On Directed Lattice Paths With Additional Vertical Steps

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Abstract

The paper is devoted to the study of lattice paths that consist of vertical steps $(0, -1)$ and non-vertical steps $(1, k)$ for some $k \in \mathbb{Z}$. Two special families of primary and free lattice paths with vertical steps are considered. It is shown that for any family of primary paths there are equinumerous families of proper weighted lattice paths that consist of only non-vertical steps. The relation between primary and free paths is established and some combinatorial and statistical properties are obtained. It is shown that the expected number of vertical steps in a primary path running from $(0, 0)$ to $(n, -1)$ is equal to the number of free paths running from $(0, 0)$ to $(n, 0)$. Enumerative results with generating functions are given. Finally, a few examples of families of paths with vertical steps are presented and related to Łukasiewicz, Motzkin, Dyck and Delannoy paths.

1 Introduction

A *lattice path* is a sequence of points from $\mathbb{Z} \times \mathbb{Z}$. A pair (i, j) from $\mathbb{Z} \times \mathbb{Z}$ is called a *step* if $(i, j) \neq (0, 0)$. An example of a lattice path is given in Fig. 1. For simplicity of notation, we represent lattice paths as the words over fixed set of steps $\mathcal{S}$. We shall say that a path is an $\mathcal{S}$-path if its steps belong to $\mathcal{S}$. Lattice paths appear in many contexts. They are used in physics [21], computer science [13], and probability theory [20]. There are a huge number of papers on enumeration of lattice paths for specified sets of steps. We refer the reader to the survey of Humphreys [12] and to the references therein.

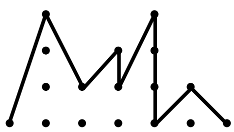

Figure 1: A lattice path running from $(0, 0)$ to $(6, 0)$.

In this paper we consider only the following types of steps. Namely, let V denote the *vertical step* $(0, -1)$ and let S_k denote the *non-vertical step* $(1, k)$, for $k \in \mathbb{Z}$. Additionally, we separate non-vertical steps into two groups. If $k \geq 0$ then S_k is called *up step* and denoted by U_k. If $k < 0$ then S_k is called *down step* and denoted by D_{-k}. For example, the path in Fig. 1 can be represented by its starting point $(0, 0)$ and the sequence $U_3 D_2 U_1 V U_2 V^3 U_1 D_1$.

There are several well-known examples of paths that consist of non-vertical steps. For instance, Dyck paths are composed of steps U_1 and D_1, Motzkin and N-Łukasiewicz paths are those for which the sets of steps are $\{U_1, U_0, D_1\}$ and $\{U_N, U_{N-1}, \ldots, U_0, D_1\}$, respectively. All of these examples are essentially one-dimensional objects. It is well known, see e.g. Deutsch [4],

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that the number of Dyck paths running from $(0,0)$ to $(2n,n)$ which never go below the x-axis is the nth Catalan number given by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

The number of Motzkin paths [6] running from $(0,0)$ to $(n,0)$ which never go below the x-axis is equal to

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k.$$

A unified enumerative and asymptotic theory of paths consisting of non-vertical steps is developed by Banderier and Flajolet [1] and concerns with the kernel method.

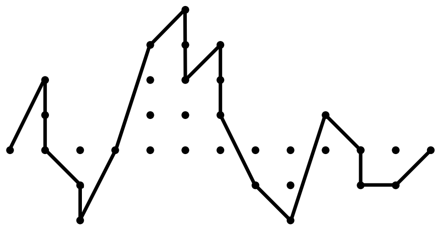

Figure 2: A free $\mathcal{S}$-path running from $(0,0)$ to $(12,0)$.

Another class of lattice paths are those which consist of vertical and non-vertical steps. The classical example is the family of Delannoy paths [2] which originally consist of steps $(1,0)$, $(1,1)$, and $(0,1)$. In our notation, a *Delannoy path* is a lattice path that consists of steps U_0, D_1, V and runs from $(0,0)$ to some (n,m) in the fourth quarter, i.e., $n \geq 0$ and $m \leq 0$. The number of Delannoy paths running from $(0,0)$ to (n,m) is equal to

$$\sum_{j=0}^{n} \binom{m}{j}\binom{n+m-j}{m}.$$

Several families of paths with steps from $\{V, U_1, U_0, D_1\}$ are considered by the author in [7].

In this paper we consider families of paths with the set of steps being an arbitrary subset of $\{V, S_N, S_{N-1}, \ldots\}$, for fixed $N \geq 0$. Throughout the paper we use the symbol Ω_N to denote the following set of steps

$$\Omega_N = \{S_k : k \leq N\}, \qquad \text{for } N \geq 0.$$

For each $m, n \in \mathbb{Z}$ and $\mathcal{S} \subset \{V\} \cup \Omega_N$, we define two families of paths. Namely, let $\mathcal{F}_{\mathcal{S}}(m,n)$ denote the set of all $\mathcal{S}$-path running from $(0,0)$ to $(n,-m)$. Let $\mathcal{P}_{\mathcal{S}}(m,n)$ denote the set of these paths from $\mathcal{F}_{\mathcal{S}}(m,n)$ for which all points except possibly the last one lie on or above the x-axis. We call a path from $\mathcal{F}_{\mathcal{S}}(m,n)$ a *free path* and a path from $\mathcal{P}_{\mathcal{S}}(m,n)$ an *m-primary path.* For instance, a free path is given in Fig. 2 and a primary path is given in Fig. 3.

In Section 3 we show that for any $\mathcal{V} \subseteq \Omega_N \cup \{V\}$ which contains U_N and V there is the corresponding set of steps

$$\mathcal{L} = \big(\mathcal{V} \setminus \{V\}\big) \cup \{D_1, U_0, U_1, \ldots, U_N\}$$

such that for any $m \geq 0$ and $n \geq 1$, we have

$$|\mathcal{P}_{\mathcal{V}}(m,n)| = \sum_{\pi \in \mathcal{P}_{\mathcal{L}}(m,n)} w(\pi), \tag{1}$$

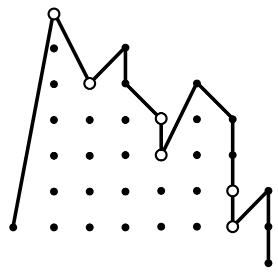

Figure 3: A 1-primary $\mathcal{S}$-path running from $(0,0)$ to $(7,-1)$. Lattice points which determine a decomposition of the path are drawn by open circles.

where w is a weight function over paths from $\mathcal{P}_{\mathcal{L}}(m,n)$. This means that additional vertical step V in primary $\mathcal{V}$-paths can be encoded by the proper weights of non-vertical steps in $\mathcal{L}$-paths. To show the above equality we define w-weighted primary $\mathcal{L}$-paths which are primary $\mathcal{L}$-paths whose steps have assigned nonnegative integers depending on the weight function w. Then we define a bijection between primary $\mathcal{V}$-paths and w-weighted primary $\mathcal{L}$-paths.

In Section 4 we establish a relation between primary and free paths. Namely, we show that for any $\mathcal{V} \subset \Omega_N \cup \{V\}$ which consists V and U_N, we have

$$\begin{aligned} |\mathcal{P}_{\mathcal{V}}(1,n)| &= \frac{1}{n}\big(|\mathcal{F}_{\mathcal{V}}(1,n)| - |\mathcal{F}_{\mathcal{V}}(0,n)|\big) \\ &= \frac{1}{n}\sum_{j=0}^{Nn+1}\binom{n+j-1}{j}|\mathcal{F}_{\mathcal{N}}(1-j,n)|, \end{aligned}$$

where $\mathcal{N} = \mathcal{V} \setminus \{V\}$. We show that

$$\begin{aligned} \#\mathit{Steps}(V \in \mathcal{P}_{\mathcal{V}}(1,n)) &= |\mathcal{F}_{\mathcal{V}}(0,n)|, \\ \#\mathit{Steps}(S_k \in \mathcal{P}_{\mathcal{V}}(1,n)) &= |\mathcal{F}_{\mathcal{V}}(1+k,n-1)|, \\ \#\mathit{Steps}(\mathcal{P}_{\mathcal{V}}(1,n)) &= |\mathcal{F}_{\mathcal{V}}(1,n)|, \end{aligned}$$

where $\#\mathit{Steps}(S \in \mathcal{P}_{\mathcal{V}}(1,n))$ denote the total number of occurrences of the step S in the set of paths from $\mathcal{P}_{\mathcal{V}}(1,n)$ and $\#\mathit{Steps}(\mathcal{P}_{\mathcal{V}}(1,n))$ denote the total number of all steps in $\mathcal{P}_{\mathcal{V}}(1,n)$. These relations shed some light on the statistical properties of the $\mathcal{V}$-paths. Any $\mathcal{N}$-path running from $(0,0)$ to (n,m) contains exactly n steps. The number of steps in a $\mathcal{V}$-path is equal to or greater than n. We show that the expected number of steps in a 1-primary $\mathcal{V}$-path running from $(0,0)$ to $(n,-1)$ is equal to

$$n\left(1 + \frac{|\mathcal{F}_{\mathcal{V}}(0,n)|}{|\mathcal{F}_{\mathcal{V}}(1,n)| - |\mathcal{F}_{\mathcal{V}}(0,n)|}\right).$$

In Section 5 we derive some enumerative results for such paths. We show that, for any $n \geq 1$ and $m \in \mathbb{Z}$, we have

$$\begin{aligned} |\mathcal{P}_{\mathcal{V}}(1,n)| &= \frac{1}{n}[y^{Nn+1}]\frac{1}{(1-y)^n}\Big(\sum_{S_k\in\mathcal{V}} y^{N-k}\Big)^n, \\ |\mathcal{F}_{\mathcal{V}}(m,n)| &= [y^{Nn+m}]\frac{1}{(1-y)^{n+1}}\Big(\sum_{S_k\in\mathcal{N}} y^{N-k}\Big)^n. \end{aligned}$$

In Section 5.3 we show that the array $(|\mathcal{F}_{\mathcal{V}}(i-(N+1)j,j)|)_{i,j\geq 0}$ is the proper Riordan array $D_{\mathcal{V}}$, where

$$D_{\mathcal{V}} = \left(\frac{1}{1-y}, \frac{y}{1-y} \sum_{S_k \in \mathcal{V}} y^{N-k} \right).$$

Let $P_m(x)$ be the generating function of the sequence $(|\mathcal{P}_{\mathcal{V}}(m,n)|)_{n\geq 0}$. As a consequence of (1), we show that $P_m(x)$ satisfies the following functional equation

$$P_0(x) = 1 + w(U_0^{0,0}) x P_0(x) + x P_0(x) \sum_{k=1}^{N} \sum_{d=1}^{k} w(U_k^{0,d}) \sum_{M} \prod_{j=1}^{d} (P_{m_j}(x) - 1),$$

$$P_m(x) = 1 + w(D_m) x + x \sum_{k=0}^{N} \sum_{d=1}^{k+1} w(U_k^{m,d}) \sum_{M} \prod_{j=1}^{d} (P_{m_j}(x) - 1),$$

for certain constants $w(D_m)$ and $w(U_k^{m,d})$.

