left.\mathbf{D}^{k} \mathbf{T}^{n} \psi(x)\right|_{k, n \in \mathbb{Z}}\right\}$ is a basis for $\boldsymbol{L}_{\mathbb{R}}^{2} \quad$ and
2. $\mathrm{f}(x)=\sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \alpha_{k, n} \mathbf{D}^{k} \mathbf{T}^{n} \psi(x) \quad \forall \mathrm{f} \in L_{\mathbb{R}}^{2} \quad$ where
3. $\quad \alpha_{n} \triangleq \int_{\mathbb{R}} \mathrm{f}(x) \mathbf{D}^{k} \mathbf{T}^{n} \psi^{*}(x) \mathrm{d} x \quad \forall \mathrm{f} \in L_{\mathbb{R}}^{2}$

100 [143], $\langle$ Chapter 3〉
$\theta$ [61], page 32, 〈Definition 1.69 $\rangle$

### 4.2 The Structure of Wavelets

In Fourier analysis, continuous dilations (Definition 4.1 page 38) of the complex exponential form a basis for the space of square integrable functions $L_{\mathbb{R}}^{2}$ such that

$$
L_{\mathbb{R}}^{2}=\operatorname{spn}\left\{\mathbf{D}_{\omega} e^{i x} \mid \omega \in \mathbb{R}\right\} .
$$

In Fourier series analysis, discrete dilations of the complex exponential form a basis for $L_{\mathbb{R}}^{2}(0,2 \pi)$ such that

$$
\mathbf{L}_{\mathbb{R}}^{2}(0,2 \pi)=\operatorname{sen}\left\{\left.\mathbf{D}_{j} e^{i x}\right|_{j \in \mathbb{Z}}\right\} .
$$

In Wavelet analysis, for some mother wavelet (Definition 4.18 page 47) $\psi(x)$,

$$
L_{\mathbb{R}}^{2}=\operatorname{spn}\left\{\mathbf{D}_{\omega} \mathbf{T}_{\tau} \psi(x) \mid \omega, \tau \in \mathbb{R}\right\} .
$$

However, the ranges of parameters $\omega$ and $\tau$ can be much reduced to the countable set $\mathbb{Z}$ resulting in a dyadic wavelet basis such that for some mother wavelet $\psi(x)$,

$$
L_{\mathbb{R}}^{2}=\operatorname{spn}\left\{\mathbf{D}^{j} \mathbf{T}^{n} \psi(x) \mid j, n \in \mathbb{Z}\right\} .
$$

Wavelets that are both dyadic and compactly supported have the attractive feature that they can be easily implemented in hardware or software by use of the Fast Wavelet Transform (Figure 10 page 49).

In 1989, Stéphane G. Mallat introduced the Multiresolution Analysis (MRA, Definition 4.12 page 43) method for wavelet construction. The MRA has since become the dominate wavelet construction method. Moreover, P.G. Lemarié has proved that all wavelets with compact support are generated by an MRA. ${ }^{101}$

The MRA is an analysis of the linear space $L_{\mathbb{R}}^{2}$. An analysis of a linear space $\boldsymbol{X}$ is any sequence $\left(V_{j}\right)_{j \in \mathbb{Z}}$ of linear subspaces of $\boldsymbol{X}$. The partial or complete reconstruction of $\boldsymbol{X}$ from $\left(V_{j}\right)_{j \in \mathbb{Z}}$ is a synthesis. ${ }^{102}$ Some analyses are completely characterized by a transform. For example, a Fourier analysis is a sequence of subspaces with sinusoidal bases. Examples of subspaces in a Fourier analysis include $V_{1}=\operatorname{spa}\left\{e^{i x}\right\}, V_{2.3}=\operatorname{spa}\left\{e^{i 2.3 x}\right\}, V_{\sqrt{2}}=$ $\operatorname{spon}\left\{e^{i \sqrt{2} x}\right\}$, etc. A transform is loosely defined as a function that maps a family of functions into an analysis. A very useful transform (a "Fourier transform") for Fourier Analysis is

$$
[\tilde{\mathbf{F}} \mathrm{f}](\omega) \triangleq \frac{1}{\sqrt{2 \pi}} \int_{\mathbb{R}} \mathrm{f}(x) e^{-i \omega x} \mathrm{~d} x
$$

[^22]

Figure 8: Examples of order structures for selected analyses (Example 4.11 page 42)

An analysis can be partially characterized by its order structure with respect to an order relation such as the set inclusion relation $\subseteq$. Most transforms have a very simple M-n order structure, as illustrated to the right. ${ }^{103}$ The M- $n$ lattices for $n \geq 3$ are modular (Lemma 1.56 page 16) but not distributive (Theorem 1.57 page 16). Analyses typically have one subspace that is a scaling subspace; and this subspace is often simply a family of constants (as is the case with Fourier
 Analysis).

An analysis can be represented using three different structures:
(1) sequence of subspaces
(2) sequence of basis vectors
(3) sequence of basis coefficients

These structures are isomorphic to each other, and can therefore be used interchangeably.

Example 4.11 ${ }^{104}$ Some examples of the order structures of some analyses are illustrated in Figure 8 (page 42).

```
'03』 [75], page 29, <$2.2\rangle
104 [75], pages 30-31
```


## 4．3 Multiresolution analysis

A multiresolution analysis provides＂coarse＂approximations of a function in a linear space $L_{\mathbb{R}}^{2}$ at multiple＂scales＂or＂resolutions＂．Key to this process is a sequence of scaling func－ tions．Most traditional transforms feature a single scaling function $\phi(x)$ set equal to one $(\phi(x)=1)$ ．This allows for convenient representation of the most basic functions，such as constants．${ }^{105}$ A multiresolution system，on the other hand，uses a generalized form of the scaling concept：${ }^{106}$
（1）Instead of the scaling function simply being set equal to unity $(\phi(x)=1)$ ，a multires－ olution analysis（Definition 4.12 page 43）is often constructed in such a way that the scaling function $\phi(x)$ forms a partition of unity such that $\sum_{n \in \mathbb{Z}} \mathbf{T}^{n} \phi(x)=1$ ．
（2）Instead of there being just one scaling function，there is an entire sequence of scaling functions $\left(\mathbf{D}^{j} \phi(x)\right)_{j \in \mathbb{Z}}$ ，each corresponding to a different＂resolution＂．

Definition 4．12 ${ }^{107}$ Let $\left(V_{j}\right)_{j \in \mathbb{Z}}$ be a sequence of subspaces on $L_{\mathbb{R}}^{2}$ ．Let $A^{-}$be the closure of a set $A$ ．The sequence $\left(V_{j}\right)_{j \in \mathbb{Z}}$ is a multiresolution analysis on $L_{\mathbb{R}}^{2}$ if

| 1． | $V_{j}=V_{j}^{-}$ | $\forall j \in \mathbb{Z}$ | （closed） <br> 2． <br> $V_{j} \subset V_{j+1}$ <br> 3． |
| :--- | :--- | :--- | :--- |
|  | $\left(\bigcup_{j \in \mathbb{Z}} V_{j}\right)^{-}=L_{\mathbb{R}}^{2}$ | $\forall j \in \mathbb{Z}$ | and |
| （linearly ordered） | and |  |  |
| 4． | $\mathrm{f} \in V_{j} \Longleftrightarrow \mathbf{D f} \in V_{j+1}$ | $\forall j \in \mathbb{Z}, f \in L_{R}^{2}$ | （self－similar） |
| 5． | $\exists \phi$ such that $\left\{\mathbf{T}^{n} \phi \mid n \in \mathbb{Z}\right\}$ is a Riesz basis for $V_{0}$. | and |  |

A multiresolution analysis is also called an MRA．An element $V_{j}$ of $\left(V_{j}\right)_{j \in \mathbb{Z}}$ is a scaling subspace of the space $L_{\mathbb{R}}^{2}$ ．The pair $\left(L_{\mathbb{R}}^{2},\left(V_{j}\right)\right)$ is a multiresolution analysis space，or MRA space．The function $\phi$ is the scaling function of the MRA space．

The traditional definition of the MRA also includes the following：

```
6． \(\mathrm{f} \in V_{j} \Longleftrightarrow \mathbf{T}^{n} \mathrm{f} \in V_{j} \quad \forall n, j \in \mathbb{Z}, \mathrm{f} \in L_{\mathbb{R}}^{2} \quad\)（translation invariant）
7．\(\bigcap_{j \in \mathbb{Z}} V_{j}=\{\mathbb{D}\} \quad\)（greatest lower bound is 0 ）
```

However，these follow from the $M R A$ as defined in Definition 4.12 （Proposition 4.13 page 44，Propo－ sition 4.14 page 44）．

[^23]Proposition $4.13{ }^{108}$ Let MRA be defined as in Definition 4.12 page 43.

$$
\left\{\left(V_{j}\right)_{j \in \mathbb{Z}} \text { is an MRA }\right\} \quad \Longrightarrow \quad \underbrace{\left\{\mathrm{f} \in V_{j} \Longleftrightarrow \mathbf{T}^{n} \mathrm{f} \in V_{j} \quad \forall n, j \in \mathbb{Z}, \mathrm{f} \in L_{\mathbb{R}}^{2}\right\}}_{\text {TRANSLATION INVARIANT }}
$$

Proposition $4.14{ }^{109}$ Let MRA be defined as in Definition 4.12 page 43.

$$
\left\{\left(v_{j}\right)_{j \in \mathbb{Z}} \text { is an MRA }\right\} \quad \Longrightarrow \quad\left\{\bigcap_{j \in \mathbb{Z}} V_{j}=\{\mathbb{O}\} \quad \text { (GREATEST LowER bound is } 0 \text { ) }\right\}
$$

The MRA（Definition 4.12 page 43）is more than just an interesting mathematical toy．Under some very＂reasonable＂conditions（next proposition），as $j \rightarrow \infty$ ，the scaling subspace $V_{j}$ is dense in $L_{\mathbb{R}}^{2}$ ．．．meaning that with the MRA we can represent any＂reasonable＂function to within an arbitrary accuracy．

Proposition $4.15{ }^{110}$

$$
\left\{\begin{array}{lll}
\text { (1). } & \left(\mathbf{T}^{n} \phi\right) \text { is } a \text { RIESZ SEQUENCE } & \text { and } \\
\text { (2). } & \tilde{\phi}(\omega) \text { is CONTINUOUS } \text { at } 0 & \text { and } \\
\text { (3). } & \tilde{\phi}(0) \neq 0
\end{array}\right\} \Longrightarrow\left\{\left(\bigcup_{j \in \mathbb{Z}} V_{j}\right)^{-}=L_{\mathbb{R}}^{2} \quad\left(\operatorname{DENSE} \text { in } L_{\mathbb{R}}^{2}\right)\right\}
$$

A multiresolution analysis（Definition 4.12 page 43）together with the set inclusion relation $\subseteq$ form the linearly ordered $\operatorname{set}$（Definition 1.4 page 4）$\left(\left(V_{j}\right), \subseteq\right)$ ，illustrated to the right by a Hasse diagram（Definition 1.6 page 4）．Subspaces $V_{j}$ increase in ＂size＂with increasing $j$ ．That is，they contain more and more vectors（functions）for larger and larger $j$－with the upper limit of this sequence being $L_{\mathbb{R}}^{2}$ ．Alternatively，we can say that approximation within a subspace $V_{j}$ yields greater＂resolution＂for increasing $j .{ }^{111}$


Remark 4．16 ${ }^{112}$ Note that the greatest lower bound（g．l．b．）of the linearly ordered set $\left(\left(V_{j}\right), \subseteq\right)$ is $\mathbf{0}$（Proposition 4.14 page 44）：All linear subspaces contain the zero vector．So the intersection of any two subspaces must at least contain $\mathbb{D}$ ．If the intersection of any two linear subspaces $\boldsymbol{X}$ and $\boldsymbol{Y}$ is exactly $\{\mathbb{O}\}$ ，then for any vector in the sum of those subspaces（ $\boldsymbol{u} \in \boldsymbol{X} \hat{+} \boldsymbol{Y}$ ） there are unique vectors $f \in X$ and $g \in Y$ such that $u=f+g$ ．This is not necessarily true if the intersection contains more than just $\{\mathbb{D}\}$ ．

[^24]

Figure 9: Example approximations of $\sin (\pi x)$ in 3 Haar scaling subspaces (see Example 4.17 page 45 )

## Example 4.17

In the Haar MRA, the scaling function $\phi(x)$ is the pulse function

$$
\phi(x)= \begin{cases}1 & \text { for } x \in[0,1) \\ 0 & \text { otherwise }\end{cases}
$$



In the subspace $V_{j}(j \in \mathbb{Z})$ the scaling functions are

$$
\mathbf{D}^{j} \phi(x)= \begin{cases}(2)^{j / 2} & \text { for } x \in\left[0,\left(2^{-j}\right)\right) \\ 0 & \text { otherwise }\end{cases}
$$



The scaling subspace $V_{0}$ is the span $V_{0} \triangleq \operatorname{spn}\left\{\left.\mathbf{T}^{n} \phi\right|_{n \in \mathbb{Z}}\right\}$. The scaling subspace $V_{j}$ is the span $V_{j} \triangleq \operatorname{sen}\left\{\mathbf{D}^{j} \mathbf{T}^{n} \phi \mid n \in \mathbb{Z}\right\}$. Note that $\left\|\mathbf{D}^{j} \mathbf{T}^{n} \phi\right\|$ for each resolution $j$ and shift $n$ is unity:

$$
\begin{array}{rlr}
\left\|\mathbf{D}^{j} \mathbf{T}^{n} \phi\right\|^{2} & =\|\phi\|^{2} \\
& =\int_{0}^{1}|1|^{2} \mathrm{~d} x \quad \text { by definition of }\|\cdot\| \text { on } L_{\mathbb{R}}^{2} \\
& =1 &
\end{array}
$$

Let $\mathrm{f}(x)=\sin (\pi x)$. Suppose we want to project $\mathrm{f}(x)$ onto the subspaces $V_{0}, V_{1}, V_{2}, \ldots$.

