

SPARSE MATRICES DESCRIBING ITERATIONS OF INTEGER-VALUED FUNCTIONS

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ABSTRACT. We consider iterations of integer-valued functions ϕ , which have no fixed points in the domain of positive integers. We define a local function ϕ_n , which is a subfunction of ϕ being restricted to the subdomain $\{0, \dots, n\}$. The iterations of ϕ_n can be described by a certain $n \times n$ sparse matrix M_n and its powers. The determinant of the related $n \times n$ matrix $\widehat{M}_n = I - M_n$, where I is the identity matrix, acts as an indicator, whether the iterations of the local function ϕ_n enter a cycle or not. If ϕ_n has no cycle, then $\det \widehat{M}_n = 1$ and the structure of the inverse \widehat{M}_n^{-1} can be characterized. Subsequently, we give applications to compute the inverse \widehat{M}_n^{-1} for some special functions. At the end, we discuss the results in connection with the $3x + 1$ and related problems.

1. INTRODUCTION

Let \mathbb{Z} and \mathbb{N} be the set of integers and positive integers, respectively. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let n, m denote positive integers in this paper. Define the finite domain $\mathbb{D}_n = \{1, \dots, n\}$ and let $\mathbb{D}_{n,0} = \mathbb{D}_n \cup \{0\}$.

We consider integer-valued functions on domains $S \subset \mathbb{N}$, where

$$\phi_S : S \rightarrow \mathbb{N}, \quad \phi(x) \neq x \quad (x \in S).$$

We may define a function ϕ induced by ϕ_S by

$$\phi(x) = \begin{cases} \phi_S(x), & \text{if } x \in S, \\ 0, & \text{else,} \end{cases}$$

which has the properties that

$$\phi : \mathbb{N}_0 \rightarrow \mathbb{N}_0, \quad \phi(0) = 0, \quad \phi(x) \neq x \quad (x \in \mathbb{N}), \quad (1.1)$$

having no fixed points in \mathbb{N} . Let Φ be the set of all such functions satisfying (1.1).

Note that a function ϕ does not have to be analytic, i.e. being an integer-valued polynomial. It can be arbitrarily defined, for example, using piecewise functions or tables of any complexity.

2010 *Mathematics Subject Classification.* 11B83 (Primary) 11C20 (Secondary).

Key words and phrases. Integer-valued functions, iterations, matrix, determinant, $3x + 1$ problem.

We construct a local function ϕ_n , that is a sub-function of $\phi \in \Phi$ being restricted to the domain $\mathbb{D}_{n,0}$, by

$$\phi_n : \mathbb{D}_{n,0} \rightarrow \mathbb{D}_{n,0}, \quad \phi_n(x) = \begin{cases} \phi(x), & \text{if } (x, \phi(x)) \in \mathbb{D}_n^2, \\ 0, & \text{else.} \end{cases}$$

We denote $\phi^m = \phi \circ \dots \circ \phi$ as an m -fold iteration of ϕ . We say that an iteration stops, if there exists an index k such that $\phi^k(x) = 0$, since all successive values of the iteration also vanish by definition. We say that ϕ has a cycle of length $m \geq 2$, if there exists $x \in \mathbb{N}$ such that $\phi^m(x) = x$ and $\phi^{m'}(x) \neq x$ for $m' < m$. If ϕ has a cycle of length m containing x , then we may define the set

$$\mathcal{C}(\phi, m, x) = \{\phi(x), \dots, \phi^m(x)\}, \quad (1.2)$$

describing all elements of this cycle. By definition this set has the properties that

$$x \in \mathcal{C}(\phi, m, x) \quad \text{and} \quad |\mathcal{C}(\phi, m, x)| = m \geq 2. \quad (1.3)$$

Regarding ϕ_n , we require that $x \in \mathbb{D}_n$ to define a cycle $\mathcal{C}(\phi_n, m, x)$ of length $m \geq 2$. Note that ϕ_n cannot have a fixed point $x \in \mathbb{D}_n$ by definition.

Lemma 1.1. *Let $\phi \in \Phi$. Assume that there exists a cycle $\mathcal{C}(\phi, m, x)$. Then*

$$\mathcal{C}(\phi, m, x) = \mathcal{C}(\phi_n, m, x) \quad \iff \quad n \geq \max \mathcal{C}(\phi, m, x).$$

Proof. Set $N = \max \mathcal{C}(\phi, m, x)$. If $n \geq N$, then we have

$$(\phi(x), \phi^2(x)), \dots, (\phi^{m-1}(x), \phi^m(x)) \in \mathbb{D}_n^2,$$

which also holds for ϕ_n having the same function values by definition. Therefore ϕ_n has the same cycle as ϕ in this case.

Conversely, if $n < N$, then there exists an index i , such that $\phi^i(x) > n$ showing that $\phi^i(x), \phi_n^i(x) \notin \mathbb{D}_n$, where $\phi_n^i(x) = 0$. Accordingly, $\mathcal{C}(\phi, m, x) = \mathcal{C}(\phi_n, m, x)$ implies that $n \geq N$ must hold. \square

Lemma 1.2. *Let $\phi \in \Phi$ and $n \geq 1$. Assume that ϕ_n has no cycle. We define the height of $x \in \mathbb{D}_n$ regarding ϕ_n by*

$$h(x) = \min \{k \in \mathbb{N} : \phi_n^k(x) = 0\}.$$

We then have

$$1 \leq h(x) \leq n \quad (x \in \mathbb{D}_n).$$

Proof. Let $x \in \mathbb{D}_n$. The property $h(x) \geq 1$ follows by definition. Next, we show that $h(x) \leq n$. Assume to the contrary that $\phi_n^n(x) \in \mathbb{D}_n$ implying that $h(x) > n$. Since ϕ_n has no cycle, we then obtain that

$$\mathcal{X} = \{x, \phi_n(x), \dots, \phi_n^n(x)\} \subset \mathbb{D}_n,$$

where \mathcal{X} must contain $n + 1$ distinct elements. This gives a contradiction to $|\mathbb{D}_n| = n$. \square