In Section 6 we present five examples of families of lattice paths with vertical steps for which we apply results obtained in the previous sections. Namely, we consider the following five sets of lattice steps:

$$\begin{aligned}
\mathcal{A} &= \{V\} \cup \{S_k : K \leq k \leq N\} = \{V, U_N, U_{N-1}, \ldots, U_0, D_1, \ldots, D_K\}, \\
\mathcal{B} &= \{V\} \cup \{S_k : 1 \leq k \leq N\} = \{V, U_N, U_{N-1}, \ldots, U_0, D_1, \}, \\
\mathcal{C} &= \{V\} \cup \Omega_N = \{V, U_N, U_{N-1}, \ldots, U_0, D_1, D_2, \ldots\}, \\
\mathcal{D} &= \{V, U_N, D_K\}, \\
\mathcal{E} &= \{V, U_1, U_0\}.
\end{aligned}$$

for any fixed $N \geq 0$ and $K \geq 1$. It is worth pointing out that $\mathcal{B}$-paths, $\mathcal{C}$-paths, $\mathcal{D}$-paths, and $\mathcal{E}$-path are generalized Łukasiewicz paths, Raney paths recently considered by the author in [8], generalized Dyck paths, and Delannoy paths, respectively.

2 Preliminaries

Let $N \geq 0$ be a nonnegative integer and let $\mathcal{S}$ be a subset of $\Omega_N \cup \{V\}$ and $m, n \in \mathbb{Z}$. Any m-primary $\mathcal{S}$-path $\mu \in \mathcal{P}_{\mathcal{S}}(m,n)$ which has at least two non-vertical steps can be uniquely decomposed into some number of vertical steps and shorter primary $\mathcal{S}$-paths. Let U_h be the first step of μ. First, suppose that $(m,h) \neq (0,0)$. The path μ passes through the points $(x_1, h), (x_2, h-1), \ldots, (x_{h+t}, -m+1) \in \mathbb{R} \times \mathbb{Z}$, such that they are chosen to be the left-most ones, i.e., $x_i = \min\{x : \mu \text{ passes through } (x, h-i+1)\}$. Note that $x_1 = 1$ and some of x_i may be not integer. Therefore, denote by $\Pi(\mu)$ the set of these points that both coordinates are integers. For instance, points from Π for the path given in Fig. 3 are marked by open circles. Cutting μ at these points we obtain a decomposition of μ into U_h and r subpaths $\alpha^{(1)}, \ldots, \alpha^{(r)}$, where $r = |\Pi(\mu)|$. Each $\alpha^{(i)}$ is either a single vertical step V or an m_i-primary $\mathcal{S}$-path for some $m_i \geq 1$. Suppose that exactly d of $\alpha^{(1)}, \ldots, \alpha^{(r)}$ are not vertical steps and denote them by $\mu^{(1)}, \ldots, \mu^{(d)}$. Hence, μ can be uniquely decomposed as

$$\mu = U_h V^{\lambda_0} \mu^{(1)} V^{\lambda_1} \mu^{(2)} V^{\lambda_2} \cdots \mu^{(d)} V^{\lambda_d},$$

where $\mu^{(i)}$ is an m_i-primary $\mathcal{S}$-path and $\lambda_0, \lambda_1, \ldots, \lambda_d$ are nonnegative integers. It is worth pointing out that $m_1, \ldots, m_{d-1} \geq 1$, $m_d \geq m$, and $\lambda_d = 0$ if $m \geq 2$. The *shape* of the path μ is the triple (m, d, k), where $k = h - \lambda_0 - \lambda_1 - \cdots - \lambda_d$. Observe that any up step in μ is the first step of the uniquely determined primary $\mathcal{S}$-subpath of μ. For this reason, we denote by $U_k^{m,d}$ the up step U_k which is the first step of an m-primary $\mathcal{S}$-path whose shape is (m, d, k).

Finally, if $(m, h) = (0, 0)$ then the path μ is decomposable into U_0 and $\mu^{(0)}$, where $\mu^{(0)}$ is a 0-primary $\mathcal{S}$-path from the set $\mathcal{P}_{\mathcal{S}}(0, n-1)$. To simplify the further consideration, we assume that the shape of μ in this case is $(0, 0, 0)$ and the first step U_0 of such path is denoted by $U_0^{0,0}$.

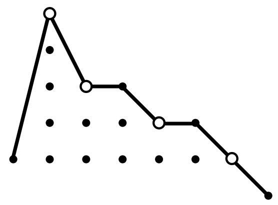

Figure 4: A 1-primary $\mathcal{S}$-path running from $(0, 0)$ to $(7, -1)$ without vertical steps. Lattice points determining a decomposition of the path are drawn by open circles.

For instance, if the set of steps $\mathcal{S}$ does not contain vertical step V, then any m-primary $\mathcal{S}$-path μ running from $(0, 0)$ to $(n, -m)$ can be uniquely decomposed as

$$\mu = U_k\, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)},$$

where $d = |\Pi(\mu)|$. See Fig. 4.

Example 1. Let $\pi = U_6 D_2 U_1 V D_1 V U_2 D_1 V^3 U_1 V^2$ be the path from Fig. 3. This path is decomposable as $U_6^{m,d} \pi^{(1)} \pi^{(2)} V \pi^{(3)} V \pi^{(4)}$, where $m = 1$, $d = 4$, and

$$\pi^{(1)} = D_2, \quad \pi^{(2)} = U_1 V D_1, \quad \pi^{(3)} = U_2 D_1 V^2, \quad \pi^{(4)} = U_1 V^2.$$

The shape of π is $(1, 4, 4)$ and $\lambda_0 = \lambda_1 = \lambda_4 = 0$, $\lambda_2 = \lambda_3 = 1$.

For $k \in \mathbb{Z}$, let $\mathcal{V}_{\geq k}$ denote the set of steps $\mathcal{V} \cap \{S_h : h \geq k\}$. For $m, k, d \geq 0$, let $\mathcal{H}_{\mathcal{V}}(m, d, k)$ denote the set of all pairs (h, λ), where h is an integer such that $S_h \in \mathcal{V}_{\geq k}$ and λ is a composition of $h - k$ into $d + 1$ parts if $m \in \{0, 1\}$, or into d parts if $m \geq 2$ (zero parts are allowed in both cases).

Proposition 1. *For any $m, d, k \geq 0$, we have*

$$|\mathcal{H}_{\mathcal{V}}(m, d, k)| = \sum_{U_h \in \mathcal{V}_{\geq k}} \binom{h - k + d - \epsilon_m}{h - k}, \tag{2}$$

where $\epsilon_m = 0$ if $m \in \{0, 1\}$, and $\epsilon_m = 1$ if $m \geq 2$.

Proof. Recall that N is the maximal integer such that $U_N \in \mathcal{V}$. Let us partition the set $\mathcal{H}_{\mathcal{V}}(m, d, k)$ into pairwise disjoint classes $A_k, A_{k+1}, \ldots, A_N$, where A_h contains these pairs whose first element is h. If $U_h \notin \mathcal{V}_{\geq k}$ then A_h is empty. If $U_h \in \mathcal{V}_{\geq k}$ then the size of A_h is the number of compositions of $h - k$ into $d + 1$ possibly zero parts, for $m \in \{0, 1\}$, or into d possibly zero parts, for $m \geq 2$, which is equal to the value of the binomial coefficient in the final formula. $\square$

3 Bijection between primary paths

Let $N \geq 0$ and $\mathcal{V}$ be a subset of the set of steps $\Omega_N \cup \{V\}$ such that $U_N \in \mathcal{V}$ and $V \in \mathcal{V}$. Now we define the corresponding set of steps $\mathcal{L}$ which does not contain V. Namely, let

$$\mathcal{L} = (\mathcal{V} \setminus \{V\}) \cup \{U_N, U_{N-1}, \ldots, U_0, D_1\}. \tag{3}$$

Let us define weighted paths. First, we define a weight function $w_{\mathcal{V}}$ over steps in primary $\mathcal{L}$-paths as follows. Let D_p be a down step from $\mathcal{L}$ and $U_k^{m,d}$ be an up step U_k which is the first step of an m-primary $\mathcal{L}$-path whose shape is (m, d, k). Then we set

$$w_{\mathcal{V}}(D_p) = \begin{cases} |\mathcal{V}_{\geq -1}| & \text{if } p = 1, \\ 1 & \text{if } p \geq 2, \end{cases} \qquad w_{\mathcal{V}}(U_k^{m,d}) = |\mathcal{H}_{\mathcal{V}}(m, d, k)|, \tag{4}$$

for all $m, d, k \geq 0$. We write $w(S)$ instead of $w_{\mathcal{V}}(S)$ for short if no confusion can arise. A *weighted m-primary $\mathcal{L}$-path* is a pair (μ, v), where μ is an m-primary $\mathcal{L}$-path consisting of n steps $\mu_1, \ldots, \mu_n$ and v is a sequence of n positive integers $v_1, \ldots, v_n$, called *weights*, such that $1 \leq v_i \leq w(\mu_i)$. A *weight* of a μ, denoted by $w_{\mathcal{V}}(\mu)$, is a product of weights of its steps. Let $\mathcal{W}_{\mathcal{L}}^{\mathcal{V}}(m, n)$ denote the set of all $w_{\mathcal{V}}$-weighted m-primary $\mathcal{L}$-paths running from $(0, 0)$ to $(n, -m)$. It is clear that we have

$$|\mathcal{W}_{\mathcal{L}}^{\mathcal{V}}(m, n)| = \sum_{\mu \in \mathcal{P}_{\mathcal{L}}(m,n)} w_{\mathcal{V}}(\mu).$$

Theorem 1. *For any $n \geq 1$ and $m \geq 0$, we have*

$$|\mathcal{P}_{\mathcal{V}}(m, n)| = |\mathcal{W}_{\mathcal{L}}(m, n)|. \tag{5}$$

Proof. In the sequel, we define a map $f_{m,n} : \mathcal{W}_{\mathcal{L}}(m, n) \to \mathcal{P}_{\mathcal{V}}(m, n)$, for any $m \geq 0$ and $n \geq 1$. In Lemma 3, given below, we show that this map is a bijection. □

The map $f_{m,n} : \mathcal{W}_{\mathcal{L}}(m, n) \to \mathcal{P}_{\mathcal{V}}(m, n)$, for $m \geq 0$, $n \geq 1$.

