DERIVED PALINTIPLE FAMILIES AND THEIR PALINOMIALS

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Abstract
We consider several families of palintiples (also known as reverse multiples) whose carries themselves are digits of lower-base palintiples and give some methods for constructing them from fundamental palintiple types. We also continue the study of palinomials introduced in an earlier paper by revealing a more direct relationship between the digits of certain palintiple types and the roots of their palinomials. We explore the consequences of this relationship for palinomials induced by palintiple families derived from lower-base palintiples. Finally, we pose some questions regarding Young graphs of derived palintiple families and consider the implications our general observations might have for relations between Young graph isomorphism classes.

1. Introduction

In a previous paper on palintiple numbers [3] (also known as reverse multiples [4, 8, 10, 11]) it is noted that "the carries [of a palintiple]...play as critical a role as the digits themselves." Indeed, the full measure of this statement is realized when one notices that the carries of a palintiple are often themselves the digits of a palintiple of a lower base. Consider the example of the $(10, 139)$-palintiple $(28, 25, 108, 113, 2)_{139}$ which has carries given by $(c_4, c_3, c_2, c_1, c_0) = (8, 7, 1, 2, 0)$. One immediately notices that the nontrivial carries are digits of the well known $(4, 10)$-palintiple 8712.

The recent work of Sloane [8] translates the palintiple problem into graph-theoretical language by means of *Young graphs* which are a succinct visualization of palintiple structure showing how the possible carries generate the possible digits of a palintiple of arbitrary length. Young graphs are a modification of tree graphs introduced by Young [10, 11] which are a representation of an efficient palintiple search method with the possible carries represented as nodes and the potential digits being associated with the edges. We note that Hoey [1, 2] presented a similar idea using finite state machines. Representations of machines which recognize palintiples bear strong resemblance to Young graphs; compare the Young graph determined by $(5, 8)$-palintiples found in [8] to the machine which recognizes $(5, 8)$-palintiples [2].

Kendrick [4] extends Sloane's work [8] by proving several of his conjectures, which includes one of Sloane's main conjectures: the (n, b)-Young graph, $Y(n, b)$, is isomorphic to the "1089-graph" if and only if $n + 1$ divides b. Kendrick goes on to list several conjectures of his own regarding other Young graph isomorphisms. We note that the above notation is slightly different from that of Sloane and Kendrick who use $Y(g, k)$ where g is the base and k is the multiplier (the order of the base and multiplier are reversed).

Other recent work includes [3] which establishes some general properties of palintiples of any base having an arbitrary number of digits using only elementary methods. As with the work of [4, 8, 10, 11], the methods therein pay particular attention to the structure of the carries which naturally separates all palintiples into three mutually exclusive and exhaustive classes. Letting $p = (d_k, d_{k-1}, \ldots, d_0)_b$ be an (n, b)-palintiple with carries c_k, c_{k-1},. . ., c_0, these classes are defined as follows: we say that p is *symmetric* if $c_j = c_{k-j}$ for all $0 \leq j \leq k$ and that p is *shifted-symmetric* if $c_j = c_{k-j+1}$ for all $0 \leq j \leq k$. A palintiple that is neither symmetric nor shifted-symmetric is called *asymmetric*. The $(4, 10)$-palintiple seen $(8, 7, 1, 2)_{10}$ has carries $(c_3, c_2, c_1, c_0) = (0, 3, 3, 0)$ making it an example of a symmetric palintiple. For more examples, the reader is referred to [3].

Comparing the above mentioned classes to Young graph isomorphism classes, it is easily shown that (n, b)-palintiples whose Young graph $Y(n, b)$ is isomorphic to $Y(9, 10)$, otherwise known as the 1089 graph [4, 8] (since $9801 = 9 \cdot 1089$ is a $(9, 10)$-palintiple), are symmetric (see the last section for a demonstration of this). Moreover, the work of [3] and [4] show that the class of shifted-symmetric palintiples is equal to the class of palintiples whose which have Young graph is complete (isomorphic to K_m for some $m > 2$ whose full definition can be found in [8]). As for those mentioned by Kendrick [4] which are neither shifted-symmetric nor isomorphic to the 1089 graph, the asymmetric class is characterized by an astonishing plurality of Young graph isomorphism classes which admits many subclassifications and isolated cases.

The above underscores a primary aim of this paper to begin to understand more fully the asymmetric class. We shall describe several families of asymmetric palintiples which are constructed, or derived, from lower-base examples. In particular, we will outline some methods for constructing new asymmetric palintiples using as carries the digits of "old" palintiples whose Young graph is isomorphic to either the 1089 graph (symmetric palintiple) or a complete graph (shifted-symmetric palintiple) in order to generate examples such as the one given in the first paragraph above.

Kendrick [4] mentions that it is still quite poorly understood how the number and graph theoretical aspects of the palintiple problem relate to one another. Our work, which relies mostly upon elementary results, gives rise to some concrete questions as to how the derived palintiple families described here can be classified according to Young graph isomorphism as well as suggest how Young graph isomorphism classes might be generated from others.

Additionally, this paper further develops the topic of palinomials introduced in [3] revealing a more intimate relationship between the digits of (n, b)-palintiples and the roots of their palinomials when $Y(n, b)$ is isomorphic $Y(9, 10)$. These results have implications for palinomials induced by palintiples derived from (n, b)-palintiples such that $Y(n, b)$ is isomorphic to either $Y(9, 10)$ or a complete graph.

2. Palintiples Whose Carries are Digits of Lower-Base Palintiples

Henceforth we shall suppose that $p = (d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b)-palintiple with carries $c_k, c_{k-1}, \ldots, c_0$. It is well-established [3, 8, 10] how the digits of a palintiple are related to the carries:

$$d_j = \frac{nbc_{k-j+1} - nc_{k-j} + bc_{j+1} - c_j}{n^2 - 1}. \tag{1}$$

We pose the general question of when the carries of a palintiple are the digits of a palintiple of a lower base as in the example given in the introduction. There are two possibilities for when this may occur:

Case 1: We find conditions under which we can construct a new $k+2$-digit $(\hat{n}, \hat{b})$-palintiple $\hat{p}$ with carries $(\hat{c}_{k+1}, \hat{c}_k, \cdots, \hat{c}_0)$ given by $(d_k, d_{k-1}, \ldots, d_0, 0)$ as in the example given in the introduction. Using Equation 1, the new digits $\hat{d}_j$ must satisfy

$$\hat{d}_j = \frac{\hat{n}\hat{b}\hat{c}_{k-j+2} - \hat{n}\hat{c}_{k-j+1} + \hat{b}\hat{c}_{j+1} - \hat{c}_j}{\hat{n}^2 - 1} = \frac{\hat{n}\hat{b}d_{k-j+1} - \hat{n}d_{k-j} + \hat{b}d_j - d_{j-1}}{\hat{n}^2 - 1}. \tag{2}$$

