Some binomial sums involving absolute values

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Abstract

We consider several families of binomial sum identities whose definition involves the absolute value function. In particular, we consider centered double sums of the form

$$S_{\alpha,\beta}(n) := \sum_{k,\,\ell} \binom{2n}{n+k}\binom{2n}{n+\ell}|k^\alpha - \ell^\alpha|^\beta,$$

obtaining new results in the cases $\alpha = 1, 2$. We show that there is a close connection between these double sums in the case $\alpha = 1$ and the single centered binomial sums considered by Tuenter.

1 Introduction

The problem of finding a closed form for the binomial sum

$$\sum_{k,\,\ell} \binom{2n}{n+k}\binom{2n}{n+\ell}|k^2-\ell^2| \tag{1}$$

arises in an application of the probabilistic method to the Hadamard maximal determinant problem [7]. Because of the double-summation and the absolute value occurring in (1), it is not obvious how to apply standard techniques [10, 15, 19]. A closed-form solution

$$2n^2\binom{2n}{n}^2 \tag{2}$$

was proved by Brent and Osborn in [6], and simpler proofs were subsequently found [5, 8, 16]. In this paper we consider a wider class of binomial sums with the distinguishing feature that an absolute value occurs in the summand.

Specifically, we consider certain d-fold binomial sums of the form

$$S(n) := \sum_{k_1,\ldots,k_d} \prod_{i=1}^{d} \binom{2n}{n+k_i} |f(k_1,\ldots,k_d)|, \tag{3}$$

where $f : \mathbb{Z}^d \to \mathbb{Z}$ is a homogeneous polynomial and $|f|$ will be called the *weight function*. For example, a simple case is $d = 1$, $f(k) = k$. This case was considered by Best [1] in an application to Hadamard matrices. The closed-form solution is

$$\sum_{k} \binom{2n}{n+k}|k| = n\binom{2n}{n}.$$

A generalization $f(k) = k^r$ (for a fixed $r \in \mathbb{N}$) was considered by Tuenter [18], and shown to be expressible using Dumont-Foata polynomials [9]. Tuenter gave an interpretation in terms of the moments of the distance to the origin in a symmetric Bernoulli random walk. It is easy to see that this interpretation generalizes: $4^{-nd}S(n)$ is the expectation of $|f(k_1,\ldots,k_d)|$ if we start at the origin and take $2n$ random steps $\pm\frac{1}{2}$ in each of d dimensions, thus arriving at the point $(k_1,\ldots,k_d) \in \mathbb{Z}^d$ with probability

$$4^{-nd}\prod_{i=1}^{d}\binom{2n}{n+k_i}.$$

A further generalization replaces $\binom{2n}{n+k_i}$ by $\binom{2n_i}{n_i+k_i}$, allowing the number of random steps $(2n_i)$ in dimension i to depend on i. With a suitable modification to the definition of S, we could also drop the restriction to an even number of steps in each dimension.[1] We briefly consider such a generalization in §2.

Tuenter's results for the case $d = 1$ were generalized by the first author [3]. In this paper we concentrate on the case $d = 2$. Generalizations of some of the results to arbitrary d are known. More specifically, the paper [4] gives closed-form solutions for the d-dimensional generalization of the sum (9) below in the cases $\alpha, \beta \in \{1, 2\}$.

There are many binomial coefficient identities in the literature, e.g. 500 are given by Gould [11]. Many such identities can be proved via generating functions [12, 19] or the Wilf-Zeilberger algorithm [15]. Nevertheless, we hope that the reader will find our results interesting, in part because of the applications mentioned above, and also because it is a challenge to generalize the results to higher values of d.

A preliminary version of this paper, with some of the results conjectural, was made available on arXiv [5]. All the conjectures have since been proved by Bostan, Lairez and Salvy [2], Krattenthaler and Schneider [14], Brent, Krattenthaler and Warnaar [4], and the present authors.

An outline of the paper follows.

