# Coefficients and roots of peak polynomials 

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#### Abstract

Given a permutation $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in \mathfrak{S}_{n}$, we say an index $i$ is a peak if $\pi_{i-1}<$ $\pi_{i}>\pi_{i+1}$. Let $P(\pi)$ denote the set of peaks of $\pi$. Given any set $S$ of positive integers, define $\mathcal{P}_{S}(n)=\left\{\pi \in \mathfrak{S}_{n}: P(\pi)=S\right\}$. Billey-Burdzy-Sagan showed that for all fixed subsets of positive integers $S$ and sufficiently large $n,\left|\mathcal{P}_{S}(n)\right|=p_{S}(n) 2^{n-|S|-1}$ for some polynomial $p_{S}(x)$ depending on $S$. They conjectured that the coefficients of $p_{S}(x)$ expanded in a binomial coefficient basis centered at $\max (S)$ are all positive. We show that this is a consequence of a stronger conjecture that bounds the modulus of the roots of $p_{S}(x)$. Furthermore, we give an efficient explicit formula for peak polynomials in the binomial basis centered at 0 , which we use to identify many integer roots of peak polynomials along with certain inequalities and identities.


## 1 Introduction

Let $\mathfrak{S}_{n}$ be the symmetric group of all permutations $\pi=\pi_{1} \pi_{2} \ldots \pi_{n}$ of $[n]:=\{1,2, \ldots, n\}$. An index $1<i<n$ of $\pi$ is a peak if $\pi_{i-1}<\pi_{i}>\pi_{i+1}$, and the peak set of $\pi$ is defined as $P(\pi):=\{i: i$ is a peak of $\pi\}$. We are interested in counting the permutations in $\mathfrak{S}_{n}$ with a fixed peak set, so let $\mathcal{P}_{S}(n):=\left\{\pi \in \mathfrak{S}_{n}: P(\pi)=S\right\}$. We say that a set $S=\left\{i_{1}<i_{2}<\cdots<i_{s}\right\}$ is $n$-admissible if $\left|\mathcal{P}_{S}(n)\right| \neq 0$. Note that we insist the elements of $S$ be listed in increasing order and that $S$ is $n$-admissible if and only if $1<i_{1}$, no two $i_{r}$

[^0]are consecutive integers, and $i_{s}<n$. If we make a statement about an admissible set $S$, we mean that $S$ is $n$-admissible for some $n$, and the statement holds for every $n$ such that $S$ is $n$-admissible. Burdzy, Sagan, and the first author recently proved the following result in Billey et al. 2013.

Theorem 1.1 ([Billey et al. 2013, Theorem 3]). If $S$ is a nonempty admissible set and $m=\max (S)$, then

$$
\left|\mathcal{P}_{S}(n)\right|=p_{S}(n) 2^{n-|S|-1}
$$

for $n \geq m$, where $p_{S}(x)$ is a polynomial of degree $m-1$ depending on $S$ such that $p_{S}(n)$ is an integer for all integral inputs $n$. If $S=\emptyset$, then $\left|\mathcal{P}_{S}(n)\right|=2^{n-1}$ and $p_{\emptyset}(n)=1$.

If $S$ is not admissible, then $\left|\mathcal{P}_{S}(n)\right|=0$ for all positive integers $n$, and we define the corresponding polynomial to be $p_{S}(x)=0$. Thus, for all finite sets $S$ of positive integers, $p_{S}(x)$ is a well-defined polynomial, which is called the peak polynomial for $S$.

In this paper, we study properties of peak polynomials such as their expansions into binomial bases, roots, and related inequalities and identities. We also enumerate permutations with a given peak set using alternating permutations and connect our results to other recent work about the peak statistic [Billey et al. 2013; Castro-Velez et al. 2013; Holroyd and Liggett 2014 Kasraoui 2012. Our primary motivation comes from combinatorics, information theory, and probability theory. Peaks sets have been studied for decades going back to [Kermack and McKendrick 1937] and used more recently in a probabilistic project concerned with mass redistribution [Burdzy et al. 2013]. Below are the principal results of this paper.

Theorem 1.2. Let $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m\right\}$ be admissible and nonempty. For $0 \leq j \leq m-1$, define the coefficients

$$
d_{j}^{S}=(-1)^{m-j-1}(-2)^{|S \cap(j, \infty)|-1} p_{S \cap[j]}(j) .
$$

If there exists an index $1 \leq r \leq s-1$ such that $i_{r+1}-i_{r}$ is odd, let $b=i_{r}$ for the largest such $r$. Then the peak polynomial $p_{S}(x)$ expands in the binomial basis centered at 0 as

$$
p_{S}(x)=\sum_{j=b}^{m-1} d_{j}^{S}\binom{x}{j}
$$

Otherwise, if there are no odd gaps, then

$$
p_{S}(x)=\left(d_{0}^{S}-(-2)^{|S|-1}\right)+\sum_{j=1}^{m-1} d_{j}^{S}\binom{x}{j} .
$$

Observe that by Theorem 1.1, $p_{S}(m)=0$ using the fact that $\mathcal{P}_{S}(m)$ is empty, but we may have $p_{S}(\ell) \neq 0$ for $\ell<m$ even though $\left|\mathcal{P}_{S}(\ell)\right|=0$. The next two results describe additional roots of $p_{S}(x)$.

Corollary 1.3. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}\right\}$ and $i_{r+1}-i_{r}$ is odd for some $1 \leq r \leq s-1$, then $0,1,2, \ldots, i_{r}$ are roots of $p_{S}(x)$.

Theorem 1.4. We have $p_{S}(i)=0$ for all $i \in S$.

Now we discuss two conjectures that inspired this paper. In the calculus of finite differences, we define the forward difference operator $\Delta$ to be $(\Delta f)(x)=f(x+1)-f(x)$. Higher order differences are given by $\left(\Delta^{n} f\right)(x)=\left(\Delta^{n-1} f\right)(x+1)-\left(\Delta^{n-1} f\right)(x)$. We use the definition of the Newton interpolating polynomial to expand $p_{S}(x)$ in the binomial basis centered at $k$ as

$$
p_{S}(x)=\sum_{j=0}^{m}\left(\Delta^{j} p_{S}\right)(k)\binom{x-k}{j} .
$$

Notice its similarity to Taylor's theorem. Below is an example of the forward differences of $p_{\{2,6,10\}}(x)$. The $k$-th column in the table is the basis vector for the expansion of $p_{\{2,6,10\}}(x)$ in the binomial basis centered at $k$. In this paper, we consider expansions centered at 0 and $m$.

| $j, k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | -8 | -4 | 0 | 2 | 4 | 6 | 0 | -18 | -72 | -196 | 0 |
| 1 | 4 | 4 | 2 | 2 | 2 | -6 | -18 | -54 | -124 | 196 | 3094 |
| 2 | 0 | -2 | 0 | 0 | -8 | -12 | -36 | -70 | 320 | 2898 | 12376 |
| 3 | -2 | 2 | 0 | -8 | -4 | -24 | -34 | 390 | 2578 | 9478 | 26564 |
| 4 | 4 | -2 | -8 | 4 | -20 | -10 | 424 | 2188 | 6900 | 17086 | 36376 |
| 5 | -6 | -6 | 12 | -24 | 10 | 434 | 1764 | 4712 | 10186 | 19290 | 33324 |
| 6 | 0 | 18 | -36 | 34 | 424 | 1330 | 2948 | 5474 | 9104 | 14034 | 20460 |
| 7 | 18 | -54 | 70 | 390 | 906 | 1618 | 2526 | 3630 | 4930 | 6426 | 8118 |
| 8 | -72 | 124 | 320 | 516 | 712 | 908 | 1104 | 1300 | 1496 | 1692 | 1888 |
| 9 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 | 196 |
| 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 1: Forward differences of $p_{\{2,6,10\}}(x)$.
We know from Theorem 1.1 that $\left(\Delta^{0} p_{S}\right)(m)=0,\left(\Delta^{m-1} p_{S}\right)(k)$ is a positive integer, and $\left(\Delta^{j} p_{S}\right)(k)=0$ for all $k \in \mathbb{Z}$ and $j \geq m$. Burdzy, Sagan, and the first author proposed the following positivity conjecture in [Billey et al. 2013].