Then $\hat{b}d_0 \equiv \hat{n}d_k \mod (\hat{n}^2 - 1)$ when $j = 0$. Therefore, in order to find a suitable higher base $\hat{b}$, it must be that $\gcd(d_0, \hat{n}^2 - 1)$ divides d_k in which case we have that $\hat{b} = s + \alpha\frac{\hat{n}^2-1}{\gcd(d_0,\hat{n}^2-1)}$ where s is the least non-negative solution of the above congruence and $\alpha \geq 1$. The above then becomes

$$\hat{d}_j = \frac{\hat{n}sd_{k-j+1} - \hat{n}d_{k-j} + sd_j - d_{j-1}}{\hat{n}^2 - 1} + \alpha\frac{\hat{n}d_{k-j+1} + d_j}{\gcd(d_0, \hat{n}^2 - 1)}. \tag{3}$$

Case 2: We now ask when we may construct a new $k+3$-digit $(\hat{n}, \hat{b})$-palintiple $\hat{p}$ with carries $(\hat{c}_{k+2}, \hat{c}_{k+1}, \cdots, \hat{c}_0)$ given by $(0, d_k, d_{k-1}, \ldots, d_0, 0)$. For $k+3$-digits we have

$$\hat{d}_j = \frac{\hat{n}\hat{b}\hat{c}_{k-j+3} - \hat{n}\hat{c}_{k-j+2} + \hat{b}\hat{c}_{j+1} - \hat{c}_j}{\hat{n}^2 - 1} = \frac{\hat{n}\hat{b}d_{k-j+2} - \hat{n}d_{k-j+1} + \hat{b}d_j - d_{j-1}}{\hat{n}^2 - 1}.$$

Then $\hat{b}d_0 \equiv 0 \mod (\hat{n}^2-1)$ when $j=0$ so that $\hat{b} = \alpha\frac{\hat{n}^2-1}{\gcd(d_0,\hat{n}^2-1)}$. It follows that

$$\hat{d}_j = \frac{\alpha}{\gcd(d_0,\hat{n}^2-1)}(\hat{n}d_{k-j+2}+d_j) - \frac{\hat{n}d_{k-j+1}+d_{j-1}}{\hat{n}^2-1}. \tag{4}$$

In order to simplify exposition, we will say that a palintiple constructed from a lower-base palintiple is a *derived* palintiple. In particular, palintiples derived in the manner described in **Case 1** and **Case 2** above will respectively be called *singly-derived* and *doubly-derived* palintiples.

3. Palintiples Derived from 1089 Palintples

Any (n,b)-palintiple for which the Young graph $Y(n,b)$ is isomorphic to $Y(9,10)$ (named the "1089 graph" by Sloane [8]) shall for the remainder of this article be called a *1089 palintiple*. Moreover, the family of palintiples derived from 1089 palintiples shall, for the purpose of less cumbersome exposition, be called *Hoey* [1] palintiples.

By the work of Kendrick [4] we may suppose that $n+1$ divides b with quotient q and that $c_j \equiv 0 \mod (n-1)$. Then by [3], $c_{k-j} = c_j$ (that is, p is a symmetric palintiple) so that by Equation 1, $d_j = nqr_{k-j+1} + qr_{j+1} - r_j$ for all $0 \le j \le k$ where $r_k, r_{k-1}, \ldots, r_0$ is a palindromic binary sequence such that $r_1 = r_{k-1} = 1$ and there are no isolated zeros or ones except $r_0 = r_k = 0$. Since $d_0 = q$ and $d_k = nq$ we have that $\gcd(d_0,\hat{n}^2-1)$ divides d_k and that $\hat{b} = n\hat{n} + \alpha\frac{\hat{n}^2-1}{\gcd(q,\hat{n}^2-1)}$. Equation 3 then yields

$$\begin{aligned}\hat{d}_j =& nqr_{j-2} + \frac{n^2\hat{n}^2q - n\hat{n} + \hat{n} - q}{\hat{n}^2-1}r_j + \frac{n^2\hat{n}q - n\hat{n}^2 - \hat{n}q + 1}{\hat{n}^2-1}r_{j-1} \\ &+ \alpha\frac{qr_{j+1} + (\hat{n}nq-1)r_j + (nq-\hat{n})r_{j-1} + \hat{n}qr_{j-2}}{\gcd(q,\hat{n}^2-1)}.\end{aligned}$$

To ensure that each term in the above is an integer for all $\alpha \ge 1$, we must have both that $\gcd(q,\hat{n}^2-1) = 1$ and $(n-1)\hat{n} \equiv (n^2-1)b \mod (\hat{n}^2-1)$. A moment's reflection reveals that $b = q(n+1)$ is the only value for $\hat{n}$ which makes the congruence statement true (it also simultaneously ensures that b and $\hat{n}^2-1$ are relatively prime). Therefore, we have $\hat{b} = nb + \alpha(b^2-1)$ with

$$\hat{d}_j = n^2qr_j + nqr_{j-2} - r_{j-1} + \alpha\left(qr_{j+1} + (qnb-1)r_j - qr_{j-1} + qbr_{j-2}\right).$$

Since $d_j < b = \hat{n}$, each $\hat{d}_j$ is less than $\hat{b}$, and since there are no singleton ones or zeros in $r_k, r_{k-1}, \ldots, r_0$ (except $r_0 = r_k$), $\hat{d}_j$ cannot be negative so that each of these is a base-$\hat{b}$ digit. Additionally, since every $\hat{d}_j$ and d_j satisfy Equation 2,

[1] In honor of D. J. Hoey, to whose memory we dedicate this work.

it follows from a routine calculation that $(\hat{d}_{k+1}, \hat{d}_k, \cdots, \hat{d}_0)_{\hat{b}} = b(\hat{d}_0, \hat{d}_1, \cdots, \hat{d}_{k+1})_{\hat{b}}$ where d_{j-1} is the jth carry for $0 < j \leq k+1$. Thus, we have the following:

Theorem 1. *Suppose $p = (d_k, d_{k-1}, \ldots, d_0)_b$ is a 1089 (n, b)-palintiple. Then for every $\hat{b} > nb$ such that $\hat{b} \equiv nb \mod (b^2-1)$ there exists an asymmetric $k+2$-digit $(b, \hat{b})$-palintiple with carries $(\hat{c}_{k+1}, \hat{c}_k, \cdots, \hat{c}_0)$ given by $(d_k, d_{k-1}, \ldots, d_0, 0)$.*

Example The table below contains several examples of Hoey palintiples constructed from the $(2,3)$-palintiple $(2,1,2,0,1)_3$ including the general form obtained from the arguments establishing Theorem 1.

