

A linear-time algorithm for the orbit problem over cyclic groups

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Abstract The orbit problem is at the heart of symmetry reduction methods for model checking concurrent systems. It asks whether two given configurations in a concurrent system (represented as finite strings over some finite alphabet) are in the same orbit with respect to a given finite permutation group (represented by their generators) acting on this set of configurations by permuting indices. It is known that the problem is in general as hard as the graph isomorphism problem, whose precise complexity (whether it is solvable in polynomial-time) is a long-standing open problem. In this paper, we consider the restriction of the orbit problem when the permutation group is cyclic (i.e. generated by a single permutation), an important restriction of the problem. It is known that this subproblem is solvable in polynomial-time. Our main result is a linear-time algorithm for this subproblem.

Keywords symmetry reductions · model checking · cyclic groups · orbits

1 Introduction

Since the inception of model checking (cf. [9]), a key challenge in verifying concurrent systems has always been how to circumvent the state explosion problem, which is exponential in the number of processes and in the number of finite-domain variables. The fundamental algorithmic problem can essentially be construed as a reachability problem in an exponentially-sized graph that is succinctly represented (e.g. in some concurrent programming language). Among others, symmetry reduction [20, 11, 16] has emerged to be an effective technique in combatting the state

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explosion problem. The essence of symmetry reduction is to identify symmetries in the system and avoid exploring states that are “similar” (under these symmetries) to previously explored states, thereby speeding up model checking.

Every symmetry reduction method has to deal with the following two computationally difficult problems: (1) how to identify symmetries in the given system, and (2) how to check that two configurations are similar under these symmetries. To simplify our discussion of Problem 1, we will restrict our discussion to *process symmetries*. [Extensions to *data symmetries* are possible, e.g., see the recent result of [28], which gives a general reduction of process and data symmetry identification in concurrent systems to symmetry identification in the solutions to constraints in the constraint-satisfaction problem.] In this case, for concurrent systems with n processes, Problem 1 amounts to searching for a group G of permutations on $[n] := \{1, \dots, n\}$ such that the system behaves in an identical way under the action of permuting the indices of the processes by any $\pi \in G$. For example, for a distributed protocol with a ring topology, the group G could be a *rotation group* generated by the “cyclical right shift” permutation RS that maps $i \mapsto i+1 \pmod n$ for each $i \in [n]$. See Example 1 for concrete examples. Although Problem 1 is computationally hard in general, a lot of research advances has been made in the past decade (e.g. see the recent survey [27], and also the recent paper [28] for a more general technique that covers both process and data symmetries). Now the group G partitions the state space of the concurrent system (i.e. Γ^n for some finite set Γ) into equivalence classes called (G -)orbits. Problem 2 is essentially the *orbit problem (over finite permutation groups)*: given G and two configurations $\mathbf{v}, \mathbf{w} \in \Gamma^n$, determine whether \mathbf{v} and \mathbf{w} are in the same G -orbit. For example, if G is generated by RS with $n = 4$, the two configurations $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$ are in the same orbit. These two computational problems can be studied independently. The focus of this paper is the second problem, i.e., the orbit problem.

Example 1 Two token-passing protocols with multiple tokens: These examples are nondeterministic versions of the randomised self-stabilising protocol of Israeli and Jalfon [21] (also see [24]).



In the first example (left figure), there are n processes P_1, \dots, P_n connected in a ring-shaped topology (i.e. the neighbours of P_i are $P_{i+1 \pmod n}$ and $P_{i-1 \pmod n}$). There are $m \leq n$ tokens in the network, each held by a unique process. At any given step, a unique process P_i holding a token is nondeterministically chosen by a scheduler and is permitted to pass the token to its neighbour P_j (i.e. either the left $P_{i-1 \pmod n}$ neighbour or the right neighbour $P_{i+1 \pmod n}$). If P_j already had a token, it will simply merge the two tokens into one, which reduces the total number of tokens in the network by 1. For

each number n of processes, this description yields a transition system. For example, configurations of the system are of the form $(\alpha_1, \dots, \alpha_n) \in \{\perp, \top\}^n$, where \top (resp. \perp) constitutes that the process holds (resp. does not hold) a token. The symmetry group G_n of the transition system is the *Dihedral group* D_n , which is generated by the cyclical right shift RS ($i \mapsto i + 1 \pmod n$) and the *reflection* ($i \mapsto n - i$, for each $i \in \{1, \dots, n\}$). In the standard composition of disjoint cycles notation, these permutations can be written as $(1, 2, \dots, n)$ and $(1, n)(2, n - 1) \cdots (\lfloor n/2 \rfloor, \lceil n/2 \rceil)$, respectively.

In the second example (right figure above), we modify the first example by disconnecting the line between P_1 and P_n , which results in a line-shaped topology. In effect, P_1 (resp. P_n) can only pass a token to P_2 (resp. P_{n-1}). The symmetry group G'_n of the system in this case is the group generated by the reflection mapping that maps $i \mapsto n - i$, for each $i \in \{1, \dots, n\}$.

The orbit problem (OP) was first studied in the context of model checking by Clarke *et al.* [11] in which it was shown to be in NP but is as hard as the graph isomorphism problem, whose precise complexity (whether it is solvable in polynomial-time) is a long-standing open problem. The difficulty of the problem is due to the fact that the input group G is represented by a set S of generators and that the size of G can be exponential in $|S|$ in the worst case. There is also a closely related variant of OP called the *constructive orbit problem (COP)*, which asks to compute the lexicographically smallest element $\mathbf{w} \in \Gamma^n$ in the orbit of a given configuration $\mathbf{v} \in \Gamma^n$ with respect to a given group G . OP is easily reducible to COP, though the reverse direction is by no means clear. COP was initially studied in the context of graph canonisation by Babai and Luks [2], in which COP was shown to be NP-hard (in contrast, OP is unlikely to be NP-hard since it would imply the collapse of the polynomial-time hierarchy to its second level¹). In the context of model checking, COP was first studied by Clarke *et al.* [10], in which a number of “easy groups” for which COP becomes solvable in P are given including polynomial-sized groups (e.g. rotation groups), the full symmetry group \mathcal{S}_n (i.e. containing all permutations on $[n]$), and disjoint/wreath products of easy groups (cf. [15]).