Conjecture 1.5 ( $\left[\right.$ Billey et al. 2013, Conjecture 14]). Each coefficient $\left(\Delta^{j} p_{S}\right)(m)$ is a positive integer for $1 \leq j \leq m-1$ and all admissible sets $S$.

It follows from Stanley's text [Stanley 2012, Corollary 1.9.3] that $p_{S}(n)$ is an integer for all integers $n$ if and only if the coefficients in the expansion of $p_{S}(n)$ in a binomial basis are integers, so we only need to prove that $\left(\Delta^{j} p_{S}\right)(m)$ is positive for $1 \leq j \leq m-1$. In the next section, we show that the positivity conjecture is a consequence of the following stronger conjecture.

Conjecture 1.6. The complex roots of $p_{S}(z)$ lie in $\{z \in \mathbb{C}:|z| \leq m$ and $\operatorname{Re}(z) \geq-3\}$ if $S$ is admissible.

Conjecture 1.6 is similar in nature to the Riemann Hypothesis. More specifically, our work fits into a bigger context of studying roots for polynomials with integer coefficients in
some basis. For example, the roots of Ehrhart polynomials |Beck et al. 2005; Braun and Develin 2008; Bump et al. 2000; Pfeifle 2010], chromatic polynomials |Brenti 1992; Brenti et al. 1994], and Hilbert polynomials Rodriguez-Villegas 2002] have all been shown to respect similar bounds on the complex plane. Additionally, we are investigating the roots of peak polynomials, because they may encode properties of their peak set, similar to how the roots of a chromatic polynomial $P(G, k)$ encode the number of connected components, blocks, and acyclic orientations of $G$.

The paper is organized as follows. In Section 2, we prove that Conjecture 1.6 implies the positivity conjecture. Section 3 proves Theorems $1.2,1.3,1.4$, and identifies some special peak polynomials. Section 4 demonstrates some behaviors of peak polynomials evaluated at nonnegative integers and patterns in the table of forward differences of $p_{S}(x)$. Section 5 develops a new method for counting the number of permutations with a given peak set using alternating permutations and the inclusion-exclusion principle. In Section 6, we relate our work to other recent results about permutations with a given peak set. We conclude with several conjectures suggested by our investigation.

## 2 An approach to the positivity conjecture

The following lemmas form a chain of arguments that proves that the positivity conjecture is a consequence of Conjecture 1.6. We write $p(x)$ or $p(z)$ when we are discussing properties of all polynomials, and we use $p_{S}(x)$ when we are discussing peak polynomials in particular.

Lemma 2.1. If $p(z)$ does not have a complex zero with real part greater than $m$, then $p^{\prime}(z), p^{\prime \prime}(z), \ldots, p^{(m-1)}(z)$ do not have a complex zero with real part greater than $m$, and thus, no real zero greater than $m$.

Proof. We use the Gauss-Lucas theorem, which states that if $p(z)$ is a (nonconstant) polynomial with complex coefficients, then all the zeros of $p^{\prime}(z)$ belong to the convex hull of the set of zeros of $p(z)$. By assumption all of the roots of $p(z)$ lie in the half-plane $\{z \in \mathbb{C}: \operatorname{Re}(z) \leq m\}$, so then by the Gauss-Lucas theorem, all of the roots of $p^{\prime}(z)$ also lie in this half-plane. Repeating this argument, we see that $p^{\prime}(z), p^{\prime \prime}(z), \ldots, p^{(m-1)}(z)$ do not have a complex zero with real part greater than $m$ and thus no real zero greater than $m$.

Lemma 2.2. If $S$ is admissible and none of $p_{S}(x), p_{S}^{\prime}(x), p_{S}^{\prime \prime}(x), \ldots, p_{S}^{(m-1)}(x)$ have a real zero greater than $m$, then $p_{S}(x), p_{S}^{\prime}(x), \ldots, p_{S}^{(m-1)}(x)$ are all positive for $x>m$.

Proof. Since $S$ is admissible, $p_{S}(m+1)$ is a positive integer. If $p_{S}(x)$ is nonpositive for some $x_{0}>m$, then $p_{S}(x)$ has a zero greater than $m$ by the intermediate value theorem, which contradicts the assumption. Therefore $p_{S}(x)$ is positive for $x>m$, so its leading coefficient is positive. It follows that the leading coefficients of $p_{S}^{\prime}(x), p_{S}^{\prime \prime}(x), \ldots, p_{S}^{(m-1)}(x)$ are also positive, so all of the derivatives of $p_{S}(x)$ are eventually positive. Again by the intermediate value theorem, the derivatives $p_{S}^{\prime}(x), p_{S}^{\prime \prime}(x), \ldots, p_{S}^{(m-1)}(x)$ are all positive for $x>m$.

We will need the following proposition which we learned from an online article by Graham Jameson. Since we don't know of a published version of this statement, we will include Jameson's proof for the sake of completeness.

Proposition 2.3 (JJameson 2014, Proposition 17]). For $n \geq 1$, there exists $\xi \in(x, x+n)$ such that $\left(\Delta^{n} p\right)(x)=p^{(n)}(\xi)$.

Proof. We induct on $n$. When $n=1$, we have the mean value theorem. Assume the statement is true for a certain $n$. Then $\left(\Delta^{n+1} p\right)(x)=\left(\Delta^{n}(\Delta p)\right)(x)=\left(\Delta^{n} q\right)(x)$, where $q(x)=(\Delta p)(x)=p(x+1)-p(x)$ is a polynomial. By the induction hypothesis, there exists $\eta \in(x, x+n)$ such that $\left(\Delta^{n} q\right)(x)=q^{(n)}(\eta)=p^{(n)}(\eta+1)-p^{(n)}(\eta)$. By the mean value theorem again, this equals $p^{(n+1)}(\xi)$ for some $\xi \in(\eta, \eta+1)$.

Lemma 2.4. If $p(x)$ is a polynomial of degree $m-1$ and $p^{\prime}(x), p^{\prime \prime}(x), \ldots, p^{(m-1)}(x)$ are positive for $x>m$, then all of the forward differences $(\Delta p)(m),\left(\Delta^{2} p\right)(m), \ldots,\left(\Delta^{m-1} p\right)(m)$ are positive.

Proof. There exists $\xi \in(m, m+n)$ such that $\left(\Delta^{n} p\right)(m)=p^{(n)}(\xi)$ using Lemma 2.3. By assumption, $p^{\prime}(x), p^{\prime \prime}(x), \ldots, p^{(m-1)}(x)$ are positive for $x>m$, so $p^{\prime}(\xi), p^{\prime \prime}(\xi), \ldots, p^{(m-1)}(\xi)$ are positive since $\xi>m$. Therefore, $(\Delta p)(m),\left(\Delta^{2} p\right)(m), \ldots,\left(\Delta^{m-1} p\right)(m)$ are positive.

Theorem 2.5. If $S$ is admissible and $p_{S}(x)$ has no zero whose real part is greater than $m$, then each coefficient $\left(\Delta^{j} p_{S}\right)(m)$ is positive for $1 \leq j \leq m-1$.

