

p -Adic Invariant Summation of Some p -Adic Functional Series

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Abstract

We consider summation of some finite and infinite functional p -adic series with factorials. In particular, we are interested in the infinite series which are convergent for all primes p , and have the same integer value for an integer argument. In this paper, we present rather large class of such p -adic summable functional series with integer coefficients which contain factorials.

1 Introduction

It is well known that the series play important role in mathematics, physics and applications. When numerical ingredients of the infinite series are rational numbers then they can be treated in any p -adic as well as in real number field, because rational numbers are endowed by real and p -adic norms simultaneously. In particular, a real divergent series deserves p -adic investigation when its p -adic sum is a rational for a rational argument.

Terms of many series in string theory, quantum field theory and quantum mechanics contain factorials. Such series are usually divergent in the real case but convergent in p -adic ones. Motivated by this reason, different p -adic aspects of the series with factorials are considered in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and many summations are performed in rational points. Also, notions of p -adic number field invariant summation, rational summation, and adelic summation are introduced.

Note that p -adic numbers and p -adic analysis have been successfully applied in modern mathematical physics (from strings to complex systems and the universe as a whole) and in some related fields (in particular in bioinformation systems, see, e.g. [15]), see [12, 13] for an early review and [14] for a recent one.

In this paper we are interested in p -adic invariant summation of a class of infinite functional series which terms contain $n!$, i.e. $\sum n!P_k(n;x)x^n$, where $P_k(n;x)$ are polynomials in x of degree k , and which coefficients depend on n . We show that there exist polynomials $P_k(n;x)$ for any degree k , such that for any $x \in \mathbb{Z}$ the corresponding sums are also rational numbers. Moreover, we have found recurrent relations how to calculate all ingredients of such $P_k(n;x)$. The obtained summa-

tion formula is a generalization of previously derived one when $x = 1$ (see [10]) in $\sum n!P_k(n; x)x^n$. Many results are illustrated by some simple examples.

All necessary general information on p -adic series can be found in standard books on p -adic analysis, see, e.g. [16].

2 p -Adic Functional Series with Factorials

We consider p -adic functional series of the form

$$S_k(x) = \sum_{n=0}^{+\infty} n! P_k(n; x) x^n = P_k(0; x) + 1! P_k(1; x) x + 2! P_k(2; x) x^2 + \dots, \quad (1)$$

where

$$P_k(n; x) = C_k(n) x^k + C_{k-1}(n) x^{k-1} + \dots + C_1(n) x + C_0(n), \quad (2)$$

$$n! = 1 \cdot 2 \cdot \dots \cdot n, \quad (0! = 1), \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \quad x \in \mathbb{Q}_p$$

and $C_j(n)$, $0 \leq j \leq k$, are some polynomials in n with integer coefficients. Mainly we are interested for which polynomials $P_k(n; x)$ we have that if $x \in \mathbb{Z}$ then the sum of the series (1) is $S_k(x) \in \mathbb{Z}$, i.e. $S_k(x)$ is also an integer the same in all p -adic cases. Since polynomials $P_k(n; x)$ are determined by polynomials $C_j(n)$, $0 \leq j \leq k$, it means that one has to find these $C_j(n)$, $0 \leq j \leq k$. Our task is to find connection between polynomial $P_k(n; x)$ and sum $S_k(x)$, which are also a polynomial.

2.1 Convergence of the p -Adic Series

Note that necessary and sufficient condition for p -adic power series to be convergent [16, 13] is

$$f(x) = \sum_{n=1}^{+\infty} a_n x^n, \quad a_n \in \mathbb{Q} \subset \mathbb{Q}_p, \quad x \in \mathbb{Q}_p, \quad |a_n x^n|_p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Necessary condition follows like in the real case, and sufficient condition is a consequence of ultrametricity. Recall that ultrametric distance satisfies the strong triangle inequality $d(x, y) \leq \max\{d(x, z), d(y, z)\}$. p -Adic distance $d_p(x, y) = |x - y|_p$ is ultrametric one, where $|\cdot|_p$ is p -adic norm. According to the Cauchy criterion, the sufficiency follows from $|a_n + a_{n+1} + \dots + a_{n+m}|_p \leq \max_{n \leq i \leq n+m} |a_i|_p = |a_n|_p \rightarrow 0$ when $n \rightarrow \infty$.