Let (μ, v) be a weighted path from $\mathcal{W}_{\mathcal{L}}(m, n)$ and $v = (v_1, \ldots, v_n)$. If $n = 1$ and $m = 0$, then $\mu = U_0$, and $1 \leq v_1 \leq |\mathcal{H}_{\mathcal{V}}(0, 0, 0)|$. Suppose that (h, λ) is the v_1th pair from $\mathcal{H}_{\mathcal{V}}(0, 0, 0)$. Note that $\lambda = (h)$. Then we set

$$f_{0,1}((U_0, v)) \stackrel{\text{def}}{=\!=} U_h V^h.$$

If $n = 1$ and $m = 1$, then $\mu = D_1$, and $1 \leq v_1 \leq |\mathcal{V}_{\geq -1}|$. Suppose that S_h is the v_1th step from $\mathcal{V}_{\geq -1}$. We set

$$f_{1,1}((D_1, v)) \stackrel{\text{def}}{=\!=} S_h V^{h+1}.$$

If $n = 1$ and $m \geq 2$, then $\mu = D_m$ and $v_1 = 1$. We set

$$f_{m,1}((D_m, v)) \stackrel{\text{def}}{=\!=} D_m.$$

If $n \geq 2$ and $m \geq 0$, then the first step of μ is an up step and the entire path can be decomposed into some number of shorter primary paths. Suppose that the shape of μ is (m, d, k) and

$$\mu = U_k^{m,d} \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)}.$$

The weight of the first step is v_1 which is an integer from $\{1, 2, \ldots, |\mathcal{H}_{\mathcal{V}}(m, d, k)|\}$, by (4). Suppose that (h, λ) is the v_1th pair from $\mathcal{H}_{\mathcal{V}}(m, d, k)$. Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_d)$. We set

$$f_{m,n}((\mu, v)) \stackrel{\text{def}}{=\!=} U_h \, V^{\lambda_0} \, f(\mu^{(1)}) V^{\lambda_1} f(\mu^{(2)}) V^{\lambda_2} \cdots f(\mu^{(d)}) V^{\lambda_d}, \tag{6}$$

where $f(\mu^{(i)}) \equiv f_{m_i,n_i}((\mu^{(i)}, v^{(i)}))$ for some m_i, n_i depending on $\mu^{(i)}$, and $v^{(i)}$ is the proper part of the sequence v, for $i = 1, 2, \ldots, d$.

Lemma 1. *If* $(\mu, v) \in \mathcal{W}_{\mathcal{L}}(m,n)$ *then* $f_{m,n}((\mu,v)) \in \mathcal{P}_{\mathcal{V}}(m,n)$*, for any* $m \geq 0$ *and* $n \geq 1$*.*

Proof. Suppose that $n \geq 2$. Observe that f changes step U_k to $U_h \in \mathcal{V}$, where $h \geq k$, and adds $\lambda_0 + \cdots + \lambda_d$ vertical steps V to the original path. The pair (h, λ) belongs to $\mathcal{H}_{\mathcal{V}}(m,d,k)$ which implies that $h - k = \lambda_0 + \cdots + \lambda_d$. Using the induction on n we show that $f_{m,n}((\mu,v))$ runs from $(0,0)$ to $(n,-m)$. The condition that $\lambda_d = 0$ for $m \geq 2$ ensures that the last step of the resulting path is not the vertical one. $\square$

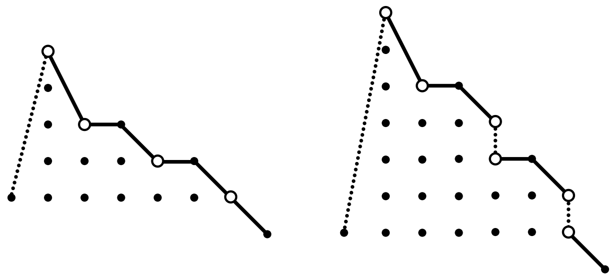

Figure 5: The map f changes U_4 to U_6 and adds two vertical steps between certain subpaths. The map g changes U_6 to U_4 and removes corresponding vertical steps.

Example 2. Let us consider lattice paths which consist of steps from the set $\mathcal{V} = \{V, U_6, U_5, \ldots, U_0, D_1, D_2\}$ and let $\mathcal{L} = \mathcal{V} \setminus \{V\}$. Let $(\mu, v) \in \mathcal{W}_{\mathcal{L}}(1,7)$, where $\mu = U_4 D_2 U_0 D_1 U_0 D_1 D_1$, see Fig. 4 and Fig. 5. Suppose that $v = (v_1, \ldots, v_7)$. The path is decomposable into U_4 and $\mu^{(1)}, \ldots, \mu^{(4)}$, where

$$\mu^{(1)} = D_2, \quad \mu^{(2)} = U_0 D_1, \quad \mu^{(3)} = U_0 D_1, \quad \mu^{(4)} = D_1.$$

By the definition of the weight function w, $v_1 \in \{1, 2, \ldots, |\mathcal{H}_{\mathcal{V}}(1,4,4)|\}$ and $|\mathcal{H}_{\mathcal{V}}(1,4,4)| = 21$, by Proposition 1. Suppose that the v_1th pair from the set $\mathcal{H}_{\mathcal{V}}(1,4,4)$ is $(6, \lambda)$, where $\lambda = (0,0,1,1,0)$ is a composition of 2 into 5 parts. We have

$$f_{1,7}((\mu,v)) = U_6 V^0 f(\mu^{(1)}) V^0 f(\mu^{(2)}) V^1 f(\mu^{(3)}) V^1 f(\mu^{(4)}) V^0.$$

The path $U_6 V^0 \mu^{(1)} V^0 \mu^{(2)} V^1 \mu^{(3)} V^1 \mu^{(4)} V^0$ is given in Fig. 5. The final path $f((\mu, v))$ for certain weight vector v is given in Fig. 3.

The map $g_{m,n} : \mathcal{P}_{\mathcal{V}}(m,n) \to \mathcal{W}_{\mathcal{L}}(m,n)$, for $m \geq 0$, $n \geq 1$.

Let π be a path from $\mathcal{P}_{\mathcal{V}}(m,n)$. If $n = 1$ and $m = 0$, then $\pi = U_h V^h$ for certain $U_h \in \mathcal{V}$. Suppose that the pair (h, λ), where $\lambda = (h)$, is the v_1th pair in $\mathcal{H}_{\mathcal{V}}(0,0,0)$ is (h, λ). Then we set

$$g_{0,1}(U_h V^h) \stackrel{\text{def}}{=\!=} (U_0, (v_1)).$$

If $n = 1$ and $m = 1$, then $\pi = S_h V^{h+1}$ for certain $S_h \in \mathcal{V}_{\geq -1}$. Suppose that S_h is the v_1th step in $\mathcal{V}_{\geq -1}$. We set

$$g_{1,1}(U_h V^{h+1}) \stackrel{\text{def}}{=\!=} (D_1, (v_1)).$$

If $n = 1$ and $m \geq 2$, then $\pi = D_m$. We set

$$g_{m,1}(D_m) \stackrel{\text{def}}{=\!=} (D_m, (1)).$$

If $n \geq 2$ then the first step of π is an up step and the entire path π can be decomposed into some number of shorter primary paths. Suppose that the shape of π is (m, d, k) and it can be decomposed as

$$\pi = U_h V^{\lambda_0} \pi^{(1)} V^{\lambda_1} \pi^{(2)} V^{\lambda_2} \cdots \pi^{(d)} V^{\lambda_d},$$

Let $\lambda = (\lambda_0, \ldots, \lambda_d)$. Suppose that (h, λ) is the v_1th pair in $\mathcal{H}_{\mathcal{V}}(m, d, k)$. We set

$$g_{m,n}(\pi) \stackrel{\text{def}}{=\!=} \Big(U_k\, g\big(\pi^{(1)}\big) g\big(\pi^{(2)}\big) \cdots g\big(\pi^{(d)}\big), v\Big), \tag{7}$$

where $v = (v_1, v_2, \ldots, v_r)$ and $v_2, \ldots, v_r$ depend on $g(\pi^{(1)}), \ldots, g(\pi^{(d)})$.

Lemma 2. *If $\pi \in \mathcal{P}_{\mathcal{V}}(m, n)$ then $g_{m,n}(\pi) \in \mathcal{W}_{\mathcal{L}}(m, n)$, for any $m \geq 0$ and $n \geq 1$.*

Proof. Suppose that $n \geq 2$. As in the proof of Lemma 1, the function g changes step U_h to $U_k \in \mathcal{L}$ and removes exactly $h-k$ vertical steps from the original path. Thus the resulting path ends at the same lattice point as the original one does. Using the induction we also show that the resulting path does not contain vertical steps. □

Example 3. As in the previous example, let us consider lattice paths which consist of steps from the set $\mathcal{V} = \{V, S_6, S_5, \ldots S_{-2}\}$ and let $\mathcal{L} = \mathcal{V} \setminus \{V\}$. Let $\pi \in \mathcal{P}_{\mathcal{V}}(1, 7)$ be the path given in Fig. 3. The decomposition of π is given in Example 1. The shape of π is $(1, 4, 4)$ and $\lambda = (0, 0, 1, 1, 0)$. In Example 2, we assume that the pair $(6, \lambda)$ is the v_1th element from $\mathcal{H}_{\mathcal{V}}(1, 4, 4)$. Thus $g_{1,7}(\pi) = (\mu, v)$, where $v = (v_1, \ldots, v_7)$ and $\mu = U_4\, g(\pi^{(1)})\, g(\pi^{(2)}) \cdots g(\pi^{(4)})$. The final path $g(\pi)$ for certain weight vector v is given in Fig. 4.

Lemma 3. *We have $f_{m,n}^{-1} = g_{m,n}$ for any $m \geq 0$ and $n \geq 1$.*

Proof. We need to show that $g(f((\mu, v))) = (\mu, v)$ for any weighted path $(\mu, v) \in \mathcal{W}_{\mathcal{L}}(m, n)$ and $f(g(\pi)) = \pi$ for any $\pi \in \mathcal{P}_{\mathcal{V}}(m, n)$. Simple verification shows that the claim is true for $n = 1$. Let us prove the first statement for $n \geq 2$. Let $\mu = U_k^{m,d} \mu^{(1)} \cdots \mu^{(d)}$ and $v = (v_1, \ldots, v_n)$. Assume that (h, λ) is the v_1th pair from $\mathcal{H}_{\mathcal{V}}(m, d, k)$ and $\lambda = (\lambda_0, \ldots, \lambda_d)$. On the one hand, by the definition of f, the path $\pi = f_{m,n}((\mu, v))$ can be rewritten as

$$\pi = U_h\, V^{\lambda_0}\, f(\mu^{(1)}) V^{\lambda_1} f(\mu^{(2)}) V^{\lambda_2} \cdots f(\mu^{(d)}) V^{\lambda_d},$$

On the other hand, π can be decomposed as

$$\pi = U_h V^{\rho_0} \pi^{(1)} V^{\rho_1} \pi^{(2)} V^{\rho_2} \cdots \pi^{(t)} V^{\rho_t},$$

for some primary $\mathcal{V}$-paths $\pi^{(1)}, \ldots, \pi^{(t)}$. First, we need to show two statements: (i) $d = t$, $\pi^{(i)} = f(\mu^{(i)})$ for $i = 1, 2, \ldots, d$, and (ii) $(\lambda_0, \ldots, \lambda_d) = (\rho_0, \ldots, \rho_d)$. The first one is due to the definition of the function f. The resulting path $\pi = f_{m,n}((\mu, v))$ is the concatenation of paths $f(\pi^{(1)}), \ldots, f(\pi^{(d)})$, which are primary $\mathcal{V}$-paths, and some number (possibly zero) of vertical steps V between these shorter primary subpaths. The second condition follows from the observation that any primary $\mathcal{V}$-path does not begin with a vertical step, thus $\lambda_i = \rho_i$ for $i = 0, 1, \ldots, d$.