Proof. The proof is a consequence of Lemma 2.1, Lemma 2.2, and Lemma 2.4.
It is clear that Conjecture 1.6 satisfies the hypothesis of Theorem 2.5, so we prove Conjecture 1.5 if we can appropriately bound the roots of $p_{S}(x)$.

In the supplemental data set Fahrbach 2013], we used Sage to verify Conjecture 1.5 and Conjecture $\sqrt{1.6}$ for all admissible sets $S$ with $\max (S) \leq 15$. For each row in the table of Fahrbach 2013, we list a peak set $S, p_{S}(x)$, the forward differences of $p_{S}(x)$ centered at $m$, and the complex roots of $p_{S}(z)$. Our Sage code is at the bottom of this document. We initially computed this data to gain insight about the positivity conjecture, but after plotting the complex roots of $p_{S}(z)$, we conjectured Corollary 1.3 and Theorem 1.4 . We also noticed repeated and predictable structure in the complex roots, which led to Conjecture 1.6, Theorem 3.15, and Corollary 3.18.

## 3 Roots of peak polynomials

Our main theorems from the introduction are proved here in Subsection 3.1. In particular, we give an explicit formula for $p_{S}(x)$ in the binomial basis centered at 0. In Subsection 3.2 we look at peak polynomials with only integral roots, and the results in Subsection 3.3 show that if $S$ has a gap of 3 , then $p_{S}(x)$ is independent of the peaks to the left of this gap up to a constant. All of the results in this section assume that $S$ is admissible, though not explicitly stated in the hypothesis. Also, note that $m \neq \max (S)$ in most of the recurrences.

### 3.1 Main results

The following recurrence relations are very efficient for computation and are the foundation of every result in this section.

Corollary 3.1 ([Billey et al. 2013, Corollary 4]). We have

$$
p_{S}(x)=p_{S_{1}}(m-1)\binom{x}{m-1}-2 p_{S_{1}}(x)-p_{S_{2}}(x)
$$

where $S_{1}=S \backslash\{m\}$ and $S_{2}=S_{1} \cup\{m-1\}$.
Lemma 3.2. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+k\right\}$ and $k \geq 2$, then

$$
p_{S}(x)=-2 p_{S_{1}}(x) \chi(k \text { even })+\sum_{j=1}^{k-1}(-1)^{k-1-j} p_{S_{1}}(m+j)\binom{x}{m+j} .
$$

Proof. We induct on $k$ and use Corollary 3.1. In the base case $k=2$, and

$$
p_{S}(x)=-2 p_{S_{1}}(x)+p_{S_{1}}(m+1)\binom{x}{m+1} .
$$

By induction,

$$
\begin{aligned}
p_{S}(x)= & p_{S_{1}}(m+k-1)\binom{x}{m+k-1}-2 p_{S_{1}}(x)-p_{S_{2}}(x) \\
= & p_{S_{1}}(m+k-1)\binom{x}{m+k-1}-2 p_{S_{1}}(x) \\
& -\left[-2 p_{S_{1}}(x) \chi(k-1 \text { even })+\sum_{j=1}^{k-2}(-1)^{k-2-j} p_{S_{1}}(m+j)\binom{x}{m+j}\right] \\
= & -2 p_{S_{1}}(x) \chi(k \text { even })+\sum_{j=1}^{k-1}(-1)^{k-1-j} p_{S_{1}}(m+j)\binom{x}{m+j} .
\end{aligned}
$$

Corollary 3.3. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+k\right\}$ and $k \geq 2$, then

$$
\left|\mathcal{P}_{S}(n)\right|=-\chi(k \text { even })\left|\mathcal{P}_{S_{1}}(n)\right|+\sum_{j=1}^{k-1}(-1)^{k-1-j}\binom{n}{m+j}\left|\mathcal{P}_{S_{1}}(m+j)\right| \cdot\left|\mathcal{P}_{\emptyset}(n-(m+j))\right| .
$$

Proof. Apply Theorem 1.1 to Lemma 3.2 .
We can interpret Corollary 3.3 combinatorially. Choose $m+k-1$ of the $n$ elements and arrange them such that their peak set is $S_{1}$. Arrange the remaining $n-(m+k-1)$ elements so that there are no peaks, and append this sequence to the previous one. In the combined sequence there is either a peak at $m+k, m+k-1$, or no peak after $m$. Since $m+k \in S$,

$$
\left|\mathcal{P}_{S}(n)\right|=\binom{n}{m+k-1}\left|\mathcal{P}_{S_{1}}(m+k-1)\right| \cdot\left|\mathcal{P}_{\emptyset}(n-(m+k-1))\right|-\left|\mathcal{P}_{S_{2}}(n)\right|-\left|\mathcal{P}_{S_{1}}(n)\right| .
$$

We repeat this procedure for $\left|\mathcal{P}_{S_{2}}(n)\right|$ to count all the permutations whose peak set is $S_{1} \cup\{m+k-1\}$, but this also counts permutations whose peak set is $S_{1} \cup\{m+k-2\}$ and $S_{1}$. We repeat this process until we count permutations whose peak set is $S_{1} \cup\{m+1\}$,
but this peak set is inadmissible and terminates the procedure. Notice that $\left|\mathcal{P}_{S_{1}}(n)\right|$ telescopes because it is included in each iteration with an alternating sign.

We now present the peak polynomial for a single peak and the proof of an explicit formula for peak polynomials with nonempty peak sets in the binomial basis centered at 0 . The results about roots due to odd gaps and peaks follow.

Theorem 3.4 ([|Billey et al. 2013, Theorem 6]). If $S=\{m\}$, then

$$
p_{S}(x)=\binom{x-1}{m-1}-1 .
$$

The following lemma is a special case of the well-known Vandermonde identity. The proof is very simple to state in this case, so we include it.

Lemma 3.5. For $m \geq 1$, we have

$$
\binom{x-1}{m-1}=\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{x}{k} .
$$

Proof. We induct on $m$. When $m=1$, both terms are 1. Assume the statement is true for any $m$. Using the induction hypothesis and the standard recurrence,

$$
\binom{x-1}{(m+1)-1}=\binom{x}{m}-\binom{x-1}{m-1}=\sum_{k=0}^{(m+1)-1}(-1)^{(m+1)-1-k}\binom{x}{k}
$$

Proof of Theorem 1.2. The proof follows by iterating Lemma 3.2. In the case that there no odd gaps, we have

$$
p_{S}(x)=(-2)^{|S|-1}\left[\binom{x-1}{i_{1}-1}-1\right]+\sum_{j=i_{1}}^{m-1} d_{j}^{S}\binom{x}{j}
$$

and then use Lemma 3.5 to shift the $p_{\left\{i_{1}\right\}}(x)$ term to the binomial basis centered at 0 .
Corollary 3.6. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}\right\}$ and $i_{r+1}-i_{r}$ is odd for some $1 \leq r \leq s-1$, then $0,1, \ldots, i_{r}$ are roots of $p_{S}(x)$.

Proof. The proof follows from Theorem 1.2.
Corollary 3.7. If $S$ contains an odd peak, then $p_{S}(0)=0$. Otherwise, $p_{S}(0)=(-2)^{|S|}$.
Proof. The proof follows from Theorem 1.2.
Theorem 3.8. We have $p_{S}(i)=0$ for $i \in S$.
Proof. We induct on $|S|$ for all nonempty admissible sets $S$. In the base case $|S|=1$, and $p_{\{m\}}(m)=0$ by Theorem 3.4. In the inductive step, let $m=\max (S)$. If $i \in S_{1}$, then $p_{S_{1}}(i)=0$ by the induction hypothesis, so $p_{S}(i)=0$ by Lemma 3.2. We also know that $p_{S}(m)=0$ by Theorem 1.1, so $p_{S}(i)=0$ for all $i \in S$.

