

A GAME WITH DIVISORS AND ABSOLUTE DIFFERENCES OF EXPONENTS

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ABSTRACT. In this work, we discuss a number game that develops in a manner similar to that on which Gilbreath's conjecture on iterated absolute differences between consecutive primes is formulated. In our case the action occurs at the exponent level and there, the evolution is reminiscent of that in a final Ducci game. We present features of the whole field of the game created by the successive generations, prove an analog of Gilbreath's conjecture, and raise some open questions.

1. INTRODUCTION

We study a cellular automaton, whose different facets display characteristics that are both multiplicative and additive in nature. Essentially, it is defined by an 'atomic transformation' function, which is given by an arithmetical operation. To any integers a, b we associate the integer

$$Z(a, b) := \frac{ab}{(\gcd(a, b))^2}.$$

Applying this transformation on any two neighbor elements of a given sequence of integer numbers, we obtain a new $Z(\cdot, \cdot)$ -generated sequence. We write it on the next row, such that each term is placed in the middle, below its parents, like in a contemporary presentation of the Pascal arithmetic triangle. One difference, however, lies in the fact that in our cases of interest the first row is long.

Our subject of study is the construction that starts with the sequence of prime numbers on the first row. We denote by $a_{m,n}$ the n -th element from the m -th row of this table, for $m \geq 0, n \geq 1$. Looking at the initial part of the honeycomb table of numbers thus generated, we get a triangle such as the one displayed in Figure 1. During the involved calculations one finds, for example:

$$\begin{aligned} Z(46189, 96577) &= Z(11 \cdot 13 \cdot 17 \cdot 19, 13 \cdot 17 \cdot 19 \cdot 23) \\ &= \frac{11 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23}{13^2 \cdot 17^2 \cdot 19^2} = 11 \cdot 23 = 253. \end{aligned}$$

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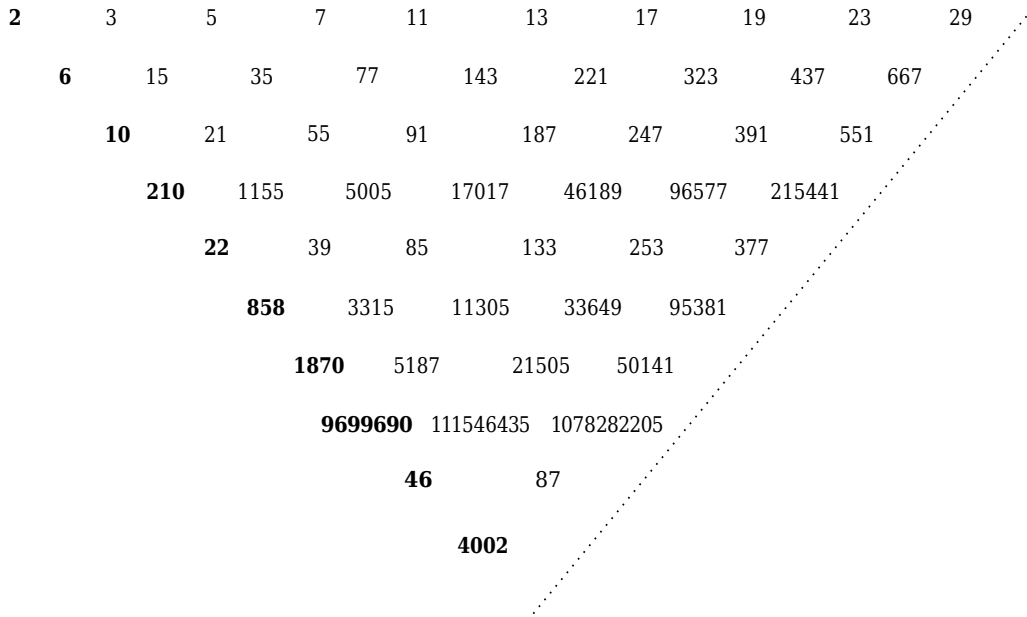


FIGURE 1. The triangular slice of order 10 generated by iterated applications of $Z(\cdot, \cdot)$ on the sequence of prime numbers.

A construction similar to that in Figure 1, but with $Z(\cdot, \cdot)$ replaced by the absolute difference function leads to a table whose appearance seems simpler (see Guy [16, Problem A10]). In that case the numbers quickly get smaller and one notices that the numbers on the left edge of the triangle are equal to one. Proth [24] thought that he proved that fact, but his argument was incorrect and the problem remained open. Later Gilbreath rediscovered the property and the statement is known as Gilbreath's Conjecture. Odlyzko [22] verified the conjecture for triangles of size $< \pi(10^{13})$. The apparent simplicity of the rows of higher generations hides deep problems involving gaps of different orders between primes, on which current results only scrape the surface of a field that remains untouchable by known techniques. Although, Odlyzko [22] observed that there are chances to overcome the involved difficulties. Reasoning heuristically, he concluded that the taming property of the numbers on the left edge of such a construction should be generally valid. Indeed, he showed that due to iterativity, this should be a characteristic property of lots of other sequences, provided their gaps to neighbor elements are random and relatively small.

