A MONOTONICITY PROPERTY FOR GENERALIZED FIBONACCI SEQUENCES

Toufik Mansour
Department of Mathematics, University of Haifa, 31905 Haifa, Israel
tmansour@univ.haifa.ac.il

Mark Shattuck
Department of Mathematics, University of Tennessee, Knoxville, TN 37996
shattuck@math.utk.edu

Abstract. Given $k \geq 2$, let a_n be the sequence defined by the recurrence $a_n = \alpha_1 a_{n-1} + \cdots + \alpha_k a_{n-k}$ for $n \geq k$, with initial values $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1$. We show under a couple of assumptions concerning the constants α_i that the ratio $\frac{\sqrt[n]{a_n}}{\sqrt[n-1]{a_{n-1}}}$ is strictly decreasing for all $n \geq N$, for some N depending on the sequence, and has limit 1. In particular, this holds in the cases when all of the α_i are unity or when all of the α_i are zero except for the first and last, which are unity. Furthermore, when $k = 3$ or $k = 4$, it is shown that one may take N to be an integer less than 12 in each of these cases.

Keywords: monotonicity, log-concavity, k-Fibonacci numbers, tribonacci numbers

2010 Mathematics Subject Classification: 05A10, 11B39, 11B75

1. Introduction

In 1982, Firoozbakht conjectured that the sequence $\{\sqrt[n]{p_n}\}_{n\geq 1}$ is strictly decreasing, where p_n denotes the n-th prime. A stronger conjecture was later made by Sun [12] that in fact

$$\frac{\sqrt[n+1]{p_{n+1}}}{\sqrt[n]{p_n}} < 1 - \frac{\log\log n}{2n^2}, \qquad n > 4,$$

which has been verified for all $n \leq 3.5 \cdot 10^6$. Inspired by this and [11], Sun posed several conjectures in [12] concerning the monotonicity of sequences of the form $\{\sqrt[n]{y_n}\}_{n\geq N}$, where $\{y_n\}_{n\geq 0}$ is a familiar number theoretic or combinatorial sequence. Partial progress has been made in this direction, including Chen et al. [3] for Bernoulli numbers, Hou et al. [4] for Fibonacci and derangement numbers, and Wang and Zhu [13] for Motzkin and (large) Schröder numbers.

Recall that a sequence $\{y_n\}_{n\geq 0}$ is said to be (*strictly*) *log concave* (see, e.g., [2, 10]) if the sequence of ratios $\{\frac{y_n}{y_{n-1}}\}_{n\geq 1}$ is (strictly) decreasing. If the sequence of ratios is increasing, then y_n is said to be *log convex* (see [6]). Suppose $A > 0$ and $B \neq 0$ are integers such that $A^2 - 4B > 0$. Let u_n denote the sequence defined by the second order recurrence $u_n = Au_{n-1} - Bu_{n-2}$ if $n \geq 2$, with initial values $u_0 = 0$ and $u_1 = 1$. In [4, Theorem 1.1], it was shown that $\sqrt[n]{u_n}$ is strictly log-concave for all $n \geq N$, for some N depending on the sequence, and has limit 1. In the special case $A = 1$ and $B = -1$, which corresponds to the Fibonacci sequence, it is shown that one may take $N = 5$. Here, we consider the question of monotonicity of $\frac{\sqrt[n]{a_n}}{\sqrt[n-1]{a_{n-1}}}$ for a class of sequences a_n defined by a more general linear recurrence.

Given $k \geq 2$, let a_n be a sequence of non-negative real numbers defined by the recurrence

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_k a_{n-k}, \qquad n \geq k, \tag{1.1}$$

with $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1$. One combinatorial interpretation for a_n, which follows from [1, Section 3.1], is that it counts the weighted linear tilings of length $n-k+1$ in which the tiles have length at most k, where a tile of length i is assigned the weight α_i. It will be shown that the sequence $\{\sqrt[n]{a_n}\}$ is strictly log-concave for all n sufficiently large under a couple of assumptions concerning the constants α_i (see Theorem 2.4 below). As a special case, one obtains the log-concavity result mentioned in the previous paragraph for the second-order sequence u_n.