In §2 we consider a special class of double sums that can be reduced to the single sums of [3, 18].

In §3 we consider a generalization of the motivating case (1) described above: $f(k, \ell) = (k^\alpha - \ell^\alpha)^\beta$. In the case $\alpha = 2$ we give recurrence relations that allow such sums to be evaluated in closed form for any given positive integer β. The recurrence relations naturally split into the cases where β is even (easy) and odd (more difficult).

Theorem 6 in §4 gives a closed form for an analogous triple sum. In [5, Conjecture 2] a closed form for the analogous quadruple sum was conjectured. This conjecture has now been proved by Brent, Krattenthaler and Warnaar [4]; in fact they give a generalization to arbitrary positive integer d.

In §5 we state several double sum identities that were proved or conjectured by us [5]. The missing proofs have now been provided by Bostan, Lairez and Salvy [2] and by Krattenthaler and Schneider [14].

[1] For example, in the case $d = 1$ we could consider $\sum_k \binom{n}{k} |f(n - 2k)|$.

Notation

The set of all integers is $\mathbb{Z}$, and the set of non-negative integers is $\mathbb{N}$,

The binomial coefficient $\binom{n}{k}$ is defined to be zero if $k < 0$ or $k > n$ (and hence always if $n < 0$). Using this convention, we often avoid explicitly specifying upper and lower limits on k or excluding cases where $n < 0$.

In the definition of the weight function $|f|$, we always interpret 0^0 as 1.

2 Some double sums reducible to single sums

Tuenter [18] considered the binomial sum

$$S_\beta(n) := \sum_k \binom{2n}{n+k} |k|^\beta, \tag{4}$$

and a generalization[2] to

$$U_\beta(n) := \sum_k \binom{n}{k} \left| \frac{n}{2} - k \right|^\beta \tag{5}$$

was given by the first author [3].

Tuenter showed that

$$S_{2\beta}(n) = Q_\beta(n) 2^{2n-\beta}, \quad S_{2\beta+1}(n) = P_\beta(n) n \binom{2n}{n}, \tag{6}$$

where $P_\beta(n)$ and $Q_\beta(n)$ are polynomials of degree β with integer coefficients, satisfying certain three-term recurrence relations, and expressible in terms of Dumont-Foata polynomials [9]. Closed-form expressions for $S_\beta(n)$, $P_\beta(n)$, $Q_\beta(n)$ are known [3].

In this section we consider the double sum

$$T_\beta(m,n) := \sum_{k,\,\ell} \binom{2m}{m+k} \binom{2n}{n+\ell} |k-\ell|^\beta \tag{7}$$

and show that it can be expressed as a single sum of the form (4).

[2]It is a generalization because $S_\beta(n) = U_\beta(2n)$, but $U_\beta(n)$ is well-defined for all $n \in \mathbb{N}$.

Theorem 1. *For all $\beta, m, n \in \mathbb{N}$, we have*

$$T_\beta(m,n) = S_\beta(m+n),$$

where T_β is defined by (7) *and S_β is defined by* (4).

Proof. If $\beta = 0$ then $T_0(m,n) = 2^{2(m+n)} = S_0(m+n)$. Hence, we may assume that $\beta > 0$ (so $0^\beta = 0$). Let $d = |k-\ell|$. We split the sum (7) defining $T_\beta(m,n)$ into three parts, corresponding to $k > \ell$, $k < \ell$, and $k = \ell$. The third part vanishes. If $k > \ell$ then $d = k - \ell$ and $k = d + \ell$; if $k < \ell$ then $d = \ell - k$ and $\ell = d + k$. Thus, we get

$$\begin{aligned} T_\beta(m,n) &= \sum_{d>0}\sum_{\ell}\binom{2m}{m+d+\ell}\binom{2n}{n+\ell}d^\beta + \sum_{d>0}\sum_{k}\binom{2m}{m+k}\binom{2n}{n+k+d}d^\beta \\ &= \sum_{d>0} d^\beta \sum_{\ell}\binom{2m}{m+d+\ell}\binom{2n}{n-\ell} + \sum_{d>0} d^\beta \sum_{k}\binom{2n}{n+k+d}\binom{2m}{m-k}. \end{aligned}$$