We define the i -th unit vector of size n by \mathbf{e}_i and the zero vector by \mathbf{e}_0 , which we shall use in an unambiguous context. Let $A, B, E_n(\cdot, \cdot), I_n$ be $n \times n$ matrices. As usual, I_n denotes the identity matrix, where we use I instead, if possible. Define the matrix

$$E_n(i, j) = \mathbf{e}_i \mathbf{e}_j^t,$$

which only has the entry 1 at row i and column j and zeros elsewhere. Let $\#A$ denote the total number of nonzero entries of A . The term $\#A^k$ should be read as $\#(A^k)$. We define for A and B that

$$A \cap B = \sum_{\substack{1 \leq i, j \leq n \\ A_{ij} B_{ij} \neq 0}} 1,$$

counting all entries, where both matrices have nonzero entries in common. We call A and B to be disjoint, if

$$A \cap B = 0$$

implying that

$$\#(A + B) = \#A + \#B.$$

The main aim of the paper is to construct $n \times n$ matrices, which are connected with the properties of a function $\phi \in \Phi$ as well as its local function ϕ_n . We define the following matrix by column vectors induced by the local function ϕ_n by

$$M_n(\phi) = \left(\mathbf{e}_{\phi_n(1)}, \dots, \mathbf{e}_{\phi_n(n)} \right) \in \mathbb{Z}^{n \times n} \quad (\phi \in \Phi, n \geq 2), \tag{1.4}$$

being a binary matrix with $\{0, 1\}$ entries. Further we define the related matrix

$$\widehat{M}_n(\phi) = I - M_n(\phi) \in \mathbb{Z}^{n \times n} \quad (\phi \in \Phi, n \geq 2),$$

which consists of entries with $\{-1, 0, 1\}$. Since ϕ_n has no fixed points in \mathbb{D}_n , the diagonal of $M_n(\phi)$ and $\widehat{M}_n(\phi)$ has only entries with 0 and 1, respectively. By construction the matrices $M_n(\phi)$ and $\widehat{M}_n(\phi)$ are sparse matrices, since $\#M_n(\phi) \leq n$ and $\#\widehat{M}_n(\phi) \leq 2n$, both being of order $O(n)$.

The main property of $M_n(\phi)$ is that for $x \in \mathbb{D}_n$ the mapping

$$x \mapsto \phi_n(x)$$

coincides with

$$M_n(\phi) \mathbf{e}_x = \mathbf{e}_{\phi_n(x)}. \tag{1.5}$$

If there exists a cycle of ϕ , then $\det \widehat{M}_n(\phi)$ acts as an indicator for this event. This is shown by the following theorems.

Theorem 1.3. *Let $\phi \in \Phi$ and $n \geq 2$. If there exists a cycle $\mathcal{C}(\phi_n, m, x)$, then*

$$\det \widehat{M}_n(\phi) = 0$$

and $M_n(\phi)$ has an eigenvector \mathbf{v} with eigenvalue 1 defined by

$$\mathbf{v} = \sum_{y \in \mathcal{C}(\phi_n, m, x)} \mathbf{e}_y.$$

Corollary 1.4. *Let $\phi \in \Phi$. If ϕ has a cycle, then there exists an integer $N \geq 2$ such that*

$$\det \widehat{M}_n(\phi) = 0 \quad (n \geq N).$$

Proof. By assumption ϕ has a cycle, say $\mathcal{C}(\phi, m, x)$. By Lemma 1.1 we can find an integer $N \geq 2$ such that

$$\mathcal{C}(\phi, m, x) = \mathcal{C}(\phi_n, m, x) \quad (n \geq N).$$

Applying Theorem 1.3 for $n \geq N$ gives the result. \square

Theorem 1.5. *Let $\phi \in \Phi$ and $n \geq 2$. If ϕ_n has no cycle, then we have the following statements:*

- (1) *The matrix $M_n(\phi)$ is nilpotent of degree at most n .*
- (2) *The powers of $M_n(\phi)$ are binary matrices satisfying*

$$\#M_n(\phi)^k \leq n - k \quad (1 \leq k \leq n).$$

They are disjoint for different exponents that

$$M_n(\phi)^k \cap M_n(\phi)^l = 0 \quad (k \neq l, k, l \geq 1).$$

- (3) *The related matrix $\widehat{M}_n(\phi)$ is invertible, where*

$$\det \widehat{M}_n(\phi) = 1.$$

- (4) *The inverse $\widehat{M}_n(\phi)^{-1}$ is a binary matrix with the properties that*

$$\widehat{M}_n(\phi)^{-1} = I + M_n(\phi) + \cdots + M_n(\phi)^{n-1}$$

and

$$\#\widehat{M}_n(\phi)^{-1} = n + \sum_{k=1}^{n-1} \#M_n(\phi)^k \leq \binom{n+1}{2}.$$

Remark 1.6. The bounds of Theorem 1.5 are sharp. If we consider the function $\phi \in \Phi$ being induced by $f(x) = x + 1$, then $M_n(\phi)$ is nilpotent of degree n , $\#M_n(\phi)^k = n - k$ for $1 \leq k \leq n$, and consequently $\#\widehat{M}_n(\phi)^{-1} = \binom{n+1}{2}$. This will be shown by Proposition 5.1 as an example. The next theorem shows that one can compute exact values for an arbitrary local function ϕ_n , in case it has no cycle.

Theorem 1.7. *Let $\phi \in \Phi$ and $n \geq 2$. Assume that ϕ_n has no cycle. Let*

$$p_\nu = |\{x \in \mathbb{D}_n : h(x) = \nu\}| \quad (1 \leq \nu \leq n)$$

and

$$m = \max_{1 \leq \nu \leq n} \{\nu : p_\nu \geq 1\}.$$

Define

$$\pi = (p_1, \dots, p_m).$$

Then π is an ordered partition of n and length m with $1 \leq m \leq n$, where $p_1, \dots, p_m \geq 1$. We have the following statements:

- (1) The matrix $M_n(\phi)$ is nilpotent of degree m .
- (2) We have

$$M_n(\phi)^k = \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^k(x), x) \quad (1 \leq k \leq m),$$

where

$$\#M_n(\phi)^k = n - \sum_{\nu=1}^k p_\nu \quad (1 \leq k \leq m).$$

- (3) We have

$$\widehat{M}_n(\phi)^{-1} = \left(\mathbf{v}_1, \dots, \mathbf{v}_n \right),$$

where the column vectors satisfy that

$$\mathbf{v}_j = \mathbf{e}_j + \sum_{\nu=1}^{h(j)-1} \mathbf{e}_{\phi_n^\nu(j)} \quad (1 \leq j \leq n).$$

Moreover,

$$\#\widehat{M}_n(\phi)^{-1} = \sum_{x \in \mathbb{D}_n} h(x) = \sum_{\nu=1}^m \nu p_\nu \leq nm - \binom{m}{2}.$$

An ordered partition of an integer, where the order of its summands is relevant, is also called a composition (cf. [2, Chap. II, p. 123]). We will prove the theorems above in the following sections. We will give some applications of the theorems in Sections 5 and 6. See the figures therein for illustrations of examples of the matrices $\widehat{M}_n(\phi)$ and $\widehat{M}_n(\phi)^{-1}$.

2. ITERATIONS

Recall that $\phi_n : \mathbb{D}_{n,0} \rightarrow \mathbb{D}_{n,0}$ can be any exotic function, i.e. ϕ_n can be composed of piecewise functions of any complexity. This situation will be reflected in the following lemmas and propositions in this section. First, we define an orbit of an element $x \in \mathbb{D}_n$, collecting the iterations $\phi_n^k(x)$. Regarding these orbits we can define an equivalence relation on \mathbb{D}_n , which leads to a disjoint decomposition of \mathbb{D}_n . Second, these disjoint sets in question can be interpreted as labeled trees, whose properties establish the results. For basic graph theory see [2, Chap. I.17, pp. 60].