We now recall two well-known classes of recurrences. Letting $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 1$ in (1.1), one gets the *k-Fibonacci* sequence, which we will denote here by $f_n^{(k)}$. The sequence $f_n^{(k)}$ was first considered by Knuth [5] and has been given interpretations in terms of linear tilings [1, Chapter 3] and k-filtering linear partitions [8]. When $\alpha_1 = \alpha_k = 1$ and all other α_i are zero, one gets a class of sequences known as the *k-bonacci* numbers (see, e.g., [1, Section 3.4]), which we will denote by $g_n^{(k)}$. Note that both $f_n^{(k)}$ and $g_n^{(k)}$ reduce to the usual Fibonacci numbers when $k = 2$. It will be shown that the ratio $\frac{\sqrt[n]{a_n}}{\sqrt[n-1]{a_{n-1}}}$ is decreasing for all $n \geq N$ for some N depending on k whenever $a_n = f_n^{(k)}$ or $g_n^{(k)}$.

In the third section, we consider the special cases of $f_n^{(k)}$ and $g_n^{(k)}$ when $k = 3$ and $k = 4$ and show that one may take N to be an integer less than 12 in each of these cases. Our method will apply to finding the best possible N for any *given* sequence a_n satisfying a recurrence of the form (1.1) for which $\sqrt[n]{a_n}$ is eventually log-concave.

2. Main results

Given $k \geq 2$, let a_n be a sequence of non-negative real numbers defined by the recurrence

$$a_n = \alpha_1 a_{n-1} + \alpha_2 a_{n-2} + \cdots + \alpha_k a_{n-k}, \qquad n \geq k, \tag{2.1}$$

with $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1$, where the α_i are fixed real numbers and $\alpha_k \neq 0$. The characteristic equation associated with the sequence a_n is defined by

$$f(x) := x^k - \alpha_1 x^{k-1} - \alpha_2 x^{k-2} - \cdots - \alpha_k = 0. \tag{2.2}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the roots of (2.2). By [7, Lemma 5.2], we have

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_k \lambda_k^n, \qquad n \geq 0, \tag{2.3}$$

where

$$c_i = \frac{1}{\prod_{j=1, j\neq i}^{k} (\lambda_i - \lambda_j)}, \qquad 1 \leq i \leq k,$$

whenever the λ_i are distinct. Upon writing

$$f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k),$$

we have by the product rule of differentiation that

$$f'(\lambda_i) = \prod_{j=1, j\neq i}^{k} (\lambda_i - \lambda_j) = \frac{1}{c_i}, \qquad 1 \leq i \leq k.$$

Definition 2.1. *A zero of a polynomial g will be called dominant if it is simple and is strictly greater in modulus than all of its other zeros.*

Note that if g has real coefficients, then a dominant zero must be real since non-real zeros come in conjugate pairs.

Lemma 2.2. *If $f(x)$ defined by (2.2) has a dominant zero λ, then $\lambda > 0$ and $f'(\lambda) > 0$.*

Proof. Suppose $\lambda = \lambda_1$. Define

$$e_n = \frac{\sum_{i=2}^{k} c_i \lambda_i^n}{c_1 \lambda_1^n}, \qquad n \geq 0. \tag{2.4}$$

Note that $a_n = c_1 \lambda_1^n (1 + e_n)$, by (2.3). Thus λ_1 and $c_1 = \frac{1}{f'(\lambda_1)}$ real implies e_n is real. Note further that $e_n \to 0$ as $n \to \infty$ since λ_1 is dominant. Taking n to be large and even implies $c_1 > 0$ and thus $f'(\lambda_1) = \frac{1}{c_1} > 0$. Taking n to be large and odd then implies λ_1 is positive. □

The following limit holds for the numbers e_n.

Lemma 2.3. *Suppose that the polynomial $f(x)$ defined by (2.2) has dominant zero λ. Then we have*

$$\lim_{n\to\infty} (1 + e_n)^{p(n)} = 1, \tag{2.5}$$

for any polynomial $p(n)$.