By Vandermonde's identity, the inner sums over k and ℓ are both equal to $\binom{2m+2n}{m+n+d}$. Thus,

$$T_\beta(m,n) = 2\sum_{d>0}\binom{2m+2n}{m+n+d}d^\beta = \sum_{d}\binom{2m+2n}{m+n+d}|d|^\beta = S_\beta(m+n).$$

□

Remark 1. If $m = n$ then, by the shift-invariance of the weight $|k-\ell|^\beta$, we have

$$T_\beta(n,n) = \sum_{k,\,\ell}\binom{2n}{k}\binom{2n}{\ell}|k-\ell|^\beta = S_\beta(2n). \tag{8}$$

There is no need for the upper argument of the binomial coefficients to be even in (8). We can adapt the proof of Theorem 1 to show that, for all $n \in \mathbb{N}$,

$$\sum_{k,\,\ell}\binom{n}{k}\binom{n}{\ell}|k-\ell|^\beta = S_\beta(n).$$

3 Centered double sums

In this section we consider the centered double binomial sums defined by[3]

$$S_{\alpha,\beta}(n) := \sum_{k,\,\ell} \binom{2n}{n+k}\binom{2n}{n+\ell}|k^\alpha - \ell^\alpha|^\beta. \tag{9}$$

Note that $S_{1,\beta}(n) = T_\beta(n,n)$, so the case $\alpha = 1$ is covered by Theorem 1. Thus, in the following we can assume that $\alpha \ge 2$. Since we mainly consider the case $\alpha = 2$, it is convenient to define

$$W_\beta(n) := S_{2,\beta}(n) = \sum_{k,\,\ell} \binom{2n}{n+k}\binom{2n}{n+\ell}|k^2 - \ell^2|^\beta. \tag{10}$$

Remark 2. The sequences $(S_{\alpha,\beta}(n))_{n\ge1}$ for $\alpha \in \{1,2\}$ and $1 \le \beta \le 4$ are in the OEIS [17]. Specifically, $(S_{1,1}(n))_{n\ge1}$ is a subsequence of A166337 (the entry corresponding to $n = 0$ must be discarded). $(S_{2,1}(n))_{n\ge0}$ is A254408, and $(S_{\alpha,\beta}(n))_{n\ge0}$ for $(\alpha,\beta) = (1,2),(2,2),(1,3),(2,3),(1,4),(2,4)$ are A268147, A268148, …, A268152 respectively.

3.1 W_β for odd β

The analysis of $W_\beta(n)$ naturally splits into two cases, depending on the parity of β. We first consider the case that β is odd. A simpler approach is possible when β is even, as we show in §3.3.

As mentioned in §1, the evaluation of $W_1(n)$ was the motivation for this paper, and is given in the following theorem.

Theorem 2 (Brent and Osborn)**.**

$$W_1(n) = \sum_{k,\,\ell} \binom{2n}{n+k}\binom{2n}{n+\ell}|k^2 - \ell^2| = 2n^2\binom{2n}{n}^2.$$

Numerical evidence suggested the following generalization of Theorem 2. It was conjectured by the present authors [5, Conjecture 2], and proved by Krattenthaler and Schneider [14].

[3]The double sum $S_{\alpha,\beta}(n)$ should not be confused with the single sum $S_\alpha(n)$ of §2.

Theorem 3 (Krattenthaler and Schneider). *For all $m, n \in \mathbb{N}$,*

$$\sum_{k,\,\ell} \binom{2m}{m+k}\binom{2n}{n+\ell}|k^2-\ell^2| \geq 2mn\binom{2m}{m}\binom{2n}{n},$$

with equality if and only if $m = n$.