Lemma 2.1. *Let $\phi \in \Phi$ and $k, l, n \in \mathbb{N}$, where $k \neq l$. If ϕ has no cycle, then*

$$\phi^k(x) \neq \phi^l(x) \quad (x, \phi^k(x), \phi^l(x) \in \mathbb{N}).$$

Accordingly, if ϕ_n has no cycle, then

$$\phi_n^k(x) \neq \phi_n^l(x) \quad (x, \phi_n^k(x), \phi_n^l(x) \in \mathbb{D}_n).$$

Proof. By symmetry we may assume that $k > l$. Set $y = \phi^l(x)$. Assume to the contrary that we have $\phi^k(x) = \phi^l(x)$. This implies that $\phi^{k-l}(y) = y$, contradicting that ϕ has no cycle or no fixed point. Similarly, if $x, \phi_n^k(x), \phi_n^l(x) \in \mathbb{D}_n$, then $\phi_n^k(x) = \phi_n^l(x)$ also would imply, that ϕ_n has a cycle or a fixed point. \square

Lemma 2.2. *Let $\phi \in \Phi$ and $n \geq 1$. Assume that ϕ_n has no cycle. Define the orbit of $x \in \mathbb{D}_n$ by*

$$\Omega(x) = \{\phi_n^k(x) : 0 \leq k < h(x)\} \subset \mathbb{D}_n,$$

where $\phi_n^0(x) = x$ is defined to be the identity function on \mathbb{D}_n and $h(x)$ is the height of x . Then

$$1 \leq |\Omega(x)| = h(x) \leq n \quad (x \in \mathbb{D}_n).$$

There exists an equivalence relation on \mathbb{D}_n induced by ϕ_n , such that

$$x \sim y \iff \Omega(x) \cap \Omega(y) \neq \emptyset \quad (x, y \in \mathbb{D}_n). \quad (2.1)$$

Let denote $[x]$ the equivalence class of x and

$$\tilde{\mathbb{D}}_n = \mathbb{D}_n / \sim$$

the set of these classes. Define the set

$$\Omega_{[x]} = \bigcup_{x' \in [x]} \Omega(x'), \quad (2.2)$$

which covers all orbits of $x' \in [x]$. Then

$$\mathbb{D}_n = \bigcup_{[x] \in \tilde{\mathbb{D}}_n} \overset{\circ}{\Omega}_{[x]} \quad (2.3)$$

gives a disjoint decomposition.

Proof. By Lemma 1.2 we have the bounds

$$1 \leq h(x) \leq n \quad (x \in \mathbb{D}_n).$$

Lemma 2.1 implies that

$$1 \leq |\Omega(x)| = h(x) \leq n \quad (x \in \mathbb{D}_n).$$

It is easy to see that

$$x \sim y \iff \phi_n^{h(x)-1}(x) = \phi_n^{h(y)-1}(y) \quad (x, y \in \mathbb{D}_n) \quad (2.4)$$

defines an equivalence relation on \mathbb{D}_n . We show that this relation transfers to (2.1) as follows. If there exists

$$z \in \Omega(x) \cap \Omega(y) \neq \emptyset,$$

then we have

$$\Omega(z) = \{z, \phi_n(z), \dots, \phi_n^{h(z)-1}(z)\} \subset \Omega(x) \cap \Omega(y).$$

By construction of $\Omega(\cdot)$ and Lemma 2.1, the elements of an orbit can be uniquely ordered by height. Thus, there exists only one element of height 1 in $\Omega(\cdot)$. We then infer that

$$\phi_n^{h(z)-1}(z) = \phi_n^{h(x)-1}(x) = \phi_n^{h(y)-1}(y),$$

showing that $x \sim y$. Conversely, if $x \sim y$, then

$$\phi_n^{h(x)-1}(x), \phi_n^{h(y)-1}(y) \in \Omega(x) \cap \Omega(y) \neq \emptyset,$$

using the same arguments as above. After all, this shows (2.1). The construction of $\Omega_{[x]}$ in (2.2) implies that

$$\Omega_{[x]} \cap \Omega_{[y]} = \emptyset, \quad \text{if } [x] \neq [y],$$

applying (2.1). This finally establishes the disjoint decomposition in (2.3). \square

Lemma 2.3. *Let $\phi \in \Phi$ and $n \geq 1$. Assume that ϕ_n has no cycle. Define*

$$\hat{\phi}_n((x, k)) = (\phi_n(x), k - 1) \quad (x \in \mathbb{D}_n, k \geq 1).$$

Let $[x] \in \tilde{\mathbb{D}}_n$. Then there exists a labeled tree

$$T_{[x]} \cong \Omega_{[x]},$$

which can be uniquely defined by its labeled nodes by

$$T_{[x]} = \{(y, h(y)) : y \in \Omega_{[x]}\}, \quad (2.5)$$

preserving the structure induced by $\hat{\phi}_n$. The tree $T_{[x]}$ has the following properties:

- (1) There exists a unique root node $(e, 1) \in T_{[x]}$, where $\hat{\phi}_n((e, 1)) = (0, 0)$.
- (2) If a node $(d, k) \in T_{[x]}$ ($k \geq 1$) has m child nodes, then they are given by $\{(c_i, k + 1)\}_{1 \leq i \leq m}$ satisfying $\hat{\phi}_n((c_i, k + 1)) = (d, k)$ ($1 \leq i \leq m$).

Proof. By construction of $\Omega_{[x]}$, we obviously have a one-to-one correspondence

$$y \longleftrightarrow (y, h(y)) \quad (y \in \Omega_{[x]}). \quad (2.6)$$

We can define a labeled tree by

$$T_{[x]} = \{(y, h(y)) : y \in \Omega_{[x]}\},$$

labeling the nodes with $(y, h(y))$ uniquely. We will show that

$$T_{[x]} \cong \Omega_{[x]}$$

having the same structure. By Lemmas 2.1 and 2.2 we infer that an orbit $\Omega(y)$ represents a simple path

$$\{y, \phi_n(y), \dots, \phi_n^{h(y)-1}(y)\}, \quad (2.7)$$

where the elements are ordered by height. The set $\Omega_{[x]}$ covers all orbits $\Omega(y)$, respectively simple paths as in (2.7), where $y \in [x]$. Since the equivalence relation on \mathbb{D}_n satisfies (2.4), all such simple paths are closed by a common element e , where

$$e = \phi_n^{h(y)-1}(y) \quad \text{for all } y \in [x], \quad (2.8)$$

being the only element with height $h(e) = 1$ in $\Omega_{[x]}$. The structure of $\Omega_{[x]}$ is induced by ϕ_n , where each element $y \in \Omega_{[x]}$ has its height and a successor $\phi_n(y) \in \Omega_{[x]}$, if $y \neq e$. By (2.6) and the definition of $\hat{\phi}_n$ the properties of $\Omega_{[x]}$ are transferred to $T_{[x]}$. As a result, the tree $T_{[x]}$ is a cover of simple paths. Each node $(y, h(y)) \in T_{[x]}$ has a parent node $\hat{\phi}_n((y, h(y))) = (\phi_n(y), h(y) - 1) \in T_{[x]}$, if $y \neq e$.