$(\hat{n}, \hat{b})$	$(\hat{d}_5, \hat{d}_4, \hat{d}_3, \hat{d}_2, \hat{d}_1, \hat{d}_0)_{\hat{b}}$	$(\hat{c}_5, \hat{c}_4, \hat{c}_3, \hat{c}_2, \hat{c}_1, \hat{c}_0)$
$(3,14)$	$(5,3,12,8,10,1)_{14}$	$(2,1,2,0,1,0)$
$(3,22)$	$(8,5,19,13,16,2)_{22}$	$(2,1,2,0,1,0)$
$(3,30)$	$(11,7,26,18,22,3)_{30}$	$(2,1,2,0,1,0)$
$(3,6+8\alpha)$	$(2+3\alpha, 1+2\alpha, 5+7\alpha, 3+5\alpha, 4+6\alpha, \alpha)_{6+8\alpha}$	$(2,1,2,0,1,0)$

Theorem 2. *No doubly-derived palintiple can be derived from a 1089 palintiple.*

Proof. Suppose there exists a doubly-derived $(\hat{n}, \hat{b})$-palintiple $\hat{p}$ constructed from a 1089 (n, b)-palintiple $p = (d_k, d_{k-1}, \ldots, d_0)_b$ with carries $c_k, c_{k-1}, \ldots, c_0$. Then by Equation 4 we have that $q(d_{k-j+1} + d_{j-1}) \equiv 0 \mod (\hat{n}-1)$. The cases $j=1$ and $j=2$ imply that $2q \equiv 0 \mod (\hat{n}-1)$. Thus, since it is well known that the carries of any palintiple must be less than the multiplier [3, 8, 9, 10], we have $q = d_0 = \hat{c}_1 \leq \hat{n}-1$. Hence, either $q = \frac{\hat{n}-1}{2}$ or $q = \hat{n}-1$. In any case, $\gcd(q, \hat{n}^2-1)$ divides $\hat{n}-1$ so that $\hat{n}+1$ divides $\hat{b}$. But this would imply by [3] that $\hat{p}$ is itself symmetric so that $qn = d_k = d_0 = q$ which is impossible. □

4. Palintiples Derived From Shifted-Symmetric Palintiples

We now consider singly-derived palintiples derived from shifted-symmetric palintiples. For brevity, such palintiples will be called *Sutcliffe* palintiples. We then suppose by arguments presented in [3] that $(b-n)c_j \equiv (nb-1)c_j \equiv 0 \mod (n^2-1)$ and $d_j = \frac{(b-n)c_{j+1}+(nb-1)c_j}{n^2-1}$. Then by the reasoning of **Case 1** we have $\hat{b}\frac{(b-n)c_1}{n^2-1} \equiv \hat{n}\frac{(nb-1)c_k}{n^2-1} \mod (\hat{n}^2-1)$. We shall suppose that $s = \frac{\hat{n}(nb-1)}{b-n}$ is an integer. Then, since $c_j = c_{k-j+1}$ for all $0 \leq j \leq k$ by definition, s is a particular solution for $\hat{b}$ to the congruence above so that in general $\hat{b} = \frac{\hat{n}(nb-1)}{b-n} + \alpha\frac{\hat{n}^2-1}{\gcd(d_0, \hat{n}^2-1)}$. Therefore, by Equation 3

$$\begin{aligned}\hat{d}_j =& \frac{s(nb-1)-(b-n)}{(\hat{n}-1)(n^2-1)}c_j + \frac{nb-1}{n^2-1}c_{j-1} \\ &+ \frac{\alpha}{\gcd(d_0, \hat{n}^2-1)}\left(\frac{b-n}{n^2-1}c_{j+1} + (\hat{n}+1)\frac{nb-1}{n^2-1}c_j + \hat{n}\frac{b-n}{n^2-1}c_{j-1}\right).\end{aligned}$$

Supposing that $\hat{n} > d_j$ guarantees by Equation 2 that $0 \le \hat{d}_j < \hat{b}$. Hence,

Theorem 3. *Suppose $(d_k, d_{k-1}, \ldots, d_0)_b$ is a shifted-symmetric (n,b)-palintiple with carries $c_k, c_{k-1}, \ldots, c_0$. If there exists a natural number $\hat{n}$ such that $s = \frac{\hat{n}(nb-1)}{b-n}$ is an integer, $\hat{n} > d_j$, and $s\frac{(nb-1)c_j}{n^2-1} \equiv \frac{(b-n)c_j}{n^2-1} \mod (\hat{n}-1)$ for all $0 \le j \le k$, then for every $\alpha \ge 1$ such that $\gcd(d_0, \hat{n}^2-1)$ divides $\alpha\frac{(b-n)(c_{j+1}+\hat{n}c_{j-1})+(\hat{n}+1)(nb-1)c_j}{n^2-1}$ for all $0 \le j \le k$, an asymmetric $k+2$-digit $(\hat{n},\hat{b})$-palintiple exists with carries $(\hat{c}_{k+1}, \hat{c}_k, \cdots, \hat{c}_0)$ given by $(d_k, d_{k-1}, \ldots, d_0, 0)$ where $\hat{b} = s + \alpha\frac{\hat{n}^2-1}{\gcd(d_0,\hat{n}^2-1)}$.*

Theorem 3 will now be applied to a two-digit (n,b)-palintiple $p = (d_1, d_0)_b$ with one non-zero carry c. p is trivially shifted-symmetric. Provided that there is an $\hat{n}$ which satisfies the conditions of Theorem 3, then $(\hat{d}_2, \hat{d}_1, \hat{d}_0)_{\hat{b}}$ given by

$$\begin{pmatrix} \hat{d}_2 \\ \hat{d}_1 \\ \hat{d}_0 \end{pmatrix} = \begin{pmatrix} \left(\frac{nb-1}{n^2-1} + \frac{\hat{n}\alpha(b-n)}{\gcd(d_0,\hat{n}^2-1)(n^2-1)}\right)c \\ \left(\frac{s(nb-1)-(b-n)}{(\hat{n}-1)(n^2-1)} + \frac{\alpha(\hat{n}+1)(nb-1)}{\gcd(d_0,\hat{n}^2-1)(n^2-1)}\right)c \\ \frac{\alpha(b-n)}{\gcd(d_0,\hat{n}^2-1)(n^2-1)}c \end{pmatrix} = \begin{pmatrix} d_1 + \alpha\frac{\hat{n}d_0}{\gcd(d_0,\hat{n}^2-1)} \\ \frac{sd_1-d_0}{\hat{n}-1} + \alpha\frac{(\hat{n}+1)d_1}{\gcd(d_0,\hat{n}^2-1)} \\ \alpha\frac{d_0}{\gcd(d_0,\hat{n}^2-1)} \end{pmatrix}$$

is a 3-digit $(\hat{n},\hat{b})$-palintiple with carries $(\hat{c}_2, \hat{c}_1, \hat{c}_0) = (d_1, d_0, 0)$ for every $\alpha \ge 1$ where $\hat{b} = \frac{\hat{n}(nb-1)}{b-n} + \alpha\frac{\hat{n}^2-1}{\gcd(d_0,\hat{n}^2-1)}$. Thus, we have the following corollary which provides conditions for the existence of asymmetric palintiples.

Corollary 4. *If $(d_1, d_0)_b$ is an (n,b)-palintiple and there is an $\hat{n} > \max\{d_0, d_1\}$ such that $s = \frac{\hat{n}(nb-1)}{b-n}$ is an integer and $sd_1 \equiv d_0 \mod (\hat{n}-1)$, then asymmetric $(\hat{n},\hat{b})$-palintiples exist where $\hat{b} = s + \alpha\frac{\hat{n}^2-1}{\gcd(d_0,\hat{n}^2-1)}$ for any $\alpha \ge 1$.*

Example Consider the $(2,5)$-palintiple $(3,1)_5$ with carries $(c,0) = (1,0)$. We see that $\hat{n} = 5$ and $\hat{n} = 9$ satisfy the conditions of Corollary 4 from which we get the $(5,39)$-palintiple $(8,29,1)_{39}$, the $(9,107)$-palintiple $(12,40,1)_{107}$, and in general the $(5, 5+24\alpha)$-palintiple $(3+5\alpha, 11+18\alpha, \alpha)_{15+24\alpha}$ and the $(9, 27+80\alpha)$-palintiple $(3+9\alpha, 10+30\alpha, \alpha)_{27+80\alpha}$ for $\alpha \ge 1$ all with carries $(\hat{c}_2, \hat{c}_1, \hat{c}_0) = (3,1,0)$.