The values of the transform coefficients for the subspace $V_{j}$ are given by

$$
\begin{aligned}
{\left[\mathbf{R}_{j} \mathrm{f}(x)\right](n) } & =\frac{1}{\left\|\mathbf{D}^{j} \mathbf{T}^{n} \phi\right\|^{2}}\left\langle\mathrm{f}(x) \mid \mathbf{D}^{j} \mathbf{T}^{n} \phi\right\rangle \\
& =\frac{1}{\|\phi\|^{2^{2}}}\left\langle\mathrm{f}(x) \mid 2^{j / 2} \phi\left(2^{j} x-n\right)\right\rangle \\
& =2^{j / 2}\left\langle\mathrm{f}(x) \mid \phi\left(2^{j} x-n\right)\right\rangle \\
& =2^{j / 2} \int_{2^{-j j_{n}}}^{2^{-j}(n+1)} \mathrm{f}(x) \mathrm{d} x \\
& =2^{j / 2} \int_{2^{-j} n}^{2^{-j}(n+1)} \sin (\pi x) \mathrm{d} x \\
& =\left.2^{j / 2}\left(-\frac{1}{\pi}\right) \cos (\pi x)\right|_{2^{-j} n} ^{2^{-j}(n+1)} \\
& =\frac{2^{j / 2}}{\pi}\left[\cos \left(2^{-j} n \pi\right)-\cos \left(2^{-j}(n+1) \pi\right)\right]
\end{aligned}
$$

by Proposition 4.4 page 39

And the projection $\mathbf{A}_{n} \mathrm{f}(x)$ of the function $\mathrm{f}(x)$ onto the subspace $V_{j}$ is

$$
\begin{aligned}
\mathbf{A}_{j} f(x) & =\sum_{n \in \mathbb{Z}}\left\langle f(x) \mid \mathbf{D}^{j} \mathbf{T}^{n} \phi\right\rangle \mathbf{D}^{j} \mathbf{T}^{n} \phi \\
& =\frac{2^{j / 2}}{\pi} \sum_{n \in \mathbb{Z}}\left[\cos \left(2^{-j} n \pi\right)-\cos \left(2^{-j}(n+1) \pi\right)\right] 2^{j / 2} \phi\left(2^{j} x-n\right) \\
& =\frac{2^{j}}{\pi} \sum_{n \in \mathbb{Z}}\left[\cos \left(2^{-j} n \pi\right)-\cos \left(2^{-j}(n+1) \pi\right)\right] \phi\left(2^{j} x-n\right)
\end{aligned}
$$

The transforms into the subspaces $V_{0}, V_{1}$, and $V_{2}$, as well as the approximations in those subspaces are as illustrated in Figure 9 (page 45).

## 4．4 Wavelet analysis

The term＂wavelet＂comes from the French word＂ondelette＂，meaning＂small wave＂．And in essence，wavelets are＂small waves＂（as opposed to the＂long waves＂of Fourier analysis） that form a basis for the Hilbert space $L_{\mathbb{R}}^{2} \cdot{ }^{113}$

Definition 4．18 ${ }^{114}$ Let $\mathbf{T}$ and $\mathbf{D}$ be as defined in Definition 4．1 page 38．A function $\psi(x)$ in $L_{\mathbb{R}}^{2}$ is a wavelet function for $L_{\mathbb{R}}^{2}$ if
$\left\{\mathbf{D}^{j} \mathbf{T}^{n} \psi \mid j, n \in \mathbb{Z}\right\}$ is a Riesz basis for $L_{\mathbb{R}}^{2}$.
In this case，$\psi$ is also called the mother wavelet of the basis $\left\{\mathbf{D}^{j} \mathbf{T}^{n} \psi \mid j, n \in \mathbb{Z}\right\}$ ．The sequence of subspaces $\left(W_{j}\right)_{j \in \mathbb{Z}}$ is the wavelet analysis induced by $\psi$ ，where each subspace $W_{j}$ is defined as

$$
W_{j} \triangleq \operatorname{spn}\left\{\left.\mathbf{D}^{j} \mathbf{T}^{n} \psi\right|_{n \in \mathbb{Z}}\right\} .
$$

A wavelet analysis $\left(W_{j}\right)$ is often constructed from a multiresolution anaysis（Definition 4.12 page 43）$\left(V_{j}\right)$ under the relationship
$V_{j+1}=V_{j} \hat{+} W_{j}, \quad$ where $\hat{+}$ is subspace addition（Minkowski addition）．
By this relationship alone，$\left(W_{j}\right)$ is in no way uniquely defined in terms of a multiresolu－ tion analysis（ $\boldsymbol{V}_{j}$ ）．In general there are many possible complements of a subspace $V_{j}$ ．To uniquely define such a wavelet subspace，one or more additional constraints are required． One of the most common additional constraints is orthogonality，such that $V_{j}$ and $W_{j}$ are orthogonal to each other．

Definition 4．19 Let $\left(L_{\mathbb{R}}^{2},\left(V_{j}\right), \phi,\left(h_{n}\right)\right)$ be a multiresolution system（Definition 4.12 page 43） and $\left(W_{j}\right)_{j \in \mathbb{Z}}$ a wavelet analysis（Definition 4.18 page 47）with respect to $\left(V_{j}\right)_{j \in \mathbb{Z}}$ ．Let $\left(g_{n}\right)_{n \in \mathbb{Z}}$ be a sequence of coefficients such that $\psi=\sum_{n \in \mathbb{Z}} g_{n} \mathbf{D T}^{n} \phi$ ．

A wavelet system is the tuple

$$
\left(L_{\mathbb{R}}^{2},\left(V_{j}\right),\left(W_{j}\right), \phi, \psi,\left(h_{n}\right),\left(g_{n}\right)\right)
$$

and the sequence $\left(g_{n}\right)_{n \in \mathbb{Z}}$ is the wavelet coefficient sequence．
Theorem 4．20 ${ }^{115} \operatorname{Let}\left(L_{\mathbb{R}}^{2},\left(V_{j}\right),\left(W_{j}\right), \phi, \psi,\left(h_{n}\right),\left(g_{n}\right)\right)$ be a wavelet system（Definition 4.19 page 47）．Let $V_{1} \hat{+} V_{2}$ represent Minkowski addition of two subspaces $V_{1}$ and $V_{2}$ of a Hilbert space $\boldsymbol{H}$ ．

$$
\begin{aligned}
& L_{\mathbb{R}}^{2}=\lim _{j \rightarrow \infty} V_{j} \quad\left(L_{\mathbb{R}}^{2}\right. \text { is equivalent to one very large scaling subspace) } \\
& =V_{j} \hat{+} W_{j} \hat{千} W_{j+1} \hat{+} W_{j+2} \hat{+} \cdots \quad\binom{L_{R}^{2} \text { is equivalent to one scaling space }}{\text { and a sequence of wavelet subspaces }} 0 \\
& =\cdots \hat{+} W_{-2} \hat{+} W_{-1} \hat{+} W_{0} \hat{+} W_{1} \hat{+} W_{2} \hat{+} \cdots \quad\left(L_{\mathbb{R}}^{2}\right. \text { is equivalent to a sequence of wavelet subspaces) }
\end{aligned}
$$

[^25](PROOF:
(1) Proof for (1):
$$
L_{\mathbb{R}}^{2}=\lim _{j \rightarrow \infty} V_{j} \quad \text { by Definition } 4.12 \text { page } 43
$$
(2) Proof for (2):
\[

$$
\begin{aligned}
\underbrace{V_{j} \hat{+} W_{j}}_{V_{j+1}} \hat{+} W_{j+1} \hat{+} W_{j+2} \hat{+} \cdots & =\underbrace{V_{j+1} \hat{+} W_{j+1}}_{V_{j+2}} \hat{千} W_{j+2} \hat{+} W_{j+3} \hat{千} \cdots \\
& =\underbrace{V_{j+2} \hat{+} W_{j+2}}_{V_{j+3}} \hat{f} W_{j+3} \hat{+} W_{j+4} \hat{f} \cdots \\
& =\underbrace{V_{j+3} \hat{+} W_{j+3}}_{V_{j+4}} \hat{} \hat{W_{j+4}} \hat{+} W_{j+5} \hat{+} \cdots \\
& =\underbrace{V_{j+5} \hat{+} W_{j+5}}_{V_{j+5}} \hat{+} W_{j+6} \hat{+} W_{j+6} \hat{+} \cdots \\
& =\lim _{j \rightarrow \infty} V_{j+5} \hat{+} W_{j+5} \hat{+} W_{j+6} \hat{+} W_{j+6} \hat{+} \cdots \\
& =L_{\mathbb{R}}^{2}
\end{aligned}
$$
\]

(3) Proof for (3):

$$
\begin{align*}
& L_{\mathbb{R}}^{2}=\underbrace{V_{0}}_{V_{-1}+W_{-1}} \hat{+} W_{0} \hat{+} W_{1} \hat{+} W_{2} \hat{+} W_{3} \hat{+} \cdots  \tag{2}\\
& =\underbrace{V_{-1}}_{V_{-2}+W_{-2}} W_{-1} \hat{+} W_{0} \hat{+} W_{1} \hat{+} W_{2} \hat{+} W_{3} \hat{+} \cdots \\
& =\underbrace{V_{-2}}_{V_{-3} \uparrow W_{-3}} W_{-2} \hat{+} W_{-1} \hat{+} W_{0} \hat{+} W_{1} \hat{+} W_{2} \hat{+} W_{3} \hat{+} \cdots \\
& =\underbrace{V_{-3}}_{V_{-4}+W_{-4}} W_{-3} \hat{+} W_{-2} \hat{+} W_{-1} \hat{+} W_{0} \hat{+} W_{1} \hat{+} W_{2} \hat{+} W_{3} \hat{+} \cdots \\
& \text { ! } \\
& =\cdots \hat{+} W_{-3} \hat{+} W_{-2} \hat{f} W_{-1} \hat{f} W_{0} \hat{+} W_{1} \hat{+} W_{2} \hat{f} W_{3} \hat{f} \cdots
\end{align*}
$$

Remark 4.21 In the special case that two subspaces $W_{1}$ and $W_{2}$ are orthogonal to each other, then the subspace addition operation $W_{1} \hat{+} W_{2}$ is frequently expressed as $W_{1} \oplus W_{2}$. In the case of an orthonormal wavelet system, the expressions in Theorem 4.20 (page 47)
could be expressed as

$$
\begin{aligned}
L_{\mathbb{R}}^{2} & =\lim _{j \rightarrow \infty} V_{j} \\
& =V_{j} \oplus W_{j} \oplus W_{j+1} \oplus W_{j+2} \oplus \cdots \\
& =\cdots \oplus W_{-2} \oplus W_{-1} \oplus W_{0} \oplus W_{1} \oplus W_{2} \oplus \cdots
\end{aligned}
$$