On the contrary, in our case, one sees that the numbers become large, growing in augmented waves. Notice that by definition it follows that they are all square free. Looking for an analogue of Gilbreath's conjecture, we consider the prime factorization of the first element of each row of the construction whose first part is presented in Figure 1. Denote by d_m the first element of row m , for $m \geq 0$. The factorization into primes of the first few of them is given in Table 1. Analyzing this table, we remark a striking phenomenon: the number of prime factors of each of these numbers is a power of 2. The following theorem that will be proved twice, in Sections 4 and 5, establishes this fact.

TABLE 1. The first 16 entries on the left edge and their prime factorization.

m	d_m	factorization of d_m
0	2	2
1	6	$2 \cdot 3$
2	10	$2 \cdot 5$
3	210	$2 \cdot 3 \cdot 5 \cdot 7$
4	22	$2 \cdot 11$
5	858	$2 \cdot 3 \cdot 11 \cdot 13$
6	1870	$2 \cdot 5 \cdot 11 \cdot 17$
7	9699690	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$
8	46	$2 \cdot 23$
9	4002	$2 \cdot 3 \cdot 23 \cdot 29$
10	7130	$2 \cdot 5 \cdot 23 \cdot 31$
11	160660290	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 23 \cdot 29 \cdot 31 \cdot 37$
12	20746	$2 \cdot 11 \cdot 23 \cdot 41$
13	1008940218	$2 \cdot 3 \cdot 11 \cdot 13 \cdot 23 \cdot 29 \cdot 41 \cdot 43$
14	2569288370	$2 \cdot 5 \cdot 11 \cdot 17 \cdot 23 \cdot 31 \cdot 41 \cdot 47$
15	32589158477190044730	$2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot 53$

Theorem 1. *The number of prime factors of d_m is a power of 2, for any $m \geq 0$.*

Here the sequence of exponents is itself remarkable. For any $m \geq 0$, let

$$\delta_m := \frac{\log(\text{the number of prime factors of } d_m)}{\log 2}.$$

Sequence $\{\delta_m\}_{m \geq 0}$ is special and has interesting properties. For example, it is a proper subsequence of itself, and as such it is a *fractal sequence* (cf. Kimberling [17]). Indeed, the subsequence of terms having even rank coincides with the original sequence. And this fact occurs for sub-sequences, also, again and again, ad infinitum. Precisely this follows by

$$\delta_{t2^s} = \delta_t, \quad \text{for any } 0 \leq t \text{ and } 1 \leq s. \quad (1)$$

We do not know a closed formula for δ_m , but we know that it verifies many other recursive formulas. Thus, given an initial group of 2^s terms for any $s \geq 1$, we obtain the next group of 2^s terms just by incrementing by 1 all the known terms, that is,

$$\delta_{2^{s+j}} = \delta_j + 1, \quad \text{for any } 0 \leq j \leq 2^s - 1 \text{ and } 1 \leq s.$$

Computer experiments suggest that the sequence $\{\delta_m\}_{m \geq 0}$ has lots of 2-adic symmetries, both local and global, but their proofs are not the object of this work. However, here we emphasize some of these properties that are related to our automaton, presenting the sequence in different forms. On the left side of Figure 2 we wrote the first part of the sequence with 16 elements on each row. Thus $\delta_0 = 0, \delta_1 = 1, \dots, \delta_{15} = 4$. Next on the