Proof. We provide a proof only in the case when the λ_i are distinct, the proof in the case when some of the λ_i are repeated being similar. We will show

$$\lim_{n\to\infty} (1 + |e_n|)^{p(n)} = \lim_{n\to\infty} (1 - |e_n|)^{p(n)} = 1, \tag{2.6}$$

from which (2.5) follows. (Note that $1 - |e_n|$ is positive for n sufficiently large, which implies that the expression $(1 - |e_n|)^{p(n)}$ is real for all such n.) Let

$$r = \frac{\max\{|\lambda_2|, |\lambda_3|, \ldots, |\lambda_k|\}}{\lambda_1}$$

and

$$M = \frac{\max\{|c_2|, |c_3|, \ldots, |c_k|\}}{c_1}.$$

Note that

$$|e_n| \leq (k-1) M r^n, \qquad n \geq 0,$$

so to show (2.6), we only need to show

$$\lim_{n\to\infty} (1 + cr^n)^{p(n)} = \lim_{n\to\infty} (1 - cr^n)^{p(n)} = 1, \tag{2.7}$$

for constants $c > 0$ and $0 < r < 1$. The limits in (2.7) can be evaluated by taking a logarithm and applying l'Hôpital's rule, which completes the proof. □

Theorem 2.4. *Suppose that the characteristic polynomial $f(x)$ associated with the sequence a_n has dominant zero λ such that $f'(\lambda) > 1$. Then the sequence of ratios $\frac{\sqrt[n]{a_n}}{\sqrt[n-1]{a_{n-1}}}$ is strictly decreasing for all $n \geq N$, for some N depending on the α_i, and has limit 1.*

Proof. We provide a proof only in the case when the λ_i are distinct. First observe that

$$\frac{\sqrt[n]{a_n}}{\sqrt[n-1]{a_{n-1}}} > \frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}$$

if and only if

$$\left[c_1\lambda_1^n(1+e_n)\right]^{2/n} > \left[c_1\lambda_1^{n+1}(1+e_{n+1})\right]^{1/(n+1)} \left[c_1\lambda_1^{n-1}(1+e_{n-1})\right]^{1/(n-1)},$$

which may be rewritten as

$$\frac{(1+e_n)^{2(n^2-1)}}{(1+e_{n-1})^{n(n+1)}(1+e_{n+1})^{n(n-1)}} > c_1^2. \tag{2.8}$$

By Lemma 2.3, we have

$$\lim_{n\to\infty}(1+e_n)^{2(n^2-1)} = \lim_{n\to\infty}(1+e_{n-1})^{n(n+1)} = \lim_{n\to\infty}(1+e_{n+1})^{n(n-1)} = 1,$$

which implies (2.8) since $c_1 = \frac{1}{f'(\lambda_1)} < 1$.

For the last statement, note that

$$\log\left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right) = \frac{1}{n+1}\log a_{n+1} - \frac{1}{n}\log a_n = \frac{\log c_1 + \log(1+e_{n+1})}{n+1} - \frac{\log c_1 + \log(1+e_n)}{n},$$

and take limits as $n\to\infty$. □

Corollary 2.5. *If a_n is a sequence such that $f(x)$ has a dominant zero λ satisfying $f'(\lambda) > 1$, then $\sqrt[n]{a_n}$ is strictly increasing for all sufficiently large n.*

Remark: If we allow the sequence a_n to contain negative terms, then modifying slightly the proof of Theorem 2.4 yields the result for $|a_n|$.

Let us exclude for now from consideration recurrences of the form

$$a_n = \alpha_d a_{n-d} + \alpha_{2d}a_{n-2d} + \cdots + \alpha_k a_{n-k}, \qquad n \geq k,$$

for some divisor $d > 1$ of k and subject to the same initial conditions. Observe that such recurrences may be reduced, upon letting $b_m = a_{dm+d-1}$, to those of the form

$$b_m = \alpha_d b_{m-1} + \alpha_{2d}b_{m-2} + \cdots + \alpha_k b_{m-\frac{k}{d}}, \qquad m \geq \frac{k}{d},$$

where $b_0 = b_1 = \cdots = b_{\frac{k}{d}-2} = 0$ and $b_{\frac{k}{d}-1} = 1$ (note that $a_{dm+r} = 0$ for all m if $0 \leq r < d-1$, by the initial conditions).

We now describe a class of recurrences frequently arising in applications for which the characteristic polynomial has a dominant zero.