Next, under the assumption at the beginning of the proof, (h, λ) is the v_1th pair from $\mathcal{H}_{\mathcal{V}}(m, d, k)$. Thus, by the definition of the function g, the resulting path $g_{m,n}(\pi)$ is

$$g_{m,n}(\pi) = \Big(U_k^{m,d} g(f(\mu^{(1)})) g(f(\mu^{(2)})) \cdots g(f(\mu^{(d)})), v\Big),$$

where $v = (v_1, \ldots, v_n)$. Using the induction on n we show that $g(f(\mu^{(i)})) = \mu^{(i)}$ which ends the proof of the first statement.

The proof of $f(g(\pi)) = \pi$ goes in much the same way. □

4 Relations between primary and free paths

In this section we establish a relation between primary and free paths that contain vertical steps. Throughout the section, we fix $\mathcal{S}$ to be a subset of $\Omega_N \cup \{V\}$ such that $U_N \in \mathcal{S}$ and $N \geq 0$. Also, we set $\mathcal{V} = \mathcal{S} \cup \{V\}$ and $\mathcal{N} = \mathcal{S} \setminus \{V\}$.

Let a be a sequence of n integers $a_1, \ldots, a_n$. A partial sum of a is the sum $a_1 + \cdots + a_k$, for $1 \leq k \leq n$. Raney [14] shows that there is only one cyclic-shift $a' = (a_k, a_{k+1}, \ldots, a_n, a_1, \ldots, a_{k-1})$ of a such that any partial sum of a' is positive (see also [10, p. 360]). This lemma appears in the literature also as the cycle lemma [3]. For our purposes, we reformulate this lemma.

Lemma 4 (Raney lamma [14])**.** *Let $b = (b_1, \ldots, b_n)$ be a sequence of integers whose sum is -1. There is only one cyclic-shift b' of b such that every partial sum of b' except the total sum is nonnegative.*

Proof. Observe that if we rearrange terms of b in reverse order and we negate them, then we obtain the sequence $(-b_n, -b_{n-1}, \ldots, -b_1)$ whose sum is $+1$, and from the Raney lemma there is only one cyclic shift of such modified sequence which has the property that any its partial sum except the total sum is nonnegative. $\square$

Therefore, the Raney lamma implies that

$$|\mathcal{P}_{\mathcal{N}}(1, n)| = \frac{1}{n} |\mathcal{F}_{\mathcal{N}}(1, n)|, \qquad (n \geq 1). \tag{8}$$

We extend this relation between 1-primary and free $\mathcal{N}$-paths to the corresponding families of $\mathcal{V}$-paths with vertical steps.

Theorem 2. *For any $n \geq 1$ we have*

$$|\mathcal{P}_{\mathcal{V}}(1, n)| = \frac{1}{n} \big(|\mathcal{F}_{\mathcal{V}}(1, n)| - |\mathcal{F}_{\mathcal{V}}(0, n)| \big). \tag{9}$$

Proof. Any path from $\mathcal{P}_{\mathcal{V}}(1, n)$ is represented as $S_{a_1} V^{b_1} S_{a_2} V^{b_2} \cdots S_{a_n} V^{b_n}$ for some $a_1, \ldots, a_n$ depending on $\mathcal{V}$ and $b_1, \ldots, b_n \geq 0$. Let $\alpha = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n)$. The total sum of members of α is -1 and any partial sum (except the total sum) is nonnegative. By the Raney lemma, the number of such sequences is equal to $1/n$ times the number of sequences $\beta = (c_1 - d_1, \ldots, c_n - d_n)$, where $d_1, \ldots, d_n \geq 0$, $c_1, \ldots, c_n$ depend on $\mathcal{V}$, and the total sum of elements of β is -1. Observe that β designates uniquely a free $\mathcal{V}$-path running from $(0, 0)$ to $(n, -1)$ whose the first step is non-vertical. The number of sequences β is $|\mathcal{F}_{\mathcal{V}}(1, n)| - |\mathcal{F}_{\mathcal{V}}(0, n)|$, which finishes the proof. $\square$

Theorem 3. *Let $n \geq 1$ and $m \in \mathbb{Z}$, then*

$$|\mathcal{F}_{\mathcal{V}}(m, n)| = \sum_{j=0}^{Nn+m} \binom{n+j}{j} |\mathcal{F}_{\mathcal{N}}(m - j, n)|, \tag{10}$$

$$|\mathcal{P}_{\mathcal{V}}(1, n)| = \frac{1}{n} \sum_{j=0}^{Nn+1} \binom{n+j-1}{j} |\mathcal{F}_{\mathcal{N}}(1 - j, n)|. \tag{11}$$

Proof. First, we show (10). The number of vertical steps in any path from $\mathcal{F}_{\mathcal{V}}(m, n)$ is an integer from $\{0, 1, \ldots, Nn + m\}$. Therefore, we partition the family $\mathcal{F}_{\mathcal{V}}(m, n)$ into pairwise disjoint subfamilies $A_0, A_1, \ldots, A_{Nn+m}$, such that A_j contains these paths whose number of vertical steps is j. To calculate the size of A_j, observe that adding j vertical steps to any free

$\mathcal{N}$-path (without vertical steps) running from $(0,0)$ to $(n, j-m)$ we obtain a free path from $\mathcal{F}_{\mathcal{V}}(m,n)$. Any such path has n non-vertical steps S_k and those j vertical steps may be added between them on s ways, where s is the number of solutions of $a_0 + a_1 + \cdots + a_n = j$, where $a_0, \ldots, a_n \geq 0$. Therefore, the size of A_j is $\binom{n+j}{j}$ times the size of $\mathcal{F}_{\mathcal{N}}(m-j,n)$.

The second equality (11) follows directly from (9) together with (10). That is,

$$|\mathcal{P}_{\mathcal{V}}(1,n)| = \frac{1}{n}\left(\sum_{j=0}^{Nn+1}\binom{n+j}{j}|\mathcal{F}_{\mathcal{N}}(1-j,n)| - \sum_{j=0}^{Nn}\binom{n+j}{j}|\mathcal{F}_{\mathcal{N}}(0-j,n)|\right).$$

Changing the range summation of the second sum and using the recurrence relation for binomial coefficients we obtain the required formula. □

Suppose that S is a step from $\mathcal{S}$. Let $\#Steps(S \in \mathcal{P}_{\mathcal{S}}(1,n))$ denote the total number of occurrences of steps S in the set of all paths from $\mathcal{P}_{\mathcal{S}}(1,n)$, and let $\#Steps(\mathcal{P}_{\mathcal{S}}(1,n))$ denote the total number of all steps in $\mathcal{P}_{\mathcal{S}}(1,n)$.

Theorem 4. *Let $n \geq 1$, then*

$$\#Steps(V \in \mathcal{P}_{\mathcal{S}}(1,n)) = |\mathcal{F}_{\mathcal{S}}(0,n)|, \tag{12a}$$

$$\#Steps(S_k \in \mathcal{P}_{\mathcal{S}}(1,n)) = |\mathcal{F}_{\mathcal{S}}(1+k,n-1)|. \tag{12b}$$

Proof. Let S be a fixed step from $\mathcal{S}$ and let us introduce the temporary notation $\mathcal{F}$ for $\mathcal{F}_{\mathcal{S}}(0,n)$ if $S = V$ or $\mathcal{F}_{\mathcal{S}}(1+k,n-1)$ if $S = S_k$ for certain $k \in \mathbb{Z}$. Further, by a *level* we mean a line $y = l$ for any $l \in \mathbb{Z}$. Take $\pi \in \mathcal{P}_{\mathcal{V}}(1,n)$ and suppose that π has exactly d steps S and $d \geq 1$. Let $1 \leq p \leq d$, then

$$\pi = \underbrace{\pi^{(1)} S\, \pi^{(2)} S \cdots S\, \pi^{(p-1)} S\, \pi^{(p)}}_{\alpha}\, S\, \underbrace{\pi^{(p+1)} S \cdots \pi^{(d)} S\, \pi^{(d+1)}}_{\beta}, \tag{13}$$

for certain possibly empty subpaths $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d+1)}$. We define a function ϕ from the set of all occurrences of steps S in paths from $\mathcal{P}_{\mathcal{S}}(1,n)$ to the set of paths from $\mathcal{F}$ as follows

$$\phi(\pi, p) = \beta\, \alpha, \tag{14}$$

where α and β are subpaths of π defined as in (13). To show that ϕ is a bijection we need to show that $\phi(\pi,p)$ is a path from $\mathcal{F}$ and any path from $\mathcal{F}$ is decomposable as (14) for certain uniquely determined $p \geq 1$. Then the procedure ϕ is reversible and ϕ is a bijection.

First, observe that $\phi(\pi,p)$ removes only one step S from π which implies that the result is a free path from $\mathcal{F}$. Next, suppose that (x,y) is the leftmost point of $\phi(\pi,p)$ such that y is the minimal level that the path reaches. We prove that the path $\phi(\pi,p)$ reaches (x,y) exactly after the last step of β in (14). Recall that π is a primary $\mathcal{S}$-path running from $(0,0)$ to $(n,-1)$ for which only its ending point lies below the x-axis. Thus π reaches the lowest level exactly after part $\pi^{(d+1)}$. It follows that α is a path that does not go below the x-axis. On the other hand, only the ending point of β reaches the lowest level. It follows that $p-1$ is the number of steps S of $\phi(\pi,p)$ that lie to the right from (x,y).

Let γ be a free $\mathcal{S}$-path γ from $\mathcal{F}$ and $\gamma = \beta\alpha$ such that the last point of the subpath β lies at the left-most minimal level that γ reaches. Then we set $\phi^{-1}(\gamma)$ to be the pair $(\alpha S \beta, p)$, where p is the number of steps S in α plus one.

□

Example 4. Let π be a path from Fig. 3 and $S = V$. The path $\phi(\pi,2)$ is given in Fig. 6.

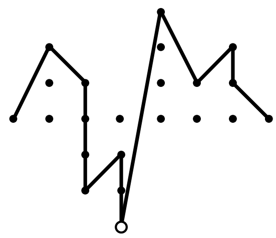

Figure 6: A free $\mathcal{V}$-path running from $(0,0)$ to $(7,0)$ which has one vertical step that lie to the right from the lowest point drawn by the open circle.