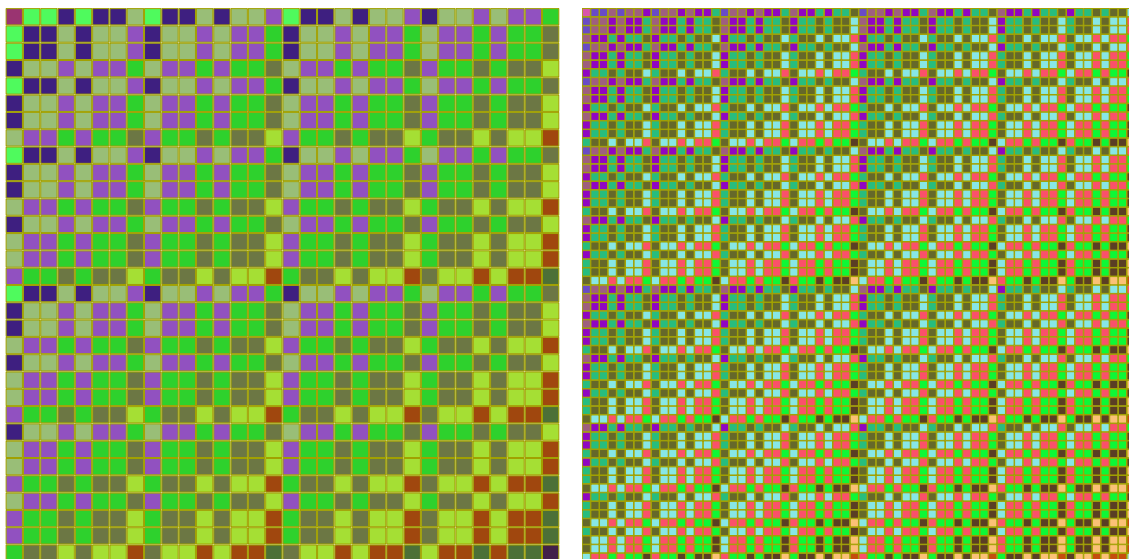


FIGURE 3. Representations of the sequence $\{\delta_m\}_{m \geq 0}$ in tables having 32 and 64 cell in each row, respectively. In each case, the colors that corresponds to the numbers δ_m are chosen randomly.

Sequence $\{\delta_m\}_{m \geq 0}$ has many other interesting properties. It is sequence A000120 from Sloane's On-line Encyclopedia of Integer Sequences and it is related to other remarkable sequences, cf. [26] and the references therein. They are mainly associated with binary representations, because δ_m equals the number of 1's from the m -th row of Pascal triangle modulo 2. Further, this links them to Fermat numbers, constructible polygons and Bernoulli numbers. A short survey on these ties is presented in [13, Section 4].

Further inspection of the factorization of entries in Figure 1 reveals that $\omega(a)$, the number of prime factors of a , is the same for all integers a situated on the same horizontal line. This is the object of our next theorem that will be proved in Section 5.

Theorem 2. *We have $\omega(d_m) = \omega(a_{m,n})$, for all $m \geq 0, n \geq 1$.*

Deeper properties of the evolution function $Z(\cdot, \cdot)$ may be investigated in other contexts, where $Z(\cdot, \cdot)$ is applied to integers that are not necessarily square free as in the present case. Various interesting questions arise for instance when the role of the starting sequence (played by the primes in this article) is taken by the sequence of natural numbers, or by others, such as the Fibonacci sequence. If the first row is a line of binomial coefficients $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n}$, from Pascal's arithmetic triangle, the generated table is a bounded triangle. Looking at their p -renderings, one sees striking differences in these pictures when one of p or n is kept fixed and the other varies, even slightly (cf. [13]).

Working only with integers that are products of distinct primes, enables one to develop techniques that may be useful in the general case. In the following sections we elaborate such a method that may be useful in all cases and the main results stated above will follow, too.

2. A RELATED GAME

In this section we apparently depart from the original problem, experimenting with a setting that is more general and more particular at the same time. For this, we lift the left edge barrier. This extends the field of the game to also allow in the new generations entries from the left side. Next, we assume here that the entries are elements of $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ and the operation $Z(\cdot, \cdot)$ is replaced by the difference in \mathbb{F}_2 . This is close to the essence of the end phase of an n -absolute difference game cf. the authors' [12]. In the last century, this automaton was discovered independently, several times, by different authors. Their attention was mainly attracted by the fact that the Ducci game enters into a null cycle when n is a power of 2. They also studied the length of the period for an arbitrary n and the number of the necessary iterations before the game enters into a cycle. For these and further problems and generalizations, the reader is referred to Ciamberlini and Marengoni [11], Meyers [21], Thwaites [27], [28], Pompilli [23], Crășmaru and the authors' [12], Andriychenko and Chamberland [1], Chamberland [9], Behn et al. [3], Thomas et al. [10], [7], Webb [29], Ehrlich [14], Ludington-Young [19], [20], Calkin et al. [7], Lidman and Thomas [18], Brown and Merzel [6] Breuer [4], [5], Caragiu et al. [2], [8] and [13].

We denote by $\mathfrak{v}: \mathbb{N} \rightarrow \mathbb{F}_2$ the sequence that lists the numbers from the first line of the game. Next, we denote by ψ the function that transforms an ascendent sequence \mathfrak{s} into the succeeding generation $\psi(\mathfrak{s})$, by taking the difference in \mathbb{F}_2 between all neighbor elements. For example, Figure 4 shows the honeycomb representation of the generations $\psi^{(0)}(\mathfrak{v}) = \mathfrak{v}, \psi^{(1)}(\mathfrak{v}), \dots, \psi^{(6)}(\mathfrak{v})$, where $\mathfrak{v}(1) = \mathfrak{v}(4) = 1$ and $\mathfrak{v}(n) = 0$, otherwise. (We denote by $\psi^{(n)}$ the n -th composition of the function ψ by itself.) There, the highlighted cells are the original positions of the numbers of interest in the game presented in the introduction and on its left is a new growing triangle. We denote by $\mathfrak{d}_m = \mathfrak{d}_m(\mathfrak{v}), 0 \leq m$, the m -th number situated on that diagonal in the game that starts with the sequence \mathfrak{v} .