We now consider doubly-derived palintiples constructed from shifted-symmetric (n,b)-palintiples. This family of palintiples will be called *Pudwell* palintiples. Letting $D = \gcd(d_0, \hat{n}^2-1)$, we have by Equation 4 that $D(\hat{n}d_{k-j+1} + d_{j-1}) \equiv 0 \mod (\hat{n}^2-1)$. Then

$$D\left(\frac{[\hat{n}(nb-1)+(b-n)]c_j + [\hat{n}(b-n)+(nb-1)]c_{j-1}}{n^2-1}\right) \equiv 0 \mod (\hat{n}^2-1).$$

By induction we must then have $D\frac{[\hat{n}(nb-1)+(b-n)]c_j}{n^2-1} \equiv 0 \mod (\hat{n}^2-1)$. It follows that $\hat{n}-1$ and $\hat{n}+1$ must respectively divide $D\frac{(b-1)c_j}{n-1}$ and $D\frac{(b+1)c_j}{n+1}$.

Using the above conclusion and a computer, we have found no examples for which $\hat{n} \neq b$. However, we have not been able to rule out this possibility. Checking all possibilities for all $b \leq 500$ yielded no Pudwell palintiples for which $\hat{n}$ and b are not equal. We shall therefore narrow our scope and consider the case $\hat{n} = b$ while leaving the $\hat{n} \neq b$ case as an open problem.

When $\hat{n} = b$ we have that $n^2 - 1$ divides Dc_j with quotient q_j for all $0 \leq j \leq k$. Expressing each d_j in terms of the carries, and then applying the above observations, Equation 4 becomes

$$\hat{d}_j = \frac{\alpha(bd_{k-j+2} + d_j) - (nq_j + q_{j-1})}{D}.$$

As argued previously, $0 \leq \hat{d}_j < \hat{b}$ since each $d_j < b = \hat{n}$. We therefore have the following.

Theorem 5. *Suppose $p = (d_k, d_{k-1}, \ldots, d_0)_b$ is an (n, b)-palintiple with carries $c_k, c_{k-1}, \ldots, c_0$ and let $D = \gcd(d_0, b^2-1)$. If n^2-1 divides Dc_j with quotient q_j for all $0 \leq j \leq k$, then for every $\alpha \geq 1$ such that D divides $\alpha(bd_{k-j+2}+d_j)-(kq_j+q_{j-1})$ for all $0 \leq j \leq k$, a $k+3$-digit asymmetric $(b, \hat{b})$-palintiple exists with carries $(\hat{c}_{k+2}, \hat{c}_{k+1}, \cdots, \hat{c}_0)$ given by $(0, d_k, d_{k-1}, \ldots, d_0, 0)$ where $\hat{b} = \alpha\frac{b^2-1}{D}$.*

The case $n = 1$ gives us another condition which guarantees the existence of asymmetric palintiples.

Corollary 6. *Suppose $(d_1, d_0)_b$ is an (n, b)-palintiple with one non-zero carry c and $D = \gcd(d_0, b^2-1)$. If n^2-1 divides Dc with quotient q and $\gcd(d_1, D)$ divides q, then there exists an asymmetric $(b, \hat{b})$-palintiple where $\hat{b} = \alpha\frac{b^2-1}{D}$.*

Proof. The arguments leading up to Theorem 5 give us that a new 4-digit palintiple $(\hat{d}_3, \hat{d}_2, \hat{d}_1, \hat{d}_0)_{\hat{b}}$ with carries $(0, d_1, d_0, 0)$ must equal $\left(\frac{\alpha b d_0}{D}, \frac{\alpha b d_1 - q}{D}, \frac{\alpha d_1 - nq}{D}, \frac{\alpha d_0}{D}\right)_{\alpha\frac{b^2-1}{D}}$ for some α where $n = 1$. Since $\gcd(d_1, D)$ divides q, there is an $\alpha \geq 1$ such that $\alpha b d_1 \equiv q \mod D$ and since $(nb-1)q = d_1 D$, we have $\alpha d_1 \equiv nq \mod D$. □

Example A family of Pudwell palintiples may be constructed from the $(6, 55)$-palintiple $(47, 7)_{55}$ with carries $(c, 0) = (5, 0)$. The conditions of Corollary 6 are satisfied and we have that $(\hat{d}_3, \hat{d}_2, \hat{d}_1, \hat{d}_0)_{\hat{b}}$ given by $\left(55\alpha, \frac{2585\alpha-1}{7}, \frac{47\alpha-6}{7}, \alpha\right)_{432\alpha}$ is an $(55, 432\alpha)$-palintiple with carries $(\hat{c}_3, \hat{c}_2, \hat{c}_1, \hat{c}_0) = (0, 47, 7, 0)$ where α is any natural number congruent to 4 modulo 7.

5. Palintiples Derived from Palintple Reversals

Digit reversals of palintiples also appear in the carries of higher-base palintiples. Therefore, we now construct asymmetric palintiples from digit-reversals of palintiples. We will not present the amount of detail as in the previous section as the arguments are essentially the same for each case. However, we will highlight points which deserve additional explanation. We shall consider both $k+2$-digit $(\hat{n},\hat{b})$-palintiples with carries $(\hat{c}_{k+1},\hat{c}_k,\cdots,\hat{c}_0)$ of the form $(d_0,d_1,\ldots,d_k,0)$ and $k+3$-digit $(\hat{n},\hat{b})$-palintiples with carries $(\hat{c}_{k+2},\hat{c}_{k+1},\cdots,\hat{c}_0)$ of the form $(0,d_0,d_1,\ldots,d_k,0)$. Such palintiples will be called respectively *singly-ρ-derived* and *doubly-ρ-derived.*

5.1. Palintiples Derived from Reversals of 1089 Palintiples

We shall now consider families of singly-ρ-derived palintiples constructed from 1089 palintiples These shall be called *ρ-Hoey* palintiples. In a manner similar to the arguments leading to Equation 3, it must be that $\hat{b}d_k \equiv \hat{n}d_0 \mod (\hat{n}^2-1)$, or $\hat{b}nq \equiv \hat{n}q \mod (\hat{n}^2-1)$. Thus, $\gcd(nq,\hat{n}^2-1)$ must divide q so that n and $\hat{n}^2-1$ must be relatively prime. We then have that $\hat{b}=m\hat{n}+\alpha\frac{\hat{n}^2-1}{\gcd(q,\hat{n}^2-1)}$ where m is the multiplicative inverse of n modulo $\hat{n}^2-1$. Re-parameterizing, we let ℓ be the least non-negative residue of $m\hat{n}$ modulo $\frac{\hat{n}^2-1}{\gcd(q,\hat{n}^2-1)}$ so that $\hat{b}=\ell+\alpha\frac{\hat{n}^2-1}{\gcd(q,\hat{n}^2-1)}$ for $\alpha\geq 1$. Then

$$\begin{aligned}\hat{d}_j =& \frac{(\ell n-\hat{n})qr_{j+1}+(\hat{n}-\ell-nq+\ell\hat{n}q)r_j+(1-\hat{n}\ell-\hat{n}nq+\ell q)r_{j-1}+(\ell n\hat{n}-1)qr_{j-2}}{\hat{n}^2-1}\\ &+\alpha\frac{nqr_{j+1}+(\hat{n}q-1)r_j+(q-\hat{n})r_{j-1}+\hat{n}nqr_{j-2}}{\gcd(q,\hat{n}^2-1)}.\end{aligned}$$