The functional series (1) contains $n!$, hence to investigate its convergence one has to know p -adic norm of $n!$. First, let us find a power $M(n)$ by which prime p is contained in $n!$ (see, e.g. [13]). Let $n = n_0 + n_1 p + \dots + n_r p^r$ and $s_n = n_0 + n_1 + \dots + n_r$ denotes the sum of digits in expansion of a natural number n in base p . We also

denote by $[x]$ the integer part of a number x . Then

$$\begin{aligned}
M(n) &= \left[\frac{n}{p} \right] + \left[\frac{n}{p^2} \right] + \cdots + \left[\frac{n}{p^r} \right] \\
&= \frac{n - n_0}{p} + \frac{n - n_0 - n_1 p}{p^2} + \cdots + \frac{n - n_0 - n_1 p - \cdots - n_{r-1} p^{r-1}}{p^r} \\
&= \frac{n}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{r-1}} \right) - \frac{n_0}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{r-1}} \right) \\
&\quad - \frac{n_1}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{r-2}} \right) - \frac{n_2}{p} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{r-3}} \right) \\
&\quad - \cdots - \frac{n_{r-1}}{p}.
\end{aligned} \tag{3}$$

Performing summation in (3), it follows

$$\begin{aligned}
M(n) &= n \frac{1 - p^{-r}}{p - 1} - n_0 \frac{1 - p^{-r}}{p - 1} - n_1 \frac{1 - p^{-r+1}}{p - 1} - n_2 \frac{1 - p^{-r+2}}{p - 1} \\
&\quad - \cdots - n_{r-1} \frac{1 - p^{-1}}{p - 1} - n_r \frac{1}{p - 1} + n_r \frac{1}{p - 1} \\
&= \frac{n - s_n}{p - 1}
\end{aligned} \tag{4}$$

Hence, one obtains

$$\begin{aligned}
n! &= m p^{M(n)} = m p^{\frac{n - s_n}{p - 1}}, \quad p \nmid m, \\
|n!|_p &= p^{-\frac{n - s_n}{p - 1}}.
\end{aligned} \tag{5}$$

Consider p -adic norm of the general term in (1) with $P_k(n; 1)$, i.e.

$$|n! P_k(n; 1) x^n|_p \leq |n! x^n|_p = p^{-\frac{n - s_n}{p - 1}} |x|_p^n = \left(p^{-\frac{n - s_n}{n(p - 1)}} |x|_p \right)^n$$

Then, because convergence requires $|n! P_k(n; 1) x^n|_p \rightarrow 0$ as $n \rightarrow \infty$, one has the following relations:

$$p^{-\frac{n - s_n}{n(p - 1)}} |x|_p < 1, \quad |x|_p < p^{\frac{n - s_n}{n(p - 1)}} \rightarrow p^{\frac{1}{p - 1}} \text{ as } n \rightarrow \infty.$$

Hence, the region of convergence D_p of the power series $\sum P_k(n; 1) n! x^n$ is $D_p = \mathbb{Z}_p$, i.e. $|x|_p \leq 1$, because norms in \mathbb{Q}_p are of the form p^ν , $\nu \in \mathbb{Z}$ and $1 < p^{\frac{1}{p - 1}}$. The region of convergence $|x|_p \leq 1$ is valid also for the series (1), because $|P_k(n; x)|_p \leq |P_k(n; 1)|_p$ when $|x|_p \leq 1$.

Since $\bigcap_p \mathbb{Z}_p = \mathbb{Z}$, it means that the infinite series $\sum P_k(n; x) n! x^n$ is simultaneously convergent for all integers and all p -adic norms.

3 Summation at Integer Points

We are interested now in determination of the polynomials $P_k(n; x)$ and the corresponding sums $S_k(x) = Q_k(x)$ of the infinite series (1), where

$$Q_k(x) = q_k x^k + q_{k-1} x^{k-1} + \cdots + q_1 x^1 + q_0 \quad (6)$$

are also some polynomials related to $P_k(n; x)$, so that $P_k(n; x)$ and $Q_k(x)$ do not depend on concrete p -adic consideration and that they are valid for all $x \in \mathbb{Z}$.