## 4．5 Fast Wavelet Transform（FWT）

Filter banks can be used to implement a＂Fast Wavelet Transform＂（FWT）．This is illustrated in Figure 10 page $49 .{ }^{116}$


Figure 10：$k$－Stage Fast Wavelet Transform（FWT）

[^26]
（A）general p－lattice

（C）p－lattice of cosets

（B）p－lattice of primorial and prime numbers

（D）p－lattice of MRA and wavelet subspaces

Figure 11：Some selected primorial lattices（see Example 5.2 page 50－Example 5.5 page 51 ）

## 5 Main Results

## 5．1 Primorial Lattices

Definition 5．1 Let $X \triangleq\left\{0, x_{0}, x_{1}, \ldots, x_{N}, y_{0}, y_{1}, \ldots, y_{N}\right\}$ be a set．
A lattice $L \triangleq(X, \vee, \wedge ; \leq)$ is primorial if
1． 0 is the least element of $L$ and
2．$L$ is atomic（Definition 1.44 page 13）and $\left\{y_{0}, x_{0}, x_{1}, \ldots, x_{N}\right\}$ are atoms of $L$ and
3．$y_{n+1}=y_{n} \vee x_{n}$ ．
A lattice that is primorial is a primorial lattice，or simply a p－lattice．

Example 5．2 A general primorial lattice is illustrated to in Figure 11 page 50 （A）．

Example 5．3 ${ }^{117}$ The set of primorial numbers and prime numbers ordered by the divides （＂｜＂）relation forms a primorial lattice，as illustrated in Figure 11 page 50 （B）．