Theorem 5. *For $n \geq 1$, we have $\#Steps(\mathcal{P}_{\mathcal{S}}(1,n)) = |\mathcal{F}_{\mathcal{S}}(1,n)|$.*

Proof. We show a bijection ψ between the set of all steps in paths from $\mathcal{P}_{\mathcal{S}}(1,n)$ and the set of paths from $\mathcal{F}_{\mathcal{S}}(1,n)$. Take a path μ from $\mathcal{P}_{\mathcal{S}}(1,n)$ and suppose that $\mu = \mu_1 \cdots \mu_r$. Let $k \in \{1,2,\ldots,r\}$, then we set

$$\psi(\mu,k) = \mu_k\,\mu_{k+1}\cdots\mu_r\,\mu_1\,\mu_2\cdots\mu_{k-1}.$$

It is clear that $\psi(\mu,k) \in \mathcal{F}_{\mathcal{S}}(1,n)$. Next, we show a map ζ from $\mathcal{F}_{\mathcal{S}}(1,n)$ to the set of all steps in paths from $\mathcal{P}_{\mathcal{S}}(1,n)$. Let π be a path from $\mathcal{F}_{\mathcal{S}}(1,n)$ and $\pi = \pi_1 \cdots \pi_r$. Let us represent π as the sequence $\hat{s} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_r)$ of integers according to the rule

$$\hat{s}_i = \begin{cases} \;\;k & \text{if } \pi_i = S_k, \\ -1 & \text{if } \pi_i = V. \end{cases}$$

The sum of terms of the sequence $\hat{s}$ is -1. Therefore, the modified Raney lemma (Lemma 4) implies that there is only one cyclic-shift $s = (s_1, \ldots, s_r)$ of $\hat{s}$ that every its partial sum except the total sum is nonnegative. Moreover, this cyclic-shift s determines uniquely an index k such that the cyclic-shift $(s_k, s_{k+1}, \ldots, s_r, s_1, \ldots, s_{k-1})$ of s is the original sequence $\hat{s}$. Now, if we change back terms of the sequence s into steps according to the above rule, we obtain a primary $\mathcal{S}$-path μ. This implies that with any free path π from $\mathcal{F}_{\mathcal{S}}(1,n)$ we have associated uniquely a primary path μ from $\mathcal{P}_{\mathcal{S}}(1,n)$ and an index k such that $\phi(\mu,k) = \pi$. $\square$

5 Enumerative results

As in previous sections, we fix $\mathcal{S}$ to be a subset of $\Omega_N \cup \{V\}$ such that $U_N \in \mathcal{S}$. Also, we set $\mathcal{V} = \mathcal{S} \cup \{V\}$ and $\mathcal{N} = \mathcal{S} \setminus \{V\}$. Recall that we denote by $\mathcal{P}_{\mathcal{S}}(m,n)$ the set of all m-primary $\mathcal{S}$-paths running from $(0,0)$ to $(n,-m)$, and by $\mathcal{F}_{\mathcal{S}}(m,n)$ the set of all free $\mathcal{S}$-paths running from $(0,0)$ to $(n,-m)$ without further restriction. In this section we derive formulas for the sizes of these families.

5.1 General case

First, we consider the case where the set of steps $\mathcal{V}$ may contain infinitely many down steps. The number of free $\mathcal{V}$-paths running from $(0,0)$ to $(n,-m)$ satisfies the following recurrence relation

$$|\mathcal{F}_{\mathcal{V}}(m,n)| = |\mathcal{F}_{\mathcal{V}}(m-1,n)| + \sum_{S_k \in \mathcal{V}} |\mathcal{F}_{\mathcal{V}}(m+k,n-1)| \tag{15}$$

with initial conditions $|\mathcal{F}_{\mathcal{V}}(-Nn,n)| = |\mathcal{F}_{\mathcal{V}}(r,0)| = 1$, for $m \geq -Nn$ and $r, n \geq 0$. For $n < 0$ or $m < -Nn$ the number of such paths is zero. Thus even $\mathcal{V}$ has infinitely many down steps, the sum on the right-hand side of (15) is finite.

Let us define a bivariate generating function in the sense that

$$F_{\mathcal{V}}(x,y) = \sum_{m\geq 0}\sum_{n\geq 0} |\mathcal{F}_{\mathcal{V}}(m-Nn,n)| x^n y^m.$$

Proposition 2. *We have*

$$F_{\mathcal{V}}(x,y) = \Big(1 - y - x\sum_{S_k\in\mathcal{V}} y^{N-k}\Big)^{-1}. \tag{16}$$

Proof. Applying standard methods of generatingfunctionology [24] to the recurrence relation (15) one can show that

$$F_{\mathcal{V}}(x,y) = 1 + yF_{\mathcal{V}}(x,y) + x\sum_{S_k\in\mathcal{S}} y^{N-k} F_{\mathcal{V}}(x,y)$$

which implies (16) immediately. □

Proposition 3. *Let $n \geq 0$ and $m \in \mathbb{Z}$, then*

$$|\mathcal{F}_{\mathcal{N}}(m,n)| = [y^{Nn+m}]\Big(\sum_{S_k\in\mathcal{N}} y^{N-k}\Big)^n, \tag{17a}$$

$$|\mathcal{F}_{\mathcal{V}}(m,n)| = [y^{Nn+m}]\frac{1}{(1-y)^{n+1}}\Big(\sum_{S_k\in\mathcal{N}} y^{N-k}\Big)^n. \tag{17b}$$

Proof. Observe that any free $\mathcal{N}$-path running from $(0,0)$ to $(n,-m)$ can be represented as $S_{N-a_1}S_{N-a_2}\cdots S_{N-a_n}$, where $a_1, a_2, \ldots, a_n$ are nonnegative integers whose sum is $Nn+m$. Thus the number of paths from $\mathcal{F}_{\mathcal{N}}(m,n)$ is the coefficient of y^{Nn+m} in the power series expansion of $(\sum_{S_k\in\mathcal{N}} y^{N-k})^n$, as claimed. On the other hand, from (16) we have

$$\sum_{m\geq 0} |\mathcal{F}_{\mathcal{V}}(m-Nn,n)| y^m = \frac{1}{(1-y)^{n+1}}\Big(\sum_{S_k\in\mathcal{V}} y^{N-k}\Big)^n, \tag{18}$$

and the formula (17b) follows. □

Proposition 4. *Let $n \geq 1$, then*

$$|\mathcal{P}_{\mathcal{V}}(1,n)| = \frac{1}{n}[y^{Nn+1}]\frac{1}{(1-y)^n}\Big(\sum_{S_k\in\mathcal{N}} y^{N-k}\Big)^n. \tag{19}$$

Proof. It follows from (9) together with (17a). Namely,

$$\begin{aligned}|\mathcal{P}_{\mathcal{V}}(1,n)| &= \frac{1}{n}\left([y^{Nn+1}]A_n(y) - [y^{Nn}]A_n(y)\right)\\ &= \frac{1}{n}\left([y^{Nn+1}]A_n(y)(1-y)\right),\end{aligned}$$

where $A_n(y) = (\sum_{S_k\in\mathcal{N}} y^{N-k})^n (1-y)^{-n-1}$. □

Corollary 1. *The expected number of vertical steps in a path from $\mathcal{P}_{\mathcal{V}}(1,n)$ is equal to*

$$n \cdot \frac{|\mathcal{F}_{\mathcal{V}}(0,n)|}{|\mathcal{F}_{\mathcal{V}}(1,n)| - |\mathcal{F}_{\mathcal{V}}(0,n)|}. \tag{20}$$

Proof. The required number is the number of all vertical steps in the set of paths from $\mathcal{P}_{\mathcal{V}}(1,n)$ divided by the number of such paths. By Theorem 4, this number is $|\mathcal{F}_{\mathcal{V}}(0,n)|/|\mathcal{P}_{\mathcal{V}}(1,n)|$. Applying (9) we obtain the formula. □

Corollary 2. *The expected number of steps in a path from $\mathcal{P}_{\mathcal{V}}(1,n)$ is equal to*

$$n\left(1 + \frac{|\mathcal{F}_{\mathcal{V}}(0,n)|}{|\mathcal{F}_{\mathcal{V}}(1,n)| - |\mathcal{F}_{\mathcal{V}}(0,n)|}\right). \tag{21}$$

Proof. The required number is the total number of steps in the set of paths from $\mathcal{P}_{\mathcal{V}}(1,n)$ divided by the number of such paths. By Theorem 5, this number is $|\mathcal{F}_{\mathcal{V}}(1,n)|/|\mathcal{P}_{\mathcal{V}}(1,n)|$. Applying (9) we obtain the formula. □

5.2 Finite set of steps

Throughout the section we assume that the sets of possible steps $\mathcal{S}, \mathcal{V}, \mathcal{N}$, and $\mathcal{L}$ are finite and K is the maximal integer such that D_K belongs to those sets. Recall that $\mathcal{L}$ is defined in (3) with respect to $\mathcal{V}$. It is worth pointing out that a unified enumerative and asymptotic theory of lattice paths consisting of steps from $\mathcal{N}$ is developed by Banderier and Flajolet [1] and is associated with the so-called kernel method.

Proposition 5. *If $K = 1$, then for $n \geq 0$ we have*

$$|\mathcal{P}_{\mathcal{V}}(0,n)| = \sum_{j=0}^{n} (-1)^{n-j} |\mathcal{P}_{\mathcal{V}}(1,j)|. \tag{22}$$

Proof. If $K = 1$ then there is no step D_p in $\mathcal{V}$ such that $p > 1$. Thus the last step of any path from $\mathcal{P}_{\mathcal{V}}(1,n)$ is either V or D_1. It follows that $|\mathcal{P}_{\mathcal{V}}(1,n)| = |\mathcal{P}_{\mathcal{V}}(1,n-1)| + |\mathcal{P}_{\mathcal{V}}(0,n)|$. Moving $|\mathcal{P}_{\mathcal{V}}(1,n-1)|$ to the left-hand side we obtain a recurrence relation for $|\mathcal{P}_{\mathcal{V}}(0,n)|$. Iterating the above gives the required sum. □

Proposition 6. *If $K = m$ and $K \geq 2$, then for $n \geq 1$ we have*

$$|\mathcal{P}_{\mathcal{V}}(K,n)| = |\mathcal{P}_{\mathcal{V}}(0,n-1)|.$$

Proof. This follows from the observation that the last step of any m-primary $\mathcal{V}$-path running from $(0,0)$ to $(n,-m)$, where $m = K$, is D_K. Removing this step we obtain 0-primary $\mathcal{V}$-path running from $(0,0)$ to $(n-1,0)$. □

Let us define two ordinary generating functions

$$P_{\mathcal{S},m}(x) = \sum_{n\geq 0} |\mathcal{P}_{\mathcal{S}}(m,n)| x^n, \quad W_{\mathcal{S},m}(x) = \sum_{n\geq 0} |\mathcal{W}_{\mathcal{S}}(m,n)| x^n. \tag{23}$$

For simplicity of notation, we write $F_m(x)$ instead of $F_{\mathcal{S},m}(x)$ for fixed $\mathcal{S}$.