Property (1): By (2.8) there exists only one element e with $h(e) = 1$. Thus, $(e, 1) \in T_{[x]}$ is the unique root node with $\hat{\phi}_n((e, 1)) = (0, 0)$.

Property (2): We assume that $(d, k) \in T_{[x]}$ has m child nodes. Since $T_{[x]}$ is a cover of simple paths, there exist m different paths, such that

$$\{(c_i, h(c_i)), \hat{\phi}_n((c_i, h(c_i))) = (d, h(c_i) - 1), \dots, (e, 1)\} \subset T_{[x]} \quad (1 \leq i \leq m),$$

where $h(c_1) = \dots = h(c_m) = k + 1$. □

Proposition 2.4. *Let $\phi \in \Phi$ and $k, n \geq 1$. Assume that ϕ_n has no cycle. Define*

$$J_{n,k}(\phi) = \{x \in \mathbb{D}_n : \phi_n^k(x) \in \mathbb{D}_n\}.$$

Then we have

$$|J_{n,k}(\phi)| \leq \max(n - k, 0).$$

In particular,

$$|J_{n,1}(\phi)| = n - |\tilde{\mathbb{D}}_n|.$$

Proof. Since ϕ_n has no cycle, we have by Lemma 1.2 that

$$\phi_n^n(x) = 0 \quad (x \in \mathbb{D}_n),$$

which implies that

$$|J_{n,k}(\phi)| = 0 \quad (k \geq n). \tag{2.9}$$

Define the complementary set

$$\overline{J_{n,k}}(\phi) = \{x \in \mathbb{D}_n : \phi_n^k(x) = 0\},$$

such that

$$J_{n,k}(\phi) \cup \overline{J_{n,k}}(\phi) = \mathbb{D}_n. \tag{2.10}$$

In view of (2.9) and (2.10), we will equivalently show for the remaining cases that

$$|\overline{J_{n,k}}(\phi)| \geq k \quad (1 \leq k < n). \tag{2.11}$$

By Lemmas 2.2 and 2.3, there exist unique labeled trees $T_{[x]}$ for $[x] \in \tilde{\mathbb{D}}_n$, where

$$\sum_{[x] \in \tilde{\mathbb{D}}_n} |T_{[x]}| = n.$$

More precisely, they build a disjoint decomposition of \mathbb{D}_n , if we only consider the first component of the elements $(y, h(y)) \in T_{[x]}$. We will count those elements in all trees, that vanish under ϕ_n^k . Since the elements can be ordered by height, we can write

$$|\overline{J_{n,k}}(\phi)| = |\{x \in \mathbb{D}_n : \phi_n^k(x) = 0\}| = \sum_{[x] \in \tilde{\mathbb{D}}_n} \sum_{\substack{(y,j) \in T_{[x]} \\ j \leq k}} 1. \tag{2.12}$$

In particular, we obtain by counting the root nodes via (2.12) and using (2.10) that

$$|\overline{J_{n,1}}(\phi)| = |\tilde{\mathbb{D}}_n| \geq 1 \quad \text{and} \quad |J_{n,1}(\phi)| = n - |\tilde{\mathbb{D}}_n|, \tag{2.13}$$

showing the special case $k = 1$. Define $\overline{J_{n,0}}(\phi) = \emptyset$. Instead of (2.12), it is more convenient to consider

$$s_k = |\overline{J_{n,k}}(\phi) \setminus \overline{J_{n,k-1}}(\phi)| = \sum_{[x] \in \tilde{\mathbb{D}}_n} \sum_{(y,k) \in T_{[x]}} 1 \quad (1 \leq k \leq n), \quad (2.14)$$

where s_k is the number of elements having height k . We derive that

$$\sum_{k=1}^n s_k = |\overline{J_{n,n}}(\phi)| = n,$$

utilizing the telescoping sum induced by (2.14), where the last equation follows by (2.9) and (2.10). Hence, $S = (s_1, \dots, s_n)$ describes an ordered partition of n , where $s_1 \geq 1$ follows by (2.13). Now, we use the properties of a tree $T_{[x]}$ given by Lemma 2.3:

- (1) A tree $T_{[x]}$, being a labeled tree, is connected.
- (2) The heights of a parent node $(d, h(d)) \in T_{[x]}$ and its child nodes $(c_i, h(c_i)) \in T_{[x]}$ differ by 1, such that $h(c_i) = h(d) + 1$.

As a consequence, we infer that the sequence $(s_\nu)_{1 \leq \nu \leq n}$, counting elements of height 1 up to n , cannot have gaps, i.e. zero elements, in the middle, such that

$$S = (s_1, \dots, s_{l-1}, 0, s_{l+1}, \dots, s_n)$$

with some index l , where $1 < l < n$, and $s_1, s_n \geq 1$. Therefore, we either must have

$$S = (1, \dots, 1) \quad \text{or} \quad S = (s_1, \dots, s_r, 0, \dots, 0), \quad (2.15)$$

where $1 \leq r < n$ and $s_1, \dots, s_r \geq 1$. In any case, we finally conclude by an easy counting argument that

$$|\overline{J_{n,k}}(\phi)| = \sum_{\nu=1}^k s_\nu \geq k \quad (1 \leq k < n), \quad (2.16)$$

since S is an ordered partition of n . This shows (2.11) completing the proof. □

Proposition 2.5. *Let $\phi \in \Phi$ and $k, n \geq 1$. Assume that ϕ_n has no cycle. Let*

$$p_\nu = |\{x \in \mathbb{D}_n : h(x) = \nu\}| \quad (1 \leq \nu \leq n)$$

and

$$m = \max_{1 \leq \nu \leq n} \{\nu : p_\nu \geq 1\}.$$

Define

$$\pi = (p_1, \dots, p_m).$$

Then π is an ordered partition of n and length m with $1 \leq m \leq n$, where $p_1, \dots, p_m \geq 1$. Moreover,

$$|J_{n,k}(\phi)| = n - \sum_{\nu=1}^k p_\nu \quad (1 \leq k \leq m)$$

and $|J_{n,k}(\phi)| = 0$ for $k \geq m$.