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117日 [75], page 30, 听 [2] 〈http://oeis.org/A002110\rangle
```



Figure 12：some MRA－wavelet systems

Example 5．4 Any partition，along with successive unions of the partition elements，gen－ erates a primorial lattice．One example of this is the cosets of $\mathbb{Z}$ ，which generate a finite primorial lattice，as illustrated in Figure 11 page 50 （C）．

Example 5．5 A special characteristic of MRA－wavelet analysis is that it＇s order structure with respect to the $\subseteq$ relation is not a simple $M_{n}$ lattice（as is with the case of Fourier and several other analyses）．Rather，it is a primorial lattice as illustrated in Figure 11 page 50 （D） and in Figure 12 page 51.

Proposition 5．6 ${ }^{118}$ Let $L \triangleq(X, \vee, \wedge ; \leq)$ be $a$ Lattice．

$$
\left\{\begin{array}{lll}
L \text { is } \\
\text { primorial }
\end{array}\right\} \Longrightarrow\left\{\begin{array}{lll}
1 . & L \text { is } \text { NONDISTRIBUTIVE } & \text { (Definition } 1.53 \text { page 15) } \\
\text { 2. } & L \text { and } \\
\text { is } \text { NONMODULAR } & \text { (Definition } 1.47 \text { page 14) } & \text { and } \\
3 . & L \text { is } \text { COMPLEMENTED } \Longleftrightarrow L \text { is FINITE } & \text { (Definition } 1.63 \text { page 17) } \\
\text { and } \\
4 . & L \text { is } \text { NOT UNIQUELY COMPLEMENTED } & \text { (Definition } 1.63 \text { page 17) } \\
\text { and } \\
5 . & L \text { is } \text { NONORTHOCOMPLEMENTED } & \text { (Definition } 1.72 \text { page 20) } \\
\text { and } \\
6 . & L \text { } \text { is NONBOOLEAN } & \text { (Definition } 1.69 \text { page 18) }
\end{array}\right\}
$$

[^27](2)Poof:

(1) Proof that $L$ is nondistributive:
(a) L contains the N5 lattice (Definition 1.49 page 14).
(b) Because $L$ contains the $N 5$ lattice, $L$ is nondistributive (Theorem 1.57 page 16).
(2) Proof that $L$ is nonmodular and nondistributive:
(a) $L$ contains the $N 5$ lattice (Definition 1.49 page 14).
(b) Because $L$ contains the $N 5$ lattice, $L$ is nonmodular (Theorem 1.50 page 14).
(3) Proof that $L$ is noncomplemented:
\[

$$
\begin{aligned}
x^{\prime} & =y^{\prime}=v^{\prime}=z \\
z^{\prime} & =\{x, y, v\} \\
x^{\prime \prime} & =\left(x^{\prime}\right)^{\prime} \\
& =z^{\prime} \\
& =\{x, y, v\} \\
& \neq x
\end{aligned}
$$
\]

(4) Proof that $L$ is nonBoolean:
(a) $L$ is nondistributive (item 1 page 52).
(b) Because $L$ is nondistributive, it is nonBoolean (Definition 1.69 page 18).

### 5.2 Reduction operator on boolean lattices

Definition 5.7 Let $\mathbb{B}$ be the set of all bounded lattices (Definition 1.39 page 12). Let $L_{2}^{N} \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ be a Boolean lattice (Definition 1.69 page 18) with $2^{N}$ elements and $N \in \mathbb{N}$ ( $N$ is a positive integer). The operator $\mathbf{R}$ is the lattice reduction operator of $L_{2}^{N}$ and $\mathbf{R} L_{2}^{N}$ is the reduction of $L_{2}^{N}$ if

$$
\mathbf{R} L_{2}^{N} \triangleq\left\{\begin{array}{l|lll}
L \in \mathbb{B} & \begin{array}{lll}
1 . & L \text { is a } 2^{N-1} \text { element Boolean lattice } & \text { and } \\
2 . & L \subseteq L_{2}^{N} & \text { and } \\
3 . & \{0,1\} \in L & \text { and } \\
4 . & \{x, y\} \text { is an orthocomplemented pair in } L \Longrightarrow & \\
& \{x, y\} \text { is an orthocomplemented pair in } L_{2}^{N}
\end{array}
\end{array}\right\}
$$

Note that in Definition 5.7, the order relation $\leq$ is the same for both $L_{2}^{N}$ and any $L$ in $\mathbf{R} L_{2}^{N}$. That is, if $x \leq y$ in $L_{2}^{N}$, then $x \leq y$ in $L$ as well.

Example 5.8 Let $L_{2}^{2}$ be a Boolean lattice (Definition 1.69 page 18) of order 2. Let $\mathbf{R}$ be the lattice reduction operator $\mathbf{R}$ and $\mathbf{R} L_{2}^{2}$ be the reduction of $L_{2}^{2}$ (Definition 5.7 page 52). Then $\mathbf{R} L_{2}^{2}$ yields a set of exactly one $2^{2-1}$ value Boolean lattice, as illustrated next:


Example 5.9 Let $L_{2}^{3}$ be a Boolean lattice (Definition 1.69 page 18) of order 3. Let $\mathbf{R}$ be the lattice reduction operator $\mathbf{R}$ and $\mathbf{R} L_{2}^{3}$ be the reduction of $L_{2}^{3}$ (Definition 5.7 page 52). The operation $\mathbf{R} L_{2}^{3}$ yields a set of three $2^{2}$ value Boolean lattices, as illustrated next:


Example 5.10 Let $L_{2}^{4}$ be a Boolean lattice (Definition 1.69 page 18) of order 4. Let $\mathbf{R}$ be the lattice reduction operator $\mathbf{R}$ and $\mathbf{R} L_{2}^{4}$ be the reduction of $L_{2}^{4}$ (Definition 5.7 page 52). The operation $\mathbf{R} L_{2}^{4}$ yields a set of ten $2^{3}$ value Boolean lattices, as illustrated in Figure 13 (page 54).

Remark 5.11 In a boolean lattice $L_{2}^{N}$ (Definition 1.69 page 18), besides the pair $\{0,1\}$, there are a total of $2^{N-1}-1$ orthocomplemented (Definition 1.72 page 20) pairs of elements. But note that any arbitrary $2^{N-1}-2$ pairs of orthocomplemented pairs does not in general generate a boolean lattice. The lattice $L_{2}^{4}$, for example, has $2^{4-1}-1=7$ orthocomplemented pairs besides $\{0,1\}$. To generate an $L_{2}^{3}$ lattice, we need 3 orthocomplemented pairs. There are $\binom{7}{3}=\frac{7!}{3!4!}=35$ ways of selecting 3 pairs from $L_{2}^{4}$, but only 10 of these ways generate a boolean lattice (Example 5.10 page 53). All other ways fail.







Figure 13: reduction of $L_{2}^{4}$ (Example 5.10 page 53)
For example, if we were to select the pairs $\left\{0, w, w^{\perp}, a, a^{\perp}, b, b^{\perp}, 1\right\}$, we would get the orthocomplemented, but non-boolean (Definition 1.69 page 18) lattice illustrated to the right; In particular, it is complemented, but nondistributive. For example, $w^{\perp} \wedge(a \vee b)=w^{\perp} \neq 0=0 \vee 0=\left(w^{\perp} \wedge a\right) \vee\left(w^{\perp} \wedge b\right)$. Alternatively, note that the set $\left\{1, a, w, 0, b^{\perp}, w^{\perp}\right\}$ together with the ordering relation $\leq$ form an $O_{6}$ sublattice (Definition 1.73 page 20), which contains an
 $N_{5}$ sublattice, which implies that the lattice to the right is non-distributive (by the Birkhoff distributivity criterion theorem 1.57 page 16).

Example 5.12 Let $L_{2}^{5}$ be a Boolean lattice (Definition 1.69 page 18) of order 5 . Let $\mathbf{R}$ be the lattice reduction operator $\mathbf{R}$ and $\mathbf{R} L_{2}^{5}$ be the reduction of $L_{2}^{5}$ (Definition 5.7 page 52). The result of the operation $\mathbf{R} L_{2}^{5}$ is partially illustrated in Figure 14 (page 55).

### 5.3 Difference operator on bounded lattices

Definition 5.13 Let $X \backslash Y$ be the standard set difference of a set $X$ and a set $Y$. Let $L_{x} \triangleq$ ( $X, \vee, \wedge, 0,1 ; \leq)$ and $L_{y} \triangleq(Y, \vee, \wedge, 0,1 ; \leq)$ be bounded lattices (Definition 1.39 page 12). The bounded lattice difference $L_{x} \otimes L_{y}$ of $L_{x}$ and $L_{y}$ is the lattice $L$ such that $L \triangleq((X \backslash Y) \cup\{0,1\}, \vee, \wedge, 0,1 ; \leq)$


Figure 14: reduction of $L_{2}^{5}$ (Example 5.12 page 54$)$

Example 5.14 Let $\theta$ be the bounded lattice difference operator (Definition 5.13 page 54).




Proposition 5.15 Let $\mathbb{B}$ be the set of all bounded lattices (Definition 1.39 page 12). Let $Q$ be the BOUNDED LATTICE DIFFERENCE operator (Definition 5.13 page 54). $(\mathbb{B}, \mathbb{Q} \subseteq)$ is a D -POSET (Definition 3.18 page 37).

Theorem 5.16 Let $L \triangleq L_{2}^{N} \otimes L_{2}^{N-1}$ be the BOUNDED LATTICE DIFFERENCE (Definition 5.13 page 54) of $a$ Boolean lattice $L_{2}^{N}$ (Definition 1.69 page 18) and $a$ Boolean lattice $L_{2}^{N-1}$ selected from the set $\mathbf{R} L_{2}^{N}$ (Definition 5.7 page 52). Let $X \triangleq\left\{L_{2}^{n} \mid n=1,2, \ldots\right\} \cup\left\{\boldsymbol{L}_{2}^{n} \otimes L_{2}^{n-1} \mid n=2,3, \ldots\right\}$.

1. $L_{2}^{N} \otimes L_{2}^{N-1}$ is an orthocomplemented lattice (Definition 1.72 page 20) and
2. The structure $\mathbb{P} \triangleq(X, \vee, \wedge ; \subseteq)$ is a primorial lattice (Definition 5.1 page 50 ).
*Proof:
(1) Proof that $L_{2}^{N} \otimes L_{2}^{N-1}$ is an orthocomplemented lattice:
(a) $L_{2}^{N}$ is a Boolean lattice by definition.
(b) $L_{2}^{N-1}$ is also a Boolean lattice (Definition 5.7 page 52).
(c) Every lattice that is Boolean is also orthocomplemented (Proposition 1.80 page 23).
(d) By definition of $L_{2}^{N} \otimes L_{2}^{N-1}$, orthocomplemented pairs are removed from $L_{2}^{N}$ and the orthocomplemented pair $\{0,1\}$ is put back in.
(e) What remains in $L_{2}^{N} \otimes L_{2}^{N-1}$ is a set of orthocomplemented pairs, ordered with the same ordering relation $\leq$ that orders $L_{2}^{N}$.
(f) All remaining orthocomplemented pairs are still involutory: $x=x^{\perp \perp} \quad \forall x \in X$
(g) All remaining orthocomplemented pairs are still antitone because the ordering relation $\leq$ in $L_{2}^{N}$ and $L_{2}^{N} \otimes L_{2}^{N-1}$ is the same.
(h) All remaining orthocomplemented pairs still have the non-contradiction property because suppose that in $L_{2}^{N} \otimes L_{2}^{N-1}$, there is an element $x$ such that $x \wedge x^{\perp}=m \neq 0$. Then in $L_{2}^{N}$, it would also be true that $x \wedge x^{\perp} \neq 0$. This cannot be true (is a contradiction); so therefore for all $x$ in $L_{2}^{N} \otimes L_{2}^{N-1}, x \wedge x^{\perp}=0$ (non-contradiction property).
(i) So $L_{2}^{N} \otimes L_{2}^{N-1}$ is an orthocomplemented lattice (Definition 1.72 page 20).
(2) Proof that $\left(X \triangleq\left\{\boldsymbol{L}_{2}^{n} \mid n=1,2, \ldots\right\} \cup\left\{\boldsymbol{L}_{2}^{n} \otimes \boldsymbol{L}_{2}^{n-1} \mid n=2,3, \ldots\right\}, \subseteq\right)$ is a primorial lattice: This follows directly from the construction of the bounded lattice difference (Definition 5.13 page 54 ) and the definition of primorial lattices (Definition 5.1 page 50).


Figure 15: a primorial lattice generated by $L_{2}^{5}$