Proposition 7. *Let $1 \leq m \leq K$, then*

$$\begin{aligned} P_{\mathcal{N},0}(x) &= 1 + \delta_0 x P_{\mathcal{N},0}(x) + x P_{\mathcal{N},0}(x) \sum_{S_k \in \mathcal{N}_1} \sum_{d=1}^{k} \sum_{M} \prod_{j=1}^{d} (P_{\mathcal{N},m_j}(x) - 1), \\ P_{\mathcal{N},m}(x) &= 1 + \delta_m x + x \sum_{U_k \in \mathcal{N}_0} \sum_{d=1}^{k+1} \sum_{M} \prod_{j=1}^{d} (P_{\mathcal{N},m_j}(x) - 1), \end{aligned} \tag{24}$$

where $\delta_m = 1$ if $S_{-m} \in \mathcal{N}$, and $\delta_m = 0$ if $S_{-m} \notin \mathcal{N}$, and the summation range M is over all solutions of $m_1 + \cdots + m_d = k + m$ such that $1 \leq m_1, \ldots, m_{d-1} \leq K$ and $\max(m, 1) \leq m_d \leq K$.

Proof. It follows from the decomposition of an m-primary $\mathcal{N}$-path. By convention, we have one path of length zero. If $S_{-m} \in \mathcal{N}$ then we have one path of length one. Let $n \geq 2$ and take any path from $\mathcal{P}_{\mathcal{N}}(m, n)$. The first step of this path is an up step, let say U_k. If $m \geq 1$ then the entire path μ is decomposable into U_k and some number, let say d, of shorter and nonempty primary $\mathcal{N}$-paths $\mu^{(1)}, \ldots, \mu^{(d)}$. Suppose that $\mu^{(i)} \in \mathcal{P}_{\mathcal{N}}(m_i, n_i)$, then the numbers $m_1, \ldots, m_d$ are positive integers no greater than K. Further, m_d is no smaller than m. Finally, if $m = 0$, then the path is decomposable as above with some number (possibly zero) of additional 0-primary $\mathcal{N}$-paths. □

Proposition 8. *Let $1 \leq m \leq K$, then*

$$\begin{aligned} W_{\mathcal{L},0}(x) &= 1 + \delta_0 x W_{\mathcal{L},0}(x) + x W_{\mathcal{L},0}(x) \sum_{k=1}^{N} \sum_{d=1}^{k} |\mathcal{H}_{\mathcal{V}}(0, d, k)| \sum_{M} \prod_{j=1}^{d} (W_{\mathcal{L},m_j}(x) - 1), \\ W_{\mathcal{L},m}(x) &= 1 + \delta_m x + x \sum_{k=0}^{N} \sum_{d=1}^{k+1} |\mathcal{H}_{\mathcal{V}}(m, d, k)| \sum_{M} \prod_{j=1}^{d} (W_{\mathcal{L},m_j}(x) - 1), \end{aligned} \tag{25}$$

where $\delta_m = |\mathcal{L}_{-m}|$ if $m \in \{0, 1\}$, $\delta_m = 1$ if $D_m \in \mathcal{L}$, and $\delta_m = 0$ if $D_m \notin \mathcal{L}$, for $m \geq 2$. Further, the summation range M is over all solutions of $m_1 + \cdots + m_d = k + m$ such that $1 \leq m_1, \ldots, m_{d-1} \leq K$ and $\max(m, 1) \leq m_d \leq K$.

Proof. The set of steps $\mathcal{L}$ contains steps $U_N, U_{N-1}, \ldots, U_0, D_1$, thus the range summation of the first sum is over k from one (for $m = 0$) or zero (for $m \neq 1$) up to N. The weight of any step $U_k^{m,d}$ is the size of $\mathcal{H}_{\mathcal{V}}(m, d, k)$. Thus substituting that numbers into the functional equation (24) we obtain the formula. □

Proposition 9. *Let $1 \leq m \leq K$, then*

$$\begin{aligned} P_{\mathcal{V},0}(x) &= 1 + \delta_0 x P_{\mathcal{V},0}(x) + x P_{\mathcal{V},0}(x) \sum_{k=1}^{N} \sum_{d=1}^{k} |\mathcal{H}_{\mathcal{V}}(0, d, k)| \sum_{M} \prod_{j=1}^{d} (P_{\mathcal{V},m_j}(x) - 1), \\ P_{\mathcal{V},m}(x) &= 1 + \delta_m x + x \sum_{k=0}^{N} \sum_{d=1}^{k+1} |\mathcal{H}_{\mathcal{V}}(m, d, k)| \sum_{M} \prod_{j=1}^{d} (P_{\mathcal{V},m_j}(x) - 1), \end{aligned} \tag{26}$$

where $\delta_m = |\mathcal{V}_{\geq -m}|$ if $m \in \{0, 1\}$, $\delta_m = 1$ if $D_m \in \mathcal{V}$, and $\delta_m = 0$ if $D_m \notin \mathcal{V}$, for $m \geq 2$. Further, the summation range M is over all solutions of $m_1 + \cdots + m_d = k + m$ such that $1 \leq m_1, \ldots, m_{d-1} \leq K$ and $\max(m, 1) \leq m_d \leq K$.

Proof. By Theorem 1, the size of $\mathcal{W}_{\mathcal{L}}(m, n)$ is equal to the size of $\mathcal{P}_{\mathcal{V}}(m, n)$, thus $P_{\mathcal{V},m}(x)$ and $W_{\mathcal{L},m}(x)$ are the same generating functions. □

5.3 Riordan arrays

The Riordan group [16, 18] is a set of infinite lower-triangular matrices defined as follows. A *proper Riordan array* is a couple $(g(x), f(x))$, where $g(x) = \sum_{n\geq 0} g_n x^n$ with $g_0 \neq 0$ and $f(x) = \sum_{n\geq 1} f_n x^n$ with $f_1 \neq 0$. With the proper Riordan array we associate the matrix, denoted by (g, f), whose (i, j)th element is given by $[x^i]g(x)f(x)^j$, for $i, j \geq 0$.

Proposition 10. *The array*

$$D_{\mathcal{V}} = \left(\frac{1}{1-y}, \frac{y}{1-y} \sum_{S_k \in \mathcal{V}} y^{N-k} \right) \tag{27}$$

is the proper Riordan array, whose (i, j)th element, denoted by $d_{i,j}$, is the number of free $\mathcal{V}$-paths running from $(0, 0)$ to $(j, (N+1)j - i)$. That is, $d_{i,j} = |\mathcal{F}_{\mathcal{V}}(i - (N+1)j, j)|$.

Proof. By (18), since $g(x) = 1/(1-x)$ and $f(x) = x(\sum_{S_k \in \mathcal{V}} x^{N-k})/(1-x)$, we conclude that (g, f) is the proper Riordan array, and

$$d_{i,j} = [y^i] \frac{1}{1-y} \left(\frac{y}{1-y} \sum_{S_p \in \mathcal{S}} y^{N-p} \right)^j = |\mathcal{F}_{\mathcal{V}}(i - (N+1)j, j)|.$$

□

Corollary 3. *Let $a(x)$ be the generating function of the sequence $(a_n)_{n\geq 0}$. Then*

$$\sum_{k\geq 0} d_{n,k} a_k = [y^n] \frac{1}{1-y} a \left(\frac{y}{1-y} \sum_{S_k \in \mathcal{V}} y^{N-k} \right),$$

where $d_{n,k} = |\mathcal{F}_{\mathcal{V}}(n - (N+1)k, k)|$. That is, $d_{n,k}$ is the number of free $\mathcal{S}$-paths running from $(0, 0)$ to $(k, Nk + k - n)$.

Proof. It follows directly from the properties of the Riordan arrays, see e.g. Sprugnoli [18, Th.1.1]. □

Example 5. If $\mathcal{V} = \{V, U_1, U_0\}$, then

$$D_{\mathcal{V}} = \left(\frac{1}{1-y}, \frac{y+y^2}{1-y} \right), \qquad d_{i,j} = [y^{i-j}] \frac{(1+y)^j}{(1-y)^{j+1}} = D(j, i-j),$$

where $D(i, j)$ is the (i, j)th Delannoy number [2, 19]. Note that $D(i, j)$ is the number of paths running from $(0, 0)$ to (i, j) consisting of steps $(1, 0)$, $(1, 1)$, and $(0, 1)$. Such paths are called Delannoy paths in the literature. By Proposition 10, we have

$$d_{i,j} = |\mathcal{F}_{\mathcal{V}}(i - 2j, j)| = D(j, i-j).$$

Therefore, there is a bijection between free $\mathcal{V}$-paths running from $(0, 0)$ to $(j, j-i)$ and Delannoy paths running from $(0, 0)$ to (j, i). See Section 6.5 for more details.

6 Examples of lattice paths with vertical steps

In this section we present five examples of lattice paths with vertical steps for which we apply results obtained in the previous sections.

6.1 The first example

Let $\mathcal{A} = \{V, U_N, U_{N-1}, \ldots, U_0, D_1, \ldots, D_K\}$ for fixed $N \geq 0$ and $K \geq 1$. Let us define the corresponding set of lattice steps without vertical step V. That is, $\mathcal{L} = \mathcal{A} \setminus \{V\}$. By Theorem 1, for $m \geq 0$, and $n \geq 1$, we have

$$|\mathcal{P}_{\mathcal{A}}(m,n)| = \sum_{\mu \in \mathcal{P}_{\mathcal{L}}(m,n)} w(\mu),$$

if the weight function w over steps from $\mathcal{L}$-paths is defined as follows

$$w(D_p) = \begin{cases} N+2 & \text{if } p = 1, \\ 1 & \text{if } p \geq 2, \end{cases}, \quad w(U_k^{m,d}) = \binom{N-k+d+1-\epsilon_m}{N-k}, \tag{28}$$

where $\epsilon_m = 0$ if $m \in \{0,1\}$ and $\epsilon_m = 1$ if $m \geq 2$.