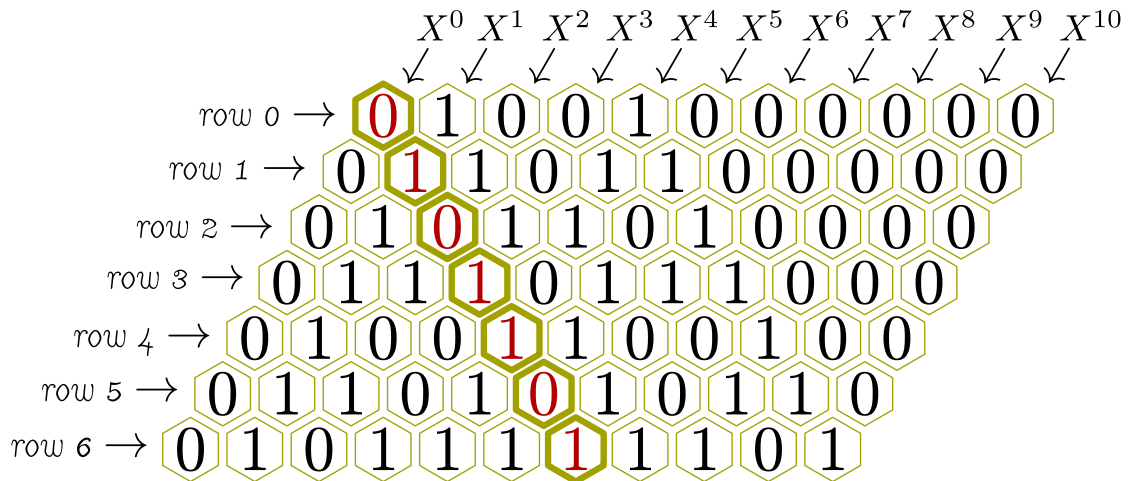


FIGURE 4. The coefficients of the first polynomials of the sequence that starts with $X + X^4$.

To any sequence \mathfrak{w} , we associate the series $P_{\mathfrak{w}}(X) \in \mathbb{F}_2[[X]]^1$ whose coefficients are the corresponding elements of the sequence \mathfrak{w} . If the number of nonzero elements is finite, then $P_{\mathfrak{w}}(X)$ is a polynomial in $\mathbb{F}_2[X]$.

For example, the polynomials associated with the first generations shown in Figure 4 are:

$$\begin{aligned} P_{\mathfrak{v}}(X) &= X + X^4, \\ P_{\psi(\mathfrak{v})}(X) &= (X + X^4)(1 + X) = X + X^2 + X^4 + X^5, \\ P_{\psi^{(2)}(\mathfrak{v})}(X) &= (X + X^4)(1 + X)^2 = X + X^3 + X^4 + X^6, \\ P_{\psi^{(3)}(\mathfrak{v})}(X) &= (X + X^4)(1 + X)^3 = X + X^2 + X^3 + X^5 + X^6 + X^7, \\ P_{\psi^{(4)}(\mathfrak{v})}(X) &= (X + X^4)(1 + X)^3 = X + X^4 + X^5 + X^8, \\ P_{\psi^{(5)}(\mathfrak{v})}(X) &= (X + X^4)(1 + X)^3 = X + X^2 + X^4 + X^6 + X^8 + X^9, \\ P_{\psi^{(6)}(\mathfrak{v})}(X) &= (X + X^4)(1 + X)^3 = X + X^3 + X^4 + X^5 + X^6 + X^7 + X^8 + X^{10}. \end{aligned}$$

The operation of passing from one generation to the next one corresponds to the polynomial side to multiplication by $1 + X$ in the ring $\mathbb{F}_2[X]$. Consequently, finding the m -th generation that starts with \mathfrak{w} , reduces to finding the associated series:

$$\begin{array}{ccc} \mathfrak{w} & \longrightarrow & P_{\mathfrak{w}}(X) \\ \downarrow & & \downarrow \times (1+X)^m \\ \psi^{(m)}(\mathfrak{w}) & \longrightarrow & P_{\psi^{(m)}(\mathfrak{w})}(X) \end{array}$$

where we have

$$P_{\psi^{(m)}(\mathfrak{w})}(X) = P_{\mathfrak{w}}(X)(1 + X)^m. \quad (2)$$

Relation (2) enables one to obtain information on the periodicity of the generating system.

Denote by $v_p(a)$ the exponent of p in the prime factorization of a . Then, if $0 < t < 2^s$, $s \geq 1$ and $t_1 = t/2^{v_2(t)}$, it follows that $v_2(2^s - t) = v_2(2^{v_2(t)}(2^{s-v_2(t)} - t_1)) = v_2(t)$. Therefore

$$\binom{2^s - 1}{k} = \prod_{t=1}^k \frac{2^s - t}{t} \quad \text{is odd for } 0 < k < 2^s,$$

and $\binom{2^s}{k} = 0$ in \mathbb{F}_2 for $0 < k < 2^s$. Using this in combination with (2), we obtain the following shifting-enhanced periodicity feature of the \mathbb{F}_2 game.