Definition 5.17 Let $L_{2}^{N}$ be a $2^{N}$ element Boolean lattice (Definition 1.69 page 18). The lattice $\mathbb{P}$ as described in Theorem 5.16 is a primorial lattice generated by $L_{2}^{N}$.

Example 5.18 Figure 15 (page 57) illustrates a primorial lattice generated by $L_{2}^{5}$.

### 5.4 Projections on primorial lattices

This section introduces three lattice projections. When performing analysis in a primorial lattice (Definition 5.1 page 50), it is necessary to project a point that exists in a lattice of "high resolution" onto a lattice $L$ of lower resolution that may or may not contain this point. The three projections introduced here are the

1. zero primorial projection (Definition 5.19 page 58) which assigns to 0 any point that does not exist in $L$
2. Sasaki primorial projection (Definition 5.20 page 58) which assigns a projection value using the Sasaki projection (Definition 2.22 page 31)
3. metric primorial projection (Definition 5.22 page 59) which assigns a projection value based on a lattice metric (Definition 2.7 page 27).

Definition 5.19 Let $\mathbb{P}$ be a primorial lattice (Definition 5.17 page 58) generated by a Boolean lattice $L_{2}^{N}$ (Definition 1.69 page 18). Let $L \triangleq(Y, \vee, \wedge, 0,1 ; \leq)$ be a lattice in $\mathbb{P}$. Let $\star \triangleq\left(x_{n}\right)$ be a sequence over the set $X$. The zero primorial projection $\Phi_{L}(x)$ of $x$ onto $L$ is defined as

$$
\Phi_{L}^{z}(x) \triangleq \bigvee_{L}[\{x, 0\} \cap Y] \quad \forall x \in X
$$

The zero primorial projection $\Phi_{L}^{z}(x)$ of $\mathbb{x}$ onto $L$ is defined as

$$
\Phi_{L}^{z}(x) \triangleq\left(y_{n}\right) \text { where } y_{n} \triangleq \Phi_{L}^{z}\left(x_{n}\right) \quad \forall x_{n} \in\left(x_{n}\right), y_{n} \in\left(y_{n}\right) \text {. }
$$

Definition 5.20 Let $\mathbb{P}$ and $\mathbb{x}$ be defined as in Definition 5.19 (page 58). Let $\mathbb{P}$ be a primorial lattice (Definition 5.17 page 58) generated by a Boolean lattice $L_{2}^{N}$ (Definition 1.69 page 18). Let $L \triangleq(Y, \vee, \wedge, 0,1 ; \leq)$ be a lattice in $\mathbb{P}$. Let $\mathbb{\otimes} \triangleq\left(x_{n}\right)$ be a sequence over the set $X$. The Sasaki primorial projection $\Phi_{L}^{s}(x)$ of $x$ onto $L$ is defined as

$$
\Phi_{L}^{s}(x) \triangleq \bigvee_{L}\left[\left\{\phi_{y}(x) \mid y \in Y\right\} \cap Y\right] \quad \forall x \in L
$$

where $\phi_{y}(x)$ is the Sasaki projection of $x$ onto $y$ (Definition 2.22 page 31) in the smallest Boolean lattice $L_{2}^{M}$ that contains both $x$ and $L$. The Sasaki primorial projection $\Phi_{L}^{s}(x)$ of $x$ onto $L$ is defined as

$$
\Phi_{L}^{s}(x) \triangleq\left(y_{n}\right) \text { where } y_{n} \triangleq \Phi_{L}^{s}\left(x_{n}\right) \quad \forall x_{n} \in\left(x_{n}\right) .
$$

The Sasaki primorial projection yields a kind of maxmini (Theorem 1.35 page 11) result:

Proposition 5.21 Let $\Phi_{L}(x)$ be the SASAKI Primorial projection of $x$ onto $L$ in a primorial lattice $\mathbb{P}$.

$$
\Phi_{L}^{s}(x)=\bigvee_{L}[\{x \wedge y \mid y \in Y\} \cap Y] \quad \forall x \in X
$$

*Proof:

$$
\begin{aligned}
\Phi_{L}^{s}(x) & \triangleq \bigvee\left[\left\{\phi_{y}(x) \mid y \in Y\right\} \cap Y\right] & & \text { by def. of Sasaki primorial projection (Definition } 5.20 \text { page 58) } \\
& \triangleq \bigvee\left[\left\{\left(x \vee y^{\perp}\right) \wedge y \mid y \in Y\right\} \cap Y\right] & & \text { by definition of Sasaki projection (Definition } 2.22 \text { page 31) } \\
& =\bigvee\left[\left\{(x \wedge y) \vee\left(y^{\perp} \wedge y\right) \mid y \in Y\right\} \cap Y\right] & & \text { by distributive prop. (Theorem } 1.70 \text { page 19) } \\
& =\bigvee[\{(x \wedge y) \vee(0) \mid y \in Y\} \cap Y] & & \text { by noncontradiction property (Theorem 1.70 page 19) } \\
& =\bigvee[\{x \wedge y \mid y \in Y\} \cap Y] & & \text { by bounded property (Theorem 1.70 page 19) }
\end{aligned}
$$

Definition 5.22 Let $\mathbb{P}$ and $x$ be defined as in Definition 5.19.
The metric primorial projection $\Phi_{L}^{m}(x)$ of $x$ onto $L$ is defined as

$$
\Phi_{L}^{m}(x) \triangleq \bigwedge_{L}[\overline{\mathrm{~B}}(x, r) \cap Y] \quad \text { where }
$$

1. $\overline{\mathrm{B}}(x, r)$ is the closed ball in $\left(L_{2}^{M}, \mathrm{~d}\right)$ with the smallest radius $r$ that contains $x$ and
2. $\left(L_{2}^{M}, \mathrm{~d}\right)$ is a metric lattice (Definition 2.7 page 27) and
3. $\quad L_{2}^{M}$ is the smallest Boolean lattice (Definition 1.69 page 18) containing $x$ and
4. the valuation function defining d is the height function on $L_{2}^{M}$.

The metric primorial projection $\Phi_{L}(x)$ of $x$ onto $L$ is defined as
$\Phi_{L}(\mathbb{x}) \triangleq\left(y_{n}\right)$ such that $y_{n} \triangleq \Phi_{L}\left(x_{n}\right)$.
Example 5.23 Here are examples of the primorial projections $\Phi_{O_{6}}^{z}(x)$ (Definition 5.19 page 58), $\Phi_{O_{6}}^{s}(x)$ (Definition 5.20 page 58), and $\Phi_{O_{6}}^{m}(x)$ (Definition 5.22 page 59) in the primorial lattice (Definition 5.1 page 50) generated by the Boolean lattice (Definition 1.69 page 18) $L_{2}^{5} \triangleq(X, \vee, \wedge, 0,1 ; \leq)$ as illustrated in Figure 15 page 57 onto the lattice $O_{6} \triangleq L_{2}^{3} \otimes L_{2}^{2} \triangleq(Y, \vee, \wedge, 0,1 ; \leq)$.

| projection | $x$ in $\mathrm{O}_{6} \triangleq L_{2}^{3} \otimes L_{2}^{2}$ |  |  |  |  |  | $x$ in $L_{2}^{3}$ |  |  | $x$ in $L_{2}^{4}$ |  |  |  |  | $x$ in $L_{2}^{5}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x=$ |  | $0 f$ | $t$ | $t^{\perp}$ | ${ }^{\perp}$ | 1 | $q$ |  | $q^{\perp}$ |  | $r$ | ${ }^{\perp}$ |  | ${ }^{\text {+ }}$ |  | $g$ |  |  | $p^{\perp}$ | d | $d^{\perp}$ |
| $\Phi_{O_{6}}^{z}(x)=$ |  | 0 f | $t$ | $t^{\perp}$ | $f^{\perp}$ | 1 | 0 |  | 0 |  | 0 | 0 |  | 0 | 0 | 0 |  |  | 0 | 0 | 0 |
| $\Phi_{O_{6}}^{s}(x)=$ |  | ) $f$ | $t$ | $t^{\perp}$ |  | 1 | 0 |  | 1 |  | $f$ | ${ }^{\perp}$ |  | $\perp$ |  | $f$ |  |  | 1 | 0 | $t$ |
| $\Phi_{O_{6}}^{m_{6}}(x)=$ |  | ) $f$ | $t$ |  | $f^{\perp}$ | 1 | 0 |  | 0 |  |  |  |  |  | $t$ | $f$ |  |  | , | 0 | $t$ |

(2)Poof:

(1) Proof for zero primorial projection values:

$$
\begin{aligned}
& \Phi_{O_{6}}^{z}(0)=\bigvee\left[(\{0\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}(f)=\bigvee\left[(\{f\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0, f\}] \quad=f \\
& \Phi_{O_{6}}^{z}(t)=\bigvee\left[(\{t\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0, t\}] \quad=t \\
& \Phi_{O_{6}}^{z}\left(t^{\perp}\right)=\bigvee\left[\left(\left\{t^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee\left[\left\{0, t^{\perp}\right\}\right] \quad=t^{\perp} \\
& \Phi_{O_{6}}^{z}\left(f^{\perp}\right)=\bigvee\left[\left(\left\{f^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee\left[\left\{0, f^{\perp}\right\}\right] \quad=f^{\perp} \\
& \Phi_{O_{6}}^{z}(1)=\bigvee\left[(\{1\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{1,0\}] \quad=1 \\
& \Phi_{O_{6}}^{z}(q)=\bigvee\left[(\{q\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}\left(q^{\perp}\right)=\bigvee\left[\left(\left\{q^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, q^{\perp}, 1\right\}\right] \quad=\bigvee\left[\left\{0, q^{\perp}\right\}\right] \quad=0 \\
& \Phi_{O_{6}}^{z}(r)=\bigvee\left[(\{r\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}\left(r^{\perp}\right)=\bigvee\left[\left(\left\{r^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, r^{\perp}, 1\right\}\right] \quad=\bigvee\left[\left\{0, r^{\perp}\right\}\right] \quad=0 \\
& \Phi_{O_{6}}^{z}(s)=\bigvee\left[(\{s\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}\left(s^{\perp}\right)=\bigvee\left[\left(\left\{s^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, r^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}(g)=\bigvee\left[(\{g\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}\left(g^{\perp}\right)=\bigvee\left[\left(\left\{g^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, r^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}(p)=\bigvee\left[(\{p\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}\left(p^{\perp}\right)=\bigvee\left[\left(\left\{p^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, r^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}(d)=\bigvee\left[(\{d\} \cup\{0\}) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0 \\
& \Phi_{O_{6}}^{z}\left(d^{\perp}\right)=\bigvee\left[\left(\left\{d^{\perp}\right\} \cup\{0\}\right) \cap\left\{0, f, t, t^{\perp}, r^{\perp}, 1\right\}\right] \quad=\bigvee[\{0\}] \quad=0
\end{aligned}
$$

(2) Proof for Sasaki primorial projection (Definition 5.20 page 58):

$$
\begin{array}{rllll}
\Phi_{O_{6}}^{s}(0) & =\bigvee[\{0 \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0,0,0,0,0,0\} \cap Y] & =\bigvee\{0\} & =0 \\
\Phi_{O_{6}}^{s}(f) & =\bigvee[\{f \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0, f, 0, f, 0, f\} \cap Y] & =\bigvee\{0, f\} & =f \\
\Phi_{O_{6}}^{s}(t) & =\bigvee[\{t \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0,0, t, 0, t, t\} \cap Y] & =\bigvee\{0, t\} & =t \\
\Phi_{O_{6}}^{s}\left(t^{\perp}\right)=\bigvee\left[\left\{t^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0, f, 0, t^{\perp}, q, t^{\perp}\right\} \cap Y\right] & =\bigvee\left\{0, f, t^{\perp}\right\} & =t^{\perp} \\
\Phi_{O_{6}}^{s}\left(f^{\perp}\right)=\bigvee\left[\left\{f^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0,0, t, q, f^{\perp}, f^{\perp}\right\} \cap Y\right] & =\bigvee\left\{0, t, f^{\perp}\right\} & =f^{\perp} \\
\Phi_{O_{6}}^{s}(1)=\bigvee[\{1 \wedge y \mid y \in Y\} \cap Y] & =\bigvee\left[\left\{0, f, t, f^{\perp}, t^{\perp}, 1\right\} \cap Y\right] & =\bigvee Y & =1 \\
\Phi_{O_{6}}^{s}(q)=\bigvee[\{q \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0,0,0, q, 0, q\} \cap Y] & =\bigvee\{0\} & =0 \\
\Phi_{O_{6}}^{s}\left(q^{\perp}\right)=\bigvee\left[\left\{q^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0, f, t, f, t, q^{\perp}\right\} \cap Y\right] & =\bigvee\{0, f, t\} & =1
\end{array}
$$



$$
\begin{array}{rllll}