Lemma 2.6. *Suppose that $\alpha_i \geq 0$ for all i in (2.1) with $\alpha_k \neq 0$ and furthermore that it is not the case that $\alpha_i = 0$ for all $i \in [k] - \{d, 2d, \ldots, k\}$ for some divisor $d > 1$ of k. Then $f(x)$ has a dominant zero.*

Proof. Let $f(x) = x^k - \alpha_1 x^{k-1} - \cdots - \alpha_k$, where the α_i satisfy the given hypotheses. By Descartes' rule of signs, the equation $f(x) = 0$ has a single (simple) positive root, which we will denote by λ. Let ρ be any root of the equation $f(x) = 0$ other than λ. We will show that the numbers $\alpha_i\rho^{k-i}$, $1 \leq i \leq k$, cannot all be non-negative real numbers. Suppose, to the contrary, that this is the case.

Let $\{i_1, i_2, \dots, i_a\}$ denote the set of indices i such that $\alpha_i \neq 0$. Let $b = \min\{i_{j+1} - i_j : 1 \leq j \leq a-1\}$ and ℓ be an index such that $i_{\ell+1} - i_\ell = b$. Then

$$\alpha_{i_{\ell+1}} \rho^{i_{\ell+1}} = r \alpha_{i_\ell} \rho^{i_\ell}$$

for some $r > 0$ implies

$$\rho = \left(\frac{r \alpha_{i_\ell}}{\alpha_{i_{\ell+1}}} \right)^{1/b} \xi,$$

where ξ denotes a primitive b'-th root of unity for some positive divisor b' of b. Note that $b' > 1$ since $f(x)$ has only one positive real zero. If b' does not divide k, then ρ^k is not a positive real since $\xi^k \neq 1$ in this case. But this contradicts the equality $\rho^k = \alpha_1 \rho^{k-1} + \cdots + \alpha_k$, since the right-hand side is a positive real. Thus b' divides k and so it must be the case that there exists some index m such that the difference $c = i_{m+1} - i_m$ is not divisible by b' (for otherwise, the second hypothesis concerning the α_i would be contradicted). But then

$$\alpha_{i_{m+1}} \rho^{\alpha_{i_{m+1}}} = s \alpha_{i_m} \rho^{\alpha_{i_m}}$$

for some $s > 0$ implies ρ^c is a positive real number and hence $\xi^c = 1$, which implies b' divides c, a contradiction.

Thus, the $\alpha_i \rho^{k-i}$ cannot all be non-negative real numbers. Suppose i' is such that $\alpha_{i'} \rho^{k-i'}$ is either negative or not real. Note that the assumption $\alpha_k > 0$ implies $i' < k$. Then we may write

$$\begin{aligned}
|\rho|^k = |\rho^k| = \left| \sum_{i=1}^{k} \alpha_i \rho^{k-i} \right| &= \left| \alpha_k + \alpha_{i'} \rho^{k-i'} + \sum_{i=1, i \neq i'}^{k-1} \alpha_i \rho^{k-i} \right| \\
&\leq \left| \alpha_k + \alpha_{i'} \rho^{k-i'} \right| + \left| \sum_{i=1, i \neq i'}^{k-1} \alpha_i \rho^{k-i} \right| \leq \left| \alpha_k + \alpha_{i'} \rho^{k-i'} \right| + \sum_{i=1, i \neq i'}^{k-1} \alpha_i |\rho|^{k-i} \\
&< \alpha_k + \alpha_{i'} |\rho|^{k-i'} + \sum_{i=1, i \neq i'}^{k-1} \alpha_i |\rho|^{k-i} = \sum_{i=1}^{k} \alpha_i |\rho|^{k-i},
\end{aligned}$$

where the last inequality is strict since $\alpha_{i'} \rho^{k-i'}$ is not a positive real number. But then we have $|\rho|^k < \sum_{i=1}^k \alpha_i |\rho|^{k-i}$, which implies $f(|\rho|) < 0$. Since $f(x) > 0$ if $x > \lambda$ and $f(x) < 0$ if $0 < x < \lambda$, it follows that $|\rho| < \lambda$, as desired. □

Remark: By Theorem 2.4, for sequences a_n defined by a recurrence of the form (2.1), where the α_i satisfy the hypotheses of Lemma 2.6, one needs only to verify the condition $f'(\lambda) > 1$ in order to establish the log-concavity of $\sqrt[n]{a_n}$ for large n.