In this paper, we consider the orbit problem over *cyclic groups* (i.e. generated by a single permutation $\pi \in \mathcal{S}_n$), which is an important subproblem of OP. In the case of rotation groups, one can do a simple enumeration of the group elements and solve the orbit problem in polynomial-time. [More precisely, if the group has m elements, this algorithm runs in time $O(mn)$, which is already quadratic over rotation groups.] However, cyclic subgroups of \mathcal{S}_n can even be of size exponential in n (see Proposition 3 below), which rules out this enumeration strategy. It turns out that the orbit problem over cyclic groups is known to be solvable in polynomial-time (e.g. see [2, 23], where this is shown for a much larger class of permutation

¹ For, if it were NP-hard, then the *coset intersection problem (for permutation groups)* would be NP-hard, owing to its polynomial-time equivalence to the orbit problem [10]. By the well-known results of [3, 17] (also see [18, Section 6.5, Chapter 27]), which essentially shows that the coset intersection problem is in the complexity class COAM (contained in the second level of the polynomial hierarchy), this would mean that the polynomial hierarchy collapses to the second level.

groups denoted as Γ_d for every *fixed* d , which contains solvable groups). Another way to see that OP over cyclic groups is solvable in polynomial-time is by a quadratic-time reduction to the classical orbit problem over rational matrices [22]: given a rational n -by- n matrix M and two rational vectors $\mathbf{v}, \mathbf{w} \in \mathbb{Q}^n$, determine if there exists $k \in \mathbb{N}$ such that $M^k \mathbf{v} = \mathbf{w}$. In fact, the two problems coincide when M is restricted to *permutation matrices* [7], i.e., 0-1 matrices with precisely one column for each row with entry 1. That OP over cyclic groups is in P follows from Kannan and Lipton’s celebrated result [22] that OP over rational matrices is in P.

Mere polynomial time-complexity is far from sufficient for the purpose of symmetry reduction methods, since a model checker will have to invoke an algorithm for the orbit problem *once each time a new configuration* in the given transition system is visited (e.g. see [27]). Recent case studies in [28] suggest that the cost of solving the orbit problem often becomes extremely prohibitive, even more so than the cost of computing the symmetries². Therefore, lightweight methods for dealing with the orbit problem are crucial for the success of symmetry reductions in model checking.

Contributions. In this paper, we provide an algorithm for the orbit problem over cyclic groups that runs in linear-time. To this end, we provide a linear-time reduction to the problem of solvability of systems of linear congruence equations. The reduction exploits subtle connections to the string searching problem.

As for the solvability of systems of linear congruence equations, there is a well-known algorithm (based on the extended Euclidean algorithm) that runs in linear-time assuming constant-time integer arithmetic operations. However, when we measure the number of bit operations (i.e. bit complexity model), it turns out that the algorithm runs in time cubic in the number of equations in the systems. To address this issue, we restrict the problem to input instances provided by our reduction from the orbit problem. We offer two solutions. Firstly, we show that the average-case complexity of the algorithm under the bit complexity model is $O(\log^5 n)$, which is sublinear [Here, n measures the size of the input to the orbit problem.] Secondly, we provide another algorithm that uses at most linearly many bit operations *in the worst case* (though on average it is worse than the first algorithm).

It turns out that permutation groups generated by two permutations already suffice to make the orbit problem as hard as the graph isomorphism problem. This is almost a direct corollary of the polynomial-time reduction in [11] from the graph isomorphism problem to the orbit problem over some group G . It turns out the group G that is produced by the reduction of [11] is *not* any arbitrary group and could easily be generated by two generators (for the same reason that the full symmetry group \mathcal{S}_n on $\{1, \dots, n\}$ can be generated by the permutations $(1, 2)$ and $(1, 2, \dots, n)$).

Organisation. Section 2 contains basic definitions, notations, and results that will be used throughout the rest of the paper. We provide our linear-time reduction from the orbit problem to equations solving in Section 3 (Algorithm 2). Thus far, we assume that arithmetic operations take constant time. We deal with the issue of bit complexity in Section 4. We conclude with future work in Section 5.

² Some examples in [28] (even with a small number of processes) require a model checker to invoke an algorithm for the orbit problem hundreds to thousands of times for one transition system.

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2 Preliminaries

General Notations: We use \log (resp. \ln) to denote logarithm in base 2 (resp. natural logarithm). We use the standard interval notations to denote a subset of integers within that interval. For example, $[i, j]$ denotes the set $\{k \in \mathbb{Z} : i \leq k < j\}$. Likewise, for each positive integer n , we use $[n]$ to denote the set $\{1, \dots, n\}$. We shall also extend arithmetic operations to sets of numbers in the usual way: whenever $S_1, S_2 \subseteq \mathbb{Z}$, we define $S_1 + S_2 := \{s_1 + s_2 : s_1 \in S_1, s_2 \in S_2\}$ and $S_1 S_2 := \{s_1 \times s_2 : s_1 \in S_1, s_2 \in S_2\}$. In the context of arithmetic over $2^{\mathbb{Z}}$, we will treat a number $n \in \mathbb{N}$ as the singleton set $\{n\}$. That way, for $a, b \in \mathbb{N}$, the notation $a + b\mathbb{Z}$ refers to the *arithmetic progression* $\{a + bc : c \in \mathbb{Z}\}$, where a (resp. b) is called the *offset* (resp. *period*) of the arithmetic progression. Likewise, for a subset $S \subseteq \mathbb{N}$, we use $\gcd(S)$ to denote the greatest common divisor of S .

We will use standard notations from formal language theory. Let Γ be an *alphabet* whose elements are called *letters*. A word (or a string) w over Γ is a finite sequence of elements from Γ . We use Γ^* to denote the set of all words over Γ . The length of w is denoted by $|w|$. Given a word $w = a_1 \dots a_n$, the notation $w[i, j]$ denotes the subword $a_i \dots a_j$. For a sequence $\sigma = i_1, \dots, i_k \in [n]^*$ of *distinct* indices of w , we write $w[\sigma]$ to denote the word $a_{i_1} \dots a_{i_k}$. We also define $\text{RS}(w)$ to be $a_n a_1 a_2 \dots a_{n-1}$, i.e., the word w cyclically right-shifted.

Number Theory: In the sequel, we will use some standard results in number theory and algorithmic number theory. The first result is Linear Congruence Theorem and its application to solving a system of linear congruences. The second result is Chinese Remainder Theorem.

Linear Congruence Theorem (e.g. see [13, Chapter 31.4] or [26, Theorem 4.5]) gives a fast method of determining whether an equation of the form $ax \equiv b \pmod{n}$ is solvable and, whenever it is solvable, the set of solutions to x .

Lemma 1 (Linear Congruence Theorem) *The equation $ax \equiv b \pmod{n}$ is solvable for the unknown x iff $d|b$, where $d = \gcd(a, n)$. Furthermore, if it is solvable, then the set of solutions equals $x_0 + (n/d)\mathbb{Z}$, for some $x_0 \in [0, n/d)$ that can be computed in time $O(\log n)$ (assuming constant-time arithmetic operations).*

An immediate application of Linear Congruence Theorem is to determine the set of solutions to a *system* of linear congruences. A *system of linear congruence equations* is a relation of the form $\bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$. [In general, a system of linear congruence equations might take an equation of the form $ax \equiv b \pmod{n}$, but we do not need this general form in the sequel.] The set of solutions $x \in \mathbb{Z}$ to this system is denoted by $\llbracket \bigwedge_{i=1}^m x \equiv a_i \pmod{b_i} \rrbracket$, which equals $\bigcap_{i=1}^m (a_i + b_i\mathbb{Z})$. The system is *solvable* if the solution set is nonempty. We use `FALSE` to denote $x \equiv 0 \pmod{2} \wedge x \equiv 1 \pmod{2}$, which is not solvable.