Corollary 4. *For $m \geq 0$ and $n \geq 1$, we have*

$$|\mathcal{F}_{\mathcal{A}}(m,n)| = \sum_{k=0}^{\lfloor \frac{Nn+m}{N+K+1} \rfloor} (-1)^k \binom{n}{k} \binom{n(N+2) - k(N+K+1) + m}{2n}, \tag{29a}$$

$$|\mathcal{P}_{\mathcal{A}}(1,n)| = \frac{1}{n} \sum_{k=0}^{\lfloor \frac{Nn+1}{N+K+1} \rfloor} (-1)^k \binom{n}{k} \binom{n(N+2) - k(N+K+1)}{2n-1}. \tag{29b}$$

Proof. By Proposition 3 and Proposition 4, we have

$$|\mathcal{F}_{\mathcal{A}}(m,n)| = [z^{Nn+m}] \frac{(1-z^{N+K+1})^n}{(1-z)^{2n+1}}, \quad |\mathcal{P}_{\mathcal{A}}(1,n)| = \frac{1}{n} [z^{Nn+1}] \frac{(1-z^{N+K+1})^n}{(1-z)^{2n}}.$$

From the binomial theorem we derive

$$\frac{(1-z^A)^B}{(1-z)^C} = \sum_{n \geq 0} \sum_{k=0}^{\lfloor \frac{n}{A} \rfloor} (-1)^k \binom{B}{k} \binom{C+n-kA-1}{C-1} z^n. \tag{30}$$

for any integers $A, B, C \geq 0$. Substituting required parameters we obtain the formulas. $\square$

Corollary 5. *Let $P_m(x) = P_{\mathcal{A},m}(x)$ and $1 \leq m \leq K$, then*

$$\begin{aligned} P_0(x) &= 1 + \delta_0 x P_0(x) + x P_0(x) \sum_{k=1}^{N} \sum_{d=1}^{k} \binom{N-k+d+1}{N-k} \sum_M \prod_{j=1}^{d} (P_{m_j}(x) - 1), \\ P_m(x) &= 1 + \delta_m x + x \sum_{k=0}^{N} \sum_{d=1}^{k+1} \binom{N-k+d+1-\epsilon_m}{N-k} \sum_M \prod_{j=1}^{d} (P_{m_j}(x) - 1), \end{aligned} \tag{31}$$

where $\delta_m = (N+1+m)$ if $m \in \{0,1\}$, $\delta_m = 1$ for $2 \leq m \leq K$. Further $\epsilon_m = 0$ if $m \in \{0,1\}$, $\epsilon_m = 1$ if $m \geq 2$, and the summation range M is over all solutions of $m_1 + \cdots + m_d = k + m$ such that $1 \leq m_1, \ldots, m_{d-1} \leq K$ and $\max(m,1) \leq m_d \leq K$.

Corollary 6. *The number of all vertical steps in the set of paths from $\mathcal{P}_{\mathcal{A}}(1,n)$ is equal to $|\mathcal{F}_{\mathcal{A}}(0,n)|$. The number of all steps in the set of paths from $\mathcal{P}_{\mathcal{A}}(1,n)$ is equal to $|\mathcal{F}_{\mathcal{A}}(1,n)|$.*

Proposition 11. *If $N = 1$ and $K = 2$, then*

$$P_1(x) = \frac{2(1-x)}{3x} - \frac{2\sqrt{\Delta}}{3x} \sin\left\{\frac{\pi}{6} + \frac{1}{3}\arccos\left(\frac{20x^3 - 6x^2 + 15x - 2}{2\Delta^{3/2}}\right)\right\},$$

where $\Delta = 1 - 5x - 2x^2$.

Proof. By Corollary 5, for $N = 1$ and $K = 2$, we obtain three functional equations

$$\begin{aligned}
P_0(x) &= \frac{1}{1 - x - xP_1(x)},\\
P_1(x) &= 1 + xP_1(x) + xP_1(x)^2 + xP_2(x),\\
P_2(x) &= \frac{1 - xP_1(x)}{1 - x - xP_1(x)} = 1 + xP_0(x),
\end{aligned}$$

which follow to the cubic equation

$$x^2P_1(x)^3 + 2x(x-1)P_1(x)^2 + (1 - x - 2x^2)P_1(x) - 1 = 0.$$

Using trigonometric methods we obtain the formula. □

Remark. Let us give some first values of the sequences considered above for $N = 1$ and $K = 2$. That is, $\mathcal{A} = \{V, U_1, U_0, D_1, D_2\}$.

$$\begin{aligned}
(|\mathcal{F}_\mathcal{A}(0,n)|)_{n\geq 0} &= (1, 3, 15, 84, 491, 2948, 18018, 111520, 696739, \dots)\\
(|\mathcal{F}_\mathcal{A}(1,n)|)_{n\geq 0} &= (1, 6, 35, 207, 1251, 7678, 47658, 298371, 1880659, \dots)\\
(|\mathcal{P}_\mathcal{A}(0,n)|)_{n\geq 0} &= (1, 2, 7, 30, 142, 716, 3771, 20502, 114194, 648276, \dots)\\
(|\mathcal{P}_\mathcal{A}(1,n)|)_{n\geq 0} &= (1, 3, 10, 41, 190, 946, 4940, 26693, 147990, 837102, \dots)\\
(|\mathcal{P}_\mathcal{A}(2,n)|)_{n\geq 0} &= (1, 1, 2, 7, 30, 142, 716, 3771, 20502, 114194, 648276, \dots)
\end{aligned}$$

6.2 Łukasiewicz paths

This section is devoted to the case where $K = 1$ from the previous example. Namely, let $\mathcal{B} = \{V, U_N, U_{N-1}, \dots, U_0, D_1\}$ for fixed $N \geq 0$. Let $\mathcal{L} = \mathcal{B} \setminus \{V\}$. Lattice paths consisting of steps from $\mathcal{L}$ are called N-Łukasiewicz paths [15, 23]. The weighted Łukasiewicz paths encode several families of combinatorial structures like involutions, permutations, and set partitions, see Varvak [22]. Now we see that the proper weighted N-Łukasiewicz paths encode m-primary $\mathcal{B}$-paths, where $m \in \{0, 1\}$.

Namely, by Theorem 1, for $m \geq 0$ and $n \geq 1$, we have

$$|\mathcal{P}_\mathcal{B}(m,n)| = \sum_{\mu \in \mathcal{P}_\mathcal{L}(m,n)} w(\mu),$$

if the weight function w over steps from $\mathcal{L}$-paths is defined as follows

$$w(D_1) = N + 2, \quad w(U_k^{1,d}) = \binom{N+2}{k+2}, \quad w(U_k^{0,d}) = \binom{N+1}{k+1}. \tag{32}$$

It is worth pointing out, that the weight function w is independent of d.

Example 6. For $N = 1$, the set $\mathcal{W}_\mathcal{L}(0,n)$ is the family of weighted Motzkin paths running from $(0,0)$ to $(n,0)$, which never go below the x-axis, and where the weight of the horizontal step $(1,0)$ is 2 if it lies on the x-axis and 3 if it lies above the x-axis, the weight $(1,1)$ is one and the weight of $(1,-1)$ is 3.

Corollary 7. *Let $m \geq 0$ and $n \geq 1$, then*

$$|\mathcal{F}_{\mathcal{B}}(m,n)| = \sum_{k=0}^{\lfloor \frac{Nn+m}{N+2} \rfloor} (-1)^k \binom{n}{k} \binom{(N+2)(n-k)+m}{2n}, \tag{33a}$$

$$|\mathcal{P}_{\mathcal{B}}(0,n)| = (-1)^n + \sum_{j=1}^{n} \sum_{k=0}^{\lfloor \frac{Nj+1}{N+2} \rfloor} \frac{(-1)^{k+n-j}}{j} \binom{j}{k} \binom{(N+2)(j-k)}{2j-1}, \tag{33b}$$

$$|\mathcal{P}_{\mathcal{B}}(1,n)| = \frac{1}{n} \sum_{k=0}^{\lfloor \frac{Nn+1}{N+2} \rfloor} (-1)^k \binom{n}{k} \binom{(N+2)(n-k)}{2n-1}. \tag{33c}$$

Corollary 8. *The number of all vertical steps in the set of paths from $\mathcal{P}_{\mathcal{B}}(1,n)$ is equal to $|\mathcal{F}_{\mathcal{B}}(0,n)|$. The number of all steps in the set of paths from $\mathcal{P}_{\mathcal{B}}(1,n)$ is equal to $|\mathcal{F}_{\mathcal{B}}(1,n)|$.*

Corollary 9. *Let $P_m(x) = P_{\mathcal{B},m}(x)$, then*

$$P_0(x) = 1 + xP_0(x) \sum_{k=0}^{N} P_1(x)^k, \qquad P_1(x) = 1 + x \sum_{k=0}^{N+1} P_1(x)^k. \tag{34}$$

Proof. Applying Proposition 9 for the weights given by (32) we obtain

$$\begin{aligned} P_0(x) &= 1 + xP_0(x) \sum_{k=0}^{N} \binom{N+1}{k+1} (P_1(x)-1)^k \\ &= 1 + \frac{xP_0(x)}{(P_1(x)-1)} \sum_{k=1}^{N+1} \binom{N+1}{k} (P_1(x)-1)^k. \end{aligned}$$

Using the binomial theorem and simplifying the result we obtain the functional equation for $P_0(x)$. In the same way one can show an analogous relation for $P_1(x)$. $\square$

For instance, if $N = 1$, then

$$P_0(x) = \frac{1 - x - \sqrt{1-6x-3x^2}}{2x(1+x)}, \quad P_1(x) = \frac{1 - x - \sqrt{1-6x-3x^2}}{2x}.$$

The generating functions of the sequences $(|\mathcal{F}_{\mathcal{B}}(m,n)|)_{n\geq 0}$, for $m \in \{0,1\}$, are derived by the author in [7] (see Eq. 18 for $m = 0$ and Eq. 30a for $m = 1$). That is,

$$\sum_{n\geq 0} |\mathcal{F}_{\mathcal{B}}(0,n)| x^n = \frac{1}{\sqrt{1-6x-3x^2}}, \quad \sum_{n\geq 0} |\mathcal{F}_{\mathcal{B}}(1,n)| x^n = \frac{1 - x - \sqrt{1-6x-3x^2}}{2x\sqrt{1-6x-3x^2}}.$$

Remark. Let $N = K = 1$.

$$\begin{aligned} (|\mathcal{F}_{\mathcal{B}}(0,n)|)_{n\geq 0} &= (1, 3, 15, 81, 459, 2673, 15849, 95175, 576963, \ldots) && (A122868) \\ (|\mathcal{F}_{\mathcal{B}}(1,n)|)_{n\geq 0} &= (1, 6, 33, 189, 1107, 6588, 39663, 240894, \ldots) && \\ (|\mathcal{P}_{\mathcal{B}}(0,n)|)_{n\geq 0} &= (1, 2, 7, 29, 133, 650, 3319, 17498, 94525, \ldots) && (A064641) \\ (|\mathcal{P}_{\mathcal{B}}(1,n)|)_{n\geq 0} &= (1, 3, 9, 36, 162, 783, 3969, 20817, 112023, \ldots) && (A156016) \end{aligned}$$

The numbers starting with A in parentheses denote corresponding sequences in OEIS [17].

6.3 Infinite number of down steps

In this section we consider the case where the set of steps contains infinitely many down steps. Namely, let $\mathcal{C} = \{V, U_N, U_{N-1}, \ldots, U_0, D_1, D_2, \ldots\}$ for fixed $N \geq 0$. Let $\mathcal{L} = \mathcal{C} \setminus \{V\}$. By Theorem 1, we have

$$|\mathcal{P}_{\mathcal{C}}(m,n)| = \sum_{\mu \in \mathcal{P}_{\mathcal{L}}(m,n)} w(\mu),$$

if the weight function w over steps from $\mathcal{L}$-paths is defined as follows

$$w(D_p) = \begin{cases} N+2 & \text{if } p = 1, \\ 1 & \text{if } p \geq 2, \end{cases} \qquad w(U_k^{m,d}) = \binom{N-k+d+1-\epsilon_m}{N-k}, \tag{35}$$

where $\epsilon_m = 0$ if $m \in \{0,1\}$ and $\epsilon_m = 1$ if $m \geq 2$.