\Phi_{O_{6}}^{s}(r)=\bigvee[\{r \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0, r, 0, r, 0, r\} \cap Y] & =\bigvee\{0\} & =0 \\
\Phi_{O_{6}}^{s}\left(r^{\perp}\right)=\bigvee\left[\left\{r^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0, s, t, e, f^{\perp}, r^{\perp}\right\} \cap Y\right] & =\bigvee\left\{0, t, f^{\perp}\right\} & =f^{\perp} \\
\Phi_{O_{6}}^{s}(s)=\bigvee[\{s \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0, s, 0, s, 0, s\} \cap Y] & =\bigvee\{0\} & =0 \\
\Phi_{O_{6}}^{s}\left(s^{\perp}\right)=\bigvee\left[\left\{s^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0,0, t, d, f^{\perp}, s^{\perp}\right\} \cap Y\right] & =\bigvee\left\{0, t, f^{\perp}\right\} & =f^{\perp} \\
\Phi_{O_{6}}^{s}(g)=\bigvee[\{g \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0,0, t, p, g, g\} \cap Y] & =\bigvee\{0, t\} & =t \\
\Phi_{O_{6}}^{s}\left(g^{\perp}\right)=\bigvee\left[\left\{g^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0, f, 0, g^{\perp}, 0, g^{\perp}\right\} \cap Y\right] & =\bigvee\{0, f\} & =f \\
\Phi_{O_{6}}^{s}(p)=\bigvee[\{p \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0,0,0, p, p, p\} \cap Y] & =\bigvee\{0\} & =0 \\
\Phi_{O_{6}}^{s}\left(p^{\perp}\right)=\bigvee\left[\left\{p^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0, f, t, g^{\perp}, t, p^{\perp}\right\} \cap Y\right] & =\bigvee\{0, f, t\} & =1 \\
\Phi_{O_{6}}^{s}(d)=\bigvee[\{d \wedge y \mid y \in Y\} \cap Y] & =\bigvee[\{0, r, 0, d, 0, d\} \cap Y] & =\bigvee\{0\} & =0 \\
\Phi_{O_{6}}^{s}\left(d^{\perp}\right)=\bigvee\left[\left\{d^{\perp} \wedge y \mid y \in Y\right\} \cap Y\right] & =\bigvee\left[\left\{0, s, t, 0, g, d^{\perp}\right\} \cap Y\right] & =\bigvee\{0, t\} & =t
\end{array}
$$

(3) Proof for metric primorial projection (Definition 5.22 page 59):

$$
\begin{aligned}
& \Phi_{O_{6}}^{m}(0)=\bigwedge[\overline{\mathrm{B}}(0,0) \cap Y]=\bigwedge\left[\{0\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right]=\bigwedge\{0\} \quad=0 \\
& \Phi_{O_{6}}^{m}(f)=\bigwedge[\overline{\mathrm{B}}(f, 0) \cap Y]=\bigwedge\left[\{f\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right]=\bigwedge\{f\}=f \\
& \Phi_{O_{6}}^{m}(t)=\bigwedge[\overline{\mathrm{B}}(t, 0) \cap Y]=\bigwedge\left[\{t\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right]=\bigwedge\{t\} \quad=t \\
& \Phi_{O_{6}}^{m}\left(t^{\perp}\right)=\bigwedge\left[\overline{\mathrm{B}}\left(t^{\perp}, 0\right) \cap Y\right]=\bigwedge\left[\left\{t^{\perp}\right\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right]=\bigwedge\left\{t^{\perp}\right\} \quad=t^{\perp} \\
& \Phi_{O_{6}}^{m}\left(f^{\perp}\right)=\bigwedge\left[\overline{\mathrm{B}}\left(f^{\perp}, 0\right) \cap Y\right]=\bigwedge\left[\left\{f^{\perp}\right\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right]=\bigwedge\left\{f^{\perp}\right\}=f^{\perp} \\
& \Phi_{O_{6}}^{m}(1)=\bigwedge[\overline{\mathrm{B}}(1,0) \cap Y]=\bigwedge\left[\{1\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right]=\bigwedge\{1\} \quad=1 \\
& \Phi_{O_{6}}^{m}(q)=\bigwedge[\overline{\mathrm{B}}(q, 1) \cap Y] \quad=\bigwedge\left[\left\{q, 0, t^{\perp}\right\} \cap Y\right] \quad=\bigwedge\left\{0, t^{\perp}\right\} \quad=0 \\
& \Phi_{O_{6}}^{m}\left(q^{\perp}\right)=\bigwedge\left[\overline{\mathrm{B}}\left(q^{\perp}, 1\right) \cap Y\right] \quad=\bigwedge\left[\left\{q^{\perp}, t, 1\right\} \cap Y\right] \quad=\bigwedge\{t, 1\} \quad=t \\
& \Phi_{O_{6}}^{m}(r)=\bigwedge[\overline{\mathrm{B}}(r, 1) \cap Y] \quad=\bigwedge[\{r, 0, d, f\} \cap Y] \quad=\bigwedge\{0, f\} \quad=0 \\
& \Phi_{O_{6}}^{m}\left(r^{\perp}\right)=\bigwedge\left[\overline{\mathrm{B}}\left(r^{\perp}, 1\right) \cap Y\right]=\bigwedge\left[\left\{r^{\perp}, d^{\perp}, f^{\perp}, 1\right\} \cap Y\right] \quad=\bigwedge\left\{f^{\perp}, 1\right\} \quad=f^{\perp} \\
& \Phi_{O_{6}}^{m}(s)=\bigwedge[\overline{\mathrm{B}}(s, 1) \cap Y] \quad=\bigwedge\left[\left\{s, 0, e, f, e^{\perp}\right\} \cap Y\right] \quad=\bigwedge\{0, f\} \quad=0 \\
& \Phi_{O_{6}}^{m}\left(s^{\perp}\right)=\bigwedge\left[\overline{\mathrm{B}}\left(s^{\perp}, 1\right) \cap Y\right]=\bigwedge\left[\left\{s^{\perp}, e^{\perp}, f^{\perp}, d, 1\right\} \cap Y\right] \quad=\bigwedge\left\{f^{\perp}, 1\right\}=f^{\perp} \\
& \Phi_{O_{6}}^{m}(g)=\bigwedge[\overline{\mathrm{B}}(g, 1) \cap Y]=\bigwedge\left[\left\{g, p, f^{\perp}, e^{\perp}, d^{\perp}, t\right\} \cap Y\right]=\bigwedge\left\{f^{\perp}, t\right\}=t \\
& \Phi_{O_{6}}^{m}\left(g^{\perp}\right)=\bigwedge\left[\overline{\mathrm{B}}\left(g^{\perp}, 1\right) \cap Y\right]=\bigwedge\left[\left\{g^{\perp}, d, e, f, p^{\perp}, t^{\perp}\right\} \cap Y\right]=\bigwedge\left\{f, t^{\perp}\right\}=f \\
& \Phi_{O_{6}}^{m}(p)=\bigwedge[\overline{\mathrm{B}}(p, 1) \cap Y]=\bigwedge[\{p, 0, p, a, b, c, g\} \cap Y]=\bigwedge\{0\} \quad=0
\end{aligned}
$$



$$
\begin{aligned}
\Phi_{O_{6}}^{m}\left(p^{\perp}\right)= & \bigwedge\left[\overline{\mathrm{B}}\left(p^{\perp}, 1\right) \cap Y\right]=\bigwedge\left[\left\{p^{\perp}, a^{\perp}, b^{\perp}, c^{\perp}, g^{\perp}, 1\right\} \cap Y\right]=\bigwedge\{1\} \\
\Phi_{O_{6}}^{m}(d) & =\bigwedge\left[\overline{\mathrm{B}}(d, 2) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \\
& =\bigwedge\left[\left\{0, a, b, d, e, f, h, i, q, r, c^{\perp}, g^{\perp}, j^{\perp}, p^{\perp}, s^{\perp}, t^{\perp}\right\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \\
& =\bigwedge\left\{0, f, t^{\perp}\right\} \\
& =0 \\
\Phi_{O_{6}}^{m}\left(d^{\perp}\right) & =\bigwedge\left[\overline{\mathrm{B}}\left(d^{\perp}, 2\right) \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \\
& =\bigwedge\left[\left\{c, g, j, p, s, t, a^{\perp}, b^{\perp}, d^{\perp}, e^{\perp}, f^{\perp}, h^{\perp}, i^{\perp}, q^{\perp}, r^{\perp}, 1\right\} \cap\left\{0, f, t, t^{\perp}, f^{\perp}, 1\right\}\right] \\
& =\bigwedge\left\{t, f^{\perp}, 1\right\} \\
& =t
\end{aligned}
$$

## 5．5 A generalized probability function

This paper introduces a new definition for a lattice－valued probability function（next）．
Definition 5．24 Let $L \triangleq(X, \vee, \wedge, \neg, 0,1 ; \leq)$ be a lattice with negation（Definition 2.16 page 30）． Let（B）be the distributivity relation（Definition 1.52 page 15）．A function $p$ in $\mathbb{R}^{X}$ is a probability on $L$ if


If p is a probability on a lattice with negation $L$ ，then $(L, p)$ is a probability space．
Remark 5．25 Definition 5.24 page 62 （previous）is not any standard definition of the prob－ ability function．On a Boolean lattice，the measure－theoretic probability function，due to A．N．Kolmogorov，is defined as ${ }^{119}$
（1）．
（2）．

$$
\begin{array}{llllll}
\text { (1). } & \mathrm{p}(1) & =1 & & \begin{array}{l}
\text { (normalized) }
\end{array} & \text { and } \\
\text { (2). } & & \geq 0 & \forall x \in X & \text { (nonnegative) } & \text { and } \\
\text { (3). } & \bigwedge_{n=1}^{\infty} x_{n}=0 \Longrightarrow \mathrm{p}\left(\bigvee_{n=1}^{\infty} x_{n}\right) & =\sum_{n=1}^{\infty} \mathrm{p}\left(x_{n}\right) & \forall x_{n} \in X & (\sigma \text {-additive) }
\end{array}
$$

${ }^{119} \boxminus$［13］，pages 22－23，〈Probability Measures $\rangle$［103］，［102］，page 16，〈field of probability $\rangle$ ， －［137］，pages 8－9，〈Definition 2．3（13）$\rangle$ ，［99］，page 27

The advantage of this definition is that p is a measure，and hence all the power of mea－ sure theory is subsequently at one＇s disposal in using p．However，it has often been ar－ gued that the requirement of $\sigma$－additivity is unnecessary for a probability function．Even as early as 1930，de Finetti argued against it，in what became a kind of polite running debate with Fréchet．${ }^{120}$ In fact，Kolmogorov himself provided some argument against $\sigma$－additivity when referring to the closely related Axiom of Continuity saying，＂Since the new axiom is essential for infinite fields of probability only，it is almost impossible to elucidate its em－ pirical meaning．．．For，in describing any observable random process we can obtain only finite fields of probability．．．．＂But in its support he added，＂This limitation has been found expedient in researches of the most diverse sort．＂${ }^{121}$

There are several other definitions of probability that only require additivity rather than $\sigma$－additivity．On a Boolean lattice，the traditional probability function is defined as ${ }^{122}$

| （1）． | $\mathrm{p}(1)$ | $=$ | 1 |  | （normalized） |
| :---: | :---: | :---: | :---: | :---: | :---: |
| （2）． | $\mathrm{p}(x)$ | $\geq$ | 0 | $\forall x \in X$ | （nonnegative） |
| （3）．$x \wedge y=0$ | $\mathrm{p}(x \vee y)$ | $=$ | $\mathrm{p}(x)+\mathrm{p}(y)$ | $\forall x, y \in X$ | （additive） |

This definition implies（on a Boolean lattice）that

| （a）． | $\mathrm{p}(0)$ | $=0$ |  | （nondegenerate） | and |
| :--- | ---: | :--- | :--- | :--- | :--- |
| （b）． | $\mathrm{p}(x)$ | $\leq 1$ | $\forall x \in X$ | （upper bounded） | and |
| （c）． | $\mathrm{p}(x)$ | $=1-\mathrm{p}(\neg x)$ | $\forall x \in X$ |  | and |
| （d）． | $\mathrm{p}(x \vee y)$ | $\leq \mathrm{p}(x)+\mathrm{p}(y)$ | $\forall x, y \in X$ | （subadditive） | and |
| （e）． | $\mathrm{p}(x \vee y)$ | $=\mathrm{p}(x)+\mathrm{p}(y)-\mathrm{p}(x \wedge y)$ | $\forall x, y \in X$ |  | and |
| （f）． | $x \leq y \leq \mathrm{p}(x)$ | $\leq \mathrm{p}(y)$ | $\forall x, y \in X$ | （monotone） | ． |

On a distributive pseudocomplemented lattice，the generalized probability function has been defined as ${ }^{123}$

| （1）． | $\mathrm{p}(0)$ | $=0$ | （nondegenerate） |
| :--- | :--- | :--- | :--- | and

On an orthomodular lattice，or a finite modular lattice，the quantum probability function is defined as ${ }^{124}$
$\begin{array}{lllll}\text {（1）．} & \mathrm{p}(0) & =0 & & \text {（nondegenerate）}\end{array}$ and $\left.\begin{array}{l}\text { and } \\ \text {（2）．} \\ \text {（3）．} x \perp y \Longrightarrow \mathrm{p}(x \vee y) \\ \text {（normalized）}\end{array}\right)$ and

However，for lattices that are not distributive，modular，or orthomodular，none of these definitions work out so well．Take for example the $O_{6}$ lattice with the＂very reasonable＂

```
120 圈 [60], 圈 [65], 屄 [59], 誊 [66], 圈 [58], 圈 [28], pages 258-260
\(121 \otimes\) [102], page 15
122 [138], pages 21-22, \(\boxminus\) [102], page 2, 〈§1. Axioms I-V〉
\({ }^{123}\) 囯 [129], page 118, - [128]
124 圈 [74], page 126, 〈DEFINITIONS \(\rangle\), 圈 [129], page 118
```

probability function given in Example 5.31 (page 66). This probability space $\left(\mathrm{O}_{6}, \mathrm{p}\right)$ fails to be any of the 4 probability functions defined in this Remark. It fails to be a measuretheoretic or traditional probability function because

$$
a \wedge b=0 \quad \text { but } \quad \mathrm{p}(a \vee b)=\mathrm{p}(1)=1 \neq \frac{1}{3}+\frac{1}{2}=\mathrm{p}(a)+\mathrm{p}(b) .
$$

It fails to be a generalized probability function because
$\mathrm{p}(a \vee b)=\mathrm{p}(1)=1 \neq \frac{1}{3}+\frac{1}{2}-0=\mathrm{p}(a)+\mathrm{p}(b)-\mathrm{p}(0)=\mathrm{p}(a)+\mathrm{p}(b)-\mathrm{p}(a \wedge b)$.
It fails to be an quantum probability function because

$$
a \perp b=0 \quad \text { but } \quad \mathrm{p}(a \vee b)=\mathrm{p}(1)=1 \neq \frac{1}{3}+\frac{1}{2}=\mathrm{p}(a)+\mathrm{p}(b) .
$$

In each of these cases, the function p fails to be additive. The solution of Definition 5.24 (page 62) is simply to "switch off" additivity when the lattice is not distributive. This method is a little "crude", but at least it allows us to define probability on a very wide class of lattices, while retaining compatibility with the Boolean case (Proposition 5.26 page 64, Proposition 5.27 page 64 , Proposition 5.28 page 65 ).

Proposition 5.26 ${ }^{125}$ Let (L, p) be a PROBABILITY SPACE (Definition 5.24 page 62).
$0 \leq \mathrm{p}(x) \leq 1 \quad \forall x \in X$

- Proof:

$$
\begin{aligned}
0 & =p(0) & & \text { by previous result } \\
& \leq p(x) & & \text { because } 0 \leq x \text { and monotone property (Definition } 5.24 \text { page 62) } \\
p(x) & \leq p(1) & & \text { because } x \leq 1 \text { and monotone property (Definition } 5.24 \text { page 62) } \\
& =1 & & \text { by property of } p \text { (Definition } 5.24 \text { page 62) }
\end{aligned}
$$

Proposition 5.27 ${ }^{126}$ Let (L, p) be a PROBABILITY SPACE (Definition 5.24 page 62).
$\left\{\begin{array}{l}L \text { is } \\ \text { ORTHOCOMPLEMENTED }\end{array}\right\} \quad \Longrightarrow \quad\{\mathrm{p}(x)=1-\mathrm{p}(\neg x) \quad \forall x \in X\}$
*Proof:

$$
\begin{aligned}
1-\mathrm{p}(\neg x) & =\mathrm{p}(1)-\mathrm{p}(\neg x) & & \text { by Definition } 5.24 \text { page } 62 \\
& =\mathrm{p}(x \vee \neg x)-\mathrm{p}(\neg x) & & \text { by excluded middle property of ortho negation (Definition } 2.14 \text { page 29) } \\
& =\mathrm{p}(x)+\mathrm{p}(\neg x)-\mathrm{p}(\neg x) & & \text { because }(x)(\neg x)=0 \text { and additive property (Definition } 5.24 \text { page 62) } \\
& =\mathrm{p}(x) & &
\end{aligned}
$$

$125 \nabla$ [138], page 21, $\langle(2-11)\rangle$
$126 \boxtimes$ [138], page 21, $\langle(2-12)\rangle$

Proposition $5.28{ }^{127}$ Let ( $L, p$ p) be a PROBABILITY SPACE (Definition 5.24 page 62).
$\left\{\begin{array}{llll}L \text { is } \\ \text { Boolean }\end{array}\right\} \Longrightarrow\left\{\begin{array}{llll}1 . & \mathrm{p}(x \vee y)=\mathrm{p}(x)+\mathrm{p}(y)-\mathrm{p}(x \wedge y) & \forall x, y \in X & \text { and } \\ 2 . & \mathrm{p}(x \vee y) \leq \mathrm{p}(x)+\mathrm{p}(y) & \forall x, y \in X & \text { (Boole's inequality) }\end{array}\right.$
*Proof:
(1) lemma: Proof that $p((\neg x) \wedge y)=p(y)-p(x \wedge y)$ :

$$
\begin{aligned}
\mathrm{p}(y)-\mathrm{p}(x y) & =\mathrm{p}(1 \wedge y)-\mathrm{p}(x y) & & \text { by definition of } 1 \text { and } \wedge \text { (Definition } 1.28 \text { page } 9) \\
& =\mathrm{p}[(x \vee \neg x) y]-\mathrm{p}(x y) & & \text { by excluded middle property of Boolean lattices } \\
& =\mathrm{p}(x y \vee \neg x y)-\mathrm{p}(x y) & & \text { by distributive property of Boolean lattices } \\
& =\mathrm{p}(x y)+\mathrm{p}(\neg x y)-\mathrm{p}(x y) & & \text { because }(x y)(\neg x y)=0 \text { and by additive property } \\
& =\mathrm{p}(\neg x y) & &
\end{aligned}
$$

(2) Proof that $\mathrm{p}(x \vee y)=\mathrm{p}(x)+\mathrm{p}(y)-\mathrm{p}(x \wedge y)$ :

$$
\begin{aligned}
\mathrm{p}(x \vee y) & =\mathrm{p}(x \vee \neg x y) & & \text { by property of Boolean lattices } \\
& =\mathrm{p}(x)+\mathrm{p}(\neg x y) & & \text { because }(x)(\neg x y)=0 \text { and by additive property } \\
& =\mathrm{p}(x)+\mathrm{p}(y)-\mathrm{p}(x \wedge y) & & \text { by item } 1 \text { (page 65) }
\end{aligned}
$$

Example 5.29 The function $\neg$ on the lattice $L$ as illustrated to the right is a Kleene negation (Definition 2.14 page 29). Together with the probability function $p$, also illustrated to the right, the pair $(L, p)$ is a probability space (Definition 5.24 page 62).

$$
\begin{aligned}
& 1=\neg 00 \mathrm{p}(1)=1 \\
& a=\neg a O p(a)=\frac{1}{2} \\
& 0=\neg 1 O p(0)=0
\end{aligned}
$$

Example 5.30 The lattice with negation $L$ (Definition 2.16 page 30) illustrated to the right is a Boolean lattice. Together with the probability function $p$, also illustrated to the right, the pair $(L, p)$ is a probability space (Definition 5.24 page 62).


[^28]Example 5.31 The lattice with negation L (Definition 2.16 page 30 ) illustrated to the right is an orthocomplemented $O_{6}$ lattice (Definition 1.73 page 20). Together with the probability function $p$, also illustrated to the right, the pair $(L, p)$ is a probability space (Defini-
 tion 5.24 page 62).

### 5.6 Applications

This section discusses some possible applications of primorial lattices.

### 5.6.1 Logic analysis

Let $L_{2}^{N}$ be a $2^{N}$-valued Boolean logic (Definition 2.27 page 33 ). Let $\mathbb{P}$ be the primorial lattice generated by $L_{2}^{N}$ (Definition 5.17 page 58). The sequence of lattices $\left(L_{2}^{N}, L_{2}^{N-1}, \ldots, L_{2}^{2}, L_{2}\right)$ in $\mathbb{P}$ are Boolean logics with decreasing "resolution" (higher values of $n$ in $L_{2}^{n}$ correspond to greater resolution). Thus, we can reduce a very complex logic in $L_{2}^{N}$ to a simpler lower resolution logic.

Moreover, the sequence of ortho logics (Definition 2.27 page 33) in $\mathbb{P}$

$$
\left(L_{2}^{N} \otimes L_{2}^{N-1}, L_{2}^{N-1} \otimes L_{2}^{N-2}, \ldots, L_{2}^{3} \otimes L_{2}^{2}, L_{2}\right)
$$

represents the Boolean logic $L_{2}^{N}$ at $N-1$ progressively lower "frequencies". Alternatively, we could say that the Boolean logic at resolution $N$ is "decomposed" into (or analyzed by) $N-1$ ortho logics. Moreover, a proposition $p$ in a higher resolution space can be projected into a lower resolution space (including the two-value classic logic space) by a projection operator (Section 5.4 page 58 ).

### 5.6.2 Fuzzy logic analysis

Fuzzy logics (Definition 2.27 page 33) can be constructed on Boolean and orthocomplemented lattices ${ }^{128}$ such that together with the subset ordering relation $\subseteq$, form of a primorial lattice $\mathbb{P}$ (Definition 5.1 page 50). A Boolean fuzzy logic $L_{2}^{N}$ can then be rendered at $N-1$ different "resolutions" using the Boolean lattices of $\mathbb{P}$ and analyzed at $N-1$ "frequencies" using the orthocomplemented lattices of $\mathbb{P}$, as described in Section 5.6.1 (page 66).

[^29]


Figure 16: primorial lattice for fuzzy subset logic (Example 5.32 page 68)

Example 5．32 Figure 16 （page 67）illustrates a fuzzy subset logic ${ }^{129}$ on a primorial lattice． The lattice $L_{2}^{3}$ contains both monotonic and non－monotonic membership functions．These are separated into lower resolution spaces $L_{2}^{2}$ containing the non－monotonic membership functions（neglecting 1 and 0 ），$L_{2}^{3} \otimes L_{2}^{2}$ containing the monotonic membership functions， and $L_{2}$ containing crisp set logic．A projection operator（Section 5.4 page 58 ）can be used to project a membership function onto any of these spaces as perhaps called for by a given application．

## 5．6．3 Probability analysis

A logic is a lattice with negation（Definition 2.16 page 30）and with an implication function defined on it．A probability is a lattice with negation and with a probability function（Definition 5.24 page 62）defined on it．
Let $L_{2}^{N}$ be the $2^{N}$－element Boolean lattice generated by an $N$－event Boolean probability space（Definition 5.24 page 62）．Let $\mathbb{P}$ be the primorial lattice（Definition 5.1 page 50 ）generated by $L_{2}^{N}$ ． Then in $\mathbb{P}$ ，the probability space can be rendered at progressively lower resolutions using the Boolean lattices of $\mathbb{P}$ ，and can be analyzed at assorted＂frequencies＂using the ortho－ complemented lattices of $\mathbb{P}$ ．

Example 5．33 A primorial lattice with a probability function is illustrated in Figure 17 （page 69）．

## 5．6．4 Symbolic sequence analysis

Definitions．Finding some properties of a sequence $\mathbb{x}$ that is constructed over a field $\mathbb{F}$ may be referred to as sequence analysis or discrete－time signal analysis．If we somehow mathematically alter $x$ with an operator $\mathbf{A}$ to produce a new sequence $y \triangleq \mathbf{A} x$ ，then this may be referred to as sequence processing，or more commonly as discrete－time signal pro－ cessing or digital signal processing（DSP）．

Basis theory．Sequence analysis and sequence processing typically make use of basis the－ ory．In basis theory in general（of which Fourier analysis and wavelet analysis are special cases），we represent some point $x$（ $x$ is a sequence）in a Banach space（a complete normed linear space）by a linear combination of a basis sequence $\left(x_{n}\right)$ such that

$$
x \stackrel{\star}{\underline{*}} \sum_{n \in Z} a_{n} x_{n}
$$

[^30]


Figure 17: primorial lattice with probability function (Example 5.33 page 68 )
where $\stackrel{\star}{=}$ represents strong convergence with respect to the norm $\|\cdot\|$ of the Banach space． Each element $a_{n}$ is a member of the field $\mathbb{F}$ of the Banach space and the sequence（ $a_{n}$ ） is often referred to as a＂transform＂（Fourier transform，discrete－time Fourier transform， wavelet transform，etc．）

In order to be able to successfully compute any transform（such as a Fourier transform or wavelet transform）in a Banach space or even a finite linear space，the sequence $\mathbb{x}$ needs to be somehow related to the field $\mathbb{F}$ over which the Banach space is constructed．

The problem．Let $\tilde{\mathbf{F}}$ be the discrete－time Fourier transform operator and $\mathbf{W}$ be a discrete－
 bolic sequence analysis and symbolic signal processing in general because of the following reasons：

1．The symbols in $x$ have no field structure；so we can＇t even add them．
2．The symbols in $x$ have no order structure；so if $A, B$ ，and $C$ are symbols，we can＇t say，for example，$A<B$ or $B<C$ ，etc．
3．The symbols in $x$ have no topology except for some arguably trivial topologies；${ }^{130}$ so we can＇t say，for example，that $A$ is＂closer＂to $B$ than it is to $C$ ，etc．
In fact，symbol sequence analysis does not just cause problems for Fourier or wavelet anal－ ysis only－it causes problems for basis theory in general because a basis is constructed in a Banach space，and symbolic sequences are in general not constructed in Banach spaces．

A kind of＂hack＂solution may be to map the symbols to points（ $p_{1}, p_{2}, \ldots, p_{N}$ ）in the com－ plex plane $\mathbb{C}$ ．If these points are chosen such that they are distinct，not on either the real or imaginary axes，and $\left|p_{1}\right|=\left|p_{2}\right|=\ldots=\left|p_{N}\right|$ ，then that would seem to be a good start， because now the mapped symbols have a field structure，and they are arguably unordered （arguably we can＇t say any one of them is greater or less than any other，just as in the orig－ inal symbol sequence）．

But we still have the topology problem．If we map，say， 4 symbols to 4 points in $\mathbb{C}$ as $p_{1}=1$ ， $p_{2}=-1, p_{3}=i$ ，and $p_{4}=-i$ ，then＂$p_{1}$＂is closer（with respect to the metric induced by the norm $|\cdot|$ ）to＂$p_{3}$＂then it is to＂$p_{2}$＂：

$$
\mathrm{d}\left(p_{1}, p_{3}\right)=\left|p_{1}-p_{3}\right|=\left(p_{1}^{2}-p_{3}^{2}\right)^{1 / 2}=\left(1^{2}-i^{2}\right)^{1 / 2}=\sqrt{2} \nless 2=\left(2^{2}-0^{2}\right)^{1 / 2}=\mathrm{d}\left(p_{1}, p_{2}\right)
$$

This unwanted topological property is introduced by the mapping，will affect the trans－ form，but yet is not a property of the original symbolic sequence．

[^31]
"Frequency" properties may be useful in symbolic sequence analysis and symbolic sequence processing. But the point here is that any kind of basis theory technique (including Fourier or wavelet techniques) may result in a kind of imperfect "hack" solution.

Proposed solution. The solution proposed here is to perform symbolic sequence analysis using primorial lattices. Suppose we have a sequence $\mathbb{x}$ over a set of $N$ symbols (each element in the sequence can be any one of $N$ different symbols). Let $\mathbb{P}$ be the primorial lattice generated by $L_{2}^{N}$. The orthogonal $N$ atoms of $L_{2}^{N}$ represent the $N$ symbols. The element $A \vee B$ in $L_{2}^{N}$, where $A$ and $B$ are 2 symbols, represents the event of a particular position in the sequence being $A$ OR $B$ (it is not possible for a particular position to be both $A$ and $B$ ).

Any symbol in $L_{2}^{N}$ can be projected onto any other Boolean or orthocomplemented lattice in $\mathbb{P}$ by use of a lattice projection (Section 5.4 page 58). The result of projecting an entire sequence onto a lattice in $\mathbb{P}$ is another sequence (Definition 5.19 page 58 ). So after projection, a sequence on $L_{2}^{N}$ results in $N-1$ sequences of lower resolution and $N-1$ sequences of assorted frequencies. This is similar in form to the Fast Wavelet Transform, as illustrated in Figure 10 (page 49).

### 5.6.5 Symbolic sequence processing (SSP)

Introduction. The previous section discusses symbolic sequence analysis-meaning we are not trying to change the properties of the sequence, we are only trying to understand its properties. This section discusses symbolic sequence processing (or symbolic signal pro-cessing)-meaning we are trying to change the properties of the sequence.
Digital signal processing (DSP) or discrete-time signal processing operates on a sequence constructed over a field $\mathbb{F}$, where $\mathbb{F}$ is typically either $\mathbb{R}$ or $\mathbb{C}$. Often by use of simple multiplication and addition operations on elements of the sequence, one can change the properties of the sequence. Often when the properties are related to Fourier analysis, the DSP operations are called "filtering".

The problem. Multiplication and addition operations commonly used in DSP require field properties. In symbolic sequence processing, we don't in general have a field.

Proposed solution. Sequence processing of, or "filtering" on, a symbolic sequence $\mathbb{x}$ can be performed by judicious selection and/or rejection of the various projections onto the logics in the primorial lattice $\mathbb{P}$.


For example，if one wants $\star$ at a lower＂resolution＂s，then simply select the sequence from a projection onto the Boolean logic at resolution lower than $N$ ．If one wants to＂filter out＂ the＂high frequency＂components of $\mathbb{x}$ ，then simply discard the projections onto the higher frequency orthocomplemented lattices before synthesizing a new sequence from the＂low frequency＂component sequences．

Synthesis of two projection sequences y and $\mathbb{z}$ into a new sequence $\mathbb{x}^{\prime}$ can be performed， for example，by pointwise join such that

$$
\begin{aligned}
\mathrm{y} \oplus \mathbb{Z} & \triangleq\left(y_{n}\right)_{n \in \mathbb{Z}} \bigvee\left(z_{n}\right)_{n \in \mathbb{Z}} \\
& \triangleq\left(y_{n} \vee z_{n}\right)_{n \in \mathbb{Z}} \\
& \triangleq\left(x_{n}\right)_{n \in \mathbb{Z}} \\
& \triangleq \mathbb{X}
\end{aligned}
$$

## 5．6．6 Genomic Signal Processing（GSP）

Genomic Signal Processing（GSP）is simply a special case of Symbolic Sequence Processing with $N=4$ ．In GSP，the 4 symbols are commonly referred to as $A, C, T$ ，and $G$ ，each of which corresponds to a nucleobase（adenine，thymine，cytosine，and guanine，respec－ tively）．${ }^{131}$ The sequence itself is called a genome．A typical genome sequence contains a large number of symbols（about 3 billion for humans， 29751 for the SARS virus）．${ }^{132}$

Example 5．34 Traditionally in GSP，the symbols $(A \vee T)$ and $(C \vee G)$ are of special interest． Portions of a genome sequence high in $(A \vee T)$ content separate at lower temperatures than do those with high $(C \vee G)$ content．${ }^{133}$ Therefore，one could construct a primorial lattice induced by $L_{2}^{4}$ that allows for convenient analysis of $A \vee T$ and／or $C \vee G$ in some lower resolution space．An example is illustrated in Figure 18 （page 73）．

Example 5．35 In some cases，genomic sequences with more than 4 symbols $(N>4)$ have been studied．${ }^{134}$ Figure 19 （page 74）illustrates a primorial lattice with an extra symbol $X$

[^32]


Figure 18: primorial lattice for genomic signal processing (GSP) with $A \vee T$ and $C \vee G$ analysis features (Example 5.34 page 72)


Figure 19: primorial lattice for genomic signal processing (GSP) with extra symbol $X$ (Example 5.35 page 72)
in the higher resolution $L_{2}^{5}$ Boolean lattice, but with only the symbols $A, C, G$, and $T$ in the lower resolution $L_{2}^{4}$ Boolean lattice. The symbol $X$ can be projected onto any of the lower resolution spaces using a projection operator (Section 5.4 page 58 ).

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Telecommunications Engineering Department, National Chiao-Tung University, Hsinchu, Taiwan dgreenhoe@gmail.com

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[^0]:    ${ }^{1} \otimes$［113］，page 470，$\theta$［12］，page 1，圈［105］，page 156，$\langle\mathrm{I}, \mathrm{II},(1)\rangle$ ，圊［38］，page 373，$\langle\mathrm{I}-\mathrm{III}\rangle$ ．An order relation is also called a partial order relation．An ordered set is also called a partially ordered set or poset．

    2 娄［139］，page 2
    

[^1]:    ${ }^{3} \boxminus$［113］，page 470，圏［133］，page 410
    ${ }^{4}$ 娄［14］，page 445

[^2]:    $5 \mapsto$ [72], page 2

[^3]:    ${ }^{16} \square$ [155], page 100

[^4]:    ${ }^{17}$ [56] , page 4

[^5]:    18 圈［148］，page 1440，〈（illustration）$\rangle$ ，誊［147］，page 498，〈partitions of a set〉
    ${ }^{19} \boxminus$［113］，page 473，$\bullet$［17］，page 16，圈［133］，圈［14］，page 442，$\bullet$［116］，page 1
    ${ }^{20} \otimes$［113］，pages 473－475，〈Lemma 1，Theorem 4〉，$\forall$［23］，pages 4－7，$\otimes$［16］，pages 795－796，圈［133］，page 409，$\langle(\alpha)\rangle$ ，圈［14］，page 442，圊［38］，pages 371－372，$\langle(1)-(4)\rangle$
    ${ }^{21} \boxminus$［88］
    

[^6]:    ${ }^{22}$［68］，page 39，圊［50］，pages 97－99，$\boxminus$［78］，$\langle § 4.2\rangle$
    ${ }^{23}$［17］，pages 19－20
    ${ }^{24} \forall$［36］，page 85，$\forall$［72］，page 38，圈［14］，page 444，圈［105］，page 157，$\theta$［125］，page 13，〈terminology〉
    ${ }^{25}$［17］，page 19，$\theta$［23］，page 11，圈［38］，page 374

[^7]:    
    ${ }^{27}$ 圈 [77], $\langle § 1.2 .2\rangle, \otimes[78],\langle § 4.5\rangle$
    ${ }^{28} \otimes$ [18], page 5

[^8]:    ${ }^{29}$ 圏［108］，page 178，圏［16］，page 800，〈see footnote $\left.\ddagger\right\rangle$

[^9]:    ${ }^{30} \Delta$［157］，page 11，$\Delta$［116］，page 1，$\langle$ Definition（1．1）$\rangle, \nabla$［117］，page 248
    ${ }^{31} \mapsto$［18］，page 82，$\bullet$［116］，page 3，$\langle$ Definition（1．7）$\rangle$
    ${ }^{32} \otimes$［136］，page 39，$\Delta$［133］，page 413，〈（2）$\rangle, \forall$［78］，〈Theorem 5．1〉
    ${ }^{33} \boxminus$［12］，pages 12－13，圈［38］，pages 391－392，〈（44）and（45）〉
    ${ }^{34} \forall$［23］，page 11，$\theta$［71］，page 70，圈［38］，$\langle$ cf Stern 1999 page 10 $\rangle, \forall$［78］，$\langle$ Theorem 5．1 $\rangle$
    

[^10]:    ${ }^{36}$［116］，page 15，〈Definition 4．1〉，圊［62］，page 67，$\theta$［130］，page 32，〈Definition 5．1〉，謄［37］， page 314 ，〈disjunctive distributive and conjunctive distributive functions〉
    $37 \boxminus$［23］，page 10，$\triangle$［17］，page 133，圊［133］，page 414，〈arithmetic axiom $\rangle$ ，圈［14］，page 453， $\theta$［9］，page 48，〈Definition II．5．1〉
    ${ }^{38}$［49］，page 237，$\otimes$［23］，page 10，圈［133］，page 416，〈（7），（8），Theorem 3〉，圈［134］，〈cf Gratzer 2003 page 159 $\rangle$［153］，page 286，〈cf Birkhoff（1948）p．133〉，圈［105］，〈cf Birkhoff（1948）p．133〉， $\theta$［78］，〈Theorem 6．1〉
    ${ }^{39} \square$［12］，pages 12－13，［105］，page $157,\left\langle p_{1} \equiv x, p_{2} \equiv y, p_{3} \equiv z, g \equiv 1,0 \equiv 0\right\rangle$

[^11]:    ${ }^{48}$ 圏［46］，page 123，$\Delta$［151］，page 51，$\bullet$［72］，page 378，〈Corollary 3．8〉
    ${ }^{49}$［113］，page 488，$\boxminus$［151］，page 30，〈Theorem 10〉
    $50 \otimes$［146］，page 103，$\otimes$［3］，page 79，$\otimes$［151］，page 40，圈［46］，page 123，$\otimes$［73］，page 698
    $51 \boxminus$［113］，page 488，$\because$［97］

[^12]:    ${ }^{52}$ 圈［89］，pages 292－293，〈＂1st set＂〉，圈［90］，page 280，〈＂4th set＂〉，$\bullet$［113］，page 488，$\bullet$［68］， page 10，$\because$［124］，pages 20－21，$\bullet$［153］，$\bullet$［167］，pages 35－37
    ${ }^{53}$ 圈［90］，page 280，〈＂4th set＂〉

[^13]:     page 830, 〈L71-L73〉
    ${ }^{55} \square$ [98], page 22, [88], page 50, [12], page 33, $\theta$ [157], page 12. The $O_{6}$ lattice is also called the Benzene ring or the hexagon.

[^14]:    

[^15]:    ${ }^{61}$［98］，page 32，〈20．$\rangle$ ，向［94］，page 57
    
    $\bullet$［43］，page 105，〈（8．1．1）$\rightarrow$［41］，page 143，$\langle\S 10.3\rangle, \bullet$［42］，page 193，$\langle\S 10.3\rangle$
    ${ }^{63} \otimes$［101］，page 119，〈\＄5．7〉
    ${ }^{64} \otimes$［45］，page 28，$\otimes$［31］，page 21，$\otimes$［82］，page 109，$\otimes$［64］，$\otimes$［63］，page 30

[^16]:    ${ }^{65}-$［55］，〈Book I Proposition 20〉
    66 ［5］，page 35
    ${ }^{67} \otimes$［43］，page 105，〈（8．1．2）$\rangle, \boxminus$［18］，pages 230－231
    ${ }^{68} \forall$［43］，page 105，$\Delta$［18］，page 231，$\langle$ §X． 2$\rangle$
    ${ }^{69}$［18］，page 230
    ${ }^{70} \boldsymbol{\nabla}$［18］，page 232，〈Theorem X．2 $\rangle, \Delta$［43］，pages 105－106，$\Delta$［22］，page 58，〈Exercise 4．25〉

[^17]:    ${ }^{82}$ 圈［159］，page 136，〈Definition 2．1 $\rangle$ ，圈［162］，page 11，〈Definition 16 ，圈［77］，〈§3．1 $\rangle$

[^18]:    ${ }^{83} \boxminus$ [157], page 12, $\ominus$ [112], page 3

[^19]:    ${ }^{85} \boxminus$［98］，page 20，$\bullet$［88］，page 79，$\langle$ A．Commutativity $\rangle, ~[115]$ ，page 227，〈Hilfssatz（Lemma） XII．1．2〉，園［152］，page 301，〈Def．5．2，cf Foulis 1962〉，圊［15］，page 833，〈＂$\left.a=(a \cap x) \cup\left(a \cap x^{\prime}\right) "\right\rangle$
    ${ }^{86} \boxminus$［87］，page 67，$\boxminus$［78］，〈Proposition 13．2〉
    ${ }^{87} \boxminus$［87］，page 68，圏［127］，page 158，$\Delta$［78］，$\langle$ Proposition 13．3 $\rangle$
    ${ }^{88} \otimes$［98］，page 20，㘣［114］
    89 ［62］，page 66，䁪［152］，〈cf Foulis 1962〉

[^20]:    ${ }^{92}$ 圈［104］，page 23，〈Proposition 1．〉
    ${ }^{93}$ 圈［104］，page 22，〈Example 1〉
    94 圈［104］，page 24，〈Example 4〉
    95 ［163］，pages 79－80，〈Definition 3．39〉，$\Delta$［27］，pages 41－42，$\Delta$［168］，page 18，〈Definitions
    〈Notation 9.4$\rangle, \forall$［25］，page 74，$\bullet$［69］，page 639，$\bullet$［34］，page $81, \otimes$［33］，page $2, \otimes$［75］，page 2

[^21]:    96 [75], page 3
    97 [75], page 3
    ${ }^{98} \square$ [75], page 4
    99 [86], page 2

[^22]:    101 圏 [109], $\Delta$ [119], page 240
     page 23 , <entry 359$\rangle$ ), which in turn means "the resolution or separation into component parts" ( $\square$ [21], http://dictionary.reference.com/browse/dissolution)

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[^23]:    105圊［95］，page 8
    ${ }^{106}$ The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan， and later as the Gaussian Pyramid by Burt and Adelson in the 1980s in the West．［118］，page 70， ［92］，［24］，［4］，［111］，［6］，［160］，〈historical survey $\rangle$
    ${ }^{107}$［85］，page 44，［119］，page 221，〈Definition 7．1〉，［118］，page 70，［122］，page 21，〈Definition 2．2．1〉，［27］，page 284，〈Definition 13．1．1〉，［8］，pages 451－452，〈Definition 7．7．6〉， －［163］，pages 300－301，〈Definition 10．16〉，［35］，pages 129－140，〈Riesz basis：page 139〉

[^24]:    108 ［85］，page 45，〈Theorem 1．6〉，［75］，pages 32－33，〈Proposition 2．1〉
    $109 \boxminus$［168］，pages 19－28，〈Proposition 2．14〉，$\Delta$［85］，page 45，〈Theorem 1．6〉，$\square$［141］， pages 313－314，$\langle$ Lemma 6．4．28〉，［75］，pages 33－35，〈Proposition 2．2〉
    $110 \boxminus$［168］，pages 28－31，〈Proposition 2．15〉，$\Delta$［75］，pages 35－37，〈Proposition 2．3〉
    $111 \theta$［123］，page 83，〈Theorem 3．2．12〉，$\theta$［106］，page 67，〈Theorem 2．14〉，$\theta$［76］，〈Theorem 7．1）
    ${ }^{112}$［75］，page 38，〈 \＄2．3．2 Order structure $\rangle$

[^25]:    ${ }^{113} \triangleq$［158］，page ix，$\theta$［7］，page 191
    ${ }^{114} \otimes$［168］，page 17，$\langle$ Definition 2．1 $\rangle, \theta$［75］，page 50，$\langle$ Definition 2．4〉
    $115 \triangleq$［75］，page 53，〈Theorem 2．8〉

[^26]:    ${ }^{116} \boldsymbol{\theta}$［119］，page 257，〈Figure 7．12〉，［75］，pages 371－372，〈Figure L．1〉
    

[^27]:    117 ［75］，page 72，〈Section 2．4．3 Order structure〉
    118 ［75］，page 52，〈Proposition 2．6〉

[^28]:    ${ }^{127}$ [138], page 21, $\langle(2-13)\rangle$, [57], pages 22-23, $\langle(7.4),(7.6)\rangle$

[^29]:    128圊 [77], $\langle \$ 2.2\rangle$

[^30]:    129童［77］，〈§3．2〉

[^31]:    ${ }^{130}$ These topologies include the indiscrete topology $\{\varnothing, X\}$ where $X \triangleq\{A, B, C\}$ ，discrete topology $2^{X}$（references：$\boxminus$［126］，page 77，$\boxminus$［107］，page 107，〈Example 3．J〉，$\boxminus$［156］， pages $42-43,\langle\mathrm{II} .4\rangle, \forall$［44］，page 18 ），and the topology induced by the discrete metric $\mathrm{d}(x, y) \triangleq$ $\{1$ for $x \neq y, 0$ for $x=y\}$（references：$\otimes$［67］，page 13，$\Delta$［31］，page 24，$\boxminus$［101］，page 19，〈Example 2．1〉）．

[^32]:    131 圈［121］，〈Mendel（1853）：gene coding uses discrete symbols〉，誊［165］，page 737，〈Watson and Crick（1953）：gene coding symbols are adenine，thymine，cytosine，and guanine $\rangle$ ，圈［164］，page 965， $\Delta$［142］，page 52

    132邑．［1］，〈http：／／www．ncbi．nlm．nih．gov／genome／guide／human／〉，〈Homo sapiens， NC＿000001－NC＿000022（22 chromosome pairs），NC＿000023（X chromosome），NC＿000024（Y chromosome），NC＿012920（mitochondria）〉，ㅁ．［1］，〈http：／／www．ncbi．nlm．nih．gov／nuccore／ $30271926\rangle$ ，〈SARS coronavirus，NC＿004718．3〉［150］，〈homo sapien chromosome 1〉，圊［149］，〈SARS coronavirus〉

    133 ［32］，page 13，〈 Remark 1．2〉
    134 誊［30］，圊［53］

[^33]:    

[^34]:    Monday $13^{\text {TH }}$ October, 2014 MRA- $\mathrm{Wa}^{\mathrm{v}^{\mathrm{e}}} \mathrm{e}_{\mathrm{t}}$ subspace architecture for logic, probability, and symbolic sequence processing $\quad$ UR