### 3.2 Peak polynomials with only integral roots

All of the peak polynomials in this subsection are completely factored and have all nonnegative integral roots. As a result, they satisfy Conjecture 1.5 by Theorem 2.5, because we have bounded the real part of their roots by $\max (S)$. In the next two lemmas, the leading coefficient is all that is recursively defined, and it depends solely on the structure of $\left\{i_{1}<i_{2}<\cdots<i_{s}\right\}$. In Conjecture 6.5, we propose a classification of all peak polynomials with only integral roots.

Lemma 3.9. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+3\right\}$, then

$$
p_{S}(x)=\frac{p_{S_{1}}(m+1)}{2(m+1)!}(x-(m+3)) \prod_{j=0}^{m}(x-j) .
$$

Proof. Using Lemma 3.2, we see that

$$
\begin{aligned}
p_{S}(x) & =\sum_{j=1}^{2}(-1)^{2-j} p_{S_{1}}(m+j)\binom{x}{m+j} \\
& =\frac{\prod_{j=0}^{m}(x-j)}{(m+1)!}\left[\frac{p_{S_{1}}(m+2)}{m+2}\left(x-\left(m+1+\frac{p_{S_{1}}(m+1)(m+2)}{p_{S_{1}}(m+2)}\right)\right)\right]
\end{aligned}
$$

but $m+3$ is also a zero of $p_{S}(x)$ by Theorem 3.8. Equating the two roots, we have

$$
p_{S_{1}}(m+2)=\frac{(m+2) p_{S_{1}}(m+1)}{2}
$$

so then

$$
p_{S}(x)=\frac{p_{S_{1}}(m+1)}{2(m+1)!}(x-(m+3)) \prod_{j=0}^{m}(x-j) .
$$

Lemma 3.10. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+3<m+5\right\}$, then

$$
p_{S}(x)=\frac{p_{S \backslash\{m+3, m+5\}}(m+1)}{12(m+1)!}(x-(m+5))(x-(m+3))(x-(m-2)) \prod_{j=0}^{m}(x-j) .
$$

Proof. The proof follows from Corollary 3.1 and Lemma 3.9 .
The next two corollaries show how $p_{S}(x)$ grows from $x_{0}$ to $x_{0}+1$ for any $x_{0} \in \mathbb{R}$, and they demonstrate how the roots shift when translating $p_{S}(x)$ to $p_{S}(x+1)$.

Corollary 3.11. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+3\right\}$, then

$$
p_{S}(x+1)=\lim _{t \rightarrow x} \frac{(t+1)(t-(m+2))}{(t-m)(t-(m+3))} p_{S}(t) .
$$

Proof. Write $p_{S}(x+1) / p_{S}(x)$ using Lemma 3.9 and apply Theorem 3.8.

Corollary 3.12. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+3<m+5\right\}$, then

$$
p_{S}(x+1)=\lim _{t \rightarrow x} \frac{(t+1)(t-(m-3))(t-(m+2))(t-(m+4))}{(t-(m-2))(t-m)(t-(m+3))(t-(m+5))} p_{S}(t)
$$

Proof. Write $p_{S}(x+1) / p_{S}(x)$ using Lemma 3.10 and apply Theorem 3.8.
A limit is needed in Corollary 3.11 and Corollary 3.12 , because $p_{S}(m+1)$ is defined and nonzero by Lemma 3.9 and Lemma 3.10, respectively. We now derive closed-form formulas for $p_{S}(x)$ when $S=\{m, m+3, \ldots, m+3 k\}$ and $S=\{m, m+3, \ldots, m+3 k, m+3 k+2\}$ for $k \geq 1$. These formulas are direct consequences of Lemma 3.9 and Lemma 3.10

Corollary 3.13. If $S=\{m, m+3, \ldots, m+3 k\}$ for $k \geq 1$, then

$$
p_{S}(x)=\frac{(m-1)(x-(m+3 k))}{2(m+1)!\left(12^{k-1}\right)} \prod_{j=0}^{m+3(k-1)}(x-j)
$$

Proof. We induct on $k$. In the base case, $k=1$ and $S=\{m, m+3\}$. Using Lemma 3.9 and Theorem 3.4, we have

$$
\begin{aligned}
p_{\{m, m+3\}}(x) & =\frac{p_{\{m\}}(m+1)}{2(m+1)!}(x-(m+3)) \prod_{j=0}^{m}(x-j) \\
& =\frac{(m-1)(x-(m+3))}{2(m+1)!} \prod_{j=0}^{m}(x-j) .
\end{aligned}
$$

In the inductive step, $S=\{m, m+3, \ldots, m+3 k\}$. We use Lemma 3.9 again, because $p_{S_{1}}(m+3 k-2)$ by the inductive hypothesis, and it follows that

$$
\begin{aligned}
p_{S}(x) & =\frac{p_{S_{1}}(m+3 k-2)}{2(m+3 k-2)!}(x-(m+3 k)) \prod_{j=0}^{m+3(k-1)}(x-j) \\
& =\frac{(m-1)(m+3 k-2)!}{2(m+1)!\left(12^{k-2}\right) 3!}\left[\frac{(x-(m+3 k))}{2(m+3 k-2)!} \prod_{j=0}^{m+3(k-1)}(x-j)\right] \\
& =\frac{(m-1)(x-(m+3 k))}{2(m+1)!\left(12^{k-1}\right)} \prod_{j=0}^{m+3(k-1)}(x-j) .
\end{aligned}
$$

Corollary 3.14. If $S=\{m, m+3, \ldots, m+3 k, m+3 k+2\}$ for $k \geq 1$, then

$$
p_{S}(x)=\frac{(m-1)(x-(m+3 k+2))(x-(m+3 k))(x-(m+3 k-5))}{(m+1)!\left(12^{k}\right)} \prod_{j=0}^{m+3(k-1)}(x-j)
$$

Proof. The proof follows from Lemma 3.10 and Theorem 3.13 .

### 3.3 Gap of three independence

The following theorem shows that if $S$ has a gap of three anywhere, then $p_{S}(x)$ is independent of the peaks to the left of that gap up to a constant. Furthermore, the complex roots of $p_{S}(z)$ depend only on the peaks to the right of the gap of three and where this gap occurs. Corollaries of this result follow.

Theorem 3.15. Let $S_{L}=\left\{i_{1}<i_{2}<\cdots<i_{\ell}=m\right\}$ and $S_{R}=\left\{j_{1}=2<j_{2}<\cdots<j_{r}\right\}$. If $S=\left\{i_{1}<i_{2}<\cdots<m<m+3<(m+1)+j_{2}<\cdots<(m+1)+j_{r}\right\}$, then

$$
p_{S}(x)=\frac{p_{S_{L}}(m+1)}{2(m+1)!} p_{S_{R}}(x-(m+1)) \prod_{k=0}^{m}(x-k)
$$

Proof. We first prove the corresponding statement in terms of permutations with a given peak set. Fix a positive integer $n>(m+1)+j_{r}$. Choose $m+1$ of the $n$ elements in $[n]$, and arrange them so that their peak set is $S_{L}$. Now arrange the remaining $n-(m+1)$ elements so that their peak set is $S_{R}$. This construction produces all of the permutations in $\mathfrak{S}_{n}$ whose peak set is $S$ without repetition, because $m+1$ and $m+2$ cannot be peaks since $m$ and $m+3$ are. Thus we have

$$
\begin{equation*}
\left|\mathcal{P}_{S}(n)\right|=\binom{n}{m+1}\left|\mathcal{P}_{S_{L}}(m+1)\right| \cdot\left|\mathcal{P}_{S_{R}}(n-(m+1))\right| . \tag{1}
\end{equation*}
$$

Using Theorem 1.1,

$$
p_{S}(n) 2^{n-|S|-1}=\binom{n}{m+1} p_{S_{L}}(m+1) 2^{(m+1)-\left|S_{L}\right|-1} p_{S_{R}}(n-(m+1)) 2^{(n-(m+1))-\left|S_{R}\right|-1} .
$$

and since $|S|=\left|S_{L}\right|+\left|S_{R}\right|$, we have

$$
p_{S}(n)=\frac{p_{S_{L}}(m+1)}{2(m+1)!} p_{S_{R}}(n-(m+1)) \prod_{k=0}^{m}(n-k) .
$$

This proves the theorem because we have shown that the polynomial on the right and the left agree on an infinite number of values.