3.2 Recurrence relations for the odd case

Theorem 2 gives $W_1(n)$. We show how $W_3(n), W_5(n), \ldots$ can be computed using recurrence relations. More precisely, we express the double sums $W_{2k+1}(n)$ in terms of certain single sums $G_k(n,m)$, and give a recurrence for the $G_k(n,m)$. We then show that $W_{2k+1}(n)$ is a linear combination of $P_k(n), \ldots, P_{2k}(n)$, where the polynomials $P_m(n)$ are as in (6), and the coefficients multiplying these polynomials satisfy another recurrence relation.

Define

$$f_q = \begin{cases} 1 \text{ if } q \neq 0; \\ \frac{1}{2} \text{ if } q = 0. \end{cases}$$

Using symmetry and the definition (10) of $W_k(n)$, we have

$$W_{2k+1}(n) = 8\sum_{q=0}^{n}\sum_{p=q}^{n}\binom{2n}{n+p}\binom{2n}{n+q}(p^2-q^2)^{2k+1}f_q; \tag{11}$$

the factor f_q allows for terms which would otherwise be counted twice.

Let $m = p - q$. Since $p^2 - q^2 = m(m+2q)$, we can write the double sum $W_{2k+1}(n)/8$ in (11) as

$$\sum_{q=0}^{n}\sum_{p=q}^{n}\binom{2n}{n+p}\binom{2n}{n+q}(p^2-q^2)^{2k+1}f_q = \sum_{m\geq 0} m^{2k+1}G_k(n,m), \tag{12}$$

where

$$G_k(n,m) := \sum_{q\geq 0}\binom{2n}{n+m+q}\binom{2n}{n+q}(m+2q)^{2k+1}f_q. \tag{13}$$

Observe that $G_k(0,m) = 0$. For convenience we define $G_k(-1,m) = 0$. We observe that $G_k(n,m)$ satisfies a recurrence relation, as follows.

Lemma 1. *For all $k, m, n \in \mathbb{N}$,*

$$\begin{aligned} G_{k+2}(n,m) &= 2(4n^2+m^2)G_{k+1}(n,m) - (4n^2-m^2)^2 G_k(n,m) \\ &\quad + 64n^2(2n-1)^2 G_k(n-1,m). \end{aligned} \tag{14}$$

Proof. If $n = 0$ the proof of (14) is trivial, since $G_k(0,m) = G_k(-1,m) = 0$. Hence, suppose that $n > 0$. We observe that

$$\begin{aligned} &[(m+2q)^4 - 2(4n^2+m^2)(m+2q)^2 + (4n^2-m^2)^2]\binom{2n}{n+m+q}\binom{2n}{n+q} \\ &= 16(n+m+q)(n-m-q)(n+q)(n-q)\binom{2n}{n+m+q}\binom{2n}{n+q} \\ &= 64n^2(2n-1)^2\binom{2n-2}{n-1+m+q}\binom{2n-2}{n-1+q}. \end{aligned}$$

Now multiply each side by $(m+2q)^{2k+1} f_q$ and sum over $q \geq 0$. □

The recurrence (14) may be used to compute $G_k(n,m)$ for given (n,m) and $k = 0, 1, 2, \ldots$, using the initial values

$$G_0(n,m) = \frac{n}{2}\binom{2n}{n}\binom{2n}{n+m}$$

and

$$G_1(n,m) = \frac{4n^2 + (2n-5)m^2}{2n-1}\, G_0(n,m).$$

These initial values may be verified from the definition (13) by standard methods [15] – we omit the details.