Proof. We use and extend the proof of Proposition 2.4. Regarding (2.15) the sequence $(s_\nu)_{1 \leq \nu \leq n}$ counts elements of height 1 up to n . Therefore, we observe with $m = r$ that

$$\pi = (p_1, \dots, p_m) = (s_1, \dots, s_r),$$

except for the case where $\pi = (1, \dots, 1)$ and $m = n$. By (2.10) and (2.16) it follows that

$$|J_{n,k}(\phi)| = n - \sum_{\nu=1}^k p_\nu \quad (1 \leq k \leq m).$$

Since π is an ordered partition of n , we have $|J_{n,m}(\phi)| = 0$ and consequently that $|J_{n,k}(\phi)| = 0$ for $k \geq m$. \square

Lemma 2.6. *Let $\pi = (p_1, \dots, p_m)$ be an ordered partition of n and length m , where $1 \leq m \leq n$. Then*

$$\sum_{\nu=1}^m \nu p_\nu \leq nm - \binom{m}{2}.$$

Proof. Let $\mathcal{P}_{n,m}$ be the set of ordered partitions of n and length m . Define

$$\varrho(\pi) = \sum_{\nu=1}^m \nu p_\nu \quad (\pi \in \mathcal{P}_{n,m}).$$

We fix $m \geq 1$ for now and use induction on n . For $n = m$, we only have $\pi = (1, \dots, 1) \in \mathcal{P}_{n,n}$. Then it follows that

$$\varrho(\pi) = \binom{m+1}{2} = m^2 - \binom{m}{2} \leq nm - \binom{m}{2}.$$

Now assume the result is true for $n \geq m$. Let $\pi = (p_1, \dots, p_m) \in \mathcal{P}_{n,m}$. We set

$$\pi'_j = (p_1, \dots, p_{j-1}, p_j + 1, p_{j+1}, \dots, p_m) \quad (1 \leq j \leq m), \quad (2.17)$$

observing that $\pi'_j \in \mathcal{P}_{n+1,m}$. We then obtain that

$$\varrho(\pi'_j) = \varrho(\pi) + j \leq nm + j - \binom{m}{2} \leq (n+1)m - \binom{m}{2} \quad (1 \leq j \leq m). \quad (2.18)$$

The set $\mathcal{P}_{n+1,m}$ can be constructed from $\mathcal{P}_{n,m}$ using (2.17). Since $\pi \in \mathcal{P}_{n,m}$ has been chosen arbitrarily, we infer that (2.18) holds for any $\pi' \in \mathcal{P}_{n+1,m}$ showing the claim for $n+1$. \square

3. MATRIX PROPERTIES

Recall the $n \times n$ matrix

$$E_n(i, j) = \mathbf{e}_i \mathbf{e}_j^t,$$

which only has the entry 1 at row i and column j and zeros elsewhere.

Lemma 3.1. *Let $n \geq 2$ and $a, b, c, d \in \mathbb{D}_n$. Then*

$$E_n(a, b)E_n(c, d) = \begin{cases} E_n(a, d), & \text{if } b = c, \\ 0, & \text{else.} \end{cases}$$

Proof. By definition we can write

$$E_n(a, b)E_n(c, d) = (\mathbf{e}_a \mathbf{e}_b^t)(\mathbf{e}_c \mathbf{e}_d^t) = \mathbf{e}_a(\mathbf{e}_b^t \mathbf{e}_c) \mathbf{e}_d^t = \delta_{bc} E_n(a, d),$$

using Kronecker's delta. \square

Lemma 3.2. *Let A be an $n \times n$ matrix with $n \geq 2$. If A is nilpotent of degree k , then we have*

$$\det(I - A) = 1 \quad \text{and} \quad (I - A)^{-1} = I + A + \cdots + A^{k-1}.$$

Proof. We first consider the decomposition

$$I = (I - A)(I + A + \cdots + A^{k-1}),$$

since $A^k = 0$. This shows that $I - A$ is invertible with the inverse as given above. Since A is nilpotent, there exists a similar matrix $U = T^{-1}AT$, that is an upper triangular matrix having zeros in its diagonal. Thus, we obtain that $I - A = T(I - U)T^{-1}$ and consequently that $\det(I - A) = 1$. \square

Proposition 3.3. *Let $\phi \in \Phi$ and $n \geq 2$. Then we have*

$$M_n(\phi)^k = \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^k(x), x) \quad (k \geq 1).$$

Proof. We use induction on k . For $k = 1$, we infer by (1.4) that

$$M_n(\phi) = \left(\mathbf{e}_{\phi_n(1)}, \cdots, \mathbf{e}_{\phi_n(n)} \right) = \sum_{(\phi_n(x), x) \in \mathbb{D}_n^2} E_n(\phi_n(x), x).$$

Now assume the result is true for k . We then obtain that

$$\begin{aligned} M_n(\phi)^{k+1} &= M_n(\phi)M_n(\phi)^k \\ &= \sum_{(\phi_n(y), y) \in \mathbb{D}_n^2} E_n(\phi_n(y), y) \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^k(x), x) \\ &= \sum_{\substack{(\phi_n(y), y) \in \mathbb{D}_n^2 \\ (\phi_n^k(x), x) \in \mathbb{D}_n^2 \\ y = \phi_n^k(x)}} E_n(\phi_n(y), y) E_n(\phi_n^k(x), x) \\ &= \sum_{(\phi_n^{k+1}(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^{k+1}(x), x). \end{aligned}$$

In the last two steps we have used Lemma 3.1 to exclude those products that provide a zero matrix. This shows the claim for $k + 1$. \square

Corollary 3.4. *Let $\phi \in \Phi$ and $n \geq 2$. If ϕ_n has no cycle, then*

$$\#M_n(\phi)^k = |J_{n,k}(\phi)| \leq n - k \quad (1 \leq k \leq n).$$

Proof. By Propositions 2.4 and 3.3 we conclude that

$$\#M_n(\phi)^k = \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} 1 = |\{(x, \phi_n^k(x)) \in \mathbb{D}_n^2\}| = |J_{n,k}(\phi)| \leq n - k \quad (1 \leq k \leq n). \quad \square$$

Proposition 3.5. *Let $\phi \in \Phi$ and $n \geq 2$. If ϕ_n has no cycle, then*

$$M_n(\phi)^k \cap M_n(\phi)^l = 0 \quad (k \neq l, k, l \geq 1).$$

Proof. Assume to the contrary that we have

$$M_n(\phi)^k \cap M_n(\phi)^l \neq 0$$

for some $k \neq l$. We then have an entry of both $M_n(\phi)^k$ and $M_n(\phi)^l$, such that

$$E_n(\phi_n^k(y), y) = E_n(\phi_n^l(x), x)$$

using Proposition 3.3. This implies that $y = x \in \mathbb{D}_n$ and consequently $\phi_n^k(x) = \phi_n^l(x) \in \mathbb{D}_n$. The last condition gives a contradiction in view of Lemma 2.1. \square

4. PROOF OF THEOREMS

Proof of Theorem 1.3. The local function ϕ_n has a cycle $\mathcal{C}(\phi_n, m, x)$ by assumption. Using (1.5) we have

$$M_n(\phi) \mathbf{e}_y = \mathbf{e}_{\phi_n(y)} \quad (y \in \mathcal{C}(\phi_n, m, x)).$$

In view of (1.2) and (1.3), ϕ_n maps $\mathcal{C}(\phi_n, m, x)$ onto itself in a cyclic way. Define

$$\mathbf{v} = \sum_{y \in \mathcal{C}(\phi_n, m, x)} \mathbf{e}_y.$$

We then infer that

$$M_n(\phi) \mathbf{v} = \mathbf{v}.$$

As a result, the vector \mathbf{v} is an eigenvector of $M_n(\phi)$ with eigenvalue 1. Consequently, we obtain that

$$\det \widehat{M}_n(\phi) = \det(I - M_n(\phi)) = 0. \quad \square$$

Proof of Theorem 1.5. By assumption $\phi_n \in \Phi$ has no cycle.