We now apply the previous results to the sequences $\sqrt[n]{f_n^{(k)}}$ and $\sqrt[n]{g_n^{(k)}}$ where $k \geq 2$.

Theorem 2.7. *The characteristic polynomial $f(x)$ associated with either the sequence $f_n^{(k)}$ or $g_n^{(k)}$ has a dominant zero λ such that $f'(\lambda) > 1$. Thus, for $k \geq 2$, the sequences $\sqrt[n]{f_n^{(k)}}$ and $\sqrt[n]{g_n^{(k)}}$ are log-concave for all $n \geq N$ for some constant N depending on k.*

Proof. We need only to verify the first statement in each case. Note that both $f_n^{(k)}$ and $g_n^{(k)}$ are defined by recurrences such that the constants α_i satisfy the conditions given in Lemma 2.6. Thus, we need only to verify $f'(\lambda) > 1$. In the case of $f_n^{(k)}$, this follows easily since

$$f'(\lambda) = k\lambda^{k-1} - (k-1)\lambda^{k-2} - \cdots - 1 = k\left(\lambda^{k-2} + \lambda^{k-3} + \cdots + \frac{1}{\lambda}\right) - (k-1)\lambda^{k-2} - \cdots - 1$$
$$= \frac{k}{\lambda} + \lambda^{k-2} + 2\lambda^{k-3} + \cdots + (k-1) > 1.$$

In the case of $g_n^{(k)}$, note that $\lambda > 1$ since $f(1) < 0$. Then $f'(\lambda) = \lambda^{k-2}(1 + k(\lambda - 1)) > 1$ since $\lambda > 1$, which completes the proof. □

3. Third and fourth order sequences

In this section, we will determine the smallest possible N in Theorem 2.4 in some particular cases. The method illustrated here can be applied to other sequences in finding the smallest N. Let us denote the $k = 3$ cases of the sequences $f_n^{(k)}$ and $g_n^{(k)}$ by t_n and r_n, respectively. The t_n and r_n are known as the *tribonacci* and 3-*bonacci* numbers, respectively. See, e.g., [1, Section 3.3] and also the sequences A000073 and A000930 in [9].

We have the following estimates for the values of the c_i and λ_i in (2.3) in the cases of t_n and r_n.

Values corresponding to the sequence t_n:

$$c_1 = 0.182803, \quad c_2 = -0.091401 + 0.340546i \quad \text{and} \quad c_3 = \overline{c_2},$$
$$\lambda_1 = 1.839286, \quad \lambda_2 = -0.419643 + 0.606290i \quad \text{and} \quad \lambda_3 = \overline{\lambda_2}.$$

Values corresponding to the sequence r_n:

$$c_1 = 0.284693, \quad c_2 = -0.142346 + 0.305033i \quad \text{and} \quad c_3 = \overline{c_2},$$
$$\lambda_1 = 1.465571, \quad \lambda_2 = -0.232785 + 0.792551i \quad \text{and} \quad \lambda_3 = \overline{\lambda_2}.$$

We will make use of these estimates in the proof of the following result.

Theorem 3.1. *The ratio* $\frac{\sqrt[n]{a_n}}{\sqrt[n-1]{a_{n-1}}}$ *is strictly decreasing for all* $n \geq 4$ *when* $a_n = t_n$ *and for all* $n \geq 8$ *when* $a_n = r_n$.

Proof. We first consider the case t_n. One can verify by direct computation that

$$\frac{\sqrt[n]{t_n}}{\sqrt[n-1]{t_{n-1}}} > \frac{\sqrt[n+1]{t_{n+1}}}{\sqrt[n]{t_n}}$$

for $4 \leq n \leq 9$, so we may assume $n \geq 10$. By (2.8), it suffices to show

$$(1 + e_n)^{2(n^2-1)} > c_1^{2/3}, \quad (1 + e_{n-1})^{n(n+1)} < c_1^{-2/3} \quad \text{and} \quad (1 + e_{n+1})^{n(n-1)} < c_1^{-2/3}, \tag{3.1}$$

for $n \geq 10$.