Proposition 1 For any solvable system of linear congruence equations $\varphi(x) := \bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$, we have $\llbracket \varphi(x) \rrbracket = \llbracket x \equiv a \pmod{b} \rrbracket$ for some $a, b \in \mathbb{Z}$. Furthermore, there exists an algorithm which computes a, b in linear time (assuming constant-time arithmetic operations).

Proposition 1 is witnessed by Algorithm 1, which is simply a repeated application of Linear Congruence Theorem.

Algorithm 1 Solving a system of modular arithmetic equations

Input: A system of modular arithmetic equations $\bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$
Output: Solution set $\llbracket \bigwedge_{i=1}^m x \equiv a_i \pmod{b_i} \rrbracket$ as \emptyset or an arithmetic progression $a + b\mathbb{Z}$.
 $a := 0; b := 1;$
for $i = 1, \dots, m$ **do**
 $\varphi(y) := by \equiv a_i - a \pmod{b_i};$
 Apply algorithm from Lemma 1 on φ returning either \emptyset or $a' + b'\mathbb{Z}$ for $\llbracket \varphi \rrbracket$;
 if $\llbracket \varphi \rrbracket = \emptyset$ **then return** NO **else** $a := a'b + a; b := bb'$ **end if**
end for
return $a + b\mathbb{Z};$

Remark 1 The number of bits that is used to maintain a and b in the worst case is linear in the size $\sum_{j=1}^m (\log a_j + \log b_j)$ of the input. This justifies treating a single arithmetic operation as a constant-time operation. We will discuss bit complexity in Section 4.

In the sequel, we will also use Chinese Remainder Theorem (e.g. see [13, Section 31.5] or [26, Theorem 2.6]).

Proposition 2 (Chinese Remainder Theorem) Let n_1, \dots, n_k be pairwise relatively prime positive integers, and $n = \prod_{i=1}^k n_i$. The ring \mathbb{Z}_n and the direct product of rings $\mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$ are isomorphic under the function $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$ with $\sigma(x) := (x \pmod{n_1}, \dots, x \pmod{n_k})$ for each $x \in \mathbb{Z}$.

Groups: We briefly recall basic concepts from group theory and permutation groups (cf. see [8]). A *group* G is a pair (S, \cdot) , where S is a set and $\cdot : (S \times S) \rightarrow S$ is a binary operator satisfying: (i) associativity (i.e. $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$), (ii) the existence of a (unique) identity element $e \in S$ such that $g \cdot e = e \cdot g = g$ for all $g \in S$, and (iii) closure under inverse (i.e. for each $g \in G$, there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$). When it is clear from the context, we will write $g \cdot g'$ as gg' . The *order* $\text{ord}(G)$ of the group G is defined to be the number $|S|$ of elements in G . This paper concerns only finite groups, i.e., groups G with $\text{ord}(G) = |S| \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define g^n by induction: (i) $g^0 = e$, and (ii) $g^n = g^{n-1} \cdot g$. The *order* $\text{ord}(g)$ of $g \in G$ is the least positive integer n such that $g^n = e$.

A *subgroup* H of $G = (S, \cdot)$ (denoted as $H \leq G$) is any group (S', \cdot_H) such that $S' \subseteq S$ and \cdot_H and \cdot agree on S' . Lagrange's Theorem states that the order $\text{ord}(H)$ of H divides the order $\text{ord}(G)$ of G . Given any subset $X \subseteq S$, the subgroup $\langle X \rangle$ of G *generated* by X consists of those elements of G which can be expressed as a finite product of elements of X and their inverses. If $H = \langle X \rangle$, then X is said to *generate* H . A *cyclic group* is a group generated by a singleton set $X = \{g\}$.

An *action* of a group $G = (S, \cdot)$ on a set Y is a function $\times : S \times Y \rightarrow Y$ such that for all $g, h \in S$ and $y \in Y$: (1) $(gh) \times y = g \times (h \times y)$, and (2) $e \times y = y$. The *stabiliser* of x by G is the subgroup $\text{Stab}_G(x) := \{g \in G : g \times x = x\}$ of G . If G is understood, $\text{Stab}(x)$ will be used to denote $\text{Stab}_G(x)$. The (G) -*orbit* containing y ,

denoted Gy , is the subset $\{g \times y : g \in G\}$ of Y . The action \times partitions the set Y into G -orbits. When the meaning is clear, we shall omit mention of the operator \times , e.g. condition (2) above becomes $ey = y$.

Permutation Groups. A *permutation* on $[n]$ is any bijection $\pi : [n] \rightarrow [n]$. The set of all permutations on $[n]$ forms the (*n*th) *full symmetry group* \mathcal{S}_n under functional composition. We shall use the notation Id to denote the identity element of each \mathcal{S}_n . A word $w = a_0 \dots a_{k-1} \in [n]^*$ containing distinct elements of $[n]$ (i.e. $a_i \neq a_j$ if $i \neq j$) can be used to denote the permutation that maps $a_i \mapsto a_{i+1 \bmod k}$ for each $i \in [0, k)$ and fixes other elements of $[n]$. In this case, w is called a *cycle* (more precisely, *k*-cycle or *transpositions* in the case when $k = 2$), which we will often write in the standard notation (a_0, \dots, a_{k-1}) so as to avoid confusion. Observe that w and $\text{RS}(w)$ represent the same cycle c . We will however fix a particular ordering to represent c (e.g. the word provided as input to the orbit problem). For this reason, if $\mathbf{v} \in \Gamma^n$ for some alphabet Γ , the notation $\mathbf{v}[c]$ is well-defined (see General Notations above), which means projections of \mathbf{v} onto elements with indices in c , e.g., if $\mathbf{v} = (1, 1, 1, 0)$ and $c = (1, 4, 2)$, then $\mathbf{v}[c] = (1, 0, 1)$. Any permutation can be written as a composition of disjoint cycles [8]. Each subgroup $G = (S, \cdot)$ of \mathcal{S}_n acts on the set Γ^n (over any finite alphabet Γ) under the group action of permuting indices, i.e., for each $\pi \in S$ and $\mathbf{v} = (a_1, \dots, a_n) \in \Gamma^n$, we define $\pi\mathbf{v} := (a_{\pi^{-1}(1)}, \dots, a_{\pi^{-1}(n)})$.