The simplest illustrative case [16] of p -adic invariant summation of the infinite series is

$$\sum_{n \geq 0} n! n = 1!1 + 2!2 + 3!3 + \dots = -1 \quad (7)$$

and obtains from (1) taking $x = 1$, $P_1(n; 1) = n$, which gives $Q_1(1) = -1$. To prove (7), one can use any one of the following two properties:

$$n!n = (n+1)! - n!, \quad \sum_{n=1}^{N-1} n!n = -1 + N!,$$

where $n! \rightarrow 0$ as $n \rightarrow \infty$.

In the sequel we shall develop and apply the corresponding method of the definite partial sums.

3.1 The Definite Partial Sums

Theorem 1 *Let*

$$A_k(n; x) = a_k(n)x^k + a_{k-1}(n)x^{k-1} + \cdots + a_1(n)x + a_0(n) \quad (8)$$

be a polynomial with coefficients $a_j(n)$, $0 \leq j \leq k$, which are polynomials in n with integer coefficients. Then there exists such polynomial $A_{k-1}(N; x)$, $N \in \mathbb{N}$, that equality

$$\sum_{n=0}^{N-1} n! [n^k x^k + U_k(x)] x^n = V_k(x) + N! x^N A_{k-1}(N; x) \quad (9)$$

holds, where

$$U_k(x) = x A_{k-1}(1; x) - A_{k-1}(0; x), \quad V_k(x) = -A_{k-1}(0; x). \quad (10)$$

Proof 1 *Let us consider summation of the following finite power series:*

$$S_k(N; x) = \sum_{n=0}^{N-1} n! n^k x^n = \delta_{0k} + \sum_{n=1}^{N-1} n! n^k x^n, \quad k \in \mathbb{N}_0.$$

We derive the following recurrent formula:

$$\begin{aligned}
S_k(N; x) &= \delta_{0k} + \sum_{n=0}^{N-2} (n+1)!(n+1)^k x^{n+1} = \delta_{0k} + x + \sum_{n=1}^{N-1} n!(n+1)^{k+1} x^{n+1} - N!N^k x^N \\
&= \delta_{0k} + x + x \sum_{n=1}^{N-1} n! x^n \sum_{\ell=0}^{k+1} \binom{k+1}{\ell} n^\ell - N!N^k x^N \\
&= \delta_{0k} + x S_0(N; x) + x \sum_{\ell=1}^{k+1} \binom{k+1}{\ell} S_\ell(N; x) - N!N^k x^N. \tag{11}
\end{aligned}$$

The above recurrent formula gives possibility to calculate sum $S_{k+1}(N; x)$ knowing all preceding sums $S_\ell(N; x)$, $\ell = 1, 2, \dots, k$ as function of $S_0(N; x)$. These sums have the form

$$\sum_{n=0}^{N-1} n! [n^k x^k + U_k(x)] x^n = V_k(x) + N! x^N A_{k-1}(N; x), \quad k \in \mathbb{N},$$

which can be rewritten as

$$S_k(N; x) = -U_k(x) S_0(N; x) x^{-k} + V_k(x) x^{-k} + N! A_{k-1}(N; x) x^{N-k}, \quad k \in \mathbb{N}, \tag{12}$$

where $U_k(x)$ and $V_k(x)$ are some polynomials in x of the degree k , and $A_k(N; x)$ is a polynomial in x which coefficients of x^n are polynomials in N of degree n .