From the factorization in (1), we clearly see that $0,1,2, \ldots, m$ are zeros of $p_{S}(z)$, and the roots of $p_{S_{R}}(z)$ are roots of $p_{S}(z)$ when translated to the right by $m+1$ in the complex plane. Note that $\operatorname{deg}\left(p_{S}(x)\right)=m+j_{r}$ because $\max (S)=(m+1)+j_{r}$, but we also see this by counting the $m+1$ leftmost integer roots and then the $j_{r}-1$ roots of $p_{S_{R}}(x)$. Theorem 3.15 also implies Lemma 3.9 when $S_{R}=\{2\}$ for all $S_{L}$, because $p_{\{2\}}(x)=x-2$. The plots and corollaries below demonstrate this independence.


Figure 1: Roots of $p_{\{2,10\}}(z)$.


Figure 2: Roots of $p_{\{4,7,15\}}(z)$.

Corollary 3.16. Let $S_{L}=\left\{i_{1}<i_{2}<\cdots<i_{\ell}=m\right\}, S_{R}=\left\{j_{1}=2<j_{2}<\cdots<j_{r}\right\}$, and $S=\left\{i_{1}<i_{2}<\cdots<m<m+3<(m+1)+j_{2}<\cdots<(m+1)+j_{r}\right\}$. If $S_{R}$ has no zero with real part greater than $j_{r}$, then $p_{S}(x)$ has no zero with real part greater than $\max (S)$.

Proof. The proof follows from Theorem 3.15.
Corollary 3.17. If $S$ has a gap of three, and $p_{S_{R}}(x)$ satisfies the positivity conjecture, then $p_{S}(x)$ satisfies the positivity conjecture.

Proof. The proof follows from Corollary 3.16
Corollary 3.18. Let $S_{L}=\left\{i_{1}<i_{2}<\cdots<i_{\ell}=m\right\}, S_{R}=\left\{j_{1}=2<j_{2}<\cdots<j_{r}\right\}$, and $S=\left\{i_{1}<i_{2}<\cdots<m<m+3<(m+1)+j_{2}<\cdots<(m+1)+j_{r}\right\}$. If we define $S+1=\{i+1: i \in S\}$, then

$$
p_{S+1}(x)=C(S) p_{S}(x-1) x,
$$

where

$$
C(S)=\frac{p_{S_{L}+1}(m+2)}{(m+2) p_{S_{L}}(m+1)}
$$

is a constant depending only on $S$.
Proof. Using Theorem 3.15, we see that

$$
p_{S}(x-1)=\frac{p_{S_{L}}(m+1)}{2(m+1)!} p_{S_{R}}(x-(m+2)) \prod_{k=0}^{m}(x-(k+1))
$$

and

$$
p_{S+1}(x)=\frac{p_{S_{L}+1}(m+2)}{2(m+2)!} p_{S_{R}}(x-(m+2)) \prod_{k=0}^{m+1}(x-k) .
$$

Solving for $p_{S+1}(x)$, we have

$$
p_{S+1}(x)=C(S) p_{S}(x-1) x,
$$

where

$$
C(S)=\frac{p_{S_{L}+1}(m+2)}{(m+2) p_{S_{L}}(m+1)}
$$

depends only on $S$.
Observe that Corollary 3.18 shifts all of the zeros of $p_{S}(z)$ in the complex plane to the right by one and then picks up a new root at 0 since $C(S)$ is a constant. The plots below illustrate this behavior.


Figure 3: Roots of $p_{\{3,5,8,14\}}(z)$.


Figure 4: Roots of $p_{\{4,6,9,15\}}(z)$.

## 4 Evaluating $p_{S}(x)$ at nonnegative integers

In the previous section, we identified integral roots of $p_{S}(x)$, so now we will try to understand the behavior of $p_{S}(x)$ at nonnegative integers $j$ when $p_{S}(j) \neq 0$. We prove that there is a curious symmetry between column and row 0 in the table of forward differences of $p_{S}(x)$ (see Table 11, and that the nonzero values of $\left|p_{S}(j)\right|$ are weakly increasing for $j \in[\max (S)-1]$ when $\min (S) \geq 4$. Again, assume that $S$ is a nonempty admissible set in the following hypotheses.

Lemma 4.1. Let $S \neq \emptyset$ and $m=\max (S)$. For $k \geq 0$, we have

$$
\sum_{j=1}^{k-1}(-1)^{k-1-j} p_{S}(m+j)\binom{m+k}{m+j}=2 p_{S}(m+k) \chi(k \text { even }) .
$$

Proof. Let $T=S \cup\{m+k\}$. We know from Theorem 1.1 that $p_{T}(m+k)=0$, and then apply Lemma 3.2.

Lemma 4.2. For $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+k\right\}$ and $\ell \in[k-1]$, we have $p_{S}(m+\ell)=-p_{S_{1}}(m+\ell)$.

Proof. Using Lemma 3.2 and Lemma 4.1, observe that

$$
\begin{aligned}
p_{S}(m+\ell)= & -2 p_{S_{1}}(m+\ell) \chi(k \text { even })+\sum_{j=1}^{k-1}(-1)^{k-1-j} p_{S_{1}}(m+j)\binom{m+\ell}{m+j} \\
= & -2 p_{S_{1}}(m+\ell) \chi(k \text { even })+(-1)^{k-\ell} \sum_{j=1}^{\ell-1}(-1)^{\ell-1-j} p_{S_{1}}(m+j)\binom{m+\ell}{m+j} \\
& +(-1)^{k-1-\ell} p_{S_{1}}(m+\ell) \\
= & -2 p_{S_{1}}(m+\ell) \chi(k \text { even })+(-1)^{k-\ell} 2 p_{S_{1}}(m+\ell) \chi(\ell \text { even })+(-1)^{k-1-\ell} p_{S_{1}}(m+\ell) .
\end{aligned}
$$

Considering all possible parities of $k$ and $\ell$, we see that $p_{S}(m+\ell)=-p_{S_{1}}(m+\ell)$.
Theorem 4.3. Let $S \neq \emptyset$ and $m=\max (S)$. If $j \in\{0,1, \ldots, m\}$, then

$$
\left(\Delta^{j} p_{S}\right)(0)=(-1)^{m+j} p_{S}(j) .
$$

Proof. We induct on $|S|$. In the base case $|S|=1$, and we use Theorem 3.4 and Lemma 3.5. It follows that

$$
\left(\Delta^{j} p_{\{m\}}\right)(0)= \begin{cases}(-1)^{m-1}-1 & \text { if } j=0 \\ (-1)^{m-1-j} & \text { if } j \in[m-1] \\ 0 & \text { if } j=m\end{cases}
$$