Write $g_k(n,m) = 0$ if $G_k(n,m) = 0$, and otherwise define $g_k(n,m)$ by

$$G_k(n,m) = \binom{2n}{n}\binom{2n}{n+m} g_k(n,m).$$

The recurrence (14) for G_k gives a corresponding recurrence for g_k:

$$\begin{aligned} g_{k+2}(n,m) &= 2(4n^2+m^2)g_{k+1}(n,m) - (4n^2-m^2)^2 g_k(n,m) \\ &\quad + 16n^2(n^2-m^2)g_k(n-1,m), \end{aligned} \tag{15}$$

with initial values

$$g_0(n,m) = \frac{n}{2}, \quad g_1(n,m) = \frac{4n^2+(2n-5)m^2}{2n-1}\, g_0(n,m).$$

Note that the $g_k(n,m)$ are rational functions in n and m; if computation with bivariate polynomials over $\mathbb{Z}$ is desired then $g_k(n,m)$ can be multiplied by $(2n-1)(2n-3)\cdots(2n-(2k-1))$. If n is fixed, then $g_k(n,m)$ is an even polynomial in m and, from the recurrence (15), the degree is $2k$. This suggests that we should define rational functions $\gamma_{k,j}(n)$ by

$$g_k(n,m) = \sum_{j=0}^{k} \gamma_{k,j}(n) m^{2j}.$$

For $j < 0$ or $j > k$ we define $\gamma_{k,j}(n) = 0$. From the recurrence (15), we obtain the following recurrence for the $\gamma_{k,j}(n)$:

$$\begin{aligned}\gamma_{k+2,j}(n) = {} & 8n^2\gamma_{k+1,j}(n) + 2\gamma_{k+1,j-1}(n) - 16n^4\gamma_{k,j}(n) + 8n^2\gamma_{k,j-1}(n) \\ & - \gamma_{k,j-2}(n) + 16n^4\gamma_{k,j}(n-1) - 16n^2\gamma_{k,j-1}(n-1).\end{aligned} \tag{16}$$

The $\gamma_{k,j}(n)$ can be computed from (16), using the initial values

$$\begin{aligned}\gamma_{0,0}(n) &= n/2, \\ \gamma_{1,0}(n) &= 2n^3/(2n-1), \\ \gamma_{1,1}(n) &= n(2n-5)/(4n-2).\end{aligned} \tag{17}$$

Using the definition of $\gamma_{k,j}(n)$ and (11)–(13), we obtain

$$W_{2k+1}(n) = 4\binom{2n}{n} \sum_{j=0}^{k} \gamma_{k,j}(n) S_{2k+2j+1}(n).$$

Since $S_{2r+1}(n) = P_r(n)n\binom{2n}{n}$, we obtain the following theorem, which shows that the double sums $W_{2k+1}(n)$ may be expressed in terms of the same polynomials $P_m(n)$ that occur in expressions for the single sums of [3, 18].

Theorem 4.

$$W_{2k+1}(n) = 4n \sum_{j=0}^{k} \gamma_{k,j}(n) P_{k+j}(n) \cdot \binom{2n}{n}^2, \tag{18}$$

where the polynomials $P_{k+j}(n)$ *are as in* (6), *and the* $\gamma_{k,j}(n)$ *may be computed from the recurrence* (16) *and the initial values given in* (17).

The factor before the binomial coefficient in (18) is a rational function $\omega_k(n)$ with denominator $(2n-1)(2n-3)\cdots(2n-2\lceil k/2\rceil+1)$. Thus, we have the following corollary of Theorem 4.

Corollary 1. *If $k \in \mathbb{N}$ and $W_k(n)$ is defined by* (10)*, then*

$$W_{2k+1}(n) = \omega_k(n)\binom{2n}{n}^2,$$

where

$$\omega_k(n)\prod_{j=1}^{\lceil k/2\rceil}(2n-2j+1)$$

is a polynomial of degree $2k+\lceil k/2\rceil+2$ over $\mathbb{Z}$. The first four cases are:

$$\begin{aligned}
\omega_0(n) &= 2n^2,\\
\omega_1(n) &= \frac{2n^3(8n^2-12n+5)}{2n-1},\\
\omega_2(n) &= \frac{2n^3(128n^4-512n^3+800n^2-568n+153)}{2n-1}, \text{ and}\\
\omega_3(n) &= \frac{2n^3\,\overline{\omega}_3(n)}{(2n-1)(2n-3)}, \text{ where}\\
\overline{\omega}_3(n) &= 9216n^7-86016n^6+350464n^5-802304n^4+\\
&\quad 1106856n^3-914728n^2+417358n-80847.
\end{aligned}$$