Substituting the above expressions of the sums (12) into the recurrence formula (11), we obtain recurrence formulas for $U_{k+1}(x)$, $V_{k+1}(x)$ and $A_k(N; x)$:

$$\sum_{\ell=1}^{k+1} \binom{k+1}{\ell} x^{k-\ell+1} U_\ell(x) - U_k(x) - x^{k+1} = 0, \tag{13}$$

$$\sum_{\ell=1}^{k+1} \binom{k+1}{\ell} x^{k-\ell+1} V_\ell(x) - V_k(x) = 0, \tag{14}$$

$$\sum_{\ell=1}^{k+1} \binom{k+1}{\ell} x^{k-\ell+1} A_{\ell-1}(N; x) - A_{k-1}(N; x) - N^k x^k = 0. \tag{15}$$

Putting $N = 0$ in (15), it follows that $V_k(x)$ is proportional to $A_{k-1}(0; x)$. After explicit calculation of $A_0(0; x)$ and $V_1(x)$, we conclude that $V_k(x) = -A_{k-1}(0; x)$ for all $k \in \mathbb{N}$. On the other hand, equality $U_k(x) = x A_{k-1}(1; x) - A_{k-1}(0; x)$ in (10) obtains if we replace $N = 1$ in (15) and multiply it by x , and then subtracting from such expression recurrence relation (15) with $N = 0$.

It is worth emphasizing that equality (9) is valid in real and all p -adic cases. The central role in this equality plays polynomial $A_k(N; x)$ which is solution of the

recurrence relation (15). When $N \rightarrow \infty$ in (9), the term with polynomial $A_{k-1}(N; x)$ p -adically disappears giving the sum of the following p -adic infinite functional series:

$$\sum_{n=0}^{\infty} n! [n^k x^k + U_k(x)] x^n = V_k(x). \quad (16)$$

This equality has the same form for any $k \in \mathbb{N}$, and polynomials $U_k(x)$ and $V_k(x)$ separately have the same values in all p -adic cases for any $x \in \mathbb{Z}$. In other words, nothing depends on p -adic properties in (16) if $x \in \mathbb{Z}$, i.e. this is p -adic invariant result. This result gives us the possibility to present a general solution of the problem posed on p -adic invariant summation of the series (1).

Theorem 2 *The functional series (1) has p -adic invariant sum*

$$\sum_{n=0}^{\infty} n! P_k(n; x) x^n = Q_k(x) \quad (17)$$

if

$$P_k(n; x) = \sum_{j=1}^k C_j [n^j x^j + U_j(x)] \quad \text{and} \quad Q_k(x) = \sum_{j=1}^k C_j U_j(x), \quad (18)$$

where $C_j, x \in \mathbb{Z}$.

Recurrent formulas (13)–(15) enable to calculate these polynomials for all $k \in \mathbb{N}$, knowing initial expressions of $U_1(x)$, $V_1(x)$ and $A_0(N; x)$, which can be obtained from (11) and they are: $U_1(x) = x - 1$, $V_1(x) = -1$ and $A_0(N; x) = 1$.

For the first five values of degree k we have obtained the following explicit expressions.

- $k = 1$

$$\begin{aligned} U_1(x) &= x - 1, \\ V_1(x) &= -1, \\ A_0(n; x) &= 1. \end{aligned} \quad (19)$$

- $k = 2$

$$\begin{aligned} U_2(x) &= -x^2 + 3x - 1, \\ V_2(x) &= 2x - 1, \\ A_1(n; x) &= (n - 2)x + 1. \end{aligned} \quad (20)$$

- $k = 3$

$$\begin{aligned} U_3(x) &= x^3 - 7x^2 + 6x - 1, \\ V_3(x) &= -3x^2 + 5x - 1, \\ A_2(n; x) &= (n^2 - 3n + 3)x^2 + (n - 5)x + 1. \end{aligned} \quad (21)$$

- $k = 4$

$$\begin{aligned}
U_4(x) &= -x^4 + 15x^3 - 25x^2 + 10x - 1, \\
V_4(x) &= 4x^3 - 17x^2 + 9x - 1, \\
A_3(n; x) &= (n^3 - 4n^2 + 6n - 4)x^3 + (n^2 - 7n + 17)x^2 + (n - 9)x + 1. \quad (22)
\end{aligned}$$

- $k = 5$

$$\begin{aligned}
U_5(x) &= x^5 - 31x^4 + 90x^3 - 65x^2 + 15x - 1, \\
V_5(x) &= -5x^4 + 49x^3 - 52x^2 + 14x - 1, \\
A_4(n; x) &= (n^4 - 5n^3 + 10n^2 - 10n + 5)x^4 + (n^3 - 9n^2 + 31n - 49)x^3 \\
&\quad + (n^2 - 12n + 52)x^2 + (n - 14)x + 1. \quad (23)
\end{aligned}$$