Since $\ell n-\hat{n}\equiv \ell n\hat{n}-1\equiv 0 \mod (\hat{n}^2-1)$, in order to ensure that the above is an integer we shall require that $\hat{n}-\ell-nq+\ell\hat{n}q\equiv 1-\hat{n}\ell-\hat{n}nq+\ell q\equiv 0 \mod (\hat{n}^2-1)$ which is equivalent to $(n-1)\hat{n}\equiv(n^2-1)q \mod (\hat{n}^2-1)$. Therefore, as before, $\hat{n}=b=q(n+1)$ so that $\gcd(q,\hat{n}^2-1)=\gcd(q,b^2-1)=1$. Thus,

$$\begin{aligned}\hat{d}_j =& \frac{(\ell n-b)qr_{j+1}+(b-\ell-nq+\ell bq)r_j+(1-b\ell-bnq+\ell q)r_{j-1}+(\ell nb-1)qr_{j-2}}{b^2-1}\\ &+\alpha(nqr_{j+1}+(bq-1)r_j+(q-b)r_{j-1}+bnqr_{j-2}).\end{aligned}$$

The above gives us the following compliment to Theorem 1.

Theorem 7. *Suppose $p=(d_k,d_{k-1},\ldots,d_0)_b$ a 1089 (n,b)-palintiple such that b^2-1 and n are relatively prime and m is the the multiplicative inverse of n modulo b^2-1. Furthermore, let ℓ be the least non-negative residue of mb modulo b^2-1. Then for every $\hat{b}>\ell$ such that $\hat{b}\equiv\ell \mod (b^2-1)$ there exists an asymmetric $k+2$-digit $(b,\hat{b})$-palintiple with carries $(\hat{c}_{k+1},\hat{c}_k,\cdots,\hat{c}_0)$ given by $(d_0,d_1,\ldots,d_k,0)$.*

Example Applying the arguments for Theorem 7 to the well known $(4,10)$-palintiple $(8,7,1,2)_{10}$ with carries $(c_3,c_2,c_1,c_0)=(0,3,3,0)$, we have $\ell=52$ which gives rise to the family of $(10,52+99\alpha)$-palintiples with digits $(\hat{d}_4,\hat{d}_3,\hat{d}_2,\hat{d}_1,\hat{d}_0)_{\hat{b}}$ given by $(42+80\alpha,37+72\alpha,5+11\alpha,14+27\alpha,4+8\alpha)_{52+99\alpha}$ all with carries $(\hat{c}_4,\hat{c}_3,\hat{c}_2,\hat{c}_1,\hat{c}_0)=(2,1,7,8,0)$ for all $\alpha\geq 1$.

Remark Although it is not always the case, the example above also yields a palintiple for $\alpha = 0$.

As shown by an argument nearly identical to that of Theorem 2, no doubly-ρ-derived palintiples can be constructed from 1089 palintiples.

5.2. Palintples Derived from Reversals of Shifted-Symmetric Palintiples

Singly-ρ-derived palintiples constructed from shifted-symmetric palintiples (called ρ-Sutcliffe palintiples) yield an argument and theorem statement nearly identical to that of Theorem 3 with the exception that the roles of $b-n$ and $nb-1$ as well as d_0 and d_k are interchanged.

Theorem 8. *Suppose* $(d_k, d_{k-1}, \ldots, d_0)_b$ *is a shifted-symmetric* (n, b)*-palintiple with carries* $c_k, c_{k-1}, \ldots, c_0$*. If there exists a natural number* $\hat{n}$ *such that* $s = \frac{\hat{n}(b-n)}{nb-1}$ *is an integer,* $\hat{n} > d_j$*, and* $s\frac{(b-n)c_j}{n^2-1} \equiv \frac{(nb-1)c_j}{n^2-1} \mod (\hat{n}-1)$ *for all* $0 \leq j \leq k$*, then for every* $\alpha \geq 1$ *such that* $\gcd(d_k, \hat{n}^2-1)$ *divides* $\alpha\frac{(nb-1)(c_{j+1}+\hat{n}c_{j-1})+(\hat{n}+1)(b-n)c_j}{n^2-1}$ *for all* $0 \leq j \leq k$*, an asymmetric* $k+2$*-digit* $(\hat{n}, \hat{b})$*-palintiple exists with carries* $(\hat{c}_{k+1}, \hat{c}_k, \cdots, \hat{c}_0)$ *given by* $(d_0, d_1, \ldots, d_k, 0)$ *where* $\hat{b} = s + \alpha\frac{\hat{n}^2-1}{\gcd(d_k, \hat{n}^2-1)}$*.*

Corollary 9. *If* $(d_1, d_0)_b$ *is an* (n, b)*-palintiple and there is an* $\hat{n} > \max\{d_0, d_1\}$ *such that* $s = \frac{\hat{n}(b-n)}{nb-1}$ *is an integer and* $sd_0 \equiv d_1 \mod (\hat{n}-1)$*, then asymmetric* $(\hat{n}, \hat{b})$*-palintiples exist where* $\hat{b} = s + \alpha\frac{\hat{n}^2-1}{\gcd(d_1, \hat{n}^2-1)}$ *for any* $\alpha \geq 1$*.*

Example Corollary 9 applies to the $(2, 5)$-palintiple $(3, 1)_5$ with one nontrivial carry $c = 1$ with $\hat{n} = 9$ satisfying its hypotheses and giving us the family of $(9, 3 + 80\alpha)$-palintiples $(1+27\alpha, 10\alpha, 3\alpha)_{3+80\alpha}$ for any $\alpha \geq 1$ each with carries $(\hat{c}_2, \hat{c}_1, \hat{c}_0) = (1, 3, 0)$.

Considering doubly-ρ-derived palintiples constructed from shifted-symmetric palintiples (ρ-Pudwell), we have

Theorem 10. *Suppose* $p = (d_k, d_{k-1}, \ldots, d_0)_b$ *is a shifted-symmetric* (n, b)*-palintiple with carries* $c_k, c_{k-1}, \ldots, c_0$ *and let* $D = \gcd(d_k, b^2-1)$*. If* n^2-1 *divides* Dc_j *with quotient* q_j *for all* $0 \leq j \leq k$*, then for every* $\alpha \geq 1$ *such that* D *divides* $\alpha(bd_{j-2} + d_{k-j}) - (q_j + nq_{j-1})$ *for all* $0 \leq j \leq k$*, a* $k+3$*-digit asymmetric* $(b, \hat{b})$*-palintiple exists with carries* $(\hat{c}_{k+2}, \hat{c}_{k+1}, \cdots, \hat{c}_0)$ *given by* $(0, d_0, d_1, \ldots, d_k, 0)$ *where* $\hat{b} = \alpha\frac{b^2-1}{D}$*.*

Corollary 11. *Suppose* $(d_1, d_0)_b$ *is an* (n, b)*-palintiple with one non-zero carry* c *and* $D = \gcd(d_1, b^2-1)$*. If* n^2-1 *divides* Dc *with quotient* q *and* $\gcd(d_0, D)$ *divides* q*, then there exists an asymmetric* $(b, \hat{b})$*-palintiple where* $\hat{b} = \alpha\frac{b^2-1}{D}$*.*

Example We again look at the $(2, 5)$-palintiple $(3, 1)_5$ with carry $r = 1$. The conditions of Corollary 11 are satisfied and we get the $(5, 8\alpha)$-palintiple $(5\alpha, \frac{5\alpha-2}{3}, \frac{\alpha-1}{3}, \alpha)_{8\alpha}$ with carries $(\hat{c}_3, \hat{c}_2, \hat{c}_1, \hat{c}_0) = (0, 1, 3, 0)$ where $\alpha \equiv 1 \mod 3$.