Proposition 1. *For any sequence \mathfrak{w} and any integer $s \geq 1$, we have*

$$P_{\psi^{(2^s)}(\mathfrak{w})}(X) = P_{\mathfrak{w}}(X) + X^{2^s} P_{\mathfrak{w}}(X).$$

In order to understand the nature of the evolution process in this game, let us treat the case of sequences having a single nonzero component, which corresponds to monomials X^k , for some nonnegative integer k . Let \mathfrak{w}_k be the sequence with $\mathfrak{w}_k(k) = 1$ and $\mathfrak{w}_k(j) = 0$ for $j \neq k$. Figure 5 displays the first six generations produced by the game that starts with \mathfrak{w}_3 .

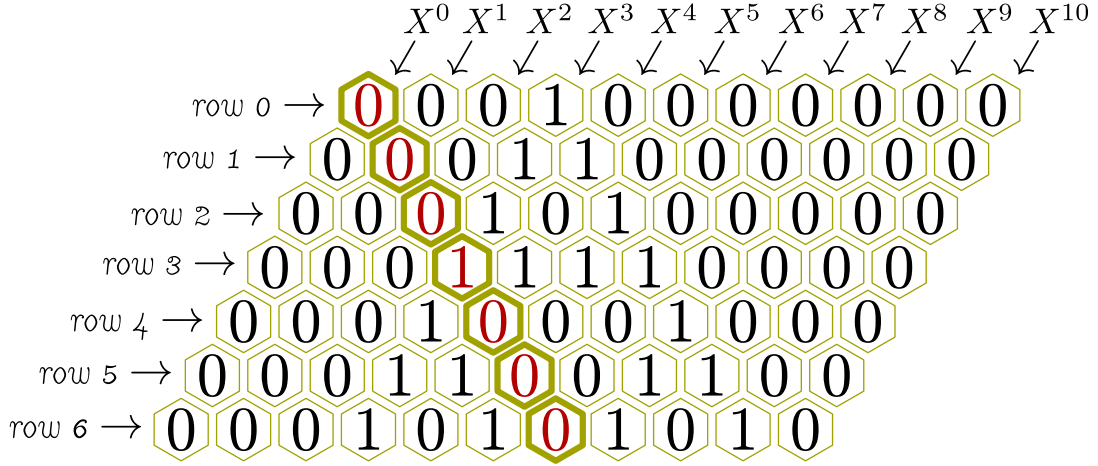


FIGURE 5. The coefficients of the first polynomials of the sequence that starts with X^3 .

By (2), it follows that for any $m \geq 1$,

$$\begin{aligned} P_{\psi(m)}(X) &= X^k(1+X)^m \\ &= X^k + \binom{m}{1}X^{k+1} + \dots + \binom{m}{j}X^{k+j} + \dots + \binom{m}{m}X^{k+m} \\ &= \sum_{j=0}^m \alpha_j X^{k+j}, \end{aligned}$$

where $\alpha_j = 1$ if $\binom{m}{j}$ is odd and $\alpha_j = 0$ otherwise. This gives a neat characterization of the entries in any generation.

Proposition 2. *For any positive integers k , m and n , we have*

$$\psi^{(m)}(\mathfrak{w}_k)(n) = \begin{cases} 1, & \text{if } \binom{m}{n-k} \text{ is odd} \\ 0, & \text{if } \binom{m}{n-k} \text{ is even} \end{cases} \quad \text{for any } n \in \mathbb{N}.$$

For the numbers on the diagonal we get the following corollary.

Corollary 1. *For any k , $n \geq 1$, we have*

$$\psi^{(n)}(\mathfrak{w}_k)(n) = \begin{cases} 1, & \text{if } \binom{n}{k} \text{ is odd} \\ 0, & \text{if } \binom{n}{k} \text{ is even} \end{cases} \quad \text{for any } n \in \mathbb{N}.$$

The parity of the binomial coefficients leads us into the wonderful world of the Sierpinski triangle. Glaisher [15] proved that on any line of the Pascal arithmetic triangle, the number of odd binomial coefficients is a power of 2. The reader is referred to Prunescu [25], the authors' survey [13, Section 2] and the references therein for classical and recent results on Pascal triangle modulo a power of a prime, and their connection

¹We denote by $\mathbb{F}_2[[X]]$ the ring of formal power series, the natural extension of $\mathbb{F}_2[X]$, the ring of polynomials of one variable over \mathbb{F}_2 .

with Fermat numbers, constructible polygons, analysis on Sierpinski gasket, etc. In the context of our game, this provides us with the following result.