Similarly, we use Theorem 3.4 to evaluate

$$
\begin{aligned}
(-1)^{m+j} p_{S}(j) & =(-1)^{m+j}\left[\binom{j-1}{m-1}-1\right] \\
& = \begin{cases}(-1)^{m+1}-1 & \text { if } j=0, \\
(-1)^{m+j+1} & \text { if } j \in[m-1], \\
0 & \text { if } j=m,\end{cases}
\end{aligned}
$$

which proves the base case.
In the inductive step $|S| \geq 2$, so let $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m<m+k\right\}$ for $k \geq 2$. Using Lemma 3.2 and expanding $p_{S_{1}}(x)$ in the binomial basis centered at 0 ,

$$
\begin{align*}
p_{S}(x) & =-2 p_{S_{1}}(x) \chi(k \text { even })+\sum_{j=m+1}^{m+k-1}(-1)^{k-1-(j-m)} p_{S_{1}}(j)\binom{x}{j} \\
& =-2\left[\sum_{j=0}^{m}\left(\Delta^{j} p_{S_{1}}\right)(0)\binom{x}{j}\right] \chi(k \text { even })+\sum_{j=m+1}^{m+k-1}(-1)^{k-1-(j-m)} p_{S_{1}}(j)\binom{x}{j} . \tag{2}
\end{align*}
$$

Assume the case that $j \in\{0,1, \ldots, m\}$. Considering both possible parities of $k$, we use (2) and the induction hypothesis to see that

$$
\begin{aligned}
\left(\Delta^{j} p_{S}\right)(0) & =-2\left(\Delta^{j} p_{S_{1}}\right)(0) \chi(k \text { even }) \\
& =-2(-1)^{m+j} p_{S_{1}}(j) \chi(k \text { even }) \\
& =(-1)^{(m+k)+j} p_{S}(j),
\end{aligned}
$$

because $p_{S}(j)=-2 p_{S_{1}}(j) \chi(k$ even) by Lemma3.2. Now let $j \in\{m+1, m+2, \ldots, m+k-1\}$. Using Lemma 4.2 and (2), we have

$$
\begin{aligned}
\left(\Delta^{j} p_{S}\right)(0) & =(-1)^{k-1-(j-m)} p_{S_{1}}(j) \\
& =(-1)^{(m+k)+j} p_{S}(j)
\end{aligned}
$$

Lastly, $\left(\Delta^{m} p_{S}\right)(0)=0$ because $\operatorname{deg}\left(p_{S}(x)\right)=m-1$, which completes the proof.
For example, if $j>0$ is between the largest odd gap and $m$, then by this symmetry property and Theorem 1.2 one can observe that

$$
p_{S}(j)=(-1)^{m+j}\left(\Delta^{j} p_{S}\right)(0)=-(-2)^{|S \cap(j, \infty)|-1} p_{S \cap[j]}(j) .
$$

If $S$ has no odd gaps, then the equation above holds for all $j \in[m]$.
Lemma 4.4. If $S \neq \emptyset$ and $m=\max (S)$, then $p_{S}(j)<p_{S}(j+1)$ for $j \geq m$.
Proof. We prove the result by splitting into two cases. When $|S|=1$, we have $p_{\{m\}}(x)$, which increases on $(m-1, \infty)$ by Theorem 3.4 and proves our claim. In the second case, let $|S| \geq 2$. We want to show that $p_{S}(j)<p_{S}(j+1)$, which is equivalent to showing $2\left|\mathcal{P}_{S}(j)\right|<\left|\mathcal{P}_{S}(j+1)\right|$, so we need to construct more than twice as many permutations in $\mathfrak{S}_{j+1}$ with peak set $S$ than there are in $\mathfrak{S}_{j}$. Note that $p_{S}(m)=0$ and $p_{S}(m+1)>0$, so we need only consider $\mathfrak{S}_{j}$ for $j \geq m+1$. First, let $\pi \in \mathfrak{S}_{j}$ and append $j+1$ to $\pi$. This gives us $\left|\mathcal{P}_{S}(j)\right|$ permutations in $\mathfrak{S}_{j+1}$. Now construct $\left|\mathcal{P}_{S}(j)\right|$ different permutations by inserting $j+1$ between positions $m-1$ and $m$, so that $j+1$ becomes the final peak. Lastly, place $j+1$ at the first peak position (reading left to right), $j$ at the next peak position, etc., and then fill the empty indices from left to right with $1,2, \ldots, j+1-|S|$, respectively. Each of the $2\left|\mathcal{P}_{S}(n)\right|+1$ constructed permutations is distinct and has peak set $S$, so $p_{S}(j)<p_{S}(j+1)$.

Theorem 4.5. Let $S=\left\{i_{1}<i_{2}<\cdots<i_{s}=m\right\}$. For integers $1 \leq j<k$, we have $\left|p_{S}(j)\right| \leq$ $\left|p_{S}(k)\right|$ provided $p_{S}(k) \neq 0$, except for the case $\{2\} \subsetneq S$ where $p_{S}(1)=2 p_{S}(3)=-(-2)^{|S|-1}$.

Proof. If $\left|p_{S}(j)\right|=0$, then the claim is trivially true, so assume that $\left|p_{S}(j)\right|>0$ which implies $S \cap(j, \infty)$ has no odd gaps. If $S=\emptyset$ or not admissible then the statement holds so assume $S \neq \emptyset$, admissible, and $m=\max (S)$. We first consider the cases where $j<k<m$. We use these assumptions along with Theorem 1.2 and Corollary 4.3 to observe that

$$
\begin{equation*}
\left|p_{S}(j)\right|=2^{|S \cap(j, \infty)|-1}\left|p_{S \cap[j]}(j)\right| . \tag{3}
\end{equation*}
$$

Consider the case $p_{S}(j+1) \neq 0$. Then $j+1 \notin S$ by Theorem 3.8, and

$$
\begin{aligned}
\left|p_{S}(j+1)\right| & =2^{|S \cap(j+1, \infty)|-1}\left|p_{S \cap[j+1]}(j+1)\right| \\
& =2^{|S \cap(j, \infty)|-1}\left|p_{S \cap[j]}(j+1)\right| .
\end{aligned}
$$

To show that $\left|p_{S}(j)\right| \leq\left|p_{S}(j+1)\right|$ it suffices to show that $\left|p_{S \cap[j]}(j)\right| \leq\left|p_{S \cap[j]}(j+1)\right|$. If $S \cap[j]=\emptyset$, then we know $p_{\emptyset}(x)=1$ from Theorem 1.1. Otherwise, we may use Lemma 4.4 because $S \neq \emptyset$ and $j \geq \max (S \cap[j])$. In both cases, $\left|p_{S}(j)\right| \leq\left|p_{S}(j+1)\right|$ when $\left|p_{S}(j+1)\right|>0$.