6. Palinomials and Derived Palintiples

We recall a definition from [3]: the (n,b)-*palinomial* induced by an (n,b)-palintiple $(d_k,\cdots,d_0)_b$ is the polynomial

$$\mathrm{Pal}(x)=\sum_{j=0}^{k}(d_j-nd_{k-j})x^j.$$

Theorem 12. *Palinomials induced by 1089 palintiples have at least one root on the unit circle.*

Proof. By [3], $\mathrm{Pal}(x)=(x-b)\sum_{j=1}^{k}c_jx^{j-1}$, and since $Y(n,b)\backsimeq Y(9,10)$, we have by [4] that $n+1$ divides b. Hence, by arguments in [3] and [4], $c_j=c_{k-j}$ is either 0 or $n-1$. Results in [6] then establish the claim. □

The next theorem reveals an even closer connection between the digits of 1089 and shifted-symmetric palintiples and the roots of their palinomials.

Theorem 13. *Let $\xi\neq b$ be a non-zero root of the palinomial induced by a 1089 or shifted-symmetric palintiple $p=(d_k,d_{k-1},\cdots d_0)_b$. Then ξ is a root of both the digit and reverse-digit polynomials. That is,*

$$\sum_{j=0}^{n}d_j\xi^j=\sum_{j=0}^{n}d_{k-j}\xi^j=0.$$

Proof. By the theorem hypothesis, $\sum_{j=1}^{k}c_jx^{j-1}$ is a palindromic polynomial. Therefore, $\mathrm{Pal}(\xi)=\mathrm{Pal}(\frac{1}{\xi})=0$. It follows that, $\sum_{j=0}^{k}d_j\xi^j=n\sum_{j=0}^{k}d_{k-j}\xi^j$ and $\sum_{j=0}^{k}d_j\xi^{-j}=n\sum_{j=0}^{k}d_{k-j}\xi^{-j}$. Multiplying the second equation by $n\xi^k$ and re-indexing the sum we have $n\sum_{j=0}^{k}d_{k-j}\xi^j=n^2\sum_{j=0}^{k}d_j\xi^j$ from which the result follows. The reverse-digit case follows in the same manner. □

Corollary 14. *Digit and reverse-digit polynomials of a 1089 palintiples have at least one root on the unit circle.*

6.1. Additional Roots of Digit Polynomials of 1089 and Shifted-Symmetric Palintiples

Since every negative or purely complex root of a palinomial induced by a 1089 palintiple is also a root of the digit polynomial, the digit and reverse-digit polynomial of a 1089 palintiple has two roots which differ from its corresponding palinomial.

Theorem 15. *Let $Pal(x)$ be the palinomial induced by a 1089 (n, b)-palintiple $p = (d_k, d_{k-1}, \cdots d_0)_b$ and let D and $\overline{D}$ respectively denote the digit and reverse-digit polynomials. Then*

$$D(x) = (d_k x^2 - x + d_0)\frac{Pal(x)}{(n-1)(x-b)} \text{ and } \overline{D}(x) = (d_0 x^2 - x + d_k)\frac{Pal(x)}{(n-1)(x-b)}.$$

Proof. Suppose $\text{Pal}(x) = (n-1)(x-b)\prod_{j=1}^{k-2}(x - \xi_j)$ is a palinomial induced by a symmetric (n, b)-palintiple. By Theorem 13 we may express these as $D(x) = d_k(x-\omega_1)(x-\omega_2)\prod_{j=1}^{k-2}(x-\xi_j)$ and $\overline{D}(x) = d_0(x-\frac{1}{\omega_1})(x-\frac{1}{\omega_2})\prod_{j=1}^{k-2}(x-\xi_j)$ where ω_1 and ω_2 are the two extra roots. Since $x = 1$ cannot be a root of any palinomial [3] and is clearly not a root of D or $\overline{D}$, we have $d_0(1-\frac{1}{\omega_1})(1-\frac{1}{\omega_2}) = d_k(1-\omega_1)(1-\omega_2)$, or $\omega_1\omega_2 = \frac{1}{n}$. Now, $D(b) = n\overline{D}(b)$ implies $(b-\omega_1)(b-\omega_2) = (b-\frac{1}{\omega_1})(b-\frac{1}{\omega_2})$ from which we have $\omega_1 + \omega_2 = \frac{1}{nq}$. Thus, ω_1 and ω_2 are the conjugate pair $\frac{1}{2nq}(1 \pm i\sqrt{4q^2n-1})$. The digit and reverse-digit polynomials may then be expressed as $D(x) = (d_k x^2 - x + d_0)\prod_{j=1}^{k-2}(x-\xi_j)$ and $\overline{D}(x) = (d_0 x^2 - x + d_k)\prod_{j=1}^{k-2}(x-\xi_j)$. □

Digit and reverse-digit polynomials of shifted-symmetric (n, b)-palintiples have one more root than their corresponding palinomials.

Theorem 16. *Let $Pal(x)$ be the palinomial induced by a shifted-symmetric (n, b)-palintiple $p = (d_k, d_{k-1}, \cdots d_0)_b$ with carries $c_k, c_{k-1}, \cdots, c_1, c_0$ and let D and $\overline{D}$ respectively denote the digit and reverse-digit polynomials. Then*

$$D(x) = (d_k x + d_0)\frac{Pal(x)}{c_k(x-b)} \text{ and } \overline{D}(x) = (d_0 x + d_k)\frac{Pal(x)}{c_k(x-b)}.$$

Proof. Suppose $\text{Pal}(x) = c_k(x-b)\prod_{j=1}^{k-1}(x-\xi_j)$. Since the digit and reverse-digit polynomials have one root that differs from $\text{Pal}(x)$ we have $D(x) = d_k(x-\omega)\prod_{j=1}^{k-1}(x-\xi_j)$ and $\overline{D}(x) = d_0(x-\frac{1}{\omega})\prod_{j=1}^{k-1}(x-\xi_j)$. $D(1) = \overline{D}(1)$ then implies that $\omega = -\frac{b-n}{nb-1}$ so that $D(x) = (d_k x + d_0)\prod_{j=1}^{k-1}(x-\xi_j)$ and $\overline{D}(x) = (d_0 x + d_k)\prod_{j=1}^{k-1}(x-\xi_j)$. □

Corollary 17. *Let $\widehat{Pal}(x)$ be the palinomial induced by a singly-derived or doubly-derived $(\hat{n}, \hat{b})$-palintiple $\hat{p}$ constructed from an (n, b)-palintiple $p = (d_k, d_{k-1}, \cdots d_0)_b$ and let $Pal(x)$ be the palinomial induced by p. Then*

$$\widehat{Pal}(x) = (x-\hat{b})(d_k x^2 - x + d_0)\frac{Pal(x)}{(n-1)(x-b)}$$

if p is a 1089 palintiple, and

$$\widehat{Pal}(x) = (x-\hat{b})(d_k x + d_0)\frac{Pal(x)}{c_k(x-b)}$$

if p is shifted-symmetric where c_k is the nth carry of p.