3.3 W_β for even β

Now we consider $W_\beta(n)$ for even β. This case is easier than the case of odd β because the absolute value in the definition (10) has no effect when β is even. Theorem 5 shows that $W_{2r}(n)$ can be expressed in terms of the single sums $S_0(n), S_2(n), \ldots, S_{4r}(n)$ or, equivalently, in terms of the polynomials $Q_0(n), Q_1(n), \ldots, Q_{2r}(n)$. It follows that $2^{2r-4n}W_{2r}(n)$ is a polynomial over $\mathbb{Z}$ of degree $2r$ in n.

Theorem 5. *For all $n \in \mathbb{N}$,*

$$\begin{aligned}
W_{2r}(n) &= \sum_k (-1)^k \binom{2r}{k} S_{2k}(n) S_{4r-2k}(n)\\
&= 2^{4n-2r}\sum_k (-1)^k \binom{2r}{k} Q_k(n) Q_{2r-k}(n),
\end{aligned}$$

where $Q_r(n)$ and $S_r(n)$ are as (4)–(6) *of* §2*, and $W_\beta(n)$ is defined by* (10)*.*

Proof. From the definition of $W_{2r}(n)$ we have

$$W_{2r}(n) = \sum_i \sum_j \binom{2n}{n+i}\binom{2n}{n+j}(i^2 - j^2)^{2r}.$$

Write

$$(i^2 - j^2)^{2r} = \sum_k (-1)^k \binom{2r}{k} i^{4r-2k} j^{2k},$$

change the order of summation in the resulting triple sum, and observe that the inner sums over i and j separate, giving $S_{4r-2k}(n)S_{2k}(n)$. This proves the first part of the theorem. The second part follows from (6). □

For example, the first four cases are

$$\begin{aligned}
W_0(n) &= 2^{4n},\\
W_2(n) &= 2^{4n-1}\, n(2n-1),\\
W_4(n) &= 2^{4n-2}\, n(2n-1)(18n^2 - 33n + 17),\\
W_6(n) &= 2^{4n-3}\, n(2n-1)(900n^4 - 4500n^3 + 8895n^2 - 8055n + 2764).
\end{aligned}$$

It follows from Theorem 5 that the coefficients of $2^{2r-4n}W_{2r}(n)$ are in $\mathbb{Z}$, but it is not obvious how to prove the stronger result, suggested by the cases above, that the coefficients of $2^{r-4n}W_{2r}(n)$ are in $\mathbb{Z}$. We leave this as a conjecture.

4 A triple sum

In Theorem 6 we give a triple sum that is analogous to the double sum of Theorem 2. A straightforward but tedious proof is given in [5, Appendix]. The result also follows from the case $d = 3$ of a more general result proved in [4, Proposition 1.1] for the analogous d-fold sum, where the weight function is generalized to the absolute value of a Vandermonde $|\Delta(i_1^2, i_2^2, \ldots, i_d^2)|$.

Theorem 6. *For all $n \in \mathbb{N}$,*

$$\begin{aligned}
&\sum_{i,\,j,\,k} \binom{2n}{n+i}\binom{2n}{n+j}\binom{2n}{n+k} |(i^2 - j^2)(i^2 - k^2)(j^2 - k^2)| \\
&\quad = \; 3n^3(n-1)\binom{2n}{n}^2 2^{2n-1}.
\end{aligned}$$

5 Further identities

In this section we give various identities that were stated in [5]. Of these, (25), (26), (27), (30) and (32) were conjectural. The conjectures have since been proved by Bostan, Lairez and Salvy [2, §7.3.2].