To do so, first note that

$$|e_n| = \left|\frac{2\mathrm{Re}(c_2\lambda_2^n)}{c_1\lambda_1^n}\right| \leq \frac{2|c_2|}{c_1}\left(\frac{|\lambda_2|}{\lambda_1}\right)^n = \frac{|\lambda_1 - \lambda_2|}{|\mathrm{Im}(\lambda_2)|}\left(\frac{|\lambda_2|}{\lambda_1}\right)^n < (3.86)(0.41)^n.$$

Thus, to show (3.1), it is enough to show

$$(1-M_n)^{2(n^2-1)} > c_1^{2/3}, \quad (1+M_{n-1})^{n(n+1)} < c_1^{-2/3} \quad \text{and} \quad (1+M_{n+1})^{n(n-1)} < c_1^{-2/3}, \tag{3.2}$$

where $M_n = (3.86)(0.41)^n$. Since M_n is a decreasing positive sequence, we have $(1+M_{n-1})^{n(n+1)} > (1+M_{n+1})^{n(n-1)}$, so we only need to show the first two inequalities in (3.2).

The first inequality in (3.2) holds if and only if $\log(1-M_n) > \frac{\log c_1}{3(n^2-1)}$. For this last inequality, we can show

$$M_n + M_n^2 < -\frac{\log(0.19)}{3(n^2-1)}, \qquad n \geq 10, \tag{3.3}$$

since $c_1 < 0.19$ and $-\log(1-y) < y + y^2$ for $0 < y < \frac{1}{2}$. To show (3.3), let $a(x) = -\frac{\log(0.19)}{3(x^2-1)}$ and $b(x) = M_x + M_x^2$, where M_x has the obvious meaning. Observe that $a(10) > b(10)$ and $\lim_{x\to\infty}(a(x)-b(x)) = 0$. Thus to prove $a(x) > b(x)$ for $x \geq 10$, it suffices to show $a'(x) < b'(x)$ for $x \geq 10$. Since $\frac{2}{3x^3} < \frac{2x}{3(x^2-1)^2}$, it is enough to show

$$\frac{(3.86)\log(0.41)}{\log(0.19)}(0.41)^x + \frac{2(3.86)^2\log(0.41)}{\log(0.19)}(0.41)^{2x} < \frac{2}{3x^3},$$

and for this, it is enough to show

$$\frac{\log(0.19)}{(3.86)\log(0.41)}(0.41)^{-x} > 3x^3, \qquad x \geq 10. \tag{3.4}$$

Note that (3.4) holds for $x = 10$, with the derivative of the difference of the two sides seen to be positive for all $x \geq 10$. This finishes the proof of the first inequality in (3.2).

We proceed in a similar manner to verify the second inequality in (3.2). Since $\log(1+y) < y$ for $y > 0$, it suffices to show $c(x) > d(x)$ for $x \geq 10$, where $c(x) = -\frac{2\log(0.19)}{3x(x+1)}$ and $d(x) = M_{x-1}$. Since $c(10) > d(10)$ and $\lim_{x\to\infty}(c(x)-d(x)) = 0$, we only need to show that $c'(x) < d'(x)$ for $x \geq 10$. Now $c'(x) < d'(x)$ if and only if

$$\frac{2(2x+1)}{3x^2(x+1)^2} > \frac{(3.86)\log(0.41)}{\log(0.19)}(0.41)^{x-1}, \qquad x \geq 10. \tag{3.5}$$

Since

$$\frac{2(2x+1)}{3x^2(x+1)^2} > \frac{2(2x+1)}{3\left(x+\frac{1}{2}\right)^4} = \frac{4}{3\left(x+\frac{1}{2}\right)^3},$$

to prove (3.5), one can show

$$(0.41)^{1-x} > \frac{3}{4}(2.08)\left(x+\frac{1}{2}\right)^3, \qquad x \geq 10,$$

which can be done by comparing the derivatives of the two sides. This establishes the second inequality in (3.2) and completes the proof in the case when $a_n = t_n$.