(1), (2): By Propositions 3.3 and 3.5 the matrices $M_n(\phi)^k$ are binary matrices for $k \geq 1$, which are disjoint for different exponents. The estimate

$$\#M_n(\phi)^k \leq n - k \quad (1 \leq k \leq n) \quad (4.1)$$

is given by Corollary 3.4. This implies that $M_n(\phi)^n$ is the zero matrix and consequently $M_n(\phi)$ is nilpotent of degree at most n .

(3), (4): By definition we have $\widehat{M}_n(\phi) = I - M_n(\phi)$. Since $M_n(\phi)$ is nilpotent of degree at most n , we obtain by Lemma 3.2 that $\det \widehat{M}_n(\phi) = 1$ and

$$\widehat{M}_n(\phi)^{-1} = I + M_n(\phi) + \cdots + M_n(\phi)^{n-1}. \quad (4.2)$$

Recall that $M_n(\phi)^k \cap I = 0$, because ϕ_n has no fixed points in \mathbb{D}_n , and that powers of $M_n(\phi)$ are disjoint for different exponents. Therefore $\widehat{M}_n(\phi)^{-1}$ is a binary matrix composed

of binary matrices given on the right-hand side of (4.2). Counting entries of these matrices above, we conclude by using the estimate in (4.1) that

$$\#\widehat{M}_n(\phi)^{-1} = n + \sum_{k=1}^{n-1} \#M_n(\phi)^k \leq \binom{n+1}{2}. \quad \square$$

Proof of Theorem 1.7. By assumption $\phi_n \in \Phi$ has no cycle. The properties of π are given by Proposition 2.5.

(1), (2): By Proposition 3.3 we have

$$M_n(\phi)^k = \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^k(x), x) \quad (k \geq 1). \quad (4.3)$$

From Proposition 2.5 and Corollary 3.4 we infer that

$$\#M_n(\phi)^k = |J_{n,k}(\phi)| = n - \sum_{\nu=1}^k p_\nu \quad (1 \leq k \leq m). \quad (4.4)$$

Since $|J_{n,m}(\phi)| = 0$, the matrix $M_n(\phi)$ is nilpotent of degree m .

(3): Since the nilpotent degree of $M_n(\phi)$ is $m \leq n$, we have by Lemma 3.2 that

$$\widehat{M}_n(\phi)^{-1} = I + M_n(\phi) + \cdots + M_n(\phi)^{m-1}. \quad (4.5)$$

As already argued in (4.2) and below, the matrices of the right-hand side of (4.5) are disjoint to each other. Counting entries we obtain by means of (4.4) that

$$\#\widehat{M}_n(\phi)^{-1} = n + \sum_{k=1}^{m-1} \#M_n(\phi)^k = \sum_{k=0}^{m-1} \sum_{\nu=k+1}^m p_\nu = \sum_{\nu=1}^m \nu p_\nu, \quad (4.6)$$

using the fact that π is a partition of n and thus

$$n - \sum_{\nu=1}^k p_\nu = \sum_{\nu=k+1}^m p_\nu \quad (0 \leq k < m).$$

Alternatively, from (4.5) and using (4.3), we derive that

$$\begin{aligned} \widehat{M}_n(\phi)^{-1} &= I + \sum_{k=1}^{m-1} \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^k(x), x) \\ &= I + \sum_{x \in \mathbb{D}_n} \sum_{\nu=1}^{h(x)-1} E_n(\phi_n^\nu(x), x). \end{aligned} \quad (4.7)$$

This implies that

$$\#\widehat{M}_n(\phi)^{-1} = \sum_{x \in \mathbb{D}_n} h(x). \quad (4.8)$$

Finally, combining (4.6) and (4.8), we achieve by applying Lemma 2.6 that

$$\#\widehat{M}_n(\phi)^{-1} = \sum_{x \in \mathbb{D}_n} h(x) = \sum_{\nu=1}^m \nu p_\nu \leq nm - \binom{m}{2}.$$

It remains to show the structure of the inverse matrix $\widehat{M}_n(\phi)^{-1}$. Let

$$\widehat{M}_n(\phi)^{-1} = \begin{pmatrix} \mathbf{v}_1, \dots, \mathbf{v}_n \end{pmatrix}.$$

From (4.7) we conclude for $j = x$ that

$$\mathbf{v}_j = \mathbf{e}_j + \sum_{\nu=1}^{h(j)-1} \mathbf{e}_{\phi_n^\nu(j)} \quad (1 \leq j \leq n). \quad \square$$

5. SIMPLE PATTERNS

In this section we shall give some applications. Define an $n \times n$ sub-diagonal matrix as

$$D_{n,k} = \sum_{j=1}^{n-k} E_n(j+k, j) \quad (0 \leq k < n)$$

and as a zero matrix otherwise.

Proposition 5.1. *Let ϕ be induced by $f(x) = x + 1$. Then $\phi \in \Phi$ and has no cycle. We have the following statements for $n \geq 2$:*

- (1) *The matrix $M_n(\phi)$ is nilpotent of degree n .*
- (2) *We have*

$$M_n(\phi)^k = D_{n,k}, \quad \#M_n(\phi)^k = n - k \quad (1 \leq k \leq n).$$

- (3) *The inverse $\widehat{M}_n(\phi)^{-1}$ is a full lower triangular matrix whose entire entries are 1. Consequently, $\#\widehat{M}_n(\phi)^{-1} = \binom{n+1}{2}$.*

Proof. It is easily seen that $\phi^k(x) = x + k$ for $x \in \mathbb{N}$. Therefore, ϕ cannot have a cycle or a fixed point in \mathbb{N} , hence $\phi \in \Phi$. By Theorem 1.7 we then obtain that

$$M_n(\phi)^k = \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^k(x), x) = D_{n,k} \quad (1 \leq k \leq n),$$

implying that $\#M_n(\phi)^k = n - k$. This shows that $M_n(\phi)$ is nilpotent of degree n . Theorem 1.5 provides that

$$\widehat{M}_n(\phi)^{-1} = I + \sum_{k=1}^{n-1} M_n(\phi)^k = \sum_{k=0}^{n-1} D_{n,k},$$

which is a full lower triangular matrix having only entries with 1. As a consequence, we have $\#\widehat{M}_n(\phi)^{-1} = \binom{n+1}{2}$. \square

Remark 5.2. If one chooses $f(x) = x + t$ with $t \in \mathbb{Z} \setminus \{0\}$, where f is defined on the subdomain $\mathbb{N}_{>|t|}$ in case $t < 0$, then one similarly obtains the following shapes of the matrix $\widehat{M}_n(\phi)^{-1}$, where ϕ is induced by f :