Complexity Analysis: We will assume that permutations will be given in the input as a composition of disjoint cycles. It is easy to see that permutations can be converted back and forth in linear time from such representations and the representations of permutations as functions. The size $\|n\|$ of a number $n \in \mathbb{N}$ is defined to be the length of the binary representation of n , which is $\lfloor \log n \rfloor + 1$. The size $\|c\|$ of a cycle $c = (a_1, \dots, a_k)$ on $[n]$ is defined to be $\sum_{i=1}^k \|a_i\|$ (in contrast, the length $|c|$ of c is k). For a permutation $\pi = c_1 \dots c_m$ where each c_i is a cycle, the size $\|\pi\|$ of π is defined to be $\sum_{i=1}^m \|c_i\|$. We will use standard asymptotic notations from analysis of algorithms (big-O and little-o), cf. [13]. We also use the standard \sim notation: $f(n) \sim g(n)$ iff $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$. We will use the standard RAM model that is commonly used when analysing the complexity of algorithms (cf. [13]). In Section 3, we will assume that integer arithmetic takes constant time. Later in Section 4, we will use the *bit complexity model* (cf. [13]), wherein the running time is measured in the number of bit operations.

3 Reducing to solving a system of linear congruence equations

The main result of the paper is:

Theorem 1 *There is a linear-time algorithm for solving the orbit problem when the acting group is cyclic.*

In this section, we will prove this theorem *assuming constant-time arithmetic operations*. In the next section, we will show that this theorem still holds for the bit complexity model.

Before we proceed to the algorithm, the following proposition shows why the naive algorithm that checks whether $g^i \mathbf{v} = \mathbf{w}$, for a given permutation $g \in \mathcal{S}_n$ and for each $i \in [0, \text{ord}(g))$, actually runs in exponential time.

Proposition 3 *There exists a sequence $\{G_i\}_{i=1}^{\infty}$ of cyclic groups $G_i = \langle g_i \rangle$ such that $\text{ord}(g_i)$ is exponential in the size $\|g_i\|$ of the permutation g_i .*

Proof Let p_n denote the n th prime. The *Prime Number Theorem* states that $p_n \sim n \log n$ (cf. [19]). For each $i \in \mathbb{Z}_{>0}$, we define a cycle c_i of length p_i by induction on i . For $i = 1$, let $c_1 = (1, 2)$. Suppose that $c_{i-1} = (j, \dots, k)$. In this case, we define c_i to be the cycle $(k+1, \dots, k+p_i)$. To define the sequence $\{g_i\}_{i=1}^{\infty}$ of permutations, simply let $g_i = \prod_{j=1}^i c_j$. For example, we have $g_3 = (1, 2)(3, 4, 5)(6, 7, 8, 9, 10)$. Since c_i 's are disjoint, the order $\text{ord}(g_i)$ of g_i is the smallest positive integer k such that $c_j^k = \text{Id}$ for all $j \in [i]$. If S_j denotes the set of integers k satisfying $c_j^k = \text{Id}$, then $\text{ord}(g_i)$ is precisely the smallest positive integer in the set $\bigcap_{j=1}^i S_j$. It is easy to see that $S_j = p_j \mathbb{Z}$, which is the set of solutions to the linear congruence equation $x \equiv 0 \pmod{p_j}$. Therefore, by the Chinese Remainder Theorem (cf. Proposition 2), the set $\bigcap_{j=1}^i S_j$ coincides with the arithmetic progression $t_i \mathbb{Z}$ with $t_i := \prod_{j=1}^i p_j$. This implies that $\text{ord}(g_i) = t_i$. Now the number t_i is also known as the *i th primorial number* [1] with $t_i \sim e^{(1+o(1))i \log i}$, which is a corollary of the Prime Number Theorem. On the other hand, the size of g_i is $\sum(i) := \sum_{j=1}^i p_j$, which is known to be $\sim \frac{1}{2}i^2 \ln i$ (cf. [4]). Therefore, $\text{ord}(g_i)$ is exponential in $\|g_i\|$ as desired. \square

Algorithm 2 Reduction to system of modular arithmetic equations

Input: A permutation $g = c_1 \cdots c_m \in \mathcal{S}_n$, a finite alphabet Γ , and $\mathbf{v}, \mathbf{w} \in \Gamma^n$.