Proposition 3. *For any integers $k \geq 1$ and $m \geq 0$, the number of nonzero entries on the line $\psi^{(m)}(\mathfrak{w}_k)$ is a power of 2.*

3. ACTION ON THE MAIN SIDE

A step nigher to the absolute difference game of the exponents requires reducing the evolution field to the region from the right side of the diagonal. For this, we look for the action on the series side.

Given a sequence $\mathfrak{w}: \mathbb{N} \rightarrow \{0, 1\}$, let $\phi(\mathfrak{w}): \mathbb{N} \rightarrow \{0, 1\}$ defined by $\phi(\mathfrak{w})(n) := |\mathfrak{w}(n+1) - \mathfrak{w}(n)|$. Then, denoting by $Q_{\mathfrak{w}}(X)$ the associated series $Q_{\mathfrak{w}}(X) = \sum_{n=0}^{\infty} \mathfrak{w}(n)X^n$, we have

$$Q_{\phi(\mathfrak{w})}(X) = \frac{(1+X)Q_{\mathfrak{w}}(X) - Q_{\mathfrak{w}}(0)}{X}.$$

In order to get the subsequent generations, applying $\phi(\mathfrak{w})$ repeatedly, we obtain

$$Q_{\phi^{(m)}(\mathfrak{w})}(X) = \left(\frac{1+X}{X}\right)^m Q_{\mathfrak{w}}(X) + O_{\alpha \geq 1}(X^{-\alpha}),$$

where the right term from right-hand side collects the sum monomials with negative exponents, only. Looking for the length of the smallest possible cycle, we put $m = 2^s$ and find that

$$\begin{aligned} Q_{\phi^{(2^s)}(\mathfrak{w})}(X) &= \left(\frac{1+X}{X}\right)^{2^s} Q_{\mathfrak{w}}(X) + O_{\alpha \geq 1}(X^{-\alpha}) \\ &= \frac{1+X^{2^s}}{X^{2^s}} Q_{\mathfrak{w}}(X) + O_{\alpha \geq 1}(X^{-\alpha}) \\ &= Q_{\mathfrak{w}}(X) + \frac{1}{X^{2^s}} Q_{\mathfrak{w}}(X) + O_{\alpha \geq 1}(X^{-\alpha}). \end{aligned}$$

This implies that

$$Q_{\phi^{(2^s)}(\mathfrak{w})}(X) = Q_{\mathfrak{w}}(X) + O_{\alpha \geq 1}(X^{-\alpha}), \quad (3)$$

provided that $Q_{\mathfrak{w}}(X)$ is a polynomial of degree $< 2^s$.

As a consequence, relation (3) explains the cyclic behavior of the game generated by the initial sequence \mathfrak{w}_k .

Theorem 3. *Let $k \geq 1$ and suppose that $2^s > k$. Then, we have $\phi^{(2^s)}(\mathfrak{w}_k) = \mathfrak{w}_k$.*

We remark that Theorem 3 also yields the length of the smallest cycle.

Corollary 2. *For any $k \geq 1$, the length of the smallest cycle is*

$$L_k := \min_{m \geq 0} \{m : m = 2^s \text{ for some } s \geq 0 \text{ and } m > k\}.$$

Example 1. We provide in Figure 7 a representation of such a game with $k = 26$. Here $s = 5$ yields the length of the smallest cycle, which is 32.

4. THE ABSOLUTE DIFFERENCE GAME OF EXPONENTS – PROOF OF THEOREM 1

The $Z(\cdot, \cdot)$ operation is in fact an absolute difference function on the exponents. Working with arbitrary numbers, this is best seen by taking their localized p -scans. Thus, we have

$$Z(p^s, p^t) = p^{(s+t)-2\min(s,t)} = p^{|s-t|}.$$

Let $\mathfrak{h} : \mathbb{N} \rightarrow \mathbb{N}$ be the initial sequence and let $\eta(\mathfrak{h}) : \mathbb{N} \rightarrow \mathbb{N}$ be the transformation defined by

$$\eta(\mathfrak{h})(n) := \frac{\mathfrak{h}(n)\mathfrak{h}(n+1)}{(\gcd(\mathfrak{h}(n), \mathfrak{h}(n+1)))^2}.$$

We shall assume that $\mathfrak{h}(n)$ is square free for $n \geq 0$. Our sequence of interest is $\eta^{(m)}(\mathfrak{h})(0)$, $m \geq 0$.