Proof. Since $\hat{p}$ is singly-derived, its carries are d_k, $d_{k-1}, \cdots, d_0, 0$ so that by [3] we have $\widehat{\text{Pal}}(x) = (x - \hat{b})\sum_{j=1}^{k+1} \hat{c}_j x^{j-1} = (x - \hat{b})\sum_{j=1}^{k+1} d_{j-1}x^{j-1} = (x - \hat{b})D(x)$. The doubly-derived case follows in a similar fashion. □

Corollary 18. *Let $\widehat{Pal}(x)$ be the palinomial induced by a singly-ρ-derived or doubly-ρ-derived $(\hat{n}, \hat{b})$-palintiple $\hat{p}$ constructed from an (n, b)-palintiple $p = (d_k, d_{k-1}, \cdots d_0)_b$ and let $Pal(x)$ be the palinomial induced by p. Then*

$$\widehat{Pal}(x) = (x - \hat{b})(d_0 x^2 - x + d_k)\frac{Pal(x)}{(n-1)(x-b)}$$

if p is a 1089 palintiple, and

$$\widehat{Pal}(x) = (x - \hat{b})(d_0 x + d_k)\frac{Pal(x)}{c_k(x-b)}$$

if p is shifted-symmetric where c_k is the nth carry of p.

Corollary 19. *Palinomials induced by Hoey and ρ-Hoey palintiples have at least one root on the unit circle.*

Corollary 20. *Palinomials induced by any two singly-derived palintiples constructed from a common palintiple which is either 1089 or shifted-symmetric, differ by only a linear factor.*

The statement of Corollary 20 also holds for doubly-derived, singly-ρ-derived, and doubly-ρ-derived palintiples.

Example The 7-digit, symmetric $(4, 10)$-palintiple $p = (8, 7, 9, 9, 9, 1, 2)_{10}$ induces the palinomial $\text{Pal}(x) = 3(x - 10)(x^4 + x^3 + x^2 + x + 1)$. The reader may also verify that, $D(x) = (8x^2 - x + 2)(x^4 + x^3 + x^2 + x + 1)$ and $\overline{D}(x) = (2x^2 - x + 8)(x^4 + x^3 + x^2 + x + 1)$. Moreover, constructing a new 8-digit palintiple from p using Theorem 1 and its supporting arguments, we take the $(10, 139)$-palintiple $\hat{p} = (28, 25, 136, 138, 138, 110, 113, 2)_{139}$ as an example. The reader may verify that the palinomial induced by $\hat{p}$ can be expressed as

$$\widehat{\text{Pal}}(x) = (x - 139)(8x^2 - x + 2)(x^4 + x^3 + x^2 + x + 1).$$

7. Open Questions and Future Work

It is still unknown if Theorems 1 and 7 and their arguments respectively give us all Hoey and ρ-Hoey palintiples. So far this seems to be the case, but remains unproven. On the other hand, however, Sutcliffe and ρ-Sutcliffe palintiples exist under conditions for which Theorems 3, 8, and their corollaries do not apply. We give as an example the $(14, 129)$-palintiple $(37, 89, 2)_{129}$ with carries $(9, 4, 0)$ derived from the $(2, 14)$-palintiple $(9, 4)_{14}$; for this particular case, $s = \frac{\hat{n}(nb-1)}{b-n}$ is not an integer. Furthermore, we have already stated that it is unknown if there are Pudwell palintiples such that $\hat{n} \neq b$. However, we must point out that ρ-Pudwell palintiples do exist for values of $\hat{n}$ other than b. As an example we present the $(34, 55)$-palintiple $(34, 1, 0, 1)_{55}$ with carries $(0, 1, 21, 0)$ which is derived from the $(11, 23)$-palintiple $(21, 1)_{23}$. Moreover, for whatever reason, ρ-Pudwell palintiples, speaking in terms of lower bases, seem to occur much more frequently than their forward counterparts. Both forward and ρ-Pudwell palintiples so far have proven to be the least well understood. In summary, finding maximal conditions for the existence of palintiples belonging to the families presented in this paper is an open topic.

Given the variety of Young graph isomorphism classes [4, 5], it is not surprising that not all asymmetric palintiples are derived palintiples. If we consider the example of the $(4, 23)$-palintiple $(6, 15, 1)_{23}$, with carries $(c_2, c_1, c_0) = (2, 1, 0)$, it is not difficult to show that no 2-digit palintiple has these carries as digits. We might then ask if there is a more general principle at work here; perhaps the carries are not the digits of a palintiple, but rather the digits of a *permutiple*. Indeed, one sees that for the above case that $(2, 1, 0)_4 = 2 \cdot (1, 0, 2)_4$. While such examples are promising, one can verify that for the the $(11, 17)$-palintiple $(14, 12, 5, 1)_{17}$, there is no permutation, base, or multiplier for which the carries $(c_3, c_2, c_1, c_0) = (3, 8, 9, 0)$ are a non-trivial permutiple. However, we do point out that there do seem to be strong connections, and naturally so, between palintiples and the more general permutiple problem. Thus, a more developed understanding of permutiples may very well provide a better understanding of palintiples.

With the above in mind, we also mention that it is unknown if singly-derived or doubly-derived palintiples can be constructed from other asymmetric palintiples (neither 1089 nor shifted-symmetric). So far none have been found.

Kendrick [4] showed that $Y(n, b) \simeq Y(9, 10)$ is equivalent b being divisible by $n + 1$. Concerning the class of symmetric palintiples, we ask if the following are equivalent for an (n, b)-palintiple p with carries $c_k, c_{k-1}, \cdots, c_1, c_0$:

1. p is symmetric
2. p is 1089
3. $c_j \equiv 0 \mod (n-1)$
4. $n + 1$ divides b

If p is 1089, the work of Kendrick [4] shows that any node of has the form $[0, 0]$, $[0, n-1]$, $[n-1, 0]$, or $[n-1, n-1]$ which establishes (2) $\Longrightarrow$ (3). (3) $\Longrightarrow$ (4) is easily established since by Equation 1, $d_0 = \frac{bc_1}{n^2-1}$ and $d_0 \neq 0$. (4) $\Longrightarrow$ (1) is demonstrated in [3]. We leave whether or not (1) $\Longrightarrow$ (2) holds as an open question.