Output: A system of modular arithmetic equations, which is satisfiable iff $\exists i \in \mathbb{N} : g^i(\mathbf{v}) = \mathbf{w}$.

```
// First solve for each individual cycle
for all  $i = 1, \dots, m$  do
  Compute the length  $|c_i|$  of the cycle  $c_i$ ;
  Compute an ordered list  $S'_i \subseteq [0, |c_i|)$  of numbers  $r$  with  $c_i^r(\mathbf{v}[c_i]) = \mathbf{w}[c_i]$ ;
  if  $S'_i = \emptyset$  then return FALSE end if
  if  $|S'_i| = 1$  then let  $a_i$  be the member of  $S'_i$ ;  $b_i := |c_i|$ ; end if
  if  $|S'_i| > 1$  then  $a_i := \min(S'_i)$ ;  $a'_i := \min(S'_i \setminus \{a_i\})$ ;  $b_i := a'_i - a_i$ ; end if
end for
// Now for each  $i \in [1, m]$  we have a modular arithmetic equation  $x \equiv a_i \pmod{b_i}$ 
return YES iff there exists  $x \in \mathbb{N}$  satisfying  $\bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$ 
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Our linear-time reduction that witnesses Theorem 1 is given in Algorithm 2. In this algorithm, the acting group is $G = \langle g \rangle$ with $g \in \mathcal{S}_n$, expressed as a composition of disjoint cycles in a standard way, say, $g = c_1 c_2 \cdots c_m$ where each c_i is a cycle. Also part of the input is two strings $\mathbf{v} = v_1 \dots v_n, \mathbf{w} = w_1 \dots w_n \in \Gamma^n$ over a finite alphabet Γ . The orbit problem is to check whether $f\mathbf{v} = \mathbf{w}$ for some $f \in G$, i.e., $f = g^r$ for some $r \in \mathbb{Z}$. Since c_i 's are pairwise disjoint cycles, the question reduces to checking if there exists $r \in \mathbb{N}$ such that

$$\forall i \in [1, m] : (c_i^r \mathbf{v})[c_i] = \mathbf{w}[c_i]$$

In other words, for each $i \in [1, m]$, applying the action c_i^r to \mathbf{v} gives us \mathbf{w} when restricted to the indices in c_i . To simplify notations in the above equation, we fix a letter $a \in \Gamma$ and, for each $i \in [1, m]$, let the notation \mathbf{v}_i (resp. \mathbf{w}_i) denote the string \mathbf{v} (resp. \mathbf{w}) in which all letters but those in positions c_i are replaced by a . The equation above, therefore, amounts to

$$\forall i \in [1, m] : c_i^r \mathbf{v}_i = \mathbf{w}_i \tag{*}$$

Essentially, Algorithm 2 sequentially goes through each cycle c_i and computes the set S_i of solutions r to $c_i^r \mathbf{v}_i = \mathbf{w}_i$ as the set of solutions to the linear congruence equation $x \equiv a_i \pmod{b_i}$. Therefore, the set of solutions to (*) is precisely the set of solutions to the system of congruence equations $\bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$, which by Proposition 1 can be solved in linear time.

To compute S_i , we first prove a simple canonical form for S_i .

Lemma 2 *For each $i = 1, \dots, m$, either $S_i = \emptyset$ or $S_i = a_i + b_i \mathbb{Z}$ for some $a_i \in [0, b_i)$ and $b_i \in (0, |c_i|]$ where b_i divides $|c_i|$.*

Proof Let $G_i = \langle c_i \rangle$ be the group generated by c_i . Consider the stabiliser $H := \text{Stab}(\mathbf{v}_i)$ of \mathbf{v}_i by G_i . Then, H is a subgroup of G_i . Since G_i is a cyclic group of order $|c_i|$, H is a cyclic group generated by some element $h = c_i^k$, where k is the smallest integer in $(0, |c_i|]$ such that $c_i^k \in H$. It is known that k must be a divisor of $|c_i|$. This implies that the orbit containing \mathbf{v}_i consists of precisely k elements $\mathbf{v}_i, c_i \mathbf{v}_i, \dots, c_i^{k-1} \mathbf{v}_i$.

Suppose that $S_i \neq \emptyset$. Let s be the smallest nonnegative integer in S_i , i.e., $c_i^s \mathbf{v}_i = \mathbf{w}_i$. Then, $s \in [0, k)$. We claim that $S_i = s + k\mathbb{Z}$. We have $s + k\mathbb{Z} \subseteq S_i$ since $c_i^{s+kn} \mathbf{v}_i = c_i^s (c_i^{kn} \mathbf{v}_i) = c_i^s \mathbf{v}_i$. Conversely, if $t \in S_i$, then $c_i^t \mathbf{v}_i = c_i^p \mathbf{v}_i$, where p is the smallest nonnegative integer such that $p \equiv t \pmod{k}$. Then, it must be the case that $p = s$ since $\mathbf{v}_i, c_i \mathbf{v}_i, \dots, c_i^{k-1} \mathbf{v}_i$ are all different. Thus, it follows that $S_i \subseteq s + k\mathbb{Z}$. Letting $a_i = s$ and $b_i = k$ completes the proof. \square

In view of Lemma 2, it suffices to show how to determine if $S_i \neq \emptyset$ and, if so, compute a_i and b_i in time $O(\|c_i\|)$. We first compute the length $|c_i|$ of the cycle c_i , which can be done in time $O(\|c_i\|)$. [This is the same as how to compute the length of a list.] We proceed by computing representatives $S'_i \subseteq [0, |c_i|)$ for S_i . This suffices to compute a_i and b_i since:

- If $S'_i = \emptyset$, then $S_i = \emptyset$.
- If $S'_i = \{a\}$, then $S_i = a + |c_i| \mathbb{Z}$.
- If $|S'_i| > 1$, then $S_i = a_i + b_i \mathbb{Z}$, whenever a_i and $a_i + b_i$ are the two smallest numbers in S'_i .

This case-by-case treatment is reflected in Line 3–Line 5 within the for-loop in Algorithm 2. To compute S'_i , we collect a subset of numbers $h \in [0, |c_i|)$ such that $c_i^h \mathbf{v}_i = \mathbf{w}_i$. A quadratic algorithm for this is easy to come up with: sequentially go through $h \in [0, |c_i|)$ while computing the current c_i^h , and save h if $c_i^h \mathbf{v}_i = \mathbf{w}_i$ holds. One way to obtain a linear-time algorithm is to reduce our problem to the *string searching problem*: given a “text” $T \in \Sigma^*$ (over some finite alphabet Σ) and a “pattern” $P \in \Sigma^*$, find all positions i in T such that $T[i, i + |P|] = P$. This problem is solvable in time $O(|T| + |P|)$ by Knuth-Morris-Pratt (KMP) algorithm (e.g. see [13]).

We now show how to reduce our problem to the string searching problem in linear time. We will also use the following running example to illustrate the reduction: $c = (6, 5, 7, 3, 2, 1)$, $\mathbf{v} = \underline{010001111}$, and $\mathbf{w} = \underline{101110001}$, where the positions in \mathbf{v} and \mathbf{w} that are modified by c are underlined. Below, we will work with the equivalent equation $(c_i^r \mathbf{v})[c_i] = \mathbf{w}[c_i]$ (i.e. instead of $c_i^r \mathbf{v}_i = \mathbf{w}_i$). Suppose that $c := c_i = (j_1, \dots, j_k)$. We have $\mathbf{v}[c] = v_{j_1} \dots v_{j_k}$ and $\mathbf{w}[c] = w_{j_1} \dots w_{j_k}$.

Lemma 3 $(c\mathbf{v})[c] = \text{RS}(\mathbf{v}[c])$.

In other words, if $\text{DOM}(c) = \{j_1, \dots, j_k\}$, the effect of c on \mathbf{v} when restricted to $\text{DOM}(c)$ coincides with applying a cyclical right shift on the string $\mathbf{v}[c]$. Following our running example, it is easy to check that $\mathbf{v}[c] = 101010$ and $(c\mathbf{v})[c] = \text{RS}(\mathbf{v}[c]) = 010101$.