Now assume that $p_{S}(j+1)=0$. Combining Theorem 1.1, Corollary 4.3, and the assumption that $\left|p_{S}(j)\right|>0$, this implies $\left|p_{S \cap[j+1]}(j+1)\right|=0$ which in turn implies $j+1 \in S$ by Lemma 4.4. Since $S$ is admissible $j+2 \notin S$ so $p_{S \cap[j+1]}(j+2)=p_{S \cap[j+2]}(j+2)>0$. By (3) this implies $\left|p_{S}(j+2)\right|>0$. To show that $\left|p_{S}(j)\right| \leq\left|p_{S}(j+2)\right|$, we will show that

$$
\begin{equation*}
2^{|S \cap(j, \infty)|-1}\left|p_{S \cap[j]}(j)\right| \leq 2^{|S \cap(j+2, \infty)|-1}\left|p_{S \cap[j+2]}(j+2)\right|, \tag{4}
\end{equation*}
$$

assuming $j+1 \in S$. Let $R=S \cap[j+2]$, and $R_{1}=R \backslash\{j+1\}$. Using Theorem 1.1, (4) is true if and only if

$$
\begin{equation*}
4\left|\mathcal{P}_{R_{1}}(j)\right| \leq\left|\mathcal{P}_{R}(j+2)\right| . \tag{5}
\end{equation*}
$$

To prove (5), observe that one can choose any $j$ elements from $[j+1]$, arrange them to have peak set $R_{1}$ in $\left|\mathcal{P}_{R_{1}}(j)\right|$ ways, and then append $j+2$ and the remaining element to this sequence in decreasing order. The resulting permutation has peak set $R$, and doing this in all possible ways yields $(j+1)\left|\mathcal{P}_{R_{1}}(j)\right|$ distinct permutations in $\mathfrak{S}_{j+2}$. If $j+1 \geq 4$, then (5) holds so $\left|p_{S}(j)\right| \leq\left|p_{S}(k)\right|$ when $\left|p_{S}(j+1)\right|=0$. Observe that the exact same argument proves the theorem for the case $m>3, j=m-1$, and $k=m+1$.

If $j+1 \in\{2,3\}$, then by (3) we can complete the proof using the fact that $p_{\emptyset}(x)=1$, and by computing the values of $p_{\{2\}}(n)$ and $p_{\{3\}}(n)$ for $n=0,1,2,3,4$, we have

$$
S=\{2\} \Longrightarrow(-2,-1,0,1,2)
$$

and

$$
S=\{3\} \Longrightarrow(0,-1,-1,0,2)
$$

In fact, using that data and Theorem 1.2 we see $p_{S}(1)=-(-2)^{|S|-1}$ for all nonempty admissible sets $S$ with no odd gaps and 0 otherwise. Similarly,

$$
p_{S}(2)= \begin{cases}0 & \text { if } 2 \in S \text { or } S \text { has an odd gap } \\ 1 & \text { if } S=\emptyset \\ -(-2)^{|S|-1} & \text { otherwise }\end{cases}
$$

and

$$
p_{S}(3)= \begin{cases}0 & \text { if } 3 \in S \text { or } S \text { has an odd gap after 3, } \\ 1 & \text { if } S \subset[2] \\ -(-2)^{|S|-2} & \text { if }\{2\} \subsetneq S \\ -(-2)^{|S|-1} & \text { otherwise }\end{cases}
$$

which proves the special case of the theorem where the inequality does Not hold. For completeness,

$$
p_{S}(4)= \begin{cases}0 & \text { if } 4 \in S \text { or } S \text { has an odd gap after 4, } \\ 1 & \text { if } S=\emptyset \\ 2 & \text { if } S=\{2\} \text { or } S=\{3\} \\ -(-2)^{|S|-1} & \text { if }\{2,3\} \cap S=\emptyset,|S|>1, \text { and } S \text { has no odd gaps } \\ (-2)^{|S|-1} & \text { otherwise }\end{cases}
$$

For $n>4$, the values of $\left|p_{S}(n)\right|$ are not typically powers of 2 .
Finally, the theorem holds for all remaining cases with $m<j<k$ by Lemma 4.4 and transitivity.

The previous proof also implies the following statement.
Corollary 4.6. Let $S$ be a set of positive integers and $j$ be a positive integer such that $p_{S}(j) \neq 0$. Let $k \geq j$ integer. If $p_{S}(k)=0$, then $k \in S$.

## 5 Connections to alternating permutations

In this section, we enumerate permutations with a given peak set using alternating permutations and tangent numbers instead of the recurrence given by Lemma 3.2. Alternating permutations allow us to easily count the number of permutations whose peak set is a superset of $S$, so we combine this idea with the inclusion-exclusion principle to evaluate $\left|\mathcal{P}_{S}(n)\right|$.

Assume that $S$ is a nonempty admissible peak set and that $m=\max (S)$. Let $\mathcal{Q}_{S}(n):=$ $\left\{\pi \in \mathfrak{S}_{n}: S \subseteq P(\pi)\right\}$ be the set of permutations $\pi \in \mathfrak{S}_{n}$ whose peak set contains $S=\left\{i_{1}<\right.$ $\left.i_{2}<\cdots<i_{s}\right\}$, and let us partition $S$ into runs of alternating substrings. An alternating substring is a maximal size subset $A_{r}$ such that $A_{r}=\left\{i_{r}, i_{r}+2, \ldots, i_{r}+2(k-1)\right\} \subseteq S$, where $i_{r}-i_{r-1} \geq 3$ if $i_{r-1} \in S$, and we call $A_{r}$ an alternating substring because

$$
\pi_{i_{r}-1}<\pi_{i_{r}}>\pi_{i_{r}+1}<\pi_{i_{r}+2}>\cdots<\pi_{i_{r}+2(k-1)}>\pi_{i_{r}+2(k-1)+1}
$$

is an alternating permutation in $\mathfrak{S}_{2 k+1}$ under an order-preserving map. Alternating permutations have peaks at every even index, and there are $E_{2 k+1}$ of them in $\mathfrak{S}_{2 k+1}$. The numbers $E_{2 k+1}$ are the tangent numbers given by the generating function

$$
\begin{aligned}
\tan x & =\sum_{k=0}^{\infty} \frac{E_{2 k+1}}{(2 k+1)!} x^{2 k+1} \\
& =x+\frac{1}{3} x^{3}+\frac{2}{15} x^{5}+\frac{17}{315} x^{7}+\ldots
\end{aligned}
$$

In 1879, André proved this result in [André 1879] using a generating function that satisfies a differential equation. See [Stanley 2010] for more background on alternating permutations.

Now let $\mathcal{A}(S)$ be the partition of an admissible set $S$ into maximal alternating substrings. For example, if $S=\{2,5,9,11,19,21,23,26\}$, then

$$
\mathcal{A}(S)=\left\{A_{1}, A_{2}, A_{3}, A_{5}, A_{8}\right\}=\{\{2\},\{5\},\{9,11\},\{19,21,23\},\{26\}\}
$$

The following results demonstrate how we can use $\mathcal{Q}_{S}(n)$ to enumerate permutations with a given peak set.

Lemma 5.1. For $n \geq m+1$, we have

$$
\left|\mathcal{Q}_{S}(n)\right|=n!\prod_{A_{r} \in \mathcal{A}(S)} \frac{E_{2\left|A_{r}\right|+1}}{\left(2\left|A_{r}\right|+1\right)!}
$$

Proof. The formula is easily checked in the case $S=\emptyset$, so assume $S \neq \emptyset$. Assume the theorem is true by induction for all sets $S^{\prime}$ such that $\left|\mathcal{A}\left(S^{\prime}\right)\right|<|\mathcal{A}(S)|$. Say $A_{1}=\left\{i_{1}, i_{1}+\right.$ $\left.2, \ldots, i_{1}+2(k-1)\right\} \in \mathcal{A}(S)$. We count the number of permutations $\pi \in \mathfrak{S}_{n}$ such that $A_{1} \subseteq P(\pi)$ by choosing $2 k+1$ of the $n$ elements, arranging them such that their peak set is $A_{1}$ in $E_{2 k+1}$ ways, then appending any permutation of the remaining $n-(2 k+1)$ elements arranged to have peak set contained in $S^{\prime}=S \backslash A_{1}$. The result now follows by induction.