For a given sequence \mathfrak{h} and prime number p , we look at the sequence of p -exponents: $\mathfrak{w}_{\mathfrak{h},p} : \mathbb{N} \rightarrow \mathbb{N}$, $\mathfrak{w}_{\mathfrak{h},p}(n) := v_p(\mathfrak{h}(n))$ and consider the diagram

$$\begin{array}{ccc} \mathfrak{h} & \longrightarrow & \mathfrak{w}_{\mathfrak{h},p} \\ \eta^{(m)} \downarrow & & \downarrow \phi^{(m)} \\ \eta^{(m)}(\mathfrak{h}) & \longrightarrow & \mathfrak{w}_{\eta^{(m)}(\mathfrak{h}),p} \end{array}$$

where

$$\mathfrak{w}_{\eta^{(m)}(\mathfrak{h}),p} = \phi^{(m)}(\mathfrak{w}_{\mathfrak{h},p}), \text{ for } m \geq 1. \quad (4)$$

We remark that (4) is correct. Indeed, if $v_p(a)$ and $v_p(b)$ are consecutive terms of the sequence $\mathfrak{w}_{\mathfrak{h},p}$, then they correspond in the next generation to $\mathfrak{w}_{\eta(\mathfrak{h}),p}$, because

$$v_p\left(\frac{ab}{(\gcd(a,b))^2}\right) = v_p(a) + v_p(b) - 2v_p(\gcd(a,b)) = |v_p(a) - v_p(b)|.$$

Then, if $p = p_k$ is the k -th prime number, by Corollary 1, it follows that

$$v_{p_k}(d_m) = \begin{cases} 1, & \text{if } \binom{m}{k} \text{ is odd} \\ 0, & \text{if } \binom{m}{k} \text{ is even} \end{cases} \quad \text{for any } m \in \mathbb{N}.$$

This implies that

$$\omega(d_m) = \#\{k : v_{p_k}(d_m) = 1\} = \#\left\{k : \binom{m}{k} \text{ is odd}\right\},$$

which is a power of 2, by a result of Glaisher [15]. This proves Theorem 1.

5. THE GAME REVISITED – PROOFS OF THEOREMS 1 AND 2

In previous sections we saw that one gets the best view of the field of the game on exponents by considering it as a superposition of p -renderings for all prime numbers p . In Figure 7, there is an example of such a rendering for $p = 101$, the 26-th prime. In Section 3 we have noticed that incomplete Sierpinski triangles repeat in a cycle of length 32 (the first number that is a power of 2 and is larger than 26). This is just an instance of an arbitrary prime and there is a rewarding way to look on it. Indeed, let T_∞ be an infinite Sierpinski triangle with one vertex on the top and having cells corresponding to

the honeycomb network of our game. We assume that T_∞ runs across the honeycomb moving from left to right, one step at a time. Furthermore, we assume that triangle T_∞ changes to a new color its nonempty cells at every step.

For $n \geq 1$, we denote by \mathcal{P}_n the image of T_∞ that lies over the honeycomb of the game when its vertex from the top lies over the n -th cell of the topmost line. Thus, \mathcal{P}_n is the p -rendering of the game corresponding to $p = p_n$, the n -th prime. For example, in Figures 6 and 7 are shown the upper parts of $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{12}$ and \mathcal{P}_{26} , respectively. With these realizations, our number game from Figure 1 corresponds to the mosaic of colors developed by the superposition of all $\mathcal{P}_n, n \geq 1$.

Now let us look at an arbitrary cell $m \geq 0, n \geq 1$. The geometrical constrains show that the prime numbers $p_k, k \geq 1$, for which a nonempty cell of \mathcal{P}_k lies over the cell of $a_{m,n}$ must satisfy the inequality $p_n \leq p_k \leq p_{m+n}$. Moreover, we need to consider only those primes p_k for which a nonempty cell of \mathcal{P}_k lies over the cell (m, n) . Let \mathcal{S}_m be the set of nonzero numbers from the m -th row of Sierpinski triangle, that is,

$$\mathcal{S}_m := \left\{ r : \binom{m}{r} \equiv 1 \pmod{2}, 0 \leq r \leq m \right\}.$$

Then, we have obtained a closed form expression for any entry in Figure 1. This is

$$a_{m,n} = \prod_{r \in \mathcal{S}_m} p_{n+r}.$$

In particular, we have proved both Theorem 1 and Theorem 2.

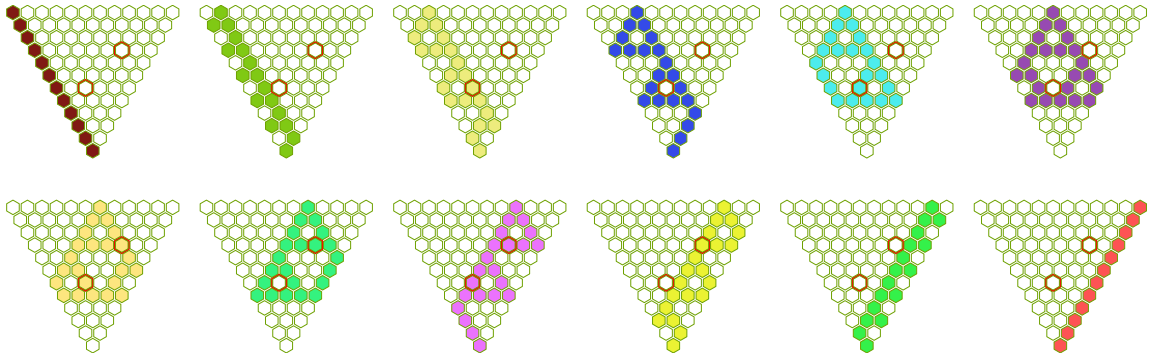


FIGURE 6. Cut-off triangles of size 12 from the the top-left of the field game covered by the pictures $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_{12}$. On each of them, focusing the sight, are highlighted the cells that correspond to the numbers $a_{3,7}$ and $a_{6,3}$.