Proof (of Lemma 3) Let $\mathbf{u} = u_1 \dots u_k := (c\mathbf{v})[c]$ and $\mathbf{u}' = u'_1 \dots u'_k := \text{RS}(\mathbf{v}[c])$. It suffices to show that $u_t = u'_t$ for all $t \in \mathbb{Z}_k$. By definition of RS, it follows that $u'_t = v_{j_{t-1}}$. Now suppose that $\mathbf{v}' = v'_1 \dots v'_n := c\mathbf{v}$. Then

$$v'_j := \begin{cases} v_j & \text{if } j \notin \text{DOM}(c) \\ v_{j'} & \text{if } j \in \text{DOM}(c) \text{ and, for some } t \in \mathbb{Z}_k, j = j_{t+1} \text{ and } j' = j_t. \end{cases}$$

So, for $t \in \text{DOM}(c)$, we have $u_t = ((c\mathbf{v})[c])[t] = (\mathbf{v}'[c])[t] = v'_{j_t} = v_{j_{t-1}}$. This proves that $u_t = u'_t$. \square

Lemma 4 For each $r \in \mathbb{N}$, we have $(c^r \mathbf{v})[c] = \text{RS}^r(\mathbf{v}[c])$.

Proof This lemma can be proved by induction on $r \in \mathbb{N}$. The base case $r = 0$ is vacuous. For the induction case, we assume the induction hypothesis: $c^{r-1} \mathbf{v} = \text{RS}^{r-1}(\mathbf{v}[c])$. It follows that

$$(c^r \mathbf{v})[c] = (c(c^{r-1} \mathbf{v})) [c] = \text{RS}((c^{r-1} \mathbf{v})[c]) = \text{RS}(\text{RS}^{r-1}(\mathbf{v}[c])) = \text{RS}^r(\mathbf{v}[c]).$$

The third equality is by Lemma 3, while the fourth equality is by the induction hypothesis. This completes the proof. \square

Define the text $T := \mathbf{v}[c]\mathbf{v}[c]$ and the pattern $P := \mathbf{w}[c]$. Observe that, for each $r \in [k]$, P is matched at position r in T iff $\text{RS}^{r-1}(\mathbf{v}[c]) = \mathbf{w}[c]$. Therefore, after running the KMP algorithm with the solution set S' , the set S'_i will be $\{r - 1 : r \in S'\}$. Observe that this step takes time $O(|T| + |P|) = O(\|c\|)$.

Example 2 Continuing with our running example, it follows that $T = \mathbf{v}[c]\mathbf{v}[c] = 101010101010$ and $P = \mathbf{w}[c] = 010101$. We see that P matches T at positions $S' = \{2, 4, 6\}$. This implies that $S'_i = \{1, 3, 5\}$ and so the set S_i of solutions $r \in \mathbb{Z}$ to the equation $c^r \mathbf{v}_i = \mathbf{w}_i$ is $1 + 2\mathbb{Z}$. \blacksquare

Summing up. To sum up, the time spent computing the linear congruence equation $x \equiv a_i \pmod{b_i}$ for each $i \in [1, m]$ is $O(\|c_i\|)$. Therefore, our reduction runs in time $O(\sum_{i=1}^m \|c_i\|) = O(\|g\|)$, which is linear in input size. Therefore, invoking Proposition 1 on the resulting system of linear congruence equations, we obtain the set of solutions to (*) in linear time.

Example 3 Let us continue with our running example. Let

$$g_1 := c(4, 8) = (6, 5, 7, 3, 2, 1)(4, 8), \quad g_2 := c(4, 8, 9) = (6, 5, 7, 3, 2, 1)(4, 8, 9).$$

Then, running Algorithm 2 on g_1 yields the system $x \equiv 1 \pmod{2} \wedge x \equiv 1 \pmod{2}$, which is equivalent to $x \equiv 1 \pmod{2}$. Running Algorithm 2 on g_2 yields the system $x \equiv 1 \pmod{2} \wedge x \equiv 1 \pmod{3}$. Both systems are solvable. \blacksquare

4 Making do with linearly many bit operations

Thus far, we have assumed that arithmetic operations take constant time. In this section, since Algorithm 1 makes a substantial use of basic arithmetic operations, we will revisit this assumption. It turns out that, although our reduction (Algorithm 2) to solving a system of linear congruence equations runs in linear time in the bit complexity model, the algorithm for solving the system of equations (Algorithm 1) uses at least a cubic number of bit-arithmetic operations. The main results in this section are two-fold: (1) on inputs given by our reduction, Algorithm 1 runs in sublinear time (more precisely, $O(\log^5 n)$) *on average* in the bit complexity model, and (2) there exists another algorithm for solving a system of linear congruence equations (with numbers in the input represented in unary) that runs in linear time in the bit complexity model in the worst case.

We begin with two lemmas that provide the running time of Algorithm 2 and Algorithm 1 in the bit complexity model.

Lemma 5 *Algorithm 2 runs in linear time in the bit complexity model.*

Proof On i th iteration, the number $|c_i|$ is stored in binary counter and can be computed by counting upwards from 0 and incrementing by 1 as we go through the elements in c_i . Although a single increment by 1 might take $O(|c_i|)$ bit operations in the worst case (since we have to propagate the carry bit), it is known (e.g. see [13, Chapter 17, p. 454]) that the entire sequence of operations actually takes time $O(|c_i|)$. Finally, since addition and subtraction of two numbers can easily be performed in $O(\beta)$ time on numbers that use at most β bits, the operation $b_i := a'_i - a_i$ on the last line of the iteration takes at most $O(\log |c_i|)$ time. Therefore, accounting for all the cycles, the algorithm takes $\sum_{i=1}^m O(\|c_i\|) = O(\sum_{i=1}^m \|c_i\|) = O(\|g\|)$, which is linear in the input size. \square

Lemma 6 *On an input $\bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$ with $N = \max\{b_i : i \in [1, m]\}$, Algorithm 1 uses at most $m \log N$ bits to store any numeric variables. Furthermore, the algorithm runs in time $O(m^3 \log^2 N)$ in the bit complexity model.*

Proof On i th iteration, the number of bits used to store a and b grow by at most $\log b_i$. On the other hand, the invariant that $a', b' \in [0, b_i)$ is always maintained on the i th iteration and so they only need at most $\log N$ bits to represent throughout the algorithm. Hence, the algorithm uses $M = O(m \log N)$ bits to store a , b , a' , and b' . Extended Euclidean Algorithm runs in time $O(M^2)$ on inputs where each number uses at most M bits (cf. [13, Problem 31-2]), which also bounds the time it takes on each iteration. Therefore, the algorithm takes at most $O(mM^2) = O(m^3 \log^2 N)$ in the bit complexity model. \square

We now provide an average case analysis of the running time of Algorithm 1 on system of linear congruence equations given by our reduction. The input to the orbit problem over cyclic groups includes a permutation $g \in S_n$ and two vectors $\mathbf{v}, \mathbf{w} \in \Gamma^n$. We briefly recall the setting of average-case analysis (cf. [25]). Let Π_N be the set of all inputs to the algorithm of size N . Likewise, let Σ_N be the sum of the *costs* (i.e. running time) of the algorithm on *all* inputs of size N . Hence, if $\Pi_{N,k}$ is the cost of the algorithm on input of size N with running time k , then $\Sigma_N = \sum_k k \Pi_{N,k}$. The *average case complexity of the algorithm* is defined to be Σ_N / Π_N .

Theorem 2 *The expected running time of Algorithm 1 in the bit complexity model on inputs provided by Algorithm 2 is $O(\log^5 n)$.*