- (1) Case $t = 1$: $\widehat{M}_n(\phi)^{-1}$ is a full lower triangular matrix shown by Proposition 5.1.
- (2) Case $t > 1$: $\widehat{M}_n(\phi)^{-1}$ is a lower triangular matrix with

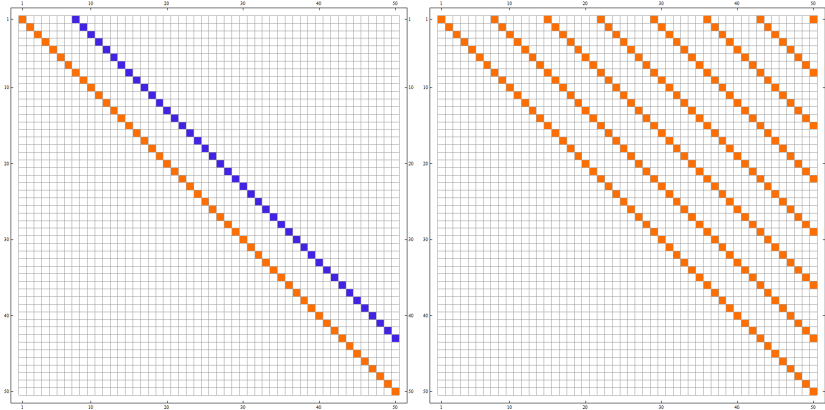
$$\widehat{M}_n(\phi)^{-1} = \sum_{j=0}^{\lfloor \frac{n-1}{t} \rfloor} D_{n,jt}.$$

For $t = 2$ this gives a checkerboard pattern.

- (3) Case $t = -1$: $\widehat{M}_n(\phi)^{-1}$ is a full upper triangular matrix.
- (4) Case $t < -1$: $\widehat{M}_n(\phi)^{-1}$ is an upper triangular matrix, which equals the transposed matrix of the case $|t| > 1$.

The cases $t \neq 1$ will be left to the reader. See Figure 5.1 for an example.

FIGURE 5.1. Matrices $\widehat{M}_n(\phi)$ and $\widehat{M}_n(\phi)^{-1}$



Case $n = 50$: ϕ is induced by $f(x) = x - 7$.
 Red entries: 1, blue entries: -1 .

For a second example, let $\mathbb{P} = \{q_1, q_2, \dots\}$ be the set of primes and $\Pi(x)$ be the prime-counting function. Define

$$\omega : \mathbb{P} \rightarrow \mathbb{P}, \quad \omega(q_j) = q_{j+1}$$

giving the next prime.

Proposition 5.3. *Let ϕ be induced by ω . Then $\phi \in \Phi$ and has no cycle. We have the following statements for $n \geq 2$:*

- (1) *The matrix $M_n(\phi)$ is nilpotent of degree $\Pi(n)$.*

(2) We have

$$M_n(\phi)^k = \sum_{j=1}^{\Pi(n)-k} E_n(q_{j+k}, q_j), \quad \#M_n(\phi)^k = \Pi(n) - k, \quad (1 \leq k \leq \Pi(n)).$$

(3) We have

$$\widehat{M}_n(\phi)^{-1} = I + \sum_{1 \leq j < i \leq \Pi(n)} E_n(q_i, q_j), \quad \#\widehat{M}_n(\phi)^{-1} = n + \binom{\Pi(n)}{2}.$$

Proof. Since $\omega^k(q_j) = q_{j+k} > q_j$ for $q_j \in \mathbb{P}$ and $j, k \geq 1$, the induced function ϕ cannot have a cycle or a fixed point in \mathbb{N} . Thus, we have $\phi \in \Phi$. By construction, $\phi(x) = 0$ for $x \in \mathbb{N}_0 \setminus \mathbb{P}$. Let $N = \Pi(n)$, then $\mathbb{D}_n \cap \mathbb{P} = \{q_1, \dots, q_N\}$. Using these properties and Theorem 1.7 we infer that

$$M_n(\phi)^k = \sum_{(\phi_n^k(x), x) \in \mathbb{D}_n^2} E_n(\phi_n^k(x), x) = \sum_{j=1}^{N-k} E_n(q_{j+k}, q_j) \quad (1 \leq k \leq N).$$

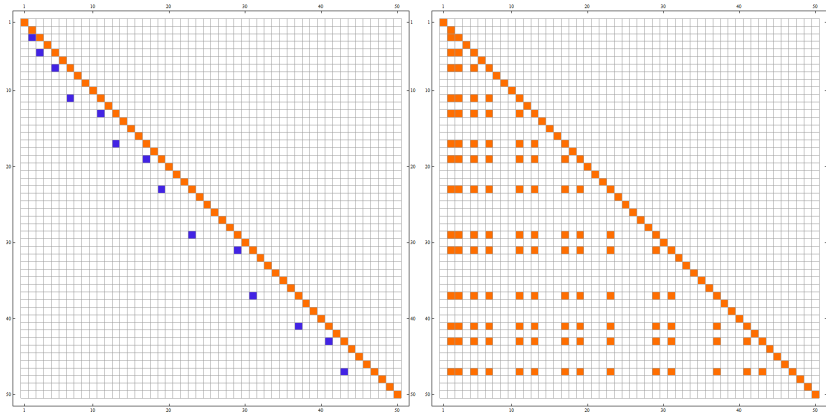
As a result, $\#M_n(\phi)^k = N - k$ for $1 \leq k \leq N$, implying that $M_n(\phi)$ is nilpotent of degree N . The last part follows by Theorem 1.5 and reordering the above sums that

$$\widehat{M}_n(\phi)^{-1} = I + \sum_{k=1}^{N-1} M_n(\phi)^k = I + \sum_{k=1}^{N-1} \sum_{j=1}^{N-k} E_n(q_{j+k}, q_j) = I + \sum_{1 \leq j < i \leq N} E_n(q_i, q_j).$$

Counting entries of $\widehat{M}_n(\phi)^{-1}$ in the equation above, we finally obtain that

$$\#\widehat{M}_n(\phi)^{-1} = n + \sum_{k=1}^{N-1} (N - k) = n + \binom{N}{2}. \quad \square$$

FIGURE 5.2. Matrices $\widehat{M}_n(\phi)$ and $\widehat{M}_n(\phi)^{-1}$



Case $n = 50$: ϕ is induced by ω .
Red entries: 1, blue entries: -1 .

6. THE $3x + 1$ PROBLEM

A variant of the Collatz function may be defined by

$$c(x) = \begin{cases} x/2, & \text{if } x \text{ is even,} \\ (3x + 1)/2, & \text{if } x \text{ is odd.} \end{cases}$$

The behavior of the iterations of this function is known as the $3x + 1$ problem. It is still an open problem to decide, whether a sequence of iterations $\mathcal{I}_c(x) = (c^k(x))_{k \geq 1}$, starting from a positive integer x , eventually returns to 1 entering a trivial cycle $\{1, 2\}$ afterwards. It is conjectured that all such iterations eventually return to 1. For a wide survey of the $3x + 1$ problem see Lagarias [3].