- $k = 6$

$$\begin{aligned}
U_6(x) &= -x^6 + 63x^5 - 301x^4 + 350x^3 - 140x^2 + 21x - 1, \\
V_6(x) &= 6x^5 - 129x^4 + 246x^3 - 121x^2 + 20x - 1, \\
A_5(n; x) &= (n^5 - 6n^4 + 15n^3 - 20n^2 + 15n - 6)x^5 + (n^4 - 11n^3 \\
&\quad + 49n^2 - 111n + 129)x^4 + (n^3 - 15n^2 + 88n - 246)x^3 \\
&\quad + (n^2 - 18n + 121)x^2 + (n - 20)x + 1. \quad (24)
\end{aligned}$$

It is already noted that polynomial $A_k(n; x)$ plays central role in p -adic invariant summation of the series (1). $A_k(n; x)$ as well as $U_k(x)$ and $V_k(x)$ can be written in the compact form

$$A_k(n; x) = \sum_{\ell=0}^k A_{k\ell}(n) x^\ell, \quad U_k(x) = \sum_{\ell=0}^k U_{k\ell} x^\ell, \quad V_k(x) = \sum_{\ell=0}^k V_{k\ell} x^\ell, \quad (25)$$

where $A_{k\ell}(n)$ is polynomial in n of degree ℓ with n^ℓ as the term of highest degree.

Then the following properties hold:

- $A_k(n; 0) = A_{k0} = 1, \quad k = 0, 1, 2, \dots$
- $A_{kk}(1) = (-1)^k, \quad k = 0, 1, 2, \dots$
- $U_{k0} = V_{k0} = -1, \quad k = 1, 2, \dots$
- $U_{kk} = (-1)^{k+1}, \quad k = 1, 2, \dots$
- $V_{kk} = (-1)^k k, \quad k = 1, 2, \dots$

3.2 Connection with Bernoulli Numbers and Some Simple Examples

Using the Volkenborn integral one can make connection of our summation formulas with the Bernoulli numbers. By definition the Volkenborn integral [16] is

$$\int_{\mathbb{Z}_p} f(x) dx = \lim_{x \rightarrow \infty} p^{-n} \sum_{j=0}^{p^n-1} f(j),$$

where f is a p -adic continuous function. One of its properties is

$$\int_{\mathbb{Z}_p} x^n dx = B_n, \quad n = 0, 1, 2, \dots, \quad (26)$$

where B_n are the Bernoulli numbers which can be defined by the recurrent relation

$$\sum_{j=0}^{n-1} \binom{n}{j} B_j = 0, \quad B_0 = 1.$$

Note that $B_1 = -\frac{1}{2}$, $B_3 = B_5 = \dots = B_{2n+1} = 0$.

As an illustration of the above summation formula (16), we present three simple ($k = 1, 2, 3$) examples including connection with the Bernoulli numbers.

- $k = 1$

$$\begin{aligned} \sum_{n=0}^{\infty} n! [(n+1)x - 1] x^n &= -1, \quad x \in \mathbb{Z}, \\ \sum_{n=0}^{\infty} n! n x^n &= -1, \quad x = 1, \\ \sum_{n=0}^{\infty} n! (-1)^n (n+2) &= 1, \quad x = -1, \\ \sum_{n=0}^{\infty} n! [(n+1)B_{n+1} - B_n] &= -1. \end{aligned} \quad (27)$$

- $k = 2$

$$\begin{aligned} \sum_{n=0}^{\infty} n! [(n^2 - 1)x^2 + 3x - 1] x^n &= 2x - 1, \quad x \in \mathbb{Z}, \\ \sum_{n=0}^{\infty} n! (n^2 + 1) &= 1, \quad x = 1, \\ \sum_{n=0}^{\infty} (-1)^n n! (n^2 - 5) &= -3, \quad x = -1, \\ \sum_{n=0}^{\infty} n! [(n^2 - 1)B_{n+2} + 3B_{n+1} - B_n] &= -2. \end{aligned} \quad (28)$$