Proof The size of a single permutation $g \in S_n$ is $O(n)$ and additionally $|S_n| = n!$. Suppose that g has k cycles (say, $g = c_1 \cdots c_k$). Then, Algorithm 2 produces a system of equations $\bigwedge_{i=1}^k x \equiv a_i \pmod{b_i}$, where $a_i, b_i \in [0, |c_i|)$. By Lemma 6, Algorithm 1 takes $O(k^3 \log^2 n)$ time in the bit complexity model, since $N := \max\{b_i : i \in [1, k]\} \leq n$. In addition, the number of permutations in S_n with k cycles is precisely the definition of the *unsigned Stirling number of the first kind* $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$. Therefore, we have $\Sigma_n = O\left(\sum_{k=1}^n (k^3 \log^2 n) \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]\right) = O\left(\log^2 n \sum_{k=1}^n k^3 \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]\right)$. Therefore, it suffices to show that $\frac{1}{n!} \sum_{k=1}^n k^3 \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \sim c \log^3 n$ for a constant c . The proof can be found in the appendix. \square

Finally, we will now give our final main result of this section.

Theorem 3 *There exists a linear-time algorithm in the bit complexity model for solving a system of linear congruence equations when the input numbers are represented in unary.*

We now provide an algorithm that witnesses the above theorem. Let $\bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$ be the given system of equations. With unary representation of numbers, the size N_i of the equation $x \equiv a_i \pmod{b_i}$ is $a_i + b_i$. We use n to denote the total number of bits in the system of equations. Initially, we compute a binary representation of all the numbers a_i 's, b_i 's, and n as in the proof of Lemma 5, which takes linear time. Next we factorise all the numbers b_i into a product of distinct prime powers $p_{j_{i1}}^{e_{i1}} \cdots p_{j_{it_i}}^{e_{it_i}}$, where p_j stands for the j th prime and all e_{ij} 's are positive integers. This can be done in time $O(\sqrt{N_i} \log^2 N_i)$. To obtain this time bound, we can use any *unconditional*³ deterministic factorisation methods like Strassen's algorithm, whose complexity was shown in [6] (cf. also see [14]) to be $O(f(N^{1/4} \log N))$ for factoring a number N , where $f(M)$ is the number of bit operations required to multiply two numbers with M bits. The standard (high-school) multiplication algorithm runs in quadratic time giving us $f(M) = O(M^2)$, which suffices for our purposes. This shows that Strassen's algorithm runs in time $O(N^{1/2} \log^2 N)$. [In practice, do factoring using the general number field sieve (cf. [13]), which performs extremely well in practice, though its complexity requires some unproven number-theoretic assumptions.]

Next, following Chinese Remainder Theorem (CRT), compute $z_{ij} := a_i \pmod{p_{ij}^{e_{ij}}}$ for each $j \in [1, t_i]$. Let us analyse the time complexity for performing this. Each z_{ij} can be computed by a standard algorithm (e.g. see [13]) in time quadratic in the number of bits used to represent a_i and $p_{ij}^{e_{ij}}$. Since each of these numbers use at most $\log N_i$ bits, each z_i can be computed in time $O(\log^2 N_i)$, which is $o(N_i)$. In addition, since $e_{ij} > 1$ for each $j \in [1, t_i]$, it follows that $t_i = O(\log N_i)$. This means that the total time it takes to compute $\{z_{ij} : j \in [1, t_i]\}$ is $O(\log^3 N_i)$, which is also $o(N_i)$. So, computing this for all $i \in [1, m]$ takes time $O(\sum_{i=1}^m \log^3 N_i)$, which is at most linear in the input size.

³ This means that the bound does not depend on any number-theoretic assumptions.

In summary, for each $i \in [1, m]$, we obtained the following system of equations, which is equivalent to $x \equiv a_i \pmod{b_i}$ by CRT:

$$x \equiv z_{i1} \pmod{p_{i1}^{e_{i1}}} \quad \wedge \quad \cdots \quad \wedge \quad x \equiv z_{it_i} \pmod{p_{it_i}^{e_{it_i}}} \quad (E_i)$$

The final step is to determine if there exists a number $x \in \mathbb{N}$ that satisfies *each* (E_i) , for all $i \in [1, m]$. Loosely, we will go through all the equations and make sure that there is no conflict between any two equations whose periods are powers of the same prime number, i.e., $x \equiv a \pmod{b}$ and $x \equiv a' \pmod{b'}$ such that $b = p^i$ and $b' = p^{i'}$ for some prime p and $i, i' \in \mathbb{Z}_{>0}$. In order to achieve this in linear-time in the bit complexity model, one has to store these equations in the memory (in the form of lookup tables) and carefully perform the lookup operations while looking for a conflict. To this end, we first compute $p_{\max} = \max\{p_{ij} : i \in [1, m], j \in [1, t_i]\}$ and $e_{\max} = \max\{e_{ij} : i \in [1, m], j \in [1, t_j]\}$.

Lemma 7 p_{\max} and e_{\max} can be computed using $O(n)$ many bit operations.

Proof The algorithm for computing p_{\max} and e_{\max} is a slight modification of the standard algorithm that computes the maximum number in a list, which sequentially goes through the list n_1, \dots, n_m while keeping the maximum number n_{\max} in the sublist explored so far. To ensure linear-time complexity, we have to make sure that when comparing the values of n_i and n_{\max} , we explore at most n_i bits of n_{\max} (since n_{\max} is possibly much larger than n_i). This is easily achievable by assuming binary representation of these numbers *without redundant leading 0s*, e.g., the number 5 will be represented as 101, not 0101 or 0000101. That way, we will only need to inspect $\log(n_i)$ bits from n_{\max} on the i th iteration, which will give a total running time of $O(\sum_{i=1}^m \log(n_i))$, which is linear in input size. \square

Next, keep one 1-dimensional array A and one 2-dimensional array B :

$$A[1, \dots, p_{\max}] \quad B[1, \dots, p_{\max}][1, \dots, e_{\max}].$$

$A[k]$ and $B[k][e]$ will not be defined when k is not a prime number. We will use $A[k]$ as a flag indicating whether some equation of the form $x \equiv z \pmod{k^e}$ has been visited, in which case $A[k]$ will contain (z, e) . In this case, we will use $B[k][e']$ (with $e' \leq e$) to store the value of $z \pmod{k^{e'}}$.

We now elaborate how A and B are used when iterating over the equations in the system. Sequentially go through each system (E_i) of equations. For each $i \in [1, m]$, sequentially go through each equation $x \equiv z_{ij} \pmod{p_{ij}^{e_{ij}}}$, for each $j \in [1, t_i]$, and check if $A[p_{ij}]$ is defined. If it is not defined, set $A[p_{ij}] := (z_{ij}, e_{ij})$ and compute $B[p_{ij}][l] = z_{ij} \pmod{p_{ij}^l}$ for each $l \in [1, e_{ij}]$. If it is defined (say, $A[p_{ij}] = (z, e)$), then we analyse the constraints $x \equiv z \pmod{p_{ij}^e}$ and $x \equiv z_{ij} \pmod{p_{ij}^{e_{ij}}}$ simultaneously. We compare e and e_{ij} resulting in three cases:

- Case 1. $e = e_{ij}$. In this case, make sure that $z = z_{ij}$ otherwise the two equations (and, hence, the entire system) cannot be satisfied simultaneously.
- Case 2. $e < e_{ij}$. In this case, make sure that $z_{ij} \equiv z \pmod{p_{ij}^e}$ (otherwise, unsatisfiable) and assign $A[p_{ij}] := (z_{ij}, e_{ij})$. For each $l \in [1, e_{ij}]$, update $B[p_{ij}][l] := z_{ij} \pmod{p_{ij}^l}$.
- Case 3. $e > e_{ij}$. In this case, make sure that $z_{ij} \equiv z \pmod{p_{ij}^{e_{ij}}}$ (otherwise, unsatisfiable).

We now analyse the running time of this final step (i.e. when scanning through the subsystem (E_i)). To this end, we measure the time it takes to process each equation $x \equiv z_{ij} \pmod{p_{ij}^{e_{ij}}}$. There are two cases, which we will analyse in turn.

(Case I): when $A[p_{ij}]$ is not defined. In this case, setting $A[p_{ij}]$ takes constant time, while setting $B[p_{ij}][l]$ for all $l \in [1, e_{ij}]$ takes $O(e_{ij} \times (\log z_{ij} + \log p_{ij}^{e_{ij}})^2)$ since computing $a \pmod b$ can be done in time quadratic in $\log(a) + \log(b)$. Since $e_{ij} \leq \log N_i$ and $z_{ij}, p_{ij} \leq N_i$, this expression can be simplified to $O(\log N_i \times \log^2(z_{ij} N_i p_{ij})) = O(\log^3 N_i)$.

(Case II): when $A[p_{ij}]$ is already defined, e.g., $A[p_{ij}] = (z, e)$. In this case, we will compare the values of e and e_{ij} . To ensure linear-time complexity, we will make sure that at most $\log(e_{ij})$ bits from e are read by using the trick from the proof of Lemma 7. For Case 1, we will need extra $O(\log z_{ij}) = O(\log N_i)$ time steps. For Case 2, we have $0 \leq z \leq p_{ij}^{e_{ij}}$ and computing $z_{ij} \pmod{p_{ij}^e}$ can be done in time $O(\log^2 N_i)$ as before. Updating $B[p_{ij}][l]$ for all $l \in [1, e_{ij}]$ takes $O(\log^3 N_i)$ as in the previous paragraph. For Case 3, since $e > e_{ij}$, we may access the value of $z \pmod{p_{ij}^{e_{ij}}}$ from $B[p_{ij}][e_{ij}]$ in constant time and compare this with the value of z_{ij} . Since $z \in [0, p_{ij}^{e_{ij}})$, this takes time $O(\log N_i)$.

In summary, either case takes time at most $O(\log^3 N_i)$. Therefore, accounting for the entire subsystem (E_i) , the algorithm incurs $O(\sum_{j=1}^{t_i} \log^3 N_i) = O(\log^4 N_i)$ time steps. Hence, accounting for *all* of the subsystems E_i ($i \in [1, m]$) the algorithm takes time $O(\sum_{i=1}^m \log^4 N_i)$, which is linear in the size of the input. This completes the proof of Theorem 3.

Remark 2 The purpose of the 2-dimensional array B above is to avoid superlinear time complexity for Case 3. We can imagine a system of linear equations $\bigwedge_{i=1}^m x \equiv a_i \pmod{b_i}$, where a_1 and b_1 are substantially larger than the other a_i 's and b_i 's ($i \in [2, m]$). In this case, without the lookup table B , checking whether $a_i \equiv a_1 \pmod{b_i}$ in Case 3 will require the algorithm to inspect the entire value of a_1 , which prevents us from bounding the time complexity in terms of a_i and will yield a superlinear time complexity for our algorithm.

5 Future work

Since an algorithm for the orbit problem will be invoked many times during an explicit-state model checking (in the worst case once each time a new state in the transition system is visited; cf. [27]), we believe that it is important to further identify efficiently solvable (preferably, in linear-time) subcases of the orbit problem. As mentioned in the Introduction, there are known classes of permutations groups whose orbit problem is polynomial-time solvable (e.g. Γ_d which contains solvable groups). We propose the question of further identifying other classes of permutations groups whose orbit problem is solvable in linear time.

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A Completing proof of Theorem 2

Let $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ denote the unsigned Stirling number of the first kind, and $\binom{n}{k}$ denote n choose k . The harmonic number H_n is defined as

$$H_n = \sum_{k=1}^n \frac{1}{k}.$$

In general, for an integer $s \geq 1$, the generalized harmonic number of order s is defined as

$$H_n^{(s)} = \sum_{k=1}^n \frac{1}{k^s}.$$

It is known that

$$\frac{1}{n!} \sum_{k=1}^n k \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = H_n.$$

Define

$$f(n) = \frac{1}{n!} \sum_{k=1}^n k^2 \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right], \quad g(n) = \frac{1}{n!} \sum_{k=1}^n k^3 \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right].$$

It is known that

$$\sum_{k=m}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \binom{k}{m} = \left[\begin{smallmatrix} n+1 \\ m+1 \end{smallmatrix} \right],$$

(see [5]). In particular, we have

$$\begin{aligned} \left[\begin{smallmatrix} n+1 \\ 3 \end{smallmatrix} \right] &= \sum_{k=2}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \binom{k}{2} = \sum_{k=1}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{k(k-1)}{2} = \frac{1}{2} n! (f(n) - H_n), \\ \left[\begin{smallmatrix} n+1 \\ 4 \end{smallmatrix} \right] &= \sum_{k=3}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \binom{k}{3} = \sum_{k=1}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] \frac{k(k-1)(k-2)}{6} = \frac{1}{6} n! (g(n) - 3f(n) + 2H_n). \end{aligned}$$

That is, we have

$$\begin{aligned} f(n) &= \frac{2}{n!} \left[\begin{smallmatrix} n+1 \\ 3 \end{smallmatrix} \right] + H_n, \quad \text{and} \\ g(n) &= \frac{6}{n!} \left[\begin{smallmatrix} n+1 \\ 4 \end{smallmatrix} \right] + 3f(n) - 2H_n = \frac{6}{n!} \left[\begin{smallmatrix} n+1 \\ 4 \end{smallmatrix} \right] + \frac{6}{n!} \left[\begin{smallmatrix} n+1 \\ 3 \end{smallmatrix} \right] + H_n. \end{aligned}$$

It is known (cf. page 217 of [12]) that

$$\begin{aligned} \frac{1}{n!} \left[\begin{smallmatrix} n+1 \\ 3 \end{smallmatrix} \right] &= \frac{1}{2} (H_n^2 - H_n^{(2)}) \\ \frac{1}{n!} \left[\begin{smallmatrix} n+1 \\ 4 \end{smallmatrix} \right] &= \frac{1}{6} (H_n^3 - 3H_n H_n^{(2)} + 2H_n^{(3)}). \end{aligned}$$

It is also known that $H_n = \gamma + \ln n$, $\lim_{n \rightarrow \infty} H_n^{(2)} = \zeta(2) = \frac{\pi^2}{6}$ and $\lim_{n \rightarrow \infty} H_n^{(3)} = \zeta(3) \approx 1.202$, where $\gamma \approx 0.577$ is Euler's constant and $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$ is the Riemann zeta function. Hence we obtain

$$f(n) = H_n^2 - H_n^{(n)} + H_n \sim \ln^2 n.$$

Putting all together, we obtain

$$g(n) = \frac{6}{n!} \left[\begin{smallmatrix} n+1 \\ 4 \end{smallmatrix} \right] + 3f(n) - 2H_n \sim \ln^3 n.$$