There are three possible cases of the behavior of a sequence $\mathcal{I}_c(x)$:

- (1) It eventually enters the trivial cycle $\{1, 2\}$.
- (2) It eventually enters a cycle other than $\{1, 2\}$.
- (3) It is unbounded.

We can establish a connection between the cases (1), (2), and the given theory in the former sections. To get rid of the trivial cycle $\{1, 2\}$, we define

$$\phi_c(x) = \begin{cases} 0, & \text{if } x \leq 2, \\ x/2, & \text{if } x > 2 \text{ is even,} \\ (3x + 1)/2, & \text{if } x > 2 \text{ is odd.} \end{cases}$$

Then we have $\phi_c \in \Phi$. As a result of Corollary 1.4, if ϕ_c has a cycle, then there exists an integer $N \geq 2$ such that

$$\det \widehat{M}_n(\phi_c) = 0 \quad (n \geq N).$$

See Figure 6.1 for the simple shape of $\widehat{M}_n(\phi_c)$ and the complicated shape of its inverse in the case $n = 50$. Regarding the local function $\phi_{c,n}$ for this case, we can compute the following parameters using Theorem 1.7:

$$\pi = (10, 4, 3, 3, 3, 3, 4, 2, 2, 2, 2, 2, 3, 2, 2, 1, 1, 1), \quad m = 18.$$

Consequently, the nilpotent degree of $M_n(\phi_c)$ is 18 and

$$\#\widehat{M}_n(\phi_c)^{-1} = \sum_{\nu=1}^m \nu p_\nu = 348.$$

Remark 6.1. Zeilberger [4] asked for an evaluation of determinants of certain $2d \times 2d$ matrices $M(d)$ occurring in an enumeration problem. Actually, this was intended as a semi-joke [5], because these matrices were disguised intentionally, hiding their close relationship to the $3x + 1$ problem at first glance.

Chapman [1] pointed out, that one obtains, after swapping columns of $M(d)$, that

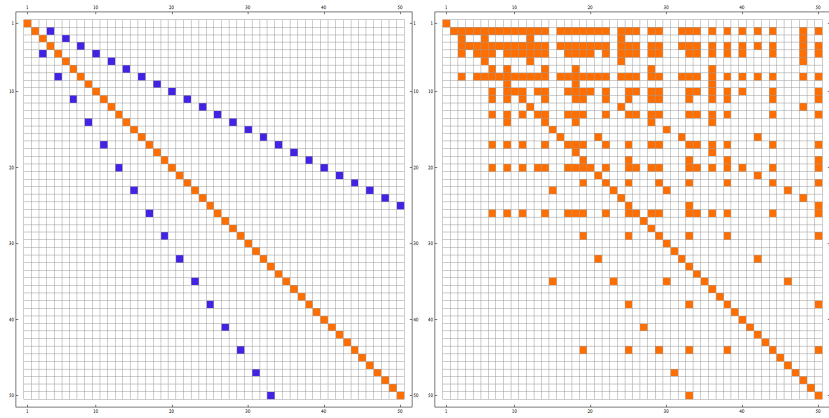
$$\det M(d) = (-1)^d \det(I - N(d)),$$

where $N(d)$ is the matrix describing the local iteration of

$$\psi(x) = \begin{cases} (x-1)/2, & \text{if } x \text{ is odd,} \\ 3x/2 + 1, & \text{if } x \text{ is even.} \end{cases}$$

Chapman showed that either $N(d)$ is nilpotent and $\det M(d) = (-1)^d$ or $N(d)$ has an eigenvector with eigenvalue 1 and $\det M(d) = 0$, induced by a cycle of ψ . Furthermore, he established the connection to the $3x+1$ problem via $c(x) = \psi(x-1) + 1$.

FIGURE 6.1. Matrices $\widehat{M}_n(\phi_c)$ and $\widehat{M}_n(\phi_c)^{-1}$



Case $n = 50$. Red entries: 1, blue entries: -1 .

As a last example, we consider a more complicated function with $r = 3$ branches:

$$\phi_r(x) = \begin{cases} 0, & \text{if } x \leq 1, \\ x/3, & \text{if } x > 1 \text{ and } x \equiv 0 \pmod{3}, \\ (2x+1)/3, & \text{if } x > 1 \text{ and } x \equiv 1 \pmod{3}, \\ (5x-1)/3, & \text{if } x > 1 \text{ and } x \equiv 2 \pmod{3}. \end{cases}$$

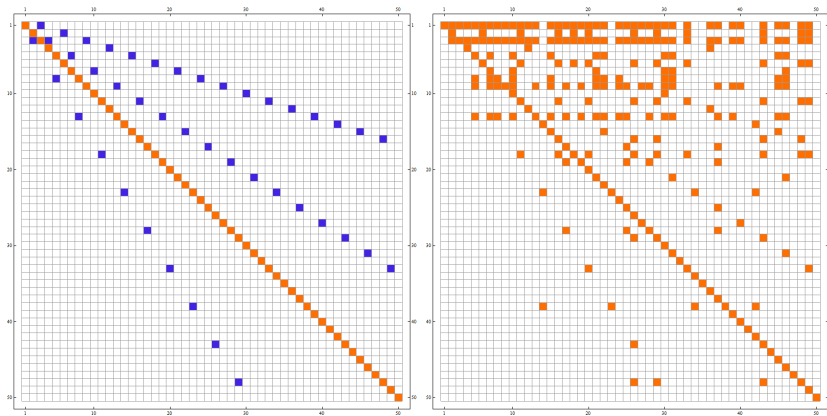
Such functions are called *generalized Collatz functions* or *residue-class-wise affine functions*, which can be defined for any $r \geq 2$, r being the number of branches, respectively, residue classes (cf. [3, (4.1), p. 12]).

The modification here, that $\phi_r(1) = 0$, is only to prevent a fixed point at $x = 1$. In this way, we have $\phi_r \in \Phi$. Again, we compute the parameters of the local function $\phi_{r,n}$ for $n = 50$:

$$\pi = (8, 2, 5, 5, 6, 4, 5, 8, 4, 2, 1), \quad m = 11.$$

Consequently, the nilpotent degree of $M_n(\phi_r)$ is 11 and

$$\#\widehat{M}_n(\phi_r)^{-1} = \sum_{\nu=1}^m \nu p_\nu = 267.$$

FIGURE 6.2. Matrices $\widehat{M}_n(\phi_r)$ and $\widehat{M}_n(\phi_r)^{-1}$ 

Case $n = 50$. Red entries: 1, blue entries: -1 .

The $3x + 1$ problem, as treated in Remark 6.1, was the starting point for the author to give a general theory here. All computations were performed using *Mathematica*.

ACKNOWLEDGMENTS

We would like to thank Pieter Moree for helpful comments.

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