6. CONCLUSION AND FUTURE DIRECTIONS

We discussed the outcome of the iterated application of a simple arithmetical function $Z(a, b)$, which operates like a recorder of the result produced by the plastic collision

of two elementary particles. Thus $Z(a, b)$ is a product of powers of primes p^{d_p} and locally, for any prime p , the exponent d_p is the absolute difference of the exponents of p in a and b . This paper leads to some further directions of research. Some open problems that naturally arise are presented below.

Sequence d_m from Table 1 increases in dyadically enhanced twists. It combines the size of primes with the selection process structured by the Sierpinski triangle sieve. Therefore, the following natural questions arise.

Question 1. Find accurate lower and upper bounds for

$$\min_{x \leq m \leq y} d_m, \quad \min_{x \leq m \leq y} \omega(d_m), \quad \max_{x \leq m \leq y} d_m \quad \text{and} \quad \max_{x \leq m \leq y} \omega(d_m).$$

Question 2. Estimate the following averages:

$$\sum_{x \leq m \leq y} \omega(d_m) \quad \text{and} \quad \sum_{x \leq m \leq y} d_m.$$

Question 3. Starting with the sequence of positive integers instead of the primes on the first line and applying the same generating $Z(\cdot, \cdot)$ -rule, we obtain a triangle that is similar to the triangle in Figure 1 (see [13, Section 9]). The sequence of numbers on the left side of this new triangle is (see [26, sequence A222311])

$$1, 2, 3, 6, 5, 15, 105, 70, 1, 5, 33, 55, 65, 273, 1001, \dots \quad (5)$$

By sorting it and removing duplicates we get a new sequence. Sloane [26, sequence A222313] makes this operations with the first 500 terms of (5) to obtain: 1, 2, 3, 5, 6, 15, 17, 33, 55, 65, 70, 105, ... The first three terms shown are certain. Sloane asks how far this list is complete up to 105 and if there is a proof that 4 cannot appear.

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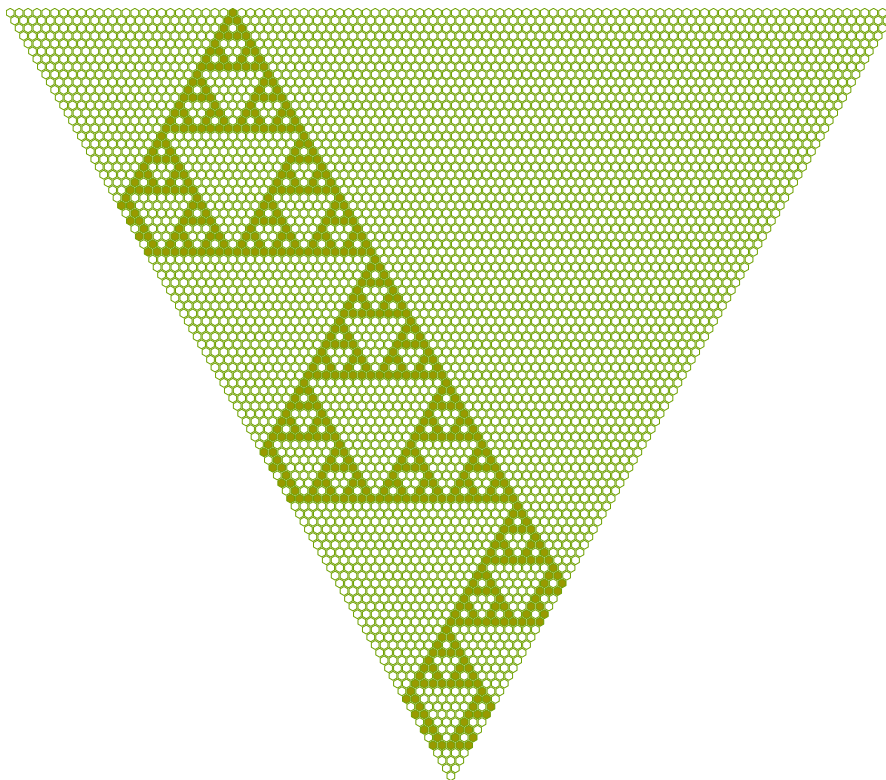


FIGURE 7. Triangular slice of size $N = 100$ cut off from the rendering produced by $p = 101$, the 26-th prime, in the table generated by iterated application of $Z(\cdot, \cdot)$ on the sequence of prime numbers.