7.1. Young Graph Isomorphism Classes of Derived Palintiples

As we have seen, the carries of a palintiple can themselves be the digits of lower-base palintiple. If we elevate our perspective to entire palintiple families (e.g. Hoey palintiples derived from 1089 palintiples), an abundance of questions present themselves:

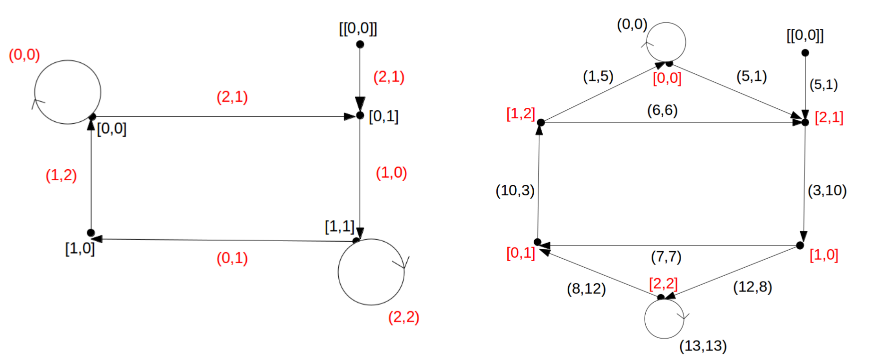
Is it possible to "derive" new Young graphs from old using the old edges as nodes?

Can we construct entire Young graph isomorphism classes from old?

How are derived palintiple families related to Young graph isomorphism classes?

Although we will not provide any complete answers to these questions, we will explore some suggestive examples which, as we shall see, give rise to other questions.

Considering $(3, 14)$-palintiples, their nontrivial carries are $(2, 3)$-palintiple digits and every $(2, 3)$-palintiple is the nontrivial carry sequence of some $(3, 14)$-palintiple. In other words the Young graph describing $(3, 14)$-palintiple structure can be "derived" from the Young graph describing $(2, 3)$-palintiple structure. The figure below compares the Young graphs $Y(2, 3)$ and $Y(3, 14)$ where the "digit-edges" of the former become the "carry-nodes" of the latter. (We note that our Young graph representation reverses the order of the the digit pairs associated with the edges since the formulation of palintiples used in this article involves finding the number that is obtained *after* multiplying by n.)



We point out that the kind of correspondence between $(2, 3)$ and $(3, 14)$-palintiples does not always exist. In particular, a $(\hat{n}, \hat{b})$-palintiple constructed from an (n, b)-palintiple does not always guarantee that the carries of any $(\hat{n}, \hat{b})$-palintiple will also be an (n, b)-palintiple. For instance, the $(9, 107)$-palintiple $(12, 40, 1)_{107}$ has carries $(3, 1, 0)$ whose nontrivial elements are the digits of the $(2, 5)$-palintiple $(3, 1)_5$ as seen in an earlier example. However, the $(9, 107)$-palintiple $(24, 80, 2)_{107}$ has carries

$(6,2,0)$ which are not the digits of a $(2,5)$-palintiple.

On the other hand, the family of $(5,39)$-palintiples can be constructed from $(2,5)$-palintiples. Consider the $(5,39)$-palintiple $(8,29,1)_{39}$ with carries $(3,1,0)$ whose nontrivial elements are again the digits of the $(2,5)$-palintiple $(3,1)_5$. The nontrivial carries of any $(5,39)$-palintiple are the digits of a $(2,5)$-palintiple and every $(2,5)$-palintiple is a nontrivial carry sequence of a $(5,39)$-palintiple.

This is all to say that, in general, the correspondence between derived palintiples and their palintiple carries can break down when $\hat{n} \neq b$. We therefore pose the question:

Suppose is a $(\hat{n},\hat{b})$-palintiple constructed from an (n,b)-palintiple. Under what conditions is it guaranteed that the carries of any $(\hat{n},\hat{b})$-palintiple will also be an (n,b)-palintiple? Is $\hat{n} = b$ such a condition?

Considering Hoey palintiples, it appears that not only $Y(3,14)$, but also $Y(3,22)$, and in general $Y(3,6+8\alpha)$ (see the example in Section 3) for all $\alpha \geq 1$ can be constructed from $Y(2,3)$. Moreover, it appears, using Kendrick's data [5], that every $Y(3,6+8\alpha)$ is isomorphic to $Y(3,14)$. In fact, not surprisingly, for every collection of 1089 (n,b)-palintiples we have checked, the Young graph of its corresponding Hoey $(b,\hat{b})$-palintiples is isomorphic to $Y(3,14)$. In this way, the isomorphism class determined by the 1089-graph, $[Y(9,10)]$, in a sense "generates" the isomorphism class $[Y(3,14)]$.

We note that not every element of $[Y(3,14)]$ is the Young graph of Hoey palintiples as Young graphs of ρ-Hoey palintiples also seem to be isomorphic to $Y(3,14)$. Furthermore, $[Y(3,14)]$ contains elements which are not Young graphs of Hoey or ρ-Hoey palintiples. The $(9,14)$-palintiple $(11,9,1,4,1)_9$ with carries $(2,1,6,7,0)$ demonstrates this. [2] These observations lead us to ask:

Are Young graphs of Hoey and ρ-Hoey palintiples always isomorphic to $Y(3,14)$? Do elements of $[Y(3,14)]$ for which the carries are not palintiple digits have any special properties?

Young graphs of Sutcliffe and ρ-Sutcliffe palintiples derived from shifted-symmetric palintiples whose Young graph is isomorphic to K_2, K_3, and K_4 all appear to be isomorphic to $Y(7,11)$. Of course, considering larger values of m and checking more cases may very well reveal other isomorphism classes. We therefore ask the following:

Are Young graphs of Sutcliffe and ρ-Sutcliffe palintiples always isomorphic to $Y(7,11)$?

[2] It is worth noting that although $(2,1,6,7)_b$ is not a palintiple in any base b, it is a base-9 permutiple: $(6,7,2,1)_9 = 4 \cdot (1,6,2,7)_9$ and $(7,2,1,6)_9 = 4 \cdot (1,7,2,6)_9$.

Additionally, for all cases we have checked, Young graphs of Pudwell and ρ-Pudwell palintiples derived from shifted-symmetric palintiples whose Young graph is isomorphic to K_2, K_3, and K_4 all seem to be isomorphic to $Y(5,8)$. Thus:

Are Young graphs of Pudwell and ρ-Pudwell palintiples always isomorphic to $Y(5,8)$?

It is not entirely unexpected that Young graphs of Hoey, Sutcliffe, and Pudwell palintiples should be isomorphic to Young graphs of their respective ρ-derived counterparts. On the other hand, it is not entirely obvious that this should always hold. In all cases considered so far it seems to be true.

Are Young graphs of derived palintiples always isomorphic to their ρ-derived counterparts?

Finally, Young graphs of $(\hat{n}, \hat{b})$-palintiples derived from (n, b)-palintiples for which $\hat{n} \neq b$ leave cases which have hardly yet been explored. We leave the reader to ponder the example of $(9, 107)$-palintiples considered earlier whose Young graph is isomorphic to $Y(25, 59)$. These palintiples are in some sense "partially" derived from $(2, 5)$-palintiples. We suspect that these nodes might make up a subgraph G which which is isomorphic to $Y(7, 11)$. The reader is likely to have noticed that other carries of $(9, 107)$-palintiples are sometimes doubles of $(2, 5)$-palintiples. Thus, $Y(9, 107)$ might contain another subgraph G' which is isomorphic to $Y(7, 11)$ but with nodes double those of G. Additional structure which may exist between these possible subgraphs is a matter of further inquiry and we leave these and other such questions to the inquisitive reader.

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