A similar proof can be given when $a_n = r_n$, which we outline as follows. We first verify by computation that

$$\frac{\sqrt[n]{r_n}}{\sqrt[n-1]{r_{n-1}}} > \frac{\sqrt[n+1]{r_{n+1}}}{\sqrt[n]{r_n}}$$

for $8 \leq n \leq 17$. Thus, we may assume $n \geq 18$ in showing (3.1) for r_n. We use the bounding function of $M_n = (2.37)(0.57)^n$ in proving the first two inequalities in (3.2). For the first inequality, instead of (3.4), one needs to show

$$\frac{\log(0.29)}{(2.37)\log(0.57)}(0.57)^{-x} > 3x^3, \qquad x \geq 18,$$

which can be done by a comparison of the derivatives of the two sides. In proving the second inequality in (3.2) above for r_n, it is enough to verify

$$(0.57)^{1-x} > \frac{3}{4}(1.08)\left(x+\frac{1}{2}\right)^3, \qquad x \geq 18.$$

This can be done by comparing derivatives of the two sides for $x \geq 18$, which completes the proof in the r_n case. □

By Theorems 2.4 and 3.1 and direct computation, we obtain the following.

Corollary 3.2. *The sequence* $\sqrt[n]{a_n}$ *is strictly increasing for* $n \geq 5$ *when* $a_n = t_n$ *or* r_n.

Let p_n and q_n denote the respective $k = 4$ cases of the $f_n^{(k)}$ and $g_n^{(k)}$. The p_n and q_n are known as the *tetranacci* and 4-*bonacci* numbers and occur, respectively, as sequences A000078 and A017898 in [9]. A proof comparable to the previous one yields the following result.

Theorem 3.3. *The ratio* $\frac{\sqrt[n]{a_n}}{\sqrt[n-1]{a_{n-1}}}$ *is strictly decreasing for all* $n \geq 5$ *when* $a_n = p_n$ *and for all* $n \geq 11$ *when* $a_n = q_n$.

Given the prior two results, one might wonder if one can find some bound for the best possible N as a function of k. In the case of $f_n^{(k)}$, such a bound seems possible in light of the fact (see [7, Lemma 5.2]) that the dominant zero of the associated characteristic polynomial approaches 2 as k approaches infinity, with all other zeros of modulus strictly less than 1 and distinct. By the present method, one would need an estimate of the magnitude of the constants c_i in (2.3). In particular, it would be useful to have a lower bound (as a function of k) for the quantity

$$m(k) := \min_{2 \leq i \leq k} \left| \prod_{j=1, j \neq i}^{k} (\lambda_i - \lambda_j) \right|.$$

If $m(k)$ can be shown, for example, to be no smaller than ab^{-k} for some constants a and b with $b > \frac{1}{2}$, then a bound for N in terms of k could probably be obtained.

References

[1] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, 2003.

[2] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, *Contemp. Math.* **178** (1994) 71–89.

[3] W. Y. C. Chen, J. J. F. Guo and L. X. W. Wang, Zeta functions and the log-behavior of combinatorial sequences, *Proc. Edinb. Math. Soc.* (2), in press.

[4] Q.-H. Hou, Z.-W. Sun and H. Wen, On monotonicity of some combinatorial sequences, *Publ. Math. Debrecen*, in press, arXiv:1208.3903.

[5] D. E. Knuth, *The Art of Computer Programming: Sorting and Searching*, Vol. 3, Addison-Wesley, 1973.

[6] L. L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, *Adv. in Appl. Math.* **39** (2007) 453–476.

[7] T. Mansour and M. Shattuck, Polynomials whose coefficients are k-Fibonacci numbers, *Ann. Math. Inform.* **40** (2012) 57–76.
[8] E. Munarini, A combinatorial interpretation of the generalized Fibonacci numbers, *Adv. in Appl. Math.* **19** (1998) 306–318.
[9] N. J. A. Sloane, The On-Line Enyclopedia of Integer Sequences, available at http://oeis.org, 2010.
[10] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Ann. New York Acad. Sci.* **576** (1989) 500–534.
[11] Z.-W. Sun, On a sequence involving sums of primes, *Bull. Aust. Math. Soc.* **88** (2013) 197–205.
[12] Z.-W. Sun, Conjectures involving arithmetical sequences, *Number Theory: Arithmetic in Shangri-La*, Proceedings of the 6th China-Japan Seminar (Shaghai, 2011), *World Scientific* (2013) 244-258.
[13] Y. Wang and B. X. Zhu, Proofs of some conjectures on monotonicity of number theoretic and combinatorial sequences, *Sci. China Math.* **57** (2014) 2429–2435.