Centered double sums

Recall that, from the definition (9), we have

$$S_{\alpha,1}(n) = \sum_{i,\,j} \binom{2n}{n+i}\binom{2n}{n+j}|i^\alpha - j^\alpha|. \tag{19}$$

We give closed-form expressions for $S_{\alpha,1}(n)$, $1 \le \alpha \le 8$. Observe that (24) follows from Theorem 1 since $S_{1,1}(n) = T_1(n,n)$, and (20) is equivalent to Theorem 2. It appears that, for even α, $S_{\alpha,1}(n)$ is a rational function of n multiplied by $\binom{2n}{n}^2$, but for odd α, it is a rational function of n multiplied by $\binom{4n}{2n}$. This was conjectured in [5], and has been proved by Krattenthaler and Schneider [14].

$$S_{2,1}(n) = 2n^2\binom{2n}{n}^2, \tag{20}$$

$$S_{4,1}(n) = \frac{2n^3(4n-3)}{2n-1}\binom{2n}{n}^2, \tag{21}$$

$$S_{6,1}(n) = \frac{2n^3(11n^2-15n+5)}{2n-1}\binom{2n}{n}^2, \tag{22}$$

$$S_{8,1}(n) = \frac{2n^3(80n^4-306n^3+428n^2-266n+63)}{(2n-1)(2n-3)}\binom{2n}{n}^2, \tag{23}$$

$$S_{1,1}(n) = 2n\binom{4n}{2n}, \tag{24}$$

$$S_{3,1}(n) = \frac{4n^2(5n-2)}{4n-1}\binom{4n-1}{2n-1}, \tag{25}$$

$$S_{5,1}(n) = \frac{8n^2(43n^3 - 70n^2 + 36n - 6)}{(4n-2)(4n-3)} \binom{4n-2}{2n-2}, \tag{26}$$

$$S_{7,1}(n) = \frac{16n^2 P_{7,1}(n)}{(4n-3)(4n-4)(4n-5)} \binom{4n-3}{2n-3}, \quad n \ge 2, \text{ where}$$

$$P_{7,1}(n) = 531n^5 - 1960n^4 + 2800n^3 - 1952n^2 + 668n - 90, \tag{27}$$

$$(S_{7,1}(1) = 12 \text{ is a special case}).$$

Following are some similar identities. We observe that, since $i^4 - j^4 = (i^2+j^2)(i^2-j^2)$, (28) is easily seen to be equivalent to (21). Similarly, since $i^6 - j^6 = (i^4 + i^2j^2 + j^4)(i^2 - j^2)$, any two of (22), (29) and (31) imply the third. Higher-dimensional generalizations of (30)–(31) are known [4].

$$\sum_{i,\,j} \binom{2n}{n+i}\binom{2n}{n+j} |i^2(i^2-j^2)| = \frac{n^3(4n-3)}{2n-1} \binom{2n}{n}^2, \tag{28}$$

$$\sum_{i,\,j} \binom{2n}{n+i}\binom{2n}{n+j} |i^4(i^2-j^2)| = \frac{n^3(10n^2-14n+5)}{2n-1} \binom{2n}{n}^2, \tag{29}$$

$$\sum_{i,\,j} \binom{2n}{n+i}\binom{2n}{n+j} |ij(i^2-j^2)| = \frac{2n^3(n-1)}{2n-1} \binom{2n}{n}^2, \tag{30}$$

$$\sum_{i,\,j} \binom{2n}{n+i}\binom{2n}{n+j} |i^2j^2(i^2-j^2)| = \frac{2n^4(n-1)}{2n-1} \binom{2n}{n}^2, \tag{31}$$

$$\sum_{i,\,j} \binom{2n}{n+i}\binom{2n}{n+j} |i^3j^3(i^2-j^2)| = \frac{2n^4(n-1)(3n^2-6n+2)}{(2n-1)(2n-3)} \binom{2n}{n}^2. \tag{32}$$

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