Corollary 10. *Let $m \geq 0$ and $n \geq 1$, then*

$$|\mathcal{F}_{\mathcal{C}}(m,n)| = \binom{(N+2)n+m}{2n}, \tag{36a}$$

$$|\mathcal{P}_{\mathcal{C}}(1,n)| = \frac{1}{n}\binom{(N+2)n}{2n-1}. \tag{36b}$$

Proof. It follows from Proposition 3 and Proposition 4. That is,

$$|\mathcal{F}_{\mathcal{C}}(m,n)| = [z^{Nn+m}]\,(1-z)^{-2n-1}, \qquad |\mathcal{P}_{\mathcal{C}}(1,n)| = \frac{1}{n}[z^{Nn+1}]\,(1-z)^{-2n}.$$

□

Corollary 11. *The expected number of vertical steps in a path from $\mathcal{P}_{\mathcal{C}}(1,n)$ is equal to $(Nn+1)/2$. The expected number of steps in a path from $\mathcal{P}_{\mathcal{C}}(1,n)$ is equal to $((N+2)n+1)/2$.*

Remark. The array $(|\mathcal{P}_{\mathcal{C}}(1,n)|)_{N,n}$ for $0 \leq N \leq 3$ and $0 \leq n \leq 7$, is

$$\begin{pmatrix} 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 3 & 10 & 42 & 198 & 1001 & 5304 & 29070 \\ 1 & 4 & 28 & 264 & 2860 & 33592 & 416024 & 5348880 \\ 1 & 5 & 60 & 1001 & 19380 & 408595 & 9104550 & 210905400 \end{pmatrix}.$$

The second row of the array is denoted by A007226 in OEIS [17]. The array $(|\mathcal{F}_{\mathcal{C}}(0,n)|)_{N,n}$ for $0 \leq N \leq 3$ and $0 \leq n \leq 7$, is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 15 & 84 & 495 & 3003 & 18564 & 116280 \\ 1 & 6 & 70 & 924 & 12870 & 184756 & 2704156 & 40116600 \\ 1 & 10 & 210 & 5005 & 125970 & 3268760 & 86493225 & 2319959400 \end{pmatrix}.$$

The second row of the array is denoted by A005809 in OEIS [17]. The third one is A001448 in OEIS [17], etc.

6.4 Dyck paths with vertical steps

Originally, a *Dyck path* [4] is a lattice path running from $(0,0)$ to $(2n,0)$ and consisting of steps U_1 and D_1, for $n \geq 0$. In this section we consider generalized Dyck paths which contain additional vertical steps. Namely, let $\mathcal{D} = \{V, U_N, D_K\}$, for fixed $N, K \geq 1$. Let $\mathcal{L} = \{U_N, U_{N-1}, \ldots, U_0, D_1, D_K\}$. By Theorem 1, we have

$$|\mathcal{P}_{\mathcal{D}}(m,n)| = \sum_{\mu \in \mathcal{P}_{\mathcal{L}}(m,n)} w(\mu),$$

if the weight function w over steps from $\mathcal{L}$-paths is defined as follows

$$w(D_p) = \begin{cases} 2 & \text{if } p = 1, \\ 1 & \text{if } p \geq 2. \end{cases}, \quad w(U_k^{m,d}) = \binom{N-k+d-\epsilon_m}{N-k},$$

where $\epsilon_m = 0$ if $m \in \{0,1\}$ and $\epsilon_m = 1$ if $m \geq 2$.

Corollary 12. *Let $m \geq 0$ and $n \geq 1$, then*

$$|\mathcal{F}_{\mathcal{D}}(m,n)| = \sum_{k=0}^{\lfloor \frac{Nn+m}{N+K} \rfloor} \binom{n}{k} \binom{n(N+1) - k(N+K) + m}{n}, \tag{37a}$$

$$|\mathcal{P}_{\mathcal{D}}(1,n)| = \frac{1}{n} \sum_{k=0}^{\lfloor \frac{Nn+1}{N+K} \rfloor} \binom{n}{k} \binom{n(N+1) - k(N+K)}{n-1}. \tag{37b}$$

Proof. It follows from Proposition 3 and Proposition 4. That is,

$$|\mathcal{F}_{\mathcal{S}}(m,n)| = [z^{Nn+m}] \frac{\left(1 + z^{N+K}\right)^n}{(1-z)^{n+1}}, \quad |\mathcal{P}_{\mathcal{S}}(1,n)| = \frac{1}{n} [z^{Nn+1}] \frac{\left(1 + z^{N+K}\right)^n}{(1-z)^n}.$$

Using (30) we obtain the required formulas. □

Corollary 13. *If $K = 1$ then for $n \geq 1$ we have*

$$|\mathcal{P}_{\mathcal{S}}(0,n)| = (-1)^n + \sum_{j=1}^{n} \sum_{k=0}^{\lfloor \frac{Nj+1}{N+1} \rfloor} \frac{(-1)^{n-j}}{j} \binom{j}{k} \binom{(N+1)(j-k)}{j-1}.$$

Proof. It follows from Proposition 5 and Corollary 12. □

Corollary 14. *Let $P_m(x) = P_{\mathcal{D},m}(x)$, then*

$$\begin{aligned} P_0(x) &= 1 + \delta_0 x P_0(x) + x P_0(x) \sum_{k=1}^{N} \sum_{d=1}^{k} \binom{N-k+d}{d} \sum_{M} \prod_{j=1}^{d} (P_{m_j}(x) - 1), \\ P_m(x) &= 1 + \delta_m x + x \sum_{k=0}^{N} \sum_{d=1}^{k+1} \binom{N-k+d-\epsilon_m}{N-k} \sum_{M} \prod_{j=1}^{d} (P_{m_j}(x) - 1), \end{aligned} \tag{38}$$

where $\delta_m = |\mathcal{D}_{-m}|$ if $m \in \{0,1\}$, $\delta_m = 1$ if $D_m \in \mathcal{D}$, and $\delta_m = 0$ if $D_m \notin \mathcal{D}$, for $m \geq 2$. Further, the summation range M is over all solutions of $m_1 + \cdots + m_d = k + m$ such that $1 \leq m_1, \ldots, m_{d-1} \leq K$ and $\max(m,1) \leq m_d \leq K$.

For instance, if $K = N = 1$, then

$$P_0(x) = \frac{1 - \sqrt{1 - 4x - 4x^2}}{2x(1+x)}, \qquad P_1(x) = \frac{1 - \sqrt{1 - 4x - 4x^2}}{2x}.$$

Remark. Let $N = K = 1$.

$$(|\mathcal{F}_{\mathcal{D}}(0,n)|)_{n\geq 0} = (1, 2, 8, 32, 136, 592, 2624, 11776, 53344, 243392, \ldots) \qquad (A006139)$$
$$(|\mathcal{F}_{\mathcal{D}}(1,n)|)_{n\geq 0} = (1, 4, 16, 68, 296, 1312, 5888, 26672, 121696, \ldots) \qquad (A179191)$$
$$(|\mathcal{P}_{\mathcal{D}}(0,n)|)_{n\geq 0} = (1, 1, 3, 9, 31, 113, 431, 1697, 6847, 28161, 117631, \ldots) \qquad (A052709)$$
$$(|\mathcal{P}_{\mathcal{D}}(1,n)|)_{n\geq 0} = (1, 2, 4, 12, 40, 144, 544, 2128, 8544, 35008, 145792, \ldots) \qquad (A025227)$$

The numbers starting with A in parentheses denote corresponding sequences in OEIS [17].

6.5 Delannoy paths

A *Delannoy path* is a lattice path from $(0,0)$ to (n,k) in $\mathbb{Z} \times \mathbb{Z}$ consisting of steps $(1,0)$, $(1,0)$, and $(0,1)$. The number of Delannoy paths running from $(0,0)$ to (n,k) is called *Delannoy number* [2] and denoted by $D(n,k)$. The number of these paths running from $(0,0)$ to (n,n) and never go below the line $y = x$ is called *central Delannoy number* [9, 11] and denoted by $D(n)$. It is well-known that

$$D(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n+k-j}{n}.$$

These numbers are denoted in OEIS [17] by A152250 and A001850. Additionaly, let us denote by $S(n)$ the number of central Delannoy paths running from $(0,0)$ to (n,n) that do not go below the line $y = x$. The numbers of such paths are called the large Schröeder numbers [5] and they are denoted by A006318 in OEIS [17].

Proposition 12. *Let $\mathcal{E} = \{V, U_1, U_0\}$ and $m, n \geq 0$. Then*

$$D(n,m) = |\mathcal{F}_{\mathcal{E}}(m-n,n)|, \qquad S(n) = |\mathcal{P}_{\mathcal{E}}(1,n)| = |\mathcal{P}_{\mathcal{E}}(0,n)|. \tag{39}$$

Proof. We obtain required bijection by transforming lattice points by the rule $(i,j) \mapsto (i-j,i)$ together with preserving connections between lattice points. Indeed, step V becomes $(1,0)$, U_1 becomes $(0,1)$, and U_0 becomes $(1,1)$. Additionally, we remove the last vertical step in every path from $\mathcal{P}_{\mathcal{E}}(1,n)$. □

Let $\mathcal{L} = \{U_1, U_0, D_1\}$. By Theorem 1, we have

$$S(n) = \sum_{\mu \in \mathcal{P}_{\mathcal{L}}(1,n)} w(\mu),$$

if the weight function w is defined as

$$w(U_1) = 1, \qquad w(U_0) = 3, \qquad w(D_1) = 2.$$

Corollary 15. *The expected number of steps $(1,1)$ in a central Delannoy path running from $(0,0)$ to (n,n) which never goes below the line $y = x$ is*

$$n\frac{D(n,n-1)}{D(n,n+1) - D(n,n)} = n\left(\sum_{j=0}^{n-1}\binom{n}{j}\binom{2n-j-1}{n}\right)\left(\sum_{j=0}^{n}\binom{n}{j}\binom{2n-j}{n-1}\right)^{-1}.$$

Proof. By Proposition 12, the expected number is the total number of steps U_0 in paths from $\mathcal{P}_{\mathcal{V}}(1,n)$ divided by the size of $\mathcal{P}_{\mathcal{V}}(1,n)$. By Theorem 4, we have $\#Steps(U_0 \in \mathcal{P}_{\mathcal{V}}(1,n)) = |\mathcal{F}_{\mathcal{V}}(1,n-1)|$. That is, required number is $D(n,n-1)/S(n)$. By (9), $S(n) = (D(n,n+1) - D(n,n))/n$. $\square$

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