- $k = 3$

$$\sum_{n=0}^{\infty} n! [(n^3 + 1)x^3 - 7x^2 + 6x - 1] x^n = -3x^2 + 5x - 1, \quad x \in \mathbb{Z}, \quad (29)$$

$$\sum_{n=0}^{\infty} n! (n^3 - 1)x^3 = 1, \quad x = 1, \quad (30)$$

$$\sum_{n=0}^{\infty} (-1)^n n! (n^3 + 15) = 9, \quad x = -1, \quad (31)$$

$$\sum_{n=0}^{\infty} n! [(n^3 + 1)B_{n+3} - 7B_{n+2} + 6B_{n+1} - B_n] = -4. \quad (32)$$

The above series with the Bernoulli numbers are p -adic convergent, because $|B_n|_p \leq p$ (see [16], p. 172).

4 Concluding Remarks

The main results presented in this paper are Theorem 1 and Theorem 2. These results are generalizations of an earlier result for $x = 1$ [10] (see also [9, 11] for some partial generalizations).

Finite series with their sums (9) are valid for real and p -adic numbers. When $n \rightarrow \infty$ the corresponding infinite series are divergent in real case, but are convergent and have the same sums in all p -adic cases. This fact can be used to extend these sums to the real case. Namely, the sum of a divergent series depends on the way of its summation and here it can be used rational sum of the series valuable in all p -adic number fields. This way of summation of real divergent series was introduced for the first time in [2] and called adelic summability. How this adelic summability is important depends on its practical future use in some concrete examples.

The simplest infinite series with $n!$ is $\sum n!$. It is convergent in all \mathbb{Z}_p , but has not p -adic invariant sum. Even it is not known so far does it has a rational sum in any \mathbb{Z}_p . Rationality of this series and $\sum n! n^k x^n$ was discussed in [9]. The series $\sum n!$ is also related to Kurepa hypothesis which states $(!n, n!) = 2$, $2 \leq n \in \mathbb{N}$, where $!n = \sum_{j=0}^{n-1} j!$. Validity of this hypothesis is still an open problem in number theory. There are many equivalent statements to the Kurepa hypothesis, see [10] and references therein. From p -adic point of view, the Kurepa hypothesis reads: $\sum_{j=0}^{\infty} j! = n_0 + n_1 p + n_2 p^2 + \dots$, where digit $n_0 \neq 0$ for all primes $p \neq 2$.

When $x = 1$, then (16) becomes

$$\sum_{n=0}^{\infty} n! [n^k x^k + u_k] x^n = v_k, \quad (33)$$

where $u_k = U_k(1)$ and $v_k = V_k(1)$ are some integers. Equality (33) was introduced in [5], and properties of u_k and v_k are investigated in series of papers by Dragovich

(see references [8, 9, 10, 11]). In [17] relationships of u_k with the Stirling and the Bell numbers are established, and p -adic irrationality of $\sum_{n \geq 0} n!n^k$ was discussed ([18, 19, 20]). Note that the following sequences are related to some real (combinatorial) problems [21]:

$$A_{k-1}(0; 1) = -V_k(1) = -v_k : 1, -1, -1, 5, -5, -21, \dots \quad \text{see A014619} \quad (34)$$

$$A_{k-1}(0; -1) = -V_k(-1) = -\bar{v}_k : 1, 3, 9, 31, 121, 523, \dots \quad \text{see A040027} \quad (35)$$

$$A_{k-1}(1; 1) - A_{k-1}(0; 1) = U_k(1) = u_k : 0, 1, -1, -2, 9, -9, \dots \quad \text{see A000587} \quad (36)$$

$$A_{k-1}(1; -1) + A_{k-1}(0; -1) = -U_k(-1) = -\bar{u}_k : 2, 5, 15, 52, 203, \dots \quad \text{see A000110.} \quad (37)$$

There are many possibilities how results obtained in this paper can be generalized in future research of p -adic invariant summation of some real divergent series. Various aspects of the sequence of polynomials $A_k(n; x)$ deserve to be analyzed. It would be also interesting to investigate relations between the Bernoulli numbers using the above finite sums of the form (9).

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