Lemma 5.2. For $n \geq m+1$, we have

$$
\left|\mathcal{P}_{S}(n)\right|=\sum_{T \supseteq S}(-1)^{|T-S|}\left|\mathcal{Q}_{T}(n)\right|
$$

Proof. The proof follows the inclusion-exclusion principle.
Call an index $i$ a free index of peak set $S$ if $i \in[m+2]$ and $i$ is neither a peak nor adjacent to a peak in $S$. The following theorem gives us a closed-form expression of tangent numbers for $|\mathcal{P}(m+1)|$ and $|\mathcal{P}(m+2)|$ when $S$ has no free indices. Note that if $S$ has no free indices, then it can be thought of as separate independent alternating permutations that are concatenated to each other, similar to the independence in Theorem 3.15.

Corollary 5.3. If $S$ has no free indices and $k \in[2]$, then

$$
\left|\mathcal{P}_{S}(m+k)\right|=(m+k)!\prod_{A_{r} \in \mathcal{A}(S)} \frac{E_{2\left|A_{r}\right|+1}}{\left(2\left|A_{r}\right|+1\right)!}
$$

Proof. We observe that $S$ is the only admissible superset of $S$ and use Lemma 5.1 and Lemma 5.2.

## 6 Related work and conjectures

In this final section, we relate our work to other recent results about permutations with a given peak set, and we also restate some conjectures that stemmed from our work. Kasraoui characterized in [Kasraoui 2012] which peak sets $S$ maximize $\left|\mathcal{P}_{S}(n)\right|$ for $n \geq 6$ and explicitly computed $\left|\mathcal{P}_{S}(n)\right|$ for such sets $S$. We compute the maximum $\left|\mathcal{P}_{S}(n)\right|$ in a different way using alternating permutations.

Theorem 6.1 (Kasraoui 2012, Theorem 1.1, 1.2]). For $n \geq 6$, the sets $S$ that maximize $\left|\mathcal{P}_{S}(n)\right|$ are

$$
S=\left\{\begin{array}{lll}
\{3,6,9, \ldots\} \cap[n-1] \text { and }\{4,7,10, \ldots\} \cap[n-1] & \text { if } n \equiv 0(\bmod 3) \\
\{3,6,9, \ldots, 3 s, 3 s+2,3 s+5, \ldots\} \cap[n-1] \text { for } 1 \leq s \leq\left\lfloor\frac{n}{3}\right\rfloor & \text { if } n \equiv 1 & (\bmod 3) \\
\{3,6,9, \ldots\} \cap[n-1] & \text { if } n \equiv 2(\bmod 3)
\end{array}\right.
$$

Theorem 6.2 (|Kasraoui 2012, Theorem 1.2]). Suppose $n \geq 6$ and $S$ maximizes $\left|\mathcal{P}_{S}(n)\right|$. Set $\ell=\left\lfloor\frac{n}{3}\right\rfloor$. Then we have

$$
\left|\mathcal{P}_{S}(n)\right|=\left\{\begin{array}{lll}
\frac{1}{5} 3^{2-\ell} n! & \text { if } n \equiv 0 & (\bmod 3) \\
\frac{2}{5} 3^{1-\ell} n! & \text { if } n \equiv 1 & (\bmod 3) \\
3^{-\ell} n! & \text { if } n \equiv 2 & (\bmod 3)
\end{array}\right.
$$

Alternative proof. We work by cases using Theorem 6.1. When $n \equiv 0(\bmod 3)$, there is only one admissible superset of $S$, which we call $T$. Using Theorem 5.1 and Lemma 5.2 ,

$$
\begin{aligned}
\left|\mathcal{P}_{S}(n)\right| & =\left|\mathcal{Q}_{S}(n)\right|-\left|\mathcal{Q}_{T}(n)\right| \\
& =n!\left(\frac{1}{3}\right)^{\ell-1}-n!\left(\frac{1}{3}\right)^{\ell-2}\left(\frac{2}{15}\right) \\
& =\frac{1}{5} 3^{2-\ell} n!,
\end{aligned}
$$

as desired. We use Corollary 5.3 to prove the cases $n \equiv 1,2(\bmod 3)$, which are simpler because there are no admissible supersets of $S$.

Another new result in Castro-Velez et al. 2013] shows that the number of permutations with the same peak set for signed permutations can be enumerated using the peak polynomial $p_{S}(x)$ for unsigned permutations. We present an alternate proof that can be used to reduce many signed permutation statistic problems to unsigned permutation statistic problems. We denote the group of signed permutations as $B_{n}$.

Theorem 6.3 (|Castro-Velez et al. 2013, Theorem 2.7]). Let $\left|\mathcal{P}_{S}^{*}(n)\right|$ be the number of signed permutations $\pi \in B_{n}$ with peak set $S$. We have $\left|\mathcal{P}_{S}^{*}(n)\right|=p_{S}(n) 2^{2 n-|S|-1}$, where $p_{S}(x)$ is the same peak polynomial used to count unsigned permutations $\pi \in \mathfrak{S}_{n}$ with peak set $S$.

Alternative proof. We naturally partition $B_{n}$ by the signage of the permutations, which gives $2^{n}$ copies of $\mathfrak{S}_{n}$ under an order-preserving map, and then we work in each copy of $\mathfrak{S}_{n}$ separately. For example, $B_{3}=\mathfrak{S}_{+++} \cup \mathfrak{S}_{++-} \cup \mathfrak{S}_{+-+} \cup \mathfrak{S}_{+--} \cup \mathfrak{S}_{-++} \cup \mathfrak{S}_{-+-} \cup \mathfrak{S}_{--+} \cup \mathfrak{S}_{---}$, where $\mathfrak{S}_{++-}$is the set of permutations of $\{1,2,-3\}$. It follows that $\left|\mathcal{P}_{S}^{*}(n)\right|=2^{n}\left|\mathcal{P}_{S}(n)\right|$, so $\left|\mathcal{P}_{S}^{*}(n)\right|=p_{S}(n) 2^{2 n-|S|-1}$ by Theorem 1.1.

Now we restate some conjectures. In the data set [Fahrbach 2013], we experimentally checked Conjecture 6.4 for all admissible peak sets $S$ where $\max (S) \leq 15$. This conjecture implies the truth of Conjecture 1.5, which we explained in Section 2. We have also shown in Subsection 3.2 that the peak sets listed in Conjecture 6.5 have only integral roots, but we have not proven the other direction. Conjecture 6.6 is an observation that is related to Conjecture 6.4, and we have proved it for all integers $x_{0}$ using Lemma 4.2 and Lemma 4.4, but not all real $x_{0}$.

Conjecture 6.4. The complex roots of $p_{S}(z)$ lie in $\{z \in \mathbb{C}:|z| \leq m$ and $\operatorname{Re}(z) \geq-3\}$ if $S$ is admissible.

Conjecture 6.5. If $S=\left\{i_{1}<i_{2}<\cdots<i_{s}\right\}$ is admissible and all of the roots of $p_{S}(x)$ are real, then all of the roots of $p_{S}(x)$ are integral. Furthermore, $p_{S}(x)$ has all real roots if and only if $S=\{2\}, S=\{2,4\}, S=\{3\}, S=\{3,5\}, S=\left\{i_{1}<i_{2}<\cdots<i_{s}<i_{s}+3\right\}$, or $S=\left\{i_{1}<i_{2}<\cdots<i_{s}<i_{s}+3<i_{s}+5\right\}$.

Conjecture 6.6. Let $S$ be admissible and $|S| \geq 2$. If $p_{S}\left(x_{0}\right)=0$ for $x_{0} \in \mathbb{R}$, then $x_{0}>\max \left(S_{1}\right)$ if and only if $x_{0}=\max (S)$.
Question 6.7. What does $p_{S}(n)$ count for $n>\max (S